Coding for T-User Multiple-Access Channels

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Abstract—**Coding schemes for the binary memoryless T-user adder channel are investigated in this paper. First upper and lower bounds on the capacity sum, which are asymptotically tight with increasing T, are derived for the noiseless case. Second, a class of T-user uniquely decodable codes with rates, asymptotically in T, equal to the maximal achievable values is constructed. A decoding algorithm for these codes is also presented. Next, a class of error-correcting codes for the noisy T-user adder channel is constructed. It is shown that these codes can be used to construct multilevel codes suitable for use on the additive white Gaussian noise channel.**

I. Introduction

MULTIPLE-ACCESS communication systems were first studied by Shannon [1] in 1961. In 1971, Ahlswede [2] determined the capacity regions for the two-user and three-user multiple-access channels with independent sources, and van der Meulen [3] put forward a limiting expression and simple inner and outer bounds on the capacity region for the two-user multiple-access channel. In 1972, Liao [4] studied the general T-user multiple-access channel with independent sources. He formulated the capacity region for this channel and proved the fundamental coding theorem. Later, Slepian and Wolf [5] considered the case with correlated sources and discussed the continuous multiple-access channel. The Gaussian multiple-access channel was considered by Cover [6] and Wyner [7]. Ahlswede [8] extended the two-input one-output multiple-access case to two-input and two-output, and Ulrey [9] generalized the previous results to the arbitrary input, arbitrary-output case in 1975. For a single-user memoryless channel, it is known that feedback will not increase capacity, but Gaarder and Wolf [10], and later Cover and Leung-Yan-Cheong [11], showed that feedback will enlarge the capacity region of the two-user multiple-access channel. An extensive survey on the information-theoretic aspects of multiple-access channels has recently been assembled by van der Meulen [12], [13].

The coding problem for multiple-access channels has been investigated by several authors [14]–[18]. This early work concentrated on code construction for the two-user adder channel. In this paper we investigate block coding for the T-user binary adder channel, both with and without noise.

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Consider the multiple-access communication system depicted in Fig. 1 in which T statistically independent sources are attempting to transmit data to T separate destinations over a common discrete memoryless channel. The T messages $U_1, U_2, \cdots, U_T$ emanating from the T sources are encoded independently according to T block codes $C_1, C_2, \cdots, C_T$ of the same length N. Assume that the T encoders maintain bit and word synchronization. The T codewords $Z_1, Z_2, \cdots, Z_T$ emanating from the T encoders are combined by the channel into a single vector Z with symbols from a certain alphabet. The single decoder at the receiver processes the received vector Z and decodes it into T estimated messages $\hat{U}_1, \hat{U}_2, \cdots, \hat{U}_T$ for the T destinations.

The T codes $C_1, C_2, \cdots, C_T$ together are called a *T-user code* $(C_1, C_2, \cdots, C_T)$; each individual code is called a *constituent code*. Let M_i be the number of distinct codewords in code C_i. Assume that these codewords are equally likely. Then, the rate of the ith constituent code is

$$R_i = \frac{\log_2 M_i}{N}.$$

The *sum rate* $R_{\text{sum}}(T)$ of the T-user code $(C_1, C_2, \cdots, C_T)$ is defined as

$$R_{\text{sum}}(T) = R_1 + R_2 + \cdots + R_T.$$

The channel considered first in this paper is the T-input noiseless adder depicted in Fig. 2. Each user's input alphabet is the integer set $\{0,1\}$, and the output z is the sum of the T inputs $z_1, z_2, \cdots z_T$, i.e.,

$$z = z_1 + z_2 + \cdots + z_T,$$

where the plus sign denotes real addition. Therefore, each output symbol is an integer from the set $\{0,1,2,\cdots,T\}$. Adding noise to this output produces the noisy adder channel shown in Fig. 3.

A T-user code $(C_1, C_2, \cdots, C_T)$ is said to be *uniquely decodable* if all sums consisting of one codeword from each constituent code are distinct. There thus exists a decoder for the T-user code that never errs on the noiseless T-user adder channel if and only if the code is uniquely decodable. In Section II we present the capacity region of the noiseless T-user binary adder channel and determine the maximal achievable sum rate for T-user uniquely decodable codes. In Section III some basic properties of T-user uniquely decodable codes are derived. In Section IV we construct a class of T-user uniquely decodable codes with rates, asymptotically in T, equal to the maximal achievable value. We also exhibit a decoding algorithm. In Section V we study the noisy T-user adder channel and the multi-user additive white Gaussian noise

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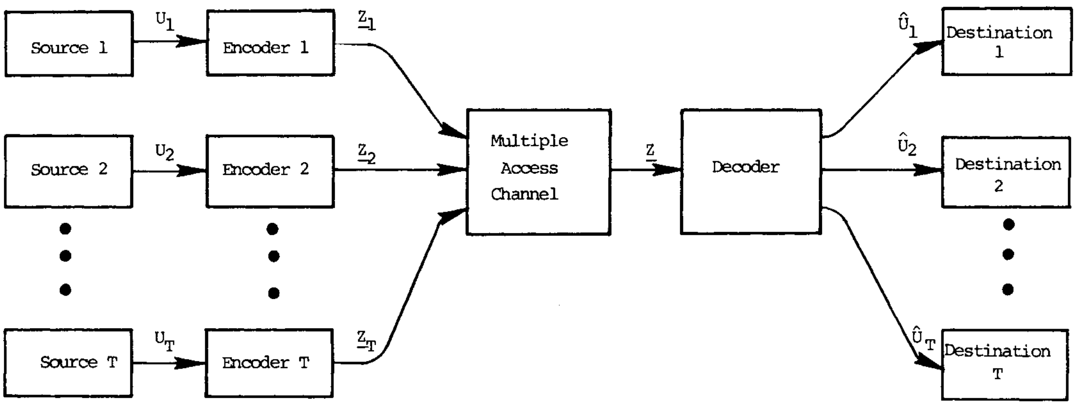

Fig. 1. Multiple access communication system.

