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LULEÅ UNIVERSITY OF TECHNOLOGY

A Discrete Model for the Efficient Analysis of Time-Varying Narrowband Communication Channels

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Riesz bases...

$L^2(\mathbb{R})$ is the set of square integrable functions, with *norm*

$$\| u \| = \left(\int_{\mathbb{R}} |u(x)|^2 \, dx \right)^{1/2} < \infty$$

and *inner product*

$$\langle u, v \rangle = \int_{\mathbb{R}} u(x)\overline{v(x)} dx.$$

A basis $\{e_k\}$ for $L^2(\mathbb{R})$ is called a *Riesz basis* for $L^2(\mathbb{R})$ if there are some $A, B > 0$ such that for all $s \in L^2(\mathbb{R})$,

$$A \sum_k |\langle s, e_k \rangle|^2 \leq \| s \|^2 \leq B \sum_k |\langle s, e_k \rangle|^2.$$

...provide numerically stable series expansions

To every Riesz basis $\{e_k\}$ corresponds a *dual/biorthogonal* basis $\{\widetilde{e_k}\}$ such that all $s \in L^2(\mathbb{R})$ have series expansions

$$s = \sum_k \langle s, e_k \rangle \widetilde{e_k} = \sum_k \langle s, \widetilde{e_k} \rangle e_k$$

Gabor basis:

$g_{q,r}(t) = e^{i2\pi bqt} g(t - ar) =$ the qth frequency component in OFDM/DMT symbol number r.

Matrix representation:

Matched filter

$$s = \sum_{q,r} c_{q,r} g_{q,r} \quad \xrightarrow{\;H\;} \quad Hs = \sum_{q',r'} \langle Hs, \gamma_{q',r'} \rangle \widetilde{\gamma_{q',r'}} = \sum_{q',r'} a_{q',r'} \widetilde{\gamma_{q',r'}}$$

$$a_{q',r'} = \langle \mathrm{H}s, \gamma_{q',r'} \rangle = \sum_{q,r} c_{q,r} \langle \mathrm{H} g_{q,r}, \gamma_{q',r'} \rangle$$

$$\mathbf{a} = \mathbf{G}\mathbf{c}, \qquad \mathbf{G}_{q',r';q,r} = \langle H g_{q,r}, \gamma_{q',r'} \rangle$$

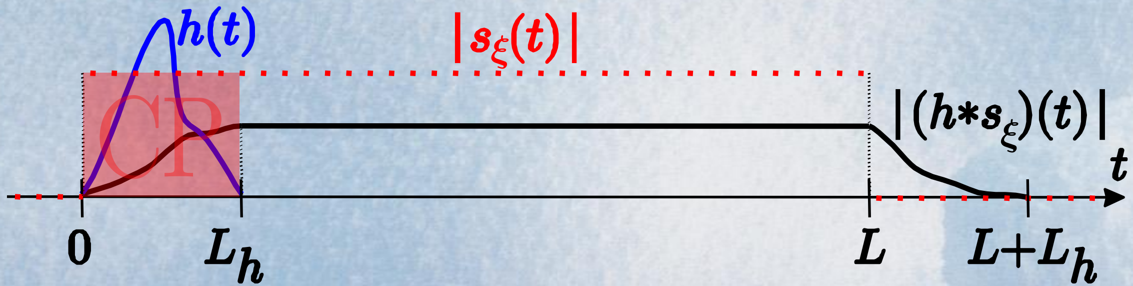
Channel matrix

Goal: approximate diagonalization

$\mathbf{a} = \mathbf{G}\mathbf{c}, \ \mathbf{G}_{q',r';q,r} = \langle H g_{q,r}, \gamma_{q',r'} \rangle$ **Channel matrix**

- For *time-invariant* channels, $H(s(\cdot - T)) = (Hs)(\cdot - T)$ so that

$$s_\xi(t) = \chi_{[0,L]}(t) e^{i2\pi\xi t} \ \Rightarrow \ H s_\xi(t) = \lambda_\xi s_\xi(t) \text{ for } t \in [L_h, L]$$



$$g = \chi_{[0,L]}, \ \gamma = \chi_{[L_h,L]}, \ L, a, b = \cdots \ \Rightarrow \ \text{diagonal channel matrix G}$$

$$\Rightarrow \ \mathbf{c} = \mathbf{G}^{-1}\mathbf{a} \text{ for all time-invariant } H$$

- *Wireless channels*: time-varying $\Rightarrow$ only approximate diagonalization
  $\Rightarrow$ need for an efficient algorithm for computing G for numerical comparisons of diagonalization properties of different g, γ on a given H.

Direct calculation of the channel matrix

Q carrier frequencies and R OFDM symbols.

- $\geq Q$ samples of each basis function $g_{q,r}$.
- Compute $Hg_{q,r}$ with $Q \times Q$ matrix.
- This takes $\mathcal{O}(Q^2)$ arithmetic operations each for QR basis functions $g_{q,r}$.
- The inner product $\langle Hg_{q,r}, \gamma_{q',r'} \rangle$ takes $\mathcal{O}(Q)$ arithmetic operations each for QR basis functions $\gamma_{q',r'}$.

Altogether: $R^2 \cdot \mathcal{O}(Q^5)$ arithmetic operations.

