PAR Reduction – Mathematical and Realization Aspects

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Summer School "Progress in Math for Comms 2007"

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- PAR computation and theoretical limits
- The error rates due to clipping (Gaussian assumption and Bahai's parabolic impulses
- Tone reservation (Tellado, Henkel)
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The PAR in DMT systems

The average power of an M-QAM point set

$$\overline{P}_{M-QAM_{(f)}} = \frac{a^2}{6} \cdot (M-1)$$

With a minimum point distance a. The peak power is defined by the edges

$$\hat{P}_{M-QAM_{(f)}} = \frac{a^2}{2} \cdot (\sqrt{M}-1)^2$$

In the following, we use the IDFT in the form

$$f_l = \sum_{m=0}^{N-1} F_m \, e^{j\,\frac{2\pi}{N}\,l\,m}$$

Splitting up the IDFT in two parts yields

$$\begin{aligned} f_l &= \sum_{m=1}^{\frac{N}{2}-1} \left(F_m e^{j\frac{2\pi}{N} l m} + F_{N-m} e^{j\frac{2\pi}{N} l (N-m)} \right) \\ &= \sum_{m=1}^{\frac{N}{2}-1} \left(|F_m|\, e^{j(\frac{2\pi}{N} l m + \varphi_m)} + |F_m|\, e^{-j(\frac{2\pi}{N} l m + \varphi_m)} \right) \\ &= \sum_{m=1}^{\frac{N}{2}-1} 2\,|F_m| \cos\Big(\underbrace{\frac{2\pi}{N} l m}_{\varphi_d} + \varphi_m \Big) \ . \end{aligned}$$

The average power in time domain will then be

$$\begin{aligned} \overline{P}_{(t)} &= \frac{1}{N} \sum_{l=0}^{N-1} |f_l|^2 \\ &= \frac{1}{N} \sum_{l=0}^{N-1} \sum_{m=1}^{\frac{N}{2}-1} 2\,|F_m| \cos\left(\frac{2\pi}{N} l m + \varphi_m \right) \sum_{n=1}^{\frac{N}{2}-1} 2\,|F_n| \cos\left(\frac{2\pi}{N} l n + \varphi_n \right) \end{aligned}$$

$$= \frac{1}{N}\sum_{l=0}^{N-1}\sum_{m=1}^{\frac{N}{2}-1}\sum_{n=1}^{\frac{N}{2}-1} 4\,|F_m|\cdot|F_n|\,\frac{1}{2}\left[\cos\left(\frac{2\pi}{N}l\,(m+n)+\varphi_m+\varphi_n\right)+\right.$$

$$\left.\underbrace{+\cos\left(\frac{2\pi}{N}l\,(m-n)+\varphi_m-\varphi_n\right)}_{=0 \text{ for all } m\neq n}\right]$$

$$= \frac{1}{N}\sum_{l=0}^{N-1}\sum_{m=1}^{\frac{N}{2}-1} 2\,|F_m|^2$$

Note that this is actually the Parseval formula for a DFT/IDFT when $1/N$ is part of the DFT. Before, we usually split it equally as $1/\sqrt{N}$ making the average power the same in time and DFT domain. Here, we instead have

$$\underbrace{\frac{1}{N}\sum_{i=0}^{N-1}|f_i|^2}_{\overline{P}_{(t)}} = N\,\underbrace{\frac{1}{N}\sum_{\substack{m=0\\F_m\neq 0}}^{N-1}|F_m|^2}_{\overline{P}_{(f)}}$$

First, let us consider a pair of 2 conjugate carriers, *i.e.*, single-carrier QAM:

$$\overline{P}_{2\,conj.compl.carr._{(t)}} = \frac{a^2}{3} \cdot (M-1) \quad .$$

$$\begin{aligned} \hat{P}_{2\,conj.compl.carr._{(t)}} &= 4\,|F_{max}|^2 \\ &= 2\,a^2 \cdot (\sqrt{M}-1)^2 \quad . \end{aligned}$$

We obtain the PAR of single-carrier modulation

$$PAR_{2\,conj.compl.carr._{(t)}} = 6 \cdot \frac{(\sqrt{M}-1)}{(\sqrt{M}+1)} \quad .$$

Often, a continuous approximation is used to model bigger QAM constellations

$$\begin{aligned}\overline{P}_{kont.QAM_{(f)}} &= \left(\frac{1}{2A_{max}}\right)^2 \int_{-A_{max}}^{A_{max}} \int_{-A_{max}}^{A_{max}} (x^2+y^2)\,dx\,dy \\ &= \frac{1}{4A_{max}^2} \int_{-A_{max}}^{A_{max}} (\frac{2}{3}A_{max}^3 + 2A_{max}y^2)\,dy \\ &= \frac{2}{3}A_{max}^2\end{aligned}$$