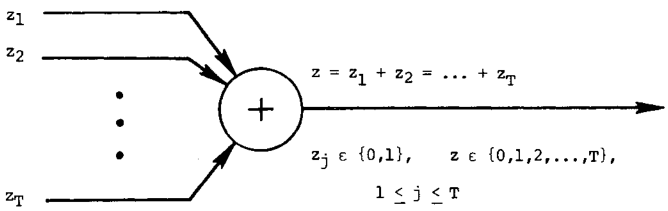

Fig. 2. Noiseless T-user binary adder channel.

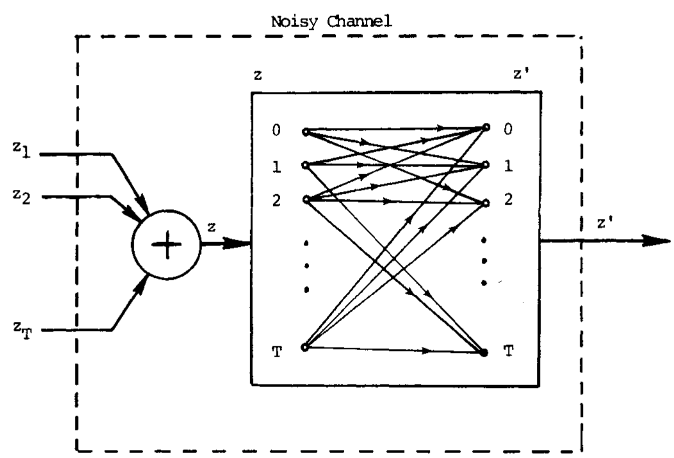

Fig. 3. Noisy T-user binary adder channel.

(AWGN) channel. Codes for these channels are constructed.

II. Capacity Calculation

The capacity region for the noiseless T-user binary adder channel can be calculated by applying the techniques proposed by Liao [4]. For this channel, the conditional probability between input and output is defined as follows:

$$P_{Z|Z_1,Z_2,\cdots,Z_T}(z|z_1,z_2,\cdots,z_T) = \begin{cases} 1, & \text{for } z=z_1+z_2+\cdots+z_T \\ 0, & \text{for } z\neq z_1+z_2+\cdots+z_T. \end{cases} \quad (2.1)$$

Using (2.1), we can calculate all conditional mutual informations between input and output. These mutual informations are simultaneously maximized when the Z_i are statistically independent and equally likely to be zero and one. This gives

$$I(Z_1;Z|Z_2,Z_3,\cdots,Z_T)=1$$

$$I(Z_1,Z_2;Z|Z_3,\cdots,Z_T)=\sum_{i=0}^{2}\frac{\binom{2}{i}}{2^2}\log_2\frac{2^2}{\binom{2}{i}} \quad (2.2)$$

$$\vdots$$

$$I(Z_1,Z_2,\cdots,Z_T;Z)=\sum_{i=0}^{T}\frac{\binom{T}{i}}{2^T}\log_2\frac{2^T}{\binom{T}{i}}.$$

From (2.2) we have the following theorem.

Theorem 2.1: The capacity region C of the noiseless T-user binary adder is

$$C=\Bigg\{(R_1,R_2,\cdots,R_T)|0\leq R_1\leq 1,$$

$$0\leq R_1+R_2\leq\sum_{i=0}^{2}\frac{\binom{2}{i}}{2^2}\log_2\frac{2^2}{\binom{2}{i}},$$

$$\vdots$$

$$0\leq R_1+R_2+\cdots+R_T\leq\sum_{i=0}^{T}\frac{\binom{T}{i}}{2^T}\log_2\frac{2^T}{\binom{T}{i}}\Bigg\}.$$

The capacity region for the three-user case is depicted in Fig. 4. Theorem 2.1 states that $R_1+R_2+\cdots+R_T=R_{\text{sum}}(T)\leq C_{\text{sum}}(T)$, where

$$C_{\text{sum}}(T)=\sum_{i=0}^{T}\frac{\binom{T}{i}}{2^T}\log_2\frac{2^T}{\binom{T}{i}}.$$

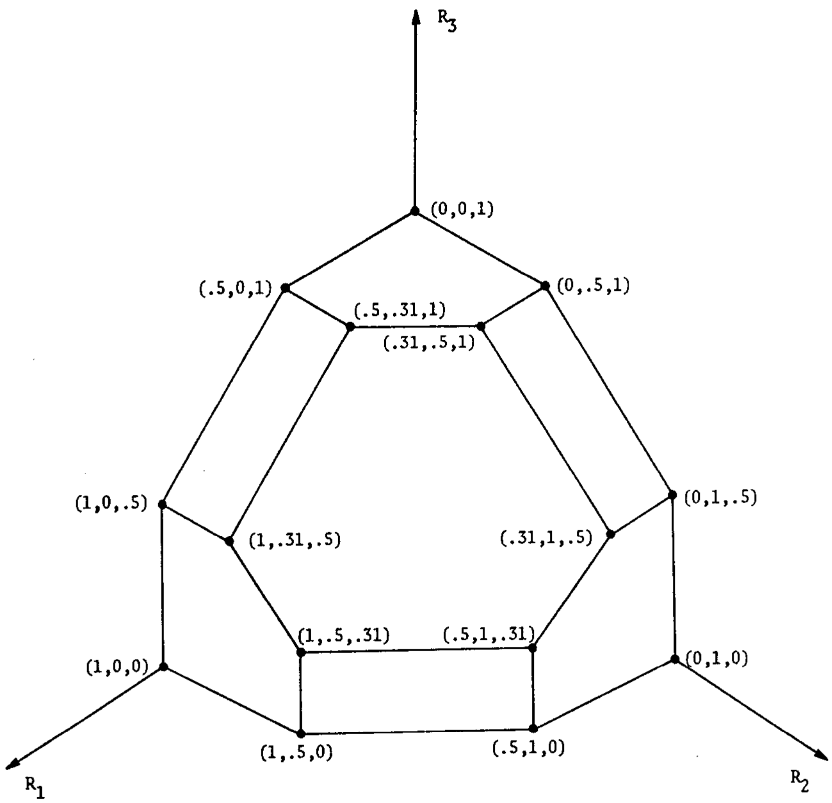

Fig. 4. Capacity region of noiseless three-user binary adder channel.