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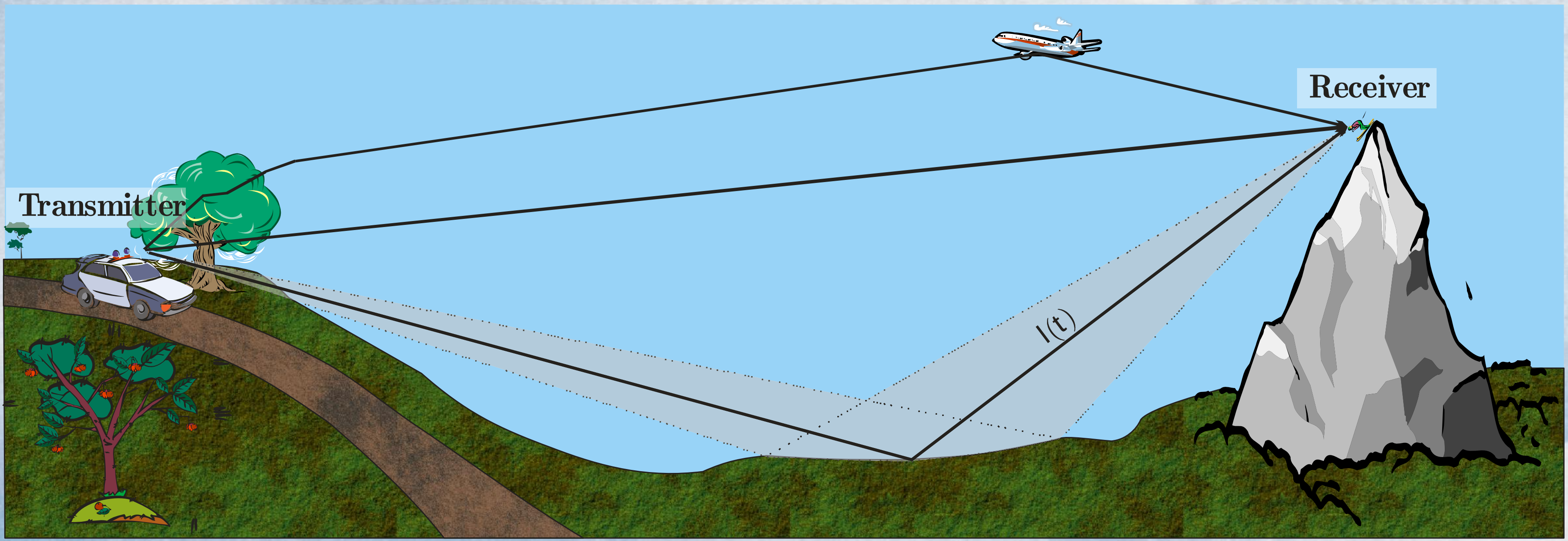
The multipath propagation model



$$Hs(t_0) = \iint_{\mathbb{R}\,\mathbb{R}} S_H(\nu, t) e^{j2\pi\nu(t_0 - t)} s(t_0 - t) d\nu dt = \int_{\mathbb{R}} h(t_0, t) s(t_0 - t) dt$$

S_H = Spreading function $\qquad$ h = time-varying impulse response

To which function(al) space should S_H belong?

- $Hg_{q,r}(t_0) = \int_{\mathbb{R}}\int_{\mathbb{R}} S_H(\nu,t)e^{j2\pi\nu(t_0-t)}g_{q,r}(t_0-t)d\nu dt = \int_{\mathbb{R}} h(t_0,t)g_{q,r}(t_0-t)dt$
- ***Common engineering practice***: Model H to be a Hilbert-Schmidt operator, that is, $S_H \in L^2(\mathbb{R}\times\mathbb{R})$. Only standard Fourier analysis necessary, hence no distribution theory and therefore more accessible to engineers.
- ***Reasonable objection***: "A good channel operator model should include (small perturbations of) the identity operator ($S_H = \partial$) and time invariant operators ($h(t_0,t) = h(0,t)$ and $S_H(\nu,t) = \delta(\nu)h(0,t)$). Such operators are not compact and therefore not Hilbert-Schmidt."

Useful channel and basis function properties

- $Hg_{q,r}(t_0) = \iint\limits_{\mathbb{R}\,\mathbb{R}} S_H(\nu,t)e^{j2\pi\nu(t_0-t)}g_{q,r}(t_0-t)d\nu dt = \int\limits_{\mathbb{R}} h(t_0,t)g_{q,r}(t_0-t)dt$
- g very well TF-localized $\Rightarrow \tilde{g}$ very well TF-localized
- We are modelling the *short - time behaviour* of the channel,

  thus we can do a smooth cut-off h to compact rectangular support.