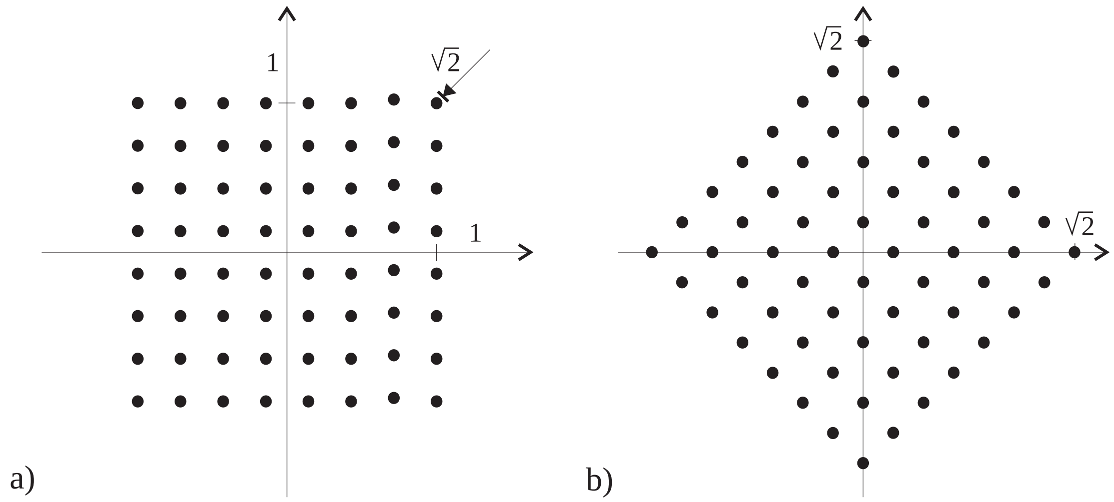
The time-domain value combining 2 carriers will be $\overline{P}_{kont.QAM_{(t)}} = 2 \cdot \frac{2}{3}A_{max}^2$

$$\hat{P}_{kont.QAM_{(f)}} = 2A_{max}^2$$

The time-domain value combining 2 carriers will be $\hat{P}_{kont.QAM_{(t)}} = 2^2 \cdot 2A_{max}^2$
and finally, the PAR of the continous quadratic ‘constellation’ results in

$$PAR_{kont.QAM_{(t)}} = \lim_{M\to\infty} PAR_{2\,conj.compl.carr._{(t)}} = 6$$

When going over to multicarrier modulation with many carriers, we approach a Gaussian distribution which would have unlimited peak values. However, due to the still limited number of carriers, also the possible peaks will be limited. Often this is not significant, since it does not have an influence on the probability (*e.g.*, 10^{-7}) of esceeding a certain threshold at, *e.g.*, 10-14 dB PAR. Nevertheless, we'd like to show that the PAR is proportionally to N, the number of carriers.



With the right constellation, the peak in time domain will easily be obtained at

$t = 0$. The left constellation will lead to a multiple of

$$1.064 \cdot \left(1 + \frac{1}{\sqrt{2}}\right) \frac{N}{4}$$

relative to single-carrier (2 conjugate carriers) modulation.

We finally obtain

$$\hat{P}_{N_{(t)}} = \left(1.064 \cdot \left(1 + \frac{1}{\sqrt{2}}\right)\right)^2 \cdot \left(\frac{N}{4}\right)^2 2a^2 (\sqrt{M} - 1)^2$$

Using Parseval's formula,

$$\overline{P}_{N_{(t)}} = N \frac{a^2}{6} (M - 1) \ ,$$

we obtain

$$PAR_{N_{(t)}} = \left(1.064 \cdot \left(1 + \frac{1}{\sqrt{2}}\right)\right)^2 \cdot 6/8 \cdot N \cdot \frac{(\sqrt{M} - 1)}{(\sqrt{M} + 1)}$$