This value is the maximal achievable sum rate for T-user uniquely decodable codes. It plays an important role in code evaluation since the sum rate is a simple measure of the efficiency of a T-user code.

We have not found a closed-form expression for $C_{\text{sum}}(T)$. Wolf [22] has shown that the maximal achievable sum rate for this channel is approximately equal to $\frac{1}{2}\log_2 \pi e T/2$. The following lemmas show that this result is asymptotically tight with increasing T.

Lemma 2.1: $C_{\text{sum}}(T)$ is lower bounded by $\frac{1}{2}\log_2 \pi T/2$.

Lemma 2.2: $C_{\text{sum}}(T)$ is upper bounded by $\frac{1}{2}\log_2 \pi e T/2$ for T even, and by $\frac{1}{2}\log_2 \pi e(T+1)/2$ for T odd.

Based on computer results, we believe that the tighter bound $\frac{1}{2}\log_2 \pi e T/2$ also holds for T odd. But proofs of this conjecture still remain open.

It follows that $C_{\text{sum}}(T)$ is asymptotically equal to $\frac{1}{2}\log_2 \pi e T/2$ as T increases. (Two quantities, $g_1(t)$ and $g_2(t)$, are said to be asymptotically equal if their ratio approaches unity as $t \to \infty$.) This can be summarized as follows.

Theorem 2.2: The maximal achievable rate of a T-user uniquely decodable code for the binary T-user noiseless adder channel is asymptotically equal to $\frac{1}{2}\log_2 \pi e T/2$ with increasing T.

III. Properties of T-User Uniquely Decodable Codes

Consider a T-user code $(C_1, C_2, \cdots, C_T)$. Let $(\boldsymbol{Z}_1, \boldsymbol{Z}_2, \cdots, \boldsymbol{Z}_T)$ and $(\boldsymbol{Z}_1', \boldsymbol{Z}_2', \cdots, \boldsymbol{Z}_T')$ be two distinct sets of vectors with $\boldsymbol{Z}_i$ and $\boldsymbol{Z}_i' \in C_i$ for $1 \leqslant i \leqslant T$. Then the T-user code $(C_1, C_2, \cdots, C_T)$ is said to be *uniquely decodable* if and only if, for every such distinct pair $(\boldsymbol{Z}_1, \boldsymbol{Z}_2, \cdots, \boldsymbol{Z}_T)$ and $(\boldsymbol{Z}_1', \boldsymbol{Z}_2', \cdots, \boldsymbol{Z}_T')$,

$$\boldsymbol{Z}_1 + \boldsymbol{Z}_2 + \cdots + \boldsymbol{Z}_T \neq \boldsymbol{Z}_1' + \boldsymbol{Z}_2' + \cdots + \boldsymbol{Z}_T'$$

where the plus sign denotes real addition and the addition operation is performed componentwise. If the channel in the communication system of Fig. 1 is a noiseless T-user binary adder channel and the constituent codes $C_1, C_2, \cdots, C_T$ employed by the system form a uniquely decodable code, then the decoder is capable of decoding every possible received vector $\boldsymbol{Z}$ without ambiguity into the T codewords that were transmitted by the T encoders. Decoding can be achieved in principle by using a decoding table.

We want to construct T-user uniquely decodable codes with maximal achievable rates $R_{\text{sum}}(T)$. An interesting (and tractable) special case occurs when the constituent codes have equal rates, i.e., $R_1 = R_2 = \cdots = R_T$. In the simplest such case, each constituent code consists of exactly two codewords. For this case, the sum rate of the T-user code is equal to T/N. Obviously in order to achieve the maximal rate for a fixed T, we must minimize the code length N.

Consider a binary T-user uniquely decodable code $(C_1, C_2, \cdots, C_T)$ for which the ith constituent code contains two words, $\boldsymbol{X}_i$ and $\boldsymbol{Y}_i$, i.e., $C_i = \{\boldsymbol{X}_i, \boldsymbol{Y}_i\}$, $i = 1, 2, \cdots, T$. We call the vector

$$\boldsymbol{d}_i = \boldsymbol{X}_i - \boldsymbol{Y}_i$$

the *difference vector* of C_i, where the minus sign denotes real subtraction. Clearly, the difference vector $\boldsymbol{d}_i$ has components from the set $\{0,1,-1\}$. Now we form the $T \times N$ ternary matrix.

$$D=\begin{bmatrix} \boldsymbol{d}_1 \\ \boldsymbol{d}_2 \\ \vdots \\ \boldsymbol{d}_T \end{bmatrix}.$$

This matrix, which will be referred to as the *difference matrix* of the T-user code $(C_1, C_2, \cdots, C_T)$, plays a central role in the construction of T-user uniquely decodable codes.