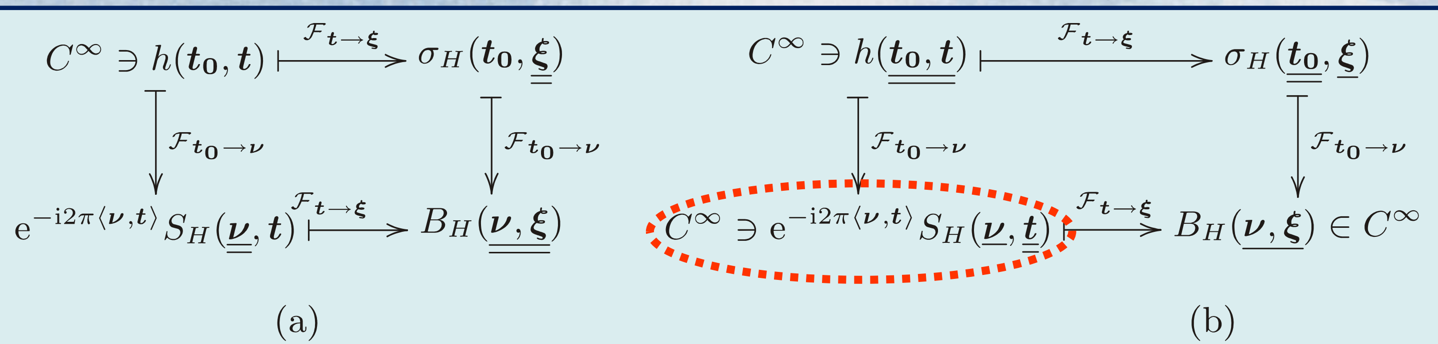


Fig. 3 (a) Bandlimiting properties of the physical channel provides a compactly supported B_H. Thus $h \in C^\infty$, but S_H may be a tempered distribution, for example, if $h(\boldsymbol{t_0},\boldsymbol{t}) = h(\boldsymbol{0},\boldsymbol{t})$. (b) We get a *finite lifelength* channel with well localized $S_H \in C^\infty$ from a smooth truncation of h such that the shape of S_H still is close to the exponential decay with $\boldsymbol{t}$ in (19c). In (a) and (b) we denote with underlining subexponential decay and compact support.

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Resulting formulas

Proposition : $\langle Hg_{q,r}, \gamma_{q',r'} \rangle = \sum_{k \in \mathcal{K}} Hg_{q,r}(kT) \overline{\mathrm{BPF}_{g_{q,r}} \gamma_{q',r'}(kT)}$

Proposition : $\mathrm{H}\, g_{q,r}(kT) = T_0 \sum_{m \in \mathcal{M}} g_{q,r}(mT) \mathcal{F}\left\{ S_{\mathrm{H}}^{\Omega_c,\Omega}(\cdot, kT - mT_0) \right\}(-mT_0)$

