BC-MAC Duality and Capacity Computation for the Binary MAC

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Outline

Duality for Gaussian SISO channels

Gaussian SISO Broadcast Channel (Downlink)

1 sender, K receiver; received symbol at user j is

$$Y_j = \sqrt{h_j} X + n_j,$$

noise $n_j \sim \mathcal{N}(0, \sigma^2)$; power constraint P

$\Rightarrow$ Capacity region $\mathcal{C}_{\text{BC}}(P, \vec{h})$

Gaussian SISO Multiple Access Channel (Uplink)

K sender, 1 receiver; received symbol is

$$Y = \sum_{j=1}^{K} \sqrt{h_j} X_j + n,$$

noise $n \sim \mathcal{N}(0, \sigma^2)$; power constraints $\vec{P} = (P_1, \ldots, P_K)$

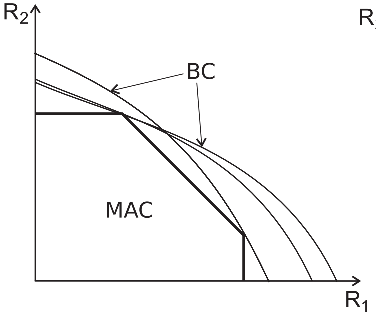
$\Rightarrow$ Capacity region $\mathcal{C}_{\text{MAC}}(\vec{P}, \vec{h})$

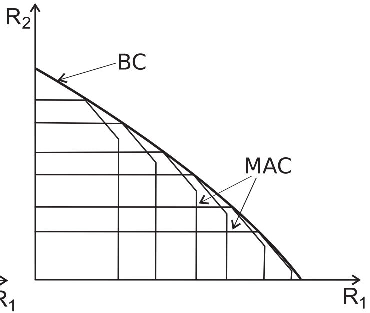
Duality for Gaussian SISO channels

Duality relations [Jindal Vishwanath Goldsmith,2004]

$$\mathcal{C}_{\text{BC}}(P,\vec{h}) = \bigcup_{\vec{P}:\sum_i P_i = P} \mathcal{C}_{\text{MAC}}(\vec{P},\vec{h})$$

$$\mathcal{C}_{\text{MAC}}(\vec{P},\vec{h}) = \bigcap_{\vec{\alpha}:\alpha_i>0} \mathcal{C}_{\text{BC}}\left(\sum_{i=1}^{K}\frac{P_i}{\alpha_i},(\alpha_1 h_1,\ldots,\alpha_K h_K)\right)$$





Duality MAC/BC

Duality extends to the fading channel, also to MIMO channels (and some others)

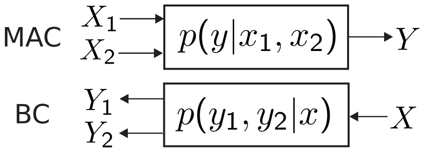
Duality is an interesting property because:

- Problems in BC often more difficult to handle $\Rightarrow$ transform to MAC, solve there, and transform back again
- Example: Computation of optimal transmit covariance matrices for MIMO BC
- Insight into structural properties of problems in network information theory?

Does duality hold for other channel models? Is there a deeper information-theoretic principle behind?

Duality for DMC?

Discrete memoryless BC/MAC:



Duality for DMC?

$$\mathcal{C}_{\mathsf{BC}}(V) = \bigcup_{W \in \mathcal{W}} \mathcal{C}_{\mathsf{MAC}}(W) \;\; ?$$

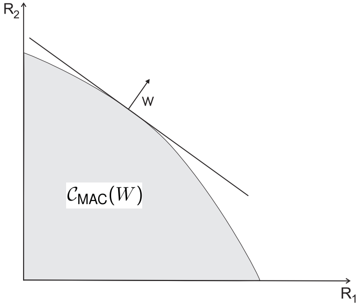
- General problem is difficult since BC region is unknown $\Rightarrow$ focus on degraded channels (where capacity region is known)
- Duality relationship was found for a class of deterministic channels
- Examples for deterministic channels where duality does not hold [Jindal Vishwanath Goldsmith]

Capacity computation for the DMC MAC

Characterization / way of calculating the capacity boundary for the DMC MAC might be useful

For a weight vector $\mathbf{w}$, solve

$$\max_{\mathbf{x} \in \mathcal{C}_{\text{MAC}}(W)} \mathbf{w}^T \mathbf{x}.$$



- Difficult nonconvex optimization problem (in general)
- Algorithms are known for computation of sum-capacity (total capacity) [Razaeian Grant,2004]

The two-user elementary MAC (EMAC)

- DMC MAC has finite decomposition into "elementary MACs"
- Capacity calculation of capacity regions of elementary MACs is sufficient [Watanabe Kamoi,2004]