It follows from the definition of unique decodability that a T-user code $(C_1, C_2, \cdots, C_T)$ is uniquely decodable if and only if, for any two distinct sets of vectors $(\boldsymbol{Z}_1, \boldsymbol{Z}_2, \cdots, \boldsymbol{Z}_T)$ and $(\boldsymbol{Z}_1', \boldsymbol{Z}_2', \cdots, \boldsymbol{Z}_T')$ in $(C_1, C_2, \cdots, C_T)$,

$$(\boldsymbol{Z}_1-\boldsymbol{Z}_1')+(\boldsymbol{Z}_2-\boldsymbol{Z}_2')+\cdots+(\boldsymbol{Z}_T-\boldsymbol{Z}_T') \neq \boldsymbol{0}^N \qquad (3.1)$$

where $\boldsymbol{0}^N$ is the all-zero N-tuple.

Let $\boldsymbol{m}=(m_1, m_2, \cdots, m_T)$ be a vector in $\{0,1,-1\}^T$. Since $\boldsymbol{Z}_i-\boldsymbol{Z}_i'$ is either $\boldsymbol{0}^N$, $\boldsymbol{d}_i$, or $-\boldsymbol{d}_i$, it follows from (3.1) that the T-user code $(C_1, C_2, \cdots, C_T)$ is uniquely decodable if and only if

$$m_1\boldsymbol{d}_1+m_2\boldsymbol{d}_2+\cdots+m_T\boldsymbol{d}_T=\boldsymbol{m}D \neq \boldsymbol{0}^N \qquad (3.2)$$

for any $\boldsymbol{m} \neq \boldsymbol{0}^T$. Thus we have immediately the following theorem.

Theorem 3.1: Let $(C_1, C_2, \cdots, C_T)$ be a T-user code of length N for which each constituent code contains two codewords. Let D be its difference matrix. Let $\boldsymbol{m}=(m_1, m_2, \cdots, m_T)$ be a T-tuple in $\{0,1,-1\}$. Then $(C_1, C_2, \cdots, C_T)$ is uniquely decodable if and only if

$$\boldsymbol{m}D=\boldsymbol{0}^N$$

implies that $\boldsymbol{m}$ is the all-zero T-tuple.

Given a $T \times N$ matrix D over $\{0,1,-1\}$ such that the rows of D are linearly independent over $\{0,1,-1\}$, it is possible to construct a T-user uniquely decodable code $(C_1, C_2, \cdots, C_T)$. In particular, the two vectors $\boldsymbol{X}_i$ and $\boldsymbol{Y}_i$ of the ith constituent code C_i are obtained from the ith row $\boldsymbol{d}_i$ of D in the following manner.

1) If the lth component of $\boldsymbol{d}_i$ is a "0", then the lth components of $\boldsymbol{X}_i$ and $\boldsymbol{Y}_i$ are arbitrarily set to "0".
2) If the lth component of $\boldsymbol{d}_i$ is a "1," then we set the lth component of $\boldsymbol{X}_i$ to "1" and the lth component of $\boldsymbol{Y}_i$ to "0".
3) If the lth component of $\boldsymbol{d}_i$ is a "-1," we set the lth component of $\boldsymbol{X}_i$ to "0" and the lth component of $\boldsymbol{Y}_i$ to "1."

From a given matrix D, we can construct more than one T-user uniquely decodable code, because the lth components of $\boldsymbol{X}_i$ and $\boldsymbol{Y}_i$ can be both set to either "0" or to "1" when the lth component of $\boldsymbol{d}_i$ is a "0". The T-user code constructed in the above manner will be said to be in *normal form*. All the T-user uniquely decodable codes constructed from a given D will be said to be *equivalent*.

Example: Consider the difference matrix

$$D_1=\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix}.$$

The three-user uniquely decodable code in normal form constructed from D_1 is

$$C_1=\{(11),(00)\}$$
$$C_2=\{(10),(01)\}$$
$$C_3=\{(10),(00)\}.$$

IV. Code Construction and Decoding for the Noiseless Adder Channel

We will now construct T-user uniquely decodable codes that achieve $C_{\text{sum}}(T)$ asymptotically. The idea of our iterative construction is based on annexing more columns (i.e., bits/word) to the difference matrix of a known uniquely decodable code, and simultaneously increasing the number of rows (i.e., users) such that the new matrix is a difference matrix for a uniquely decodable code with larger T.

The first (trivial) difference matrix is $D_0=[1]$, and the second is

$$D_1=\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{bmatrix},$$

which was seen in Section III to be the difference matrix for a three-user uniquely decodable code.

It happens that D_1 can be represented in terms of D_0 as follows:

$$D_1=\begin{bmatrix} D_0 & D_0 \\ D_0 & -D_0 \\ I_0 & 0_0 \end{bmatrix},$$

with $I_0=[1]$, $0_0=[0]$. The following theorem proves that this iterative construction works for any j.

Theorem 4.1: For any nonnegative integer j, the matrix

$$D_j=\begin{bmatrix} D_{j-1} & D_{j-1} \\ D_{j-1} & -D_{j-1} \\ I_{j-1} & 0_{j-1} \end{bmatrix} \qquad (4.1)$$

defines a $(j+2)\cdot 2^{j-1}$-user uniquely decodable code of length 2^j, where I_{j-1} is the 2^{j-1}-order identity matrix, 0_{j-1} is the $2^{j-1} \times 2^{j-1}$ zero matrix, and $D_0=[1]$.

Proof: The proof is by induction on j. For $j=0$, $D_0=[1]$ which specifies a trivial single-user (uniquely decodable) code of length 1. Assume that D_{j-1} defines a $2^{j-2}\cdot(j+1)$-user uniquely decodable code of length 2^{j-1}, $j \geqslant 1$. Now consider the $T_j \times N_j$ matrix D_j of (4.1). Clearly, $T_j=2\cdot[2^{j-2}\cdot(j+1)]+2^{j-1}=(j+2)\cdot 2^{j-1}$, and $N_j=2^{j-1}\cdot 2=2^j$. We must show that D_j is a difference matrix for a T_j-user uniquely decodable code of length N_j.