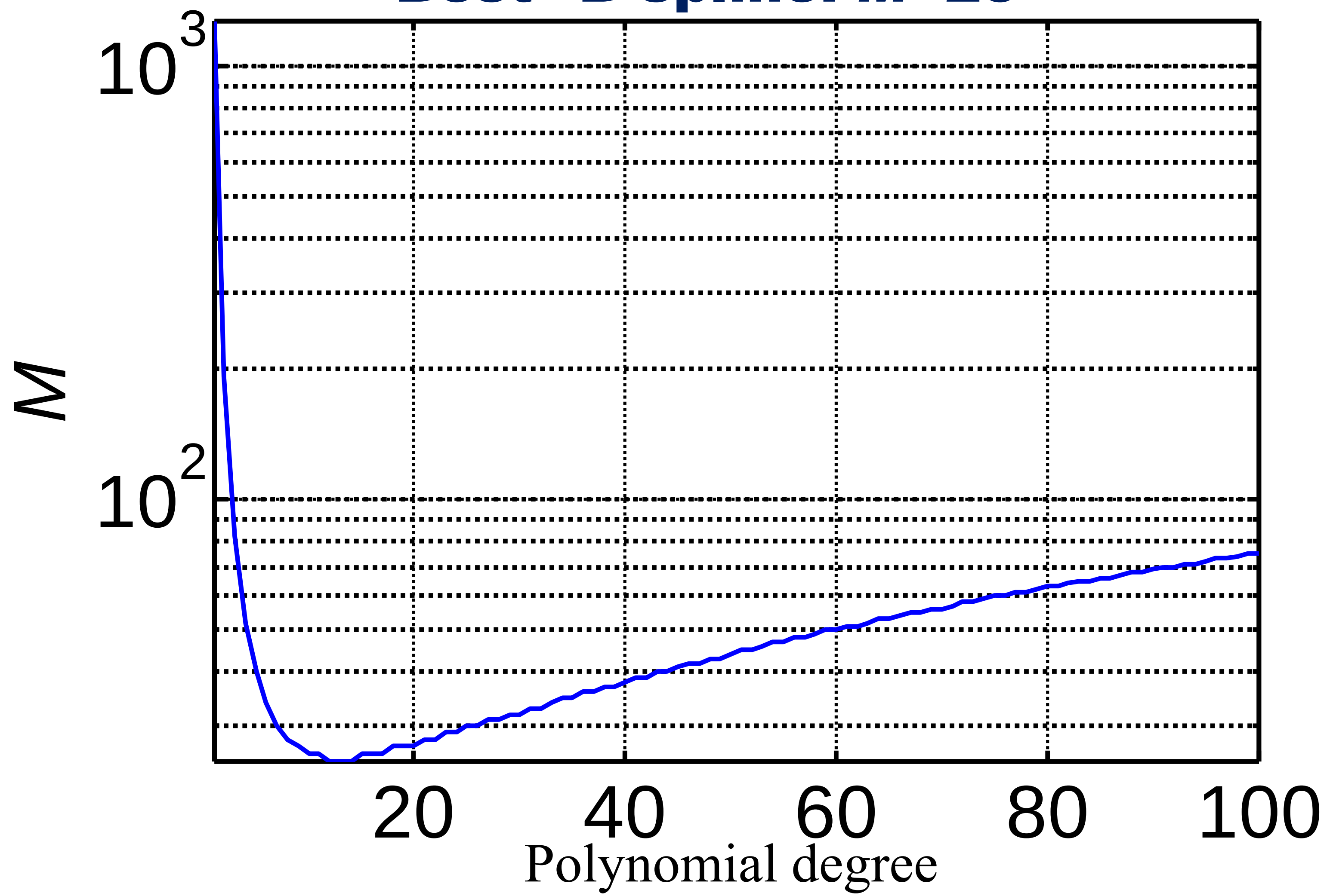
Proposition

$$\mathcal{F}\left\{ S_{\mathrm{H}}^{\Omega_c,\Omega}(\cdot, t) \right\}(t_0) = \omega_0 T'' \chi_{I_{C_0,L_0}}(t - t_0) \sum_{p \in \mathcal{P}} e^{i2\pi \langle \Omega_{c,q} + \omega_c, t - pT'' \rangle} \mathrm{sinc}_{\Omega}(t - pT'') \sum_{\mathrm{n} \in \mathcal{N}} S_{n,p} e^{i2\pi(t - t_0 - p \cdot T'')n\omega_0}$$

Complexity :

If g and γ have M nonzero Nyquist frequency samples, then the channel matrix for Q carrier frequencies and R OFDM symbols can be computed in $R^2 \cdot \mathcal{O}(M^2 \cdot Q^2)$ arithmetic operations ($< R^2 \cdot \mathcal{O}(Q^5)$ with the "naive approach").

“Best” B-spline: *M*=25



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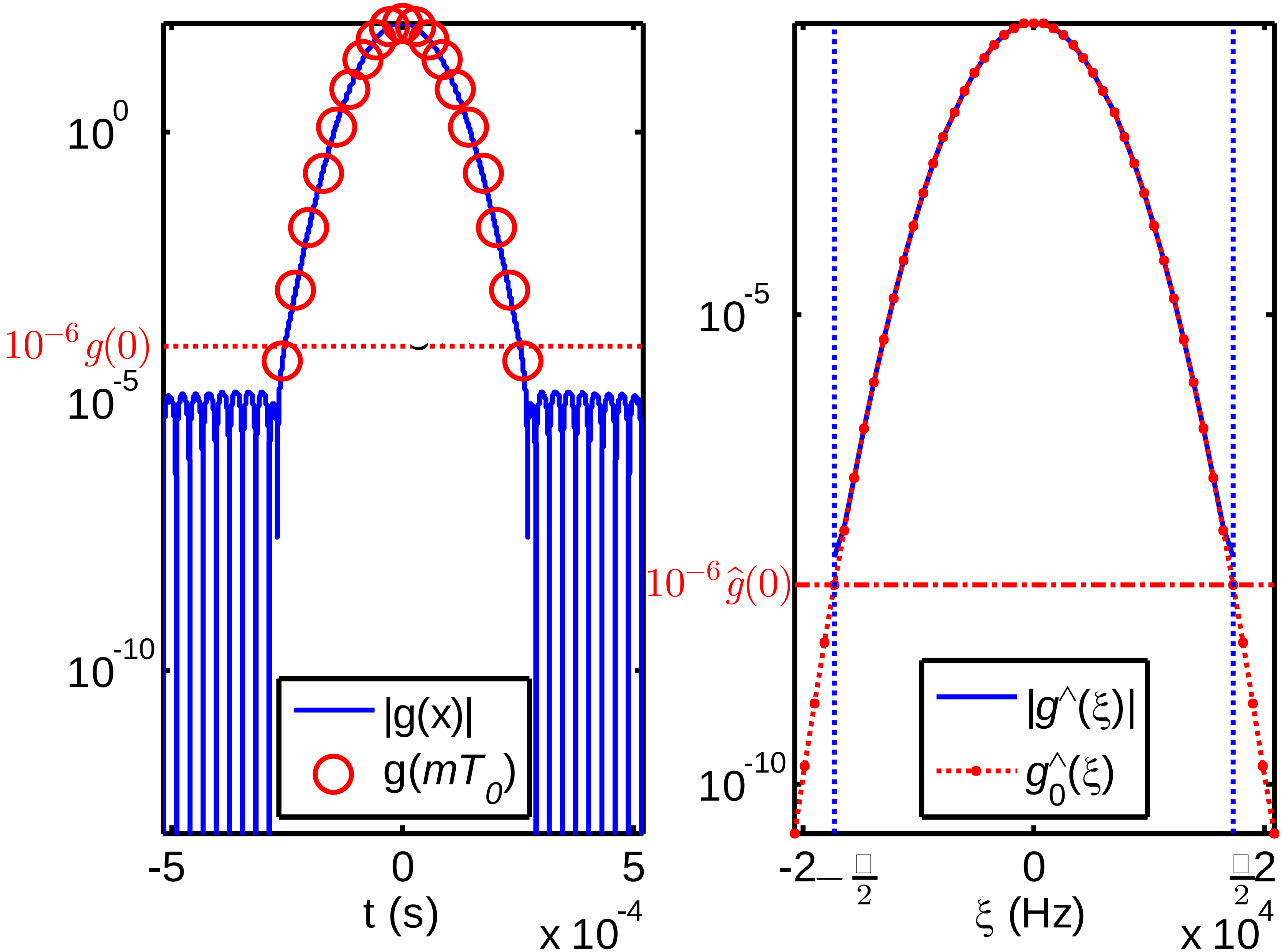
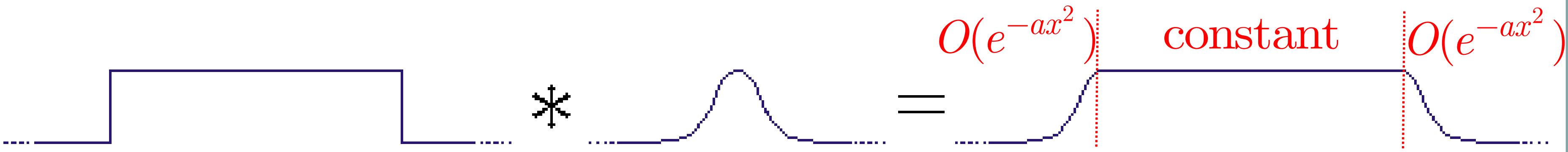
"Gaussian" window: *M*=19