EMAC

Two-user case: elementary MAC has binary input alphabet and binary output alphabet (EMAC) $\Rightarrow$ channel matrix

$$W = \begin{pmatrix} a & b & c & d \\ 1-a & 1-b & 1-c & 1-d \end{pmatrix}$$

where

$a = p(0|0,0), b = p(0|0,1), c = p(0|1,0), d = p(0|1,1)$

The two-user elementary MAC (EMAC)

Capacity region of the EMAC

$$\mathcal{C}_{\text{EMAC}}(W) = \text{Co}\left(\bigcup_{0 \le p_1, p_2 \le 1} R_{\text{p}}(W, p_1, p_2) \right)$$

where $R_{\text{p}}(W, p_1, p_2)$ is the set of all rate pairs (R_1, R_2) that satisfy the "pentagon" equations

$$\begin{aligned} R_1 &\le I(X_1; Y | X_2) \\ R_2 &\le I(X_2; Y | X_1) \\ R_1 + R_2 &\le I(Y; X_1, X_2) \end{aligned}$$

The two-user elementary MAC (EMAC)

Corner points of pentagon (successive interference cancellation points):

$$C_1(p_1,p_2) = (I(Y;X_1), I(Y;X_2|X_1))^T,$$
$$C_2(p_1,p_2) = (I(Y;X_1|X_2), I(Y;X_2))^T.$$

EMAC Optimization Problem

For $0 \leq \theta \leq \frac{1}{2}$:

$$\max_{0 \leq p_1,p_2 \leq 1} \Psi(\theta, p_1, p_2),$$

where

$$\Psi(\theta, p_1, p_2) = \theta I(Y;X_1)_{p_1,p_2} + (1-\theta) I(Y;X_2|X_1)_{p_1,p_2}$$

The two-user elementary MAC (EMAC)

EMAC Optimization Problem

$$\max_{0 \leq p_1, p_2 \leq 1} \Psi(\theta, p_1, p_2),$$

For channels of the form

$$W = \begin{pmatrix} a & a & c & d \\ 1-a & 1-a & 1-c & 1-d \end{pmatrix}$$

and for $0 \leq \theta \leq \frac{1}{2}$:

Objective function has at most one stationary point

Problem can be reduced to a one-dimensional pseudoconcave maximization problem

General EMAC: Difficulties when pentagon union is not convex; occurence of **saddle points**

The two-user elementary MAC (EMAC)

2EMAC

Two-parameter EMAC (2EMAC):

$$W = \begin{pmatrix} 0 & 0 & c & d \\ 1 & 1 & 1-c & 1-d \end{pmatrix}$$

$\Rightarrow$ we can find **closed-form optimal solution**

BSBC

Binary symmetric BC with error probability parameters $0 \le s < t < 1/2$: Boundary of capacity region $\mathcal{C}_{\text{SBBC}}(s,t)$ given by

$$\left\{ (H(\beta * s) - H(s), 1 - H(\beta * t))^T : \beta \in [0, 0.5] \right\}.$$

where $a * b := a(1-b) + (1-a)b$

Duality BSBC -> 2EMAC

Employing Newton-Raphson method indicates that

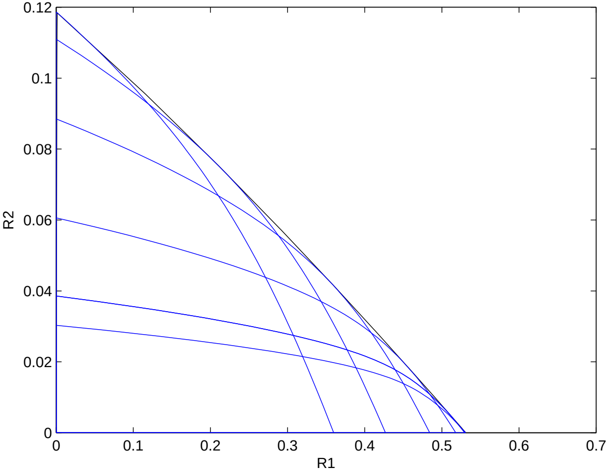
Duality BSBC -> 2EMAC

For every symmetric BBC with parameters $0 \leq s < t < 1/2$, there exists a set $\mathcal{D}$ of 2EMAC channel parameters such that

$$\bigcup_{(c,d)\in\mathcal{D}} \mathcal{C}_{\text{2EMAC}}(c,d) = \mathcal{C}_{\text{SBBC}}(s,t).$$

Duality BSBC -> 2EMAC

Employing Newton-Raphson method indicates that
Example: $s = 0.1, t = 0.2$



Summary

- Duality of BC and MAC is an interesting property, both from information-theoretic and algorithmic view
- More channel models should be investigated for duality properties, especially the DMC
- Algorithms calculating capacity boundaries might help
- For the two-user DMC MAC, the EMAC (elementary MAC) is of special interest in terms of boundary calculations
- Despite the simplicity of the channel model, the problem is difficult, but partial solutions exist already

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Thank you for your attention