Let $m=(m_1,m_2,m_3)$ be a solution vector of $mD_j=0^{N_j}$ over $\{0,1,-1\}$, where $m_1,m_2\in\{0,1,-1\}^{T_{j-1}}$, $m_3\in\{0,1,-1\}^{N_{j-1}}$. From (4.1), we have

$$m_1D_{j-1}+m_2D_{j-1}+m_3=0^{N_{j-1}}, \tag{4.2a}$$

and

$$m_1D_{j-1}-m_2D_{j-1}=0^{N_{j-1}}, \tag{4.2b}$$

which reduces to

$$2m_1D_{j-1}=-m_3, \tag{4.3}$$

and

$$m_1D_{j-1}=m_2D_{j-1}.$$

The components of $2m_1D_{j-1}$ must be even integers or zero. Since the components of m_3 are elements of the set $\{0,1,-1\}$, it follows that $m_3=0^{N_{j-1}}$. Then (4.3) gives

$$m_1D_{j-1}=0^{N_{j-1}},$$
$$m_2D_{j-1}=0^{N_{j-1}}. \tag{4.4}$$

Since D_{j-1} is a difference matrix for a T_{j-1}-user uniquely decodable code, it follows from Theorem 3.1 that $m_1=m_2=0^{T_{j-1}}$. Thus $m=(m_1,m_2,m_3)=0^{T_j}$ and, by Theorem 3.1, D_j is a difference matrix of a T_j-user uniquely decodable code of length N_j. Q.E.D.

The rate $R_{\text{sum}}(T_j)$ of a code in the class of T_j-user uniquely decodable codes described in Theorem 4.1 is

$$R_{\text{sum}}(T_j)=\frac{T_j}{N_j}=1+\frac{j}{2}=1+\frac{1}{2}\log_2 N_j.$$

Since

$$\frac{1}{2}\log_2 T_j=\frac{1}{2}\log_2 N_j+\frac{1}{2}\log_2((\log_2 N_j)+2)-\frac{1}{2},$$

it follows that

$$\lim_{N_j\to\infty}\frac{R_{\text{sum}}(T_j)}{C_{\text{sum}}(T_j)}=1. \tag{4.5}$$

This implies the following.

Corollary: The T_j-user uniquely decodable code specified by Theorem 4.1 has a sum rate $R_{\text{sum}}(T_j)$ asymptotically equal to the maximal achievable sum rate $C_{\text{sum}}(T_j)$ as T_j increases.

Fig. 5 shows the sum rate $R_{\text{sum}}(T_j)$, plotted as a function of j, $j\geq 2$, as well as the upper and lower bounds on $C_{\text{sum}}(T)$, for T even.

Decoding a T-user uniquely decodable code can be accomplished in principle with a decoding table since there is a one-to-one correspondence between each received N-tuple and the only possible set of T transmitted codewords. As in the single user situation, however, the decoding table becomes unmanageably large even for modest values of N and T. What is needed is a simple and systematic means of calculating the transmitted vectors from the received vector. For the iterative codes of Theorem 4.1 this can be done as shown below.

Consider a T-user uniquely decodable iterative code $(C_1,C_2,\cdots,C_T)$. Let $Z_1\in C_1, Z_2\in C_2,\cdots,Z_T\in C_T$ be T inputs to the noiseless T-user adder channel, and let $Z=Z_1+Z_2+\cdots+Z_T$ be the corresponding channel output. Again, let $Y=Y_1+Y_2+\cdots+Y_T$; then we can represent the difference $S=Z-Y$ in terms of the T difference vectors, $d_i=X_i-Y_i$, $i=1,2,\cdots,T$,

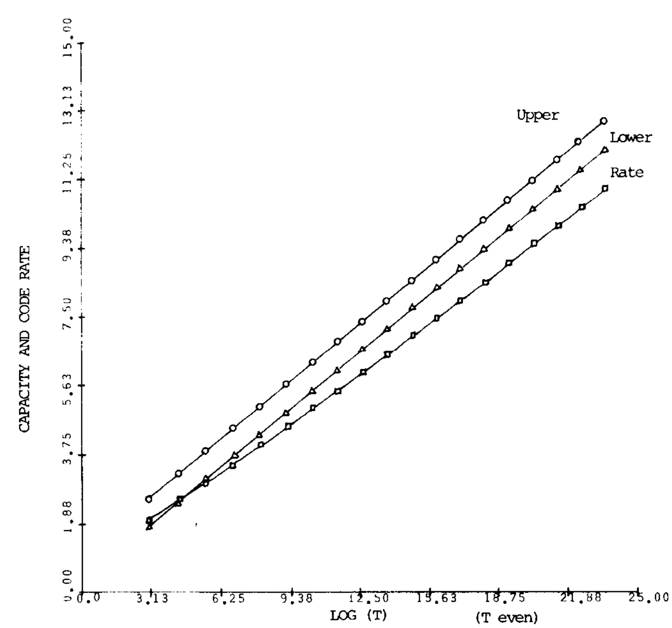

Fig. 5. Capacity bounds and T-user code rates of Theorem 4.1 for T even.

$$S=Z-Y=\sum_{i=1}^{T}(Z_i-Y_i)=\sum_{i=1}^{T}\lambda_i d_i$$

where $C_i=\{X_i,Y_i\}$ and where

$$\lambda_i=\begin{cases}1, & \text{if } Z_i=X_i,\\ 0, & \text{if } Z_i=Y_i.\end{cases}$$

This can be expressed in terms of the difference matrix D as follows,

$$S=Z-Y=\lambda D, \tag{4.6}$$

where $\lambda=(\lambda_1,\lambda_2,\cdots,\lambda_T)$.