Box convoluted with Gaussian window



In the following plots: ”Gaussian” window

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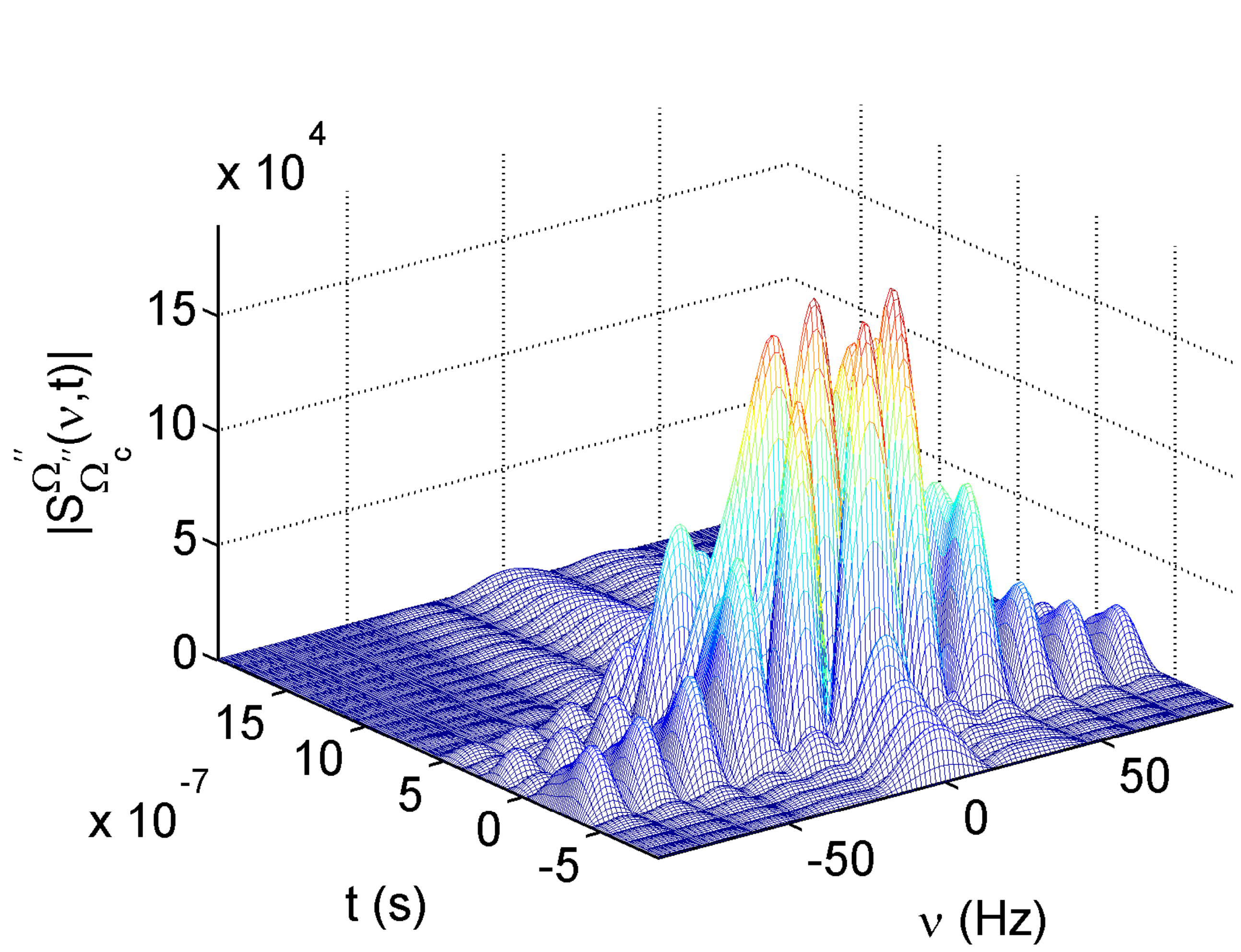
OFDM 1990 band example: Spreading function