Since Y is a fixed vector, each output Z has a vector S uniquely specified by Z. The role of S in decoding the multiple-access channel output is similar to that of a syndrome in the single-user situation; hence we will call S a "syndrome." Now the decoding problem is to find the solution vector λ over $\{0,1\}$ satisfying the equation $S=\lambda D$.

In Theorem 4.1 we used the iteration (4.1) to define a class of asymptotically good T-user codes. The decoding procedure presented here takes advantage of the structural symmetry of these codes. The basic idea is to decode the code with index j by first decoding the two subcodes with index $j-1$. Repeating this process j times decodes the T_j-user code.

Assume Z is a channel output and $S=Z-Y$ is the corresponding syndrome. Then Z can be decoded by using the following procedure to solve the equation $S=\lambda^{(j)}D_j$.

Let $S^{(j)}=S$ be the initial syndrome, i.e.,

$$S^{(j)}=S=Z-Y.$$

Then we partition $S^{(j)}$ into two 2^{j-1}-tuples as

$$S^{(j)}=(S_1^{(j-1)},S_2^{(j-1)}). \tag{4.7}$$

Let $\lambda^{(j)}$ be the solution vector of

$$\lambda^{(j)}D_j=S^{(j)}. \tag{4.8}$$

We partition the T_j-tuple $\lambda^{(j)}$ into three parts as

$$\lambda^{(j)}=(\lambda_1^{(j-1)},\lambda_2^{(j-1)},\lambda_3^{(j-1)}), \tag{4.9}$$

whose lengths are T_{j-1}, T_{j-1}, and 2^{j-1}, respectively.

Decoding Level 1: Equations (4.6), (4.7), (4.8), and (4.9) yield the following three key equations:

$$\lambda_3^{(j-1)}\equiv S_1^{(j-1)}+S_2^{(j-1)} \pmod 2 \tag{4.10}$$

$$\lambda_1^{(j-1)}D_{j-1}=\frac{S_1^{(j-1)}+S_2^{(j-1)}-\lambda_3^{(j-1)}}{2} \tag{4.11}$$

$$\lambda_2^{(j-1)}D_{j-1}=\frac{S_1^{(j-1)}-S_2^{(j-1)}-\lambda_3^{(j-1)}}{2}. \tag{4.12}$$

From (4.10) we can solve for $\lambda_3^{(j-1)}$. Then the right sides of (4.11) and (4.12) are known, and we will call them the lower order syndromes for the separate branches.

Decoding Level 2:

Branch 1: Let the new syndrome be $S^{(j-1)}$. According to (4.11), we have

$$S^{(j-1)}=\frac{S_1^{(j-1)}+S_2^{(j-1)}-\lambda_3^{(j-1)}}{2}=(S_1^{(j-2)},S_2^{(j-2)}).$$

Let

$$\lambda_1^{(j-1)}=(\lambda_1^{(j-2)},\lambda_2^{(j-2)},\lambda_3^{(j-2)})=\lambda^{(j-1)}.$$

Then (4.11) becomes

$$\lambda^{(j-1)}D_{j-1}=S^{(j-1)},$$

which is identical to (4.8) with j replaced by $j-1$.

Branch 2: Similarly, let the new syndrome be $S^{(j-1)}$. This time we use (4.12) to obtain

$$S^{(j-1)}=\frac{S_1^{(j-1)}-S_2^{(j-1)}-\lambda_3^{(j-1)}}{2}.$$

Proceeding as in Branch 1 again gives an equation identical to (4.8) with j replaced by $j-1$.

Decoding Level i ($2\leqslant i\leqslant j-1$): In general, we can repeat the above procedure by applying (4.10), (4.11), (4.12) to every branch with suitable replacements of the superscript i. The total number of decoding branches with known syndromes at the ith decoding level is equal to 2^{i-1}. Any branch at this level always has a known syndrome, let us say $S^{(j-i+1)}=(S_1^{(j-i)},S_2^{(j-i)})$, which is derived from the preceding level. Based on this syndrome, we can always decode a known 2^{j-i}-tuple $\lambda_3^{(j-i)}$ by the equation

$$\lambda_3^{(j-i)}\equiv S_1^{(j-i)}+S_2^{(j-i)} \pmod 2,$$

and derive two new branches. Each branch has its own syndrome; the first branch has a new syndrome

$$\frac{s_1^{(j-i)}+S_2^{(j-i)}-\lambda_3^{(j-i)}}{2}$$

and the second one has a new syndrome

$$\frac{S_1^{(i-j)}-S_2^{(i-j)}-\lambda_3^{(i-j)}}{2}.$$

Decoding Level j (The Final Level): At this level $\lambda_3^{(0)}$ has a single component, $\lambda_1^{(0)}$ and $\lambda_2^{(0)}$ are obtained from the following equations:

$$\lambda_1^{(0)}=\frac{S_1^{(0)}+S_2^{(0)}-\lambda_3^{(0)}}{2}$$

$$\lambda_2^{(0)}=\frac{S_1^{(0)}-S_2^{(0)}-\lambda_3^{(0)}}{2}.$$

Knowing $\lambda_1^{(0)}$ and $\lambda_2^{(0)}$, we can determine all of the higher order λ_1 and λ_2, thus concluding the decoding process.

V. Code Construction for the Noisy Adder Channel

So far we have concentrated on the construction of T-user uniquely decodable codes for the noiseless adder channel. Now we introduce noise. The noisy adder channel can be regarded as a noiseless adder channel cascaded with a discrete memoryless channel (DMC) having nonzero transition probability for all possible input–output pairs (i,j), $0\leqslant i\leqslant T$, $0\leqslant j\leqslant T$.

Let

$$Z=(z_1,z_2,\cdots,z_N)=\sum_{i=1}^{T} Z_i$$

be an N-tuple over the subset $\{0,1,2,\cdots,T\}$ of the real field R. Define Z' similarly with the constraint that the set of constituent binary codewords of Z', that is, $(Z_1',Z_2',\cdots,Z_T')$ are distinct from those of Z, that is, $(Z_1,Z_2,\cdots,Z_T)$. The N-tuples Z_j and Z_j' are codewords in C_j, the jth constituent code. The L-distance between Z and Z' is defined as follows:

$$d_L(Z,Z')=\sum_{i=1}^{N}|z_i-z_i'|=\|Z-Z'\| \tag{5.1}$$

where the minus sign denotes real subtraction and $|z_i-z_i'|$ denotes the absolute value of z_i-z_i'. Equation (5.1) defines the symbol $\|x\|$. It is easy to show that the L-distance is a metric [21].

The *minimum L-distance*, $d_{\min}$, of a T-user code is the smallest value of $d_L(Z,Z')$ over all $Z\neq Z'$.

The number of transmission errors is defined as the L-distance between the sum of the transmitted codewords Z and the channel output Z''. That is,

$$e(Z,Z'')=\|Z''-Z\|.$$

If a code has minimum distance $d_{\min}$ and distance is a metric, its error correcting capability is $[(d_{\min}-1)/2]$, cf. [19].

The error-correcting capability of a T-user code with two-word constituent codes is specified by the difference matrix of the T-user code. The following theorem relates the error correcting capability of such codes to the structure of the matrix D.

Theorem 5.1: A T-user code with 2 codewords per constituent code has minimum distance

$$d_{\min} = \min_{m} \|mD\| \tag{5.2}$$

where m is a nonzero T-tuple over $\{0,1,-1\}$.

Proof: By definition,

$$d_{\min} = \min_{Z\neq Z'} d_L(Z,Z') = \min_{Z\neq Z'} \|Z-Z'\|.$$

But from the definition of the matrix D, $Z-Z'=mD$ for some m over $\{0,1,-1\}$, $m\neq 0$, and conversely for any m there are vectors Z and Z' such that $Z-Z'=mD$. Q.E.D.

We now construct a class of T-user error-correcting codes for which the rate vector is above the time-sharing [1] hyperplane, i.e., $R_{\text{sum}}(T)>1$. A proof is furnished in the Appendix.

Theorem 5.2: Let $D_i^{(j)}$ be a matrix over $\{0,1,-1\}$ formed as follows:

$$D_i^{(j)} = \begin{bmatrix} D_{i-1}^{(j)} & D_{i-1}^{(j)} \\ D_{i-1}^{(j)} & -D_{i-1}^{(j)} \end{bmatrix}, \tag{5.3}$$

where $D_0^{(j)}$ is the matrix D_j of Theorem 4.1. Then $D_i^{(j)}$ defines a T-user code of length N and distance $d_{\min}$ where $T=(j+2)\cdot 2^{j-1+i}$, $N=2^{j+i}$, and $d_{\min}=2^i$. The sum rate of the code constructed by this theorem is

$$R_{\text{sum}}(T) = \frac{T}{N} = 1 + \frac{1}{2}\log\frac{N}{d_{\min}}. \tag{5.4}$$

Any binary T-user error-correcting code can be used to construct an S-user code, $S<T$, by grouping sets of binary codes together to form codes over larger alphabets. In particular, the codes of Theorem 5.2 can be used to form nonbinary codes as follows. Let $T=T_L\cdot L$ where T_L and L are integers. Then the T constituent codes can be partitioned into T_L sets, each having L binary codes. Let $C_{j1}, C_{j2}, \cdots, C_{jL}$ be the constituent codes of the jth set. Then these codes can be used to form an $(L+1)$-ary code F_j by taking a codeword in F_j to be the real sum of one codeword from each constituent code $C_{j1}, C_{j2}, \cdots, C_{jL}$. Each code F_j has 2^L codewords. Both the sum rate and minimum L-distance of the code are the same as those of the binary parent code. The above can be summarized as follows.

Theorem 5.3: The binary codes constructed in Theorem 5.2 can be used to construct an $(L+1)$-ary T_L-user code of length N with the following properties:

$$d_{\min} = 2^i$$

$$R_{\text{sum}} = 1 + \frac{1}{2}\log\frac{N}{d_{\min}}$$

provided $T=T_L\cdot L$, where T_L and L are both integers. The symbols i and N are defined in Theorem 5.2. These codes are suitable for use on the multi-user AWGN channels. Also, when $T_L=1$ we have a very interesting special case which will be discussed elsewhere.

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Appendix

Proof of Lemma 2.1

Since $\binom{T}{[T/2]} \geq \binom{T}{i}$ for all i, we have

$$\sum_{i=0}^{T} \frac{\binom{T}{i}}{2^T}\log_2\frac{2^T}{\binom{T}{i}} \geq \sum_{i=0}^{T}\frac{\binom{T}{i}}{2^T}\log\frac{2^T}{\binom{T}{\left[\frac{T}{2}\right]}} = \log_2\frac{2^T}{\binom{T}{\left[\frac{T}{2}\right]}}, \tag{A.1}$$

where $[T/2]$ denotes the greatest integer less than or equal to $T/2$. However,

$$\frac{2^T}{\binom{T}{\left[\frac{T}{2}\right]}} > \sqrt{\frac{\pi T}{2}} \qquad [19, \text{p. } 466]. \tag{A.2}$$

It follows from (A.1) and (A.2) that

$$\sum_{i=0}^{T}\frac{\binom{T}{i}}{2^T}\log_2\frac{2^T}{\binom{T}{i}} \geq \frac{1}{2}\log_2\frac{\pi T}{2}. \qquad \text{Q.E.D.}$$