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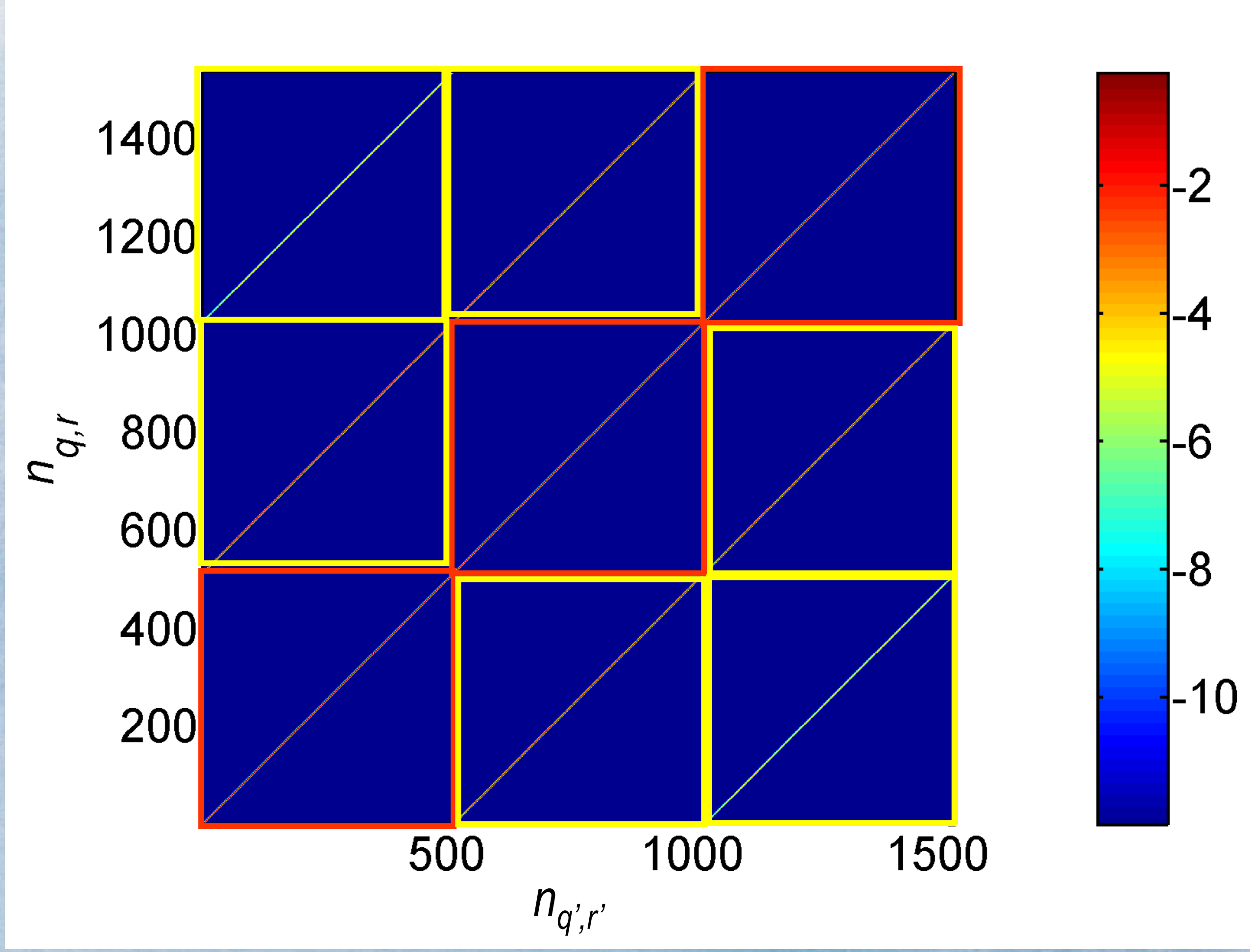
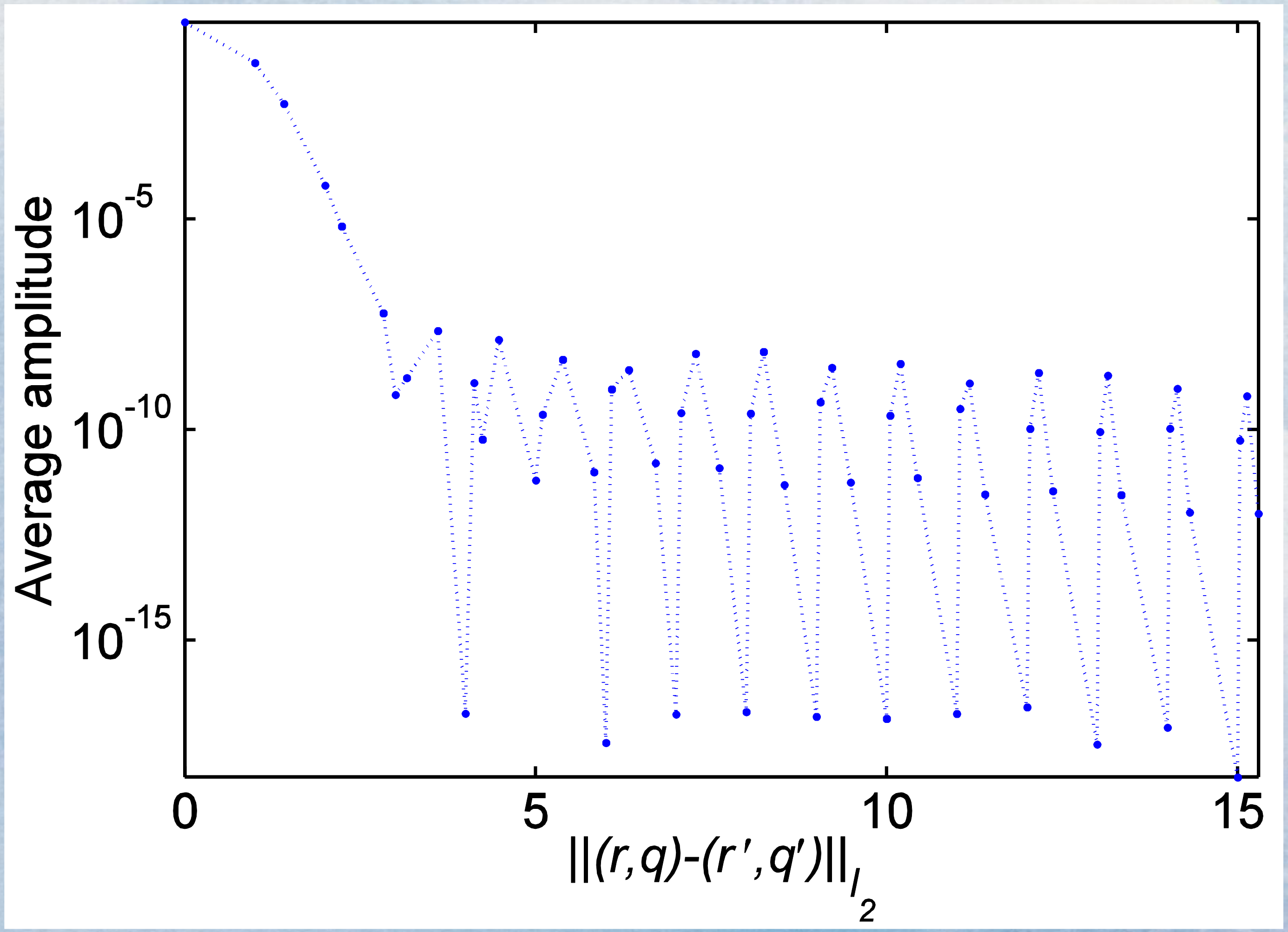
OFDM 1990 band example: $\log_{10}$ Channel matrix