Proof of Lemma 2.2

Let

$$P_i = \frac{\binom{T}{i}}{2^T}, \qquad a = \left[\frac{T+1}{2}\right],$$

and

$$L_i = \frac{1}{\sqrt{\pi a}}e^{-(i-a)^2/a}.$$

Then,

$$-\sum_{i=0}^{T} P_i\log_2 L_i = -\sum_{i=0}^{T} P_i\left\{-\frac{1}{2}\log_2(\pi a) - \frac{(i-a)^2}{a}\log_2 e\right\}$$
$$= \frac{1}{2}\log_2(\pi a) + \frac{1}{a}(\log_2 e)\left(\frac{a}{2}\right) = \frac{1}{2}\log_2(\pi e a). \tag{A.3}$$

Since

$$C_{\text{sum}}(T) = \sum_{i=0}^{T} P_i\log_2\frac{1}{P_i},$$

we have

$$C_{\text{sum}}(T)-\frac{1}{2}\log_2(\pi e a)=\sum_{i=0}^{T} P_i \log_2 \frac{L_i}{P_i}$$
$$\leq \sum_{i=0}^{T} P_i\left(\frac{L_i}{P_i}-1\right)\log_2 e$$
$$=(\log_2 e)\left\{\sum_{i=0}^{T} L_i - 1\right\}. \tag{A.4}$$

Consider all T and let

$$A=\sum_{i=0}^{T} L_i=\sum_{i=-a}^{T-a}\frac{e^{-i^2/a}}{\sqrt{\pi a}}$$
$$\leq \sum_{i=-a}^{a}\frac{e^{-i^2/a}}{\sqrt{\pi a}}. \tag{A.5}$$

It remains to show that $A<1$. From [20, (9.2)], we have

$$\frac{1}{\sqrt{\pi a}}\sum_{i=-\infty}^{\infty} e^{-i^2/a}=\sum_{i=-\infty}^{\infty} e^{-a\pi^2 i^2}$$
$$=1+2\sum_{i=1}^{\infty} e^{-a\pi^2 i^2}. \tag{A.6}$$

Therefore, from (A.5) and (A.6),

$$A\leq 1+2\sum_{i=1}^{\infty} e^{-a\pi^2 i^2}-\frac{2}{\sqrt{\pi a}}\sum_{i=a+1}^{\infty} e^{-i^2/a}$$
$$=1+2\sum_{i=1}^{\infty}\left\{e^{-a\pi^2 i^2}-e^{-\left[((i+a)^2/a)+\log_e\sqrt{\pi a}\,\right]}\right\}$$
$$=1+2\sum_{i=1}^{\infty}\left(e^{-E_{1i}}-e^{-E_{2i}}\right).$$

For $a\geq 1$, the value of i which minimizes $E_{1i}-E_{2i}$ is 1. Then for $i=1$, $E_{1i}-E_{2i}$ increases monotonically with a. Hence $E_{1i}-E_{2i}$ is minimized for $a=1$; here we have used

$$\pi^2>2^2+\log_e\sqrt{\pi}\,.$$

Hence, for all $i\geq 1$ and $a\geq 1$, $E_{1i}>E_{2i}$. Therefore, the right side of (A.4) is negative, and the lemma holds. Q.E.D.

Proof of Theorem 5.2

By Theorem 4.1, the matrix $D_0^{(j)}$ defines a code with parameters

$$T=(j+2)2^{j-1}$$
$$N=2^j$$
$$d_{\min}=2^0.$$

Now, we assume that the matrix $D_{i-1}^{(j)}$ defines a code with parameters

$$T=(j+2)2^{j-1+(i-1)}$$
$$N=2^{j+(i-1)}$$
$$d_{\min}=2^{i-1}.$$

Clearly, the matrix $D_i^{(j)}$ defines a $(j+2)2^{j-1+i}$ -user code with length 2^{j+i}. To complete the inductive proof, we must show that this code has minimum distance equal to 2^i.

Let $\boldsymbol{m}$ be a $(j+2)2^{j-1+(i-1)}$-tuple over $\{0,1,-1\}$. Partition $\boldsymbol{m}$ into two equal parts as $\boldsymbol{m}=(\boldsymbol{m}_1,\boldsymbol{m}_2)$. Consider the 2^{j+i}-tuple,

$$\boldsymbol{m}D_i^{(j)}=\left(\boldsymbol{m}_1 D_{i-1}^{(j)}+\boldsymbol{m}_2 D_{i-1}^{(j)},\boldsymbol{m}_1 D_{i-1}^{(j)}-\boldsymbol{m}_2 D_{i-1}^{(j)}\right).$$

Thus

$$\|\boldsymbol{m}D_i^{(j)}\|=\|\boldsymbol{m}_1 D_{i-1}^{(j)}+\boldsymbol{m}_2 D_{i-1}^{(j)}\|+\|\boldsymbol{m}_1 D_{i-1}^{(j)}-\boldsymbol{m}_2 D_{i-1}^{(j)}\|.$$

But since $\|a+b\|+\|a-b\|\geq 2\max(\|a\|,\|b\|)$ and since $\|\boldsymbol{m}_l D_{i-1}^{(j)}\|\geq 2^{i-1}$ for $l=1,2$, it follows that

$$\|\boldsymbol{m}D_i^{(j)}\|\geq 2^i.$$

It remains to be shown that there is a vector $\boldsymbol{m}$ such that $\|\boldsymbol{m}D_i^{(j)}\|=2^i$. It is easy to see that the bottom row of $D_0^{(j)}$ has a single one, that of $D_1^{(j)}$ has two ones, and that of D_i^j has 2^i ones. Hence the vector $\boldsymbol{m}=(00\cdots 01)$ has the desired property. Q.E.D.

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