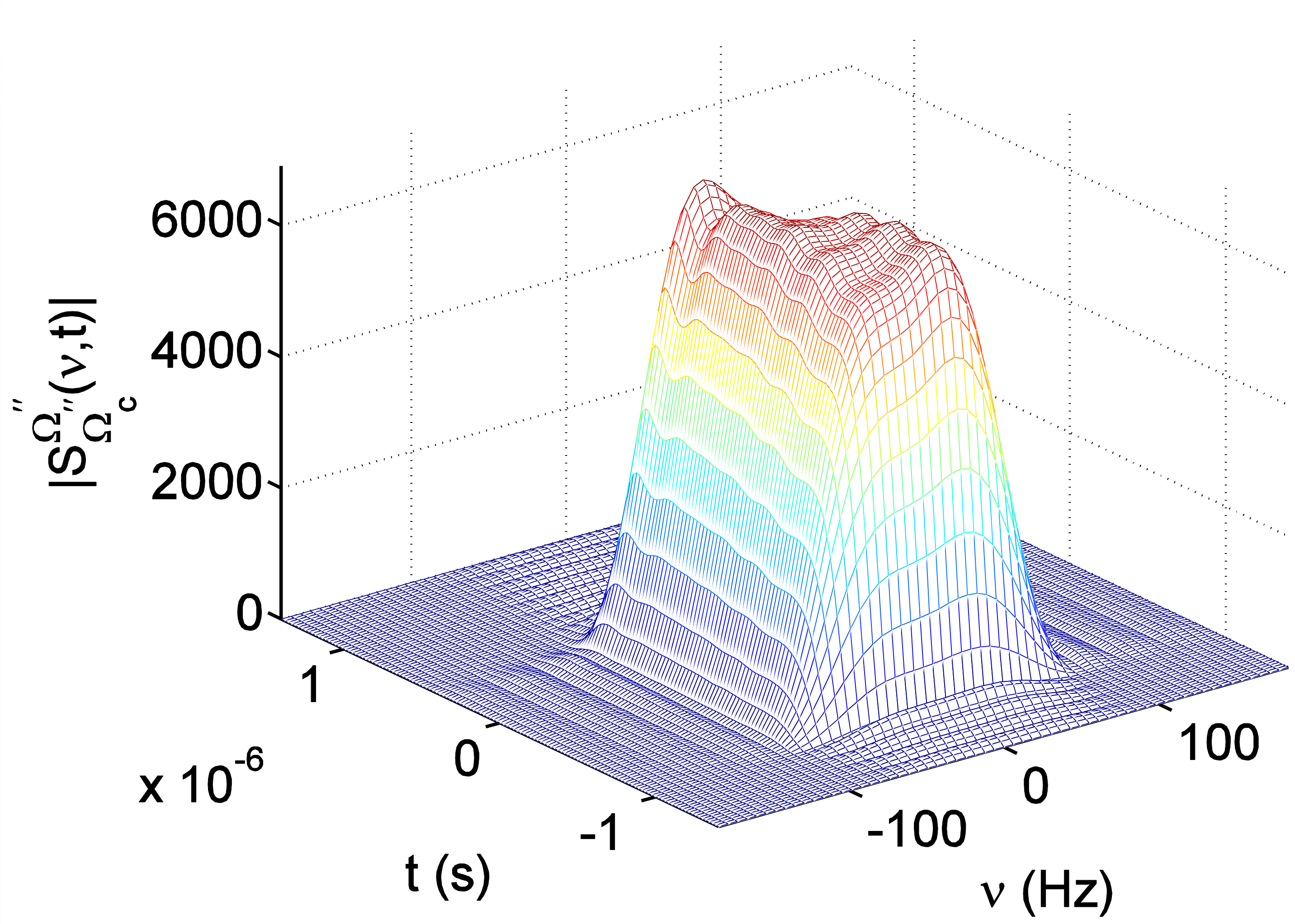
$$G_{(q,r);(q',r')} = \langle Hg_{q,r}, \gamma_{q',r'} \rangle$$



OFDM 1990 band example: Off-diagonal decay



OFDM 1990 example: Smoother spreading function



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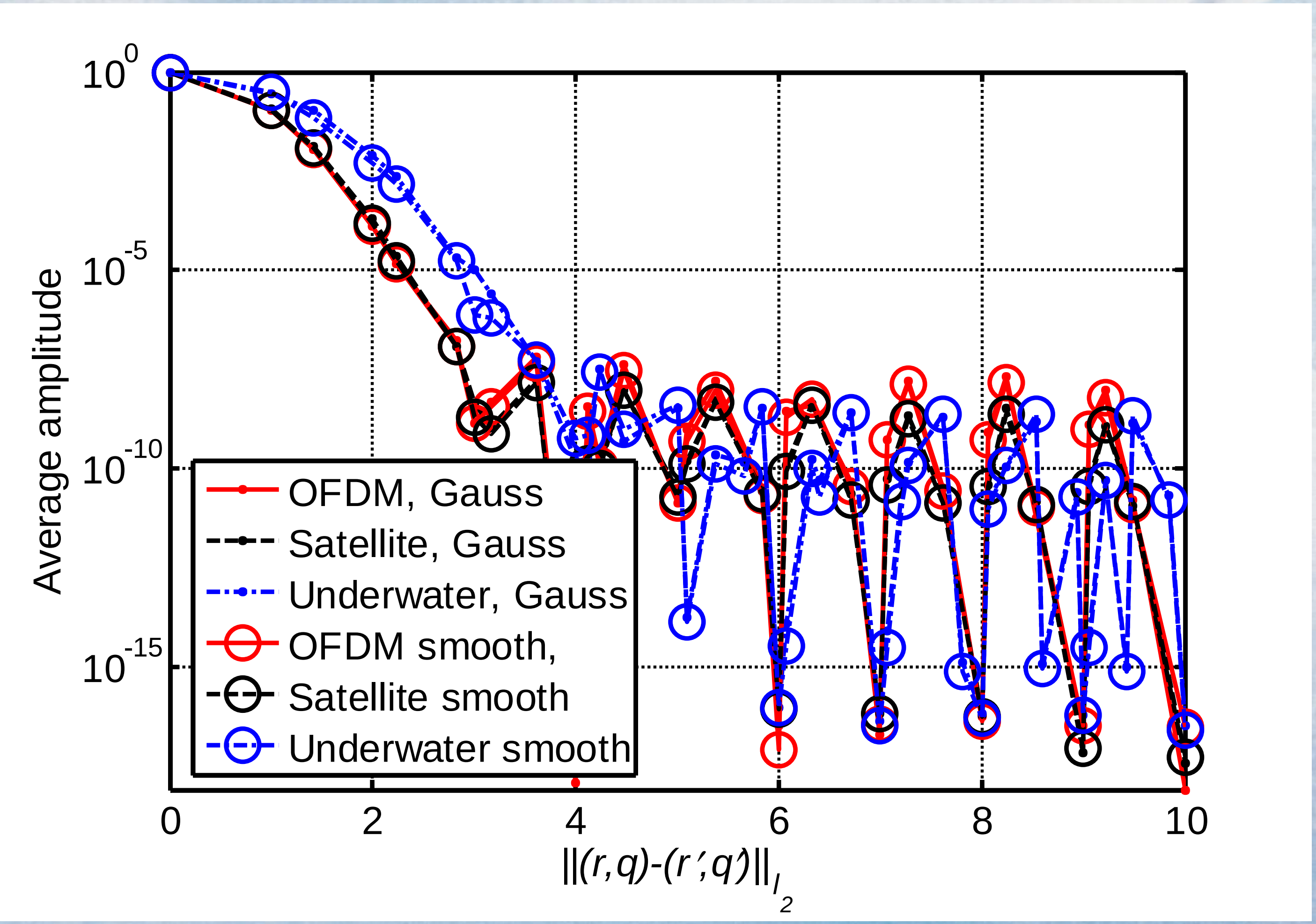
Off-diagonal decay




"That's all Folks!"