Theoretical PAR limits

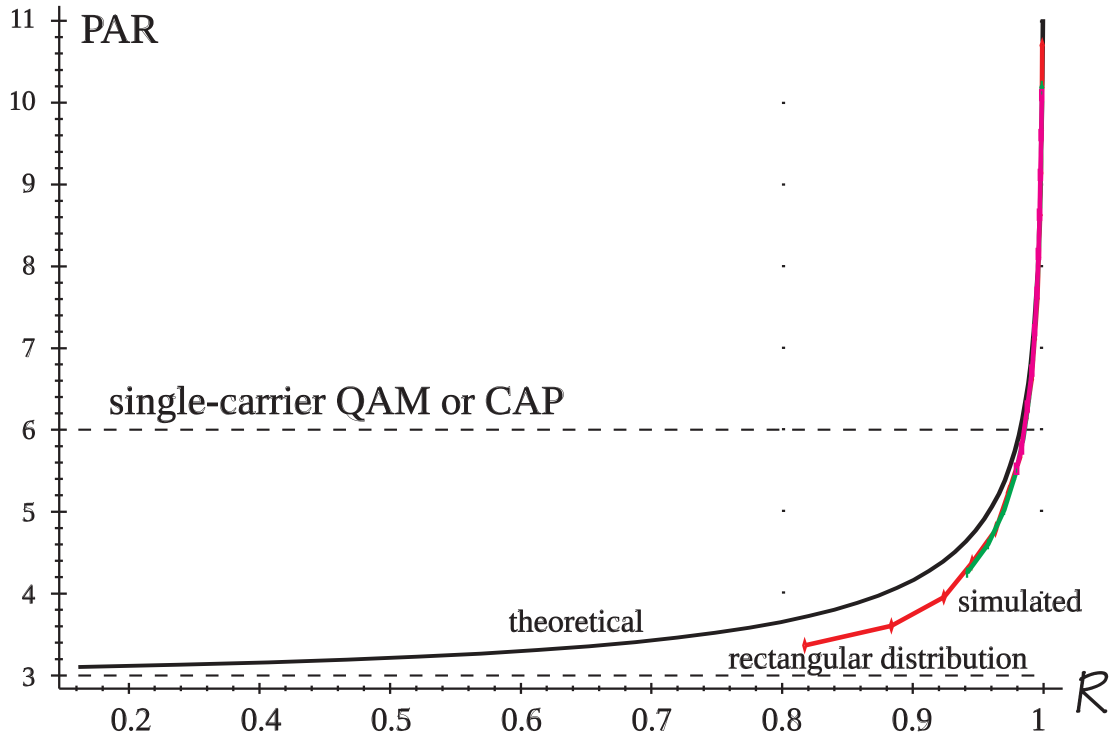
Let $\hat{s}$ be the voltage limit. The average power belonging to a limited Gaussian density is given by

$$\bar{P} = \frac{2 \cdot \int_0^{\hat{s}} x^2 \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} dx}{2 \cdot \int_0^{\hat{s}} \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma} dx} = \frac{2\sigma^2}{\frac{\sqrt{\pi}}{2} \mathrm{erf}(\frac{\hat{s}}{\sqrt{2}\sigma})} \int_0^{\frac{\hat{s}}{\sqrt{2}\sigma}} x^2 e^{-x^2} dx \,. \tag{1}$$

The probability of an amplitude lower than $\hat{s}$ is $\mathcal{P}_i = \mathrm{erf}(\frac{\hat{s}}{\sqrt{2}\sigma})$. The probability that this is the case for all N time-domain samples is $\prod_{i=0}^{N-1} \mathcal{P}_i = \mathrm{erf}^N(\frac{\hat{s}}{\sqrt{2}\sigma})$. The code rate R follows to be

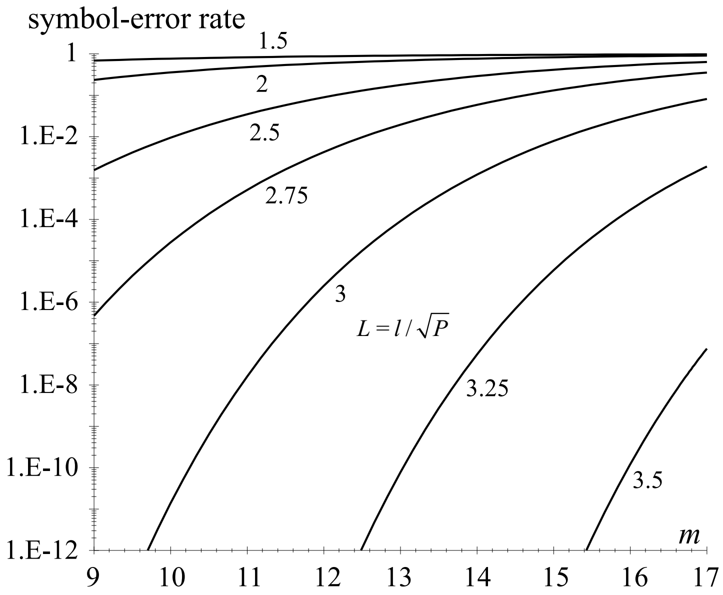
$$R = \frac{\log_2(2^{mN_{act}} \prod_{i=0}^{N-1} \mathcal{P}_i)}{mN_{act}} = 1 + \frac{N \cdot \log_2(\mathrm{erf}(\frac{\hat{s}}{\sqrt{2}\sigma}))}{N_{act} \cdot m} \,, \tag{2}$$

N/N_{act} =constant, being the ratio of the number of DFT components relative to the number of independently usable carriers ($N/N_{act} \approx 2$ for baseband transmission due to conjugacy constraints)


PAR
11
10
9
8
7
6
5
4
3
single-carrier QAM or CAP
theoretical
simulated
rectangular distribution
0.2
0.4
0.5
0.6
0.7
0.8
0.9
1
R

Results with Gaussian assumption

$$N = 2\int_{\mathcal{L}_c\sqrt{P}}^{\infty} (u - \mathcal{L}_c\sqrt{P})^2 \frac{1}{\sqrt{2\pi P}} e^{-\frac{u^2}{2P}} du \qquad p_d = \frac{1}{2}\,\mathrm{erfc}\left(\frac{a/2}{\sqrt{2N}}\right)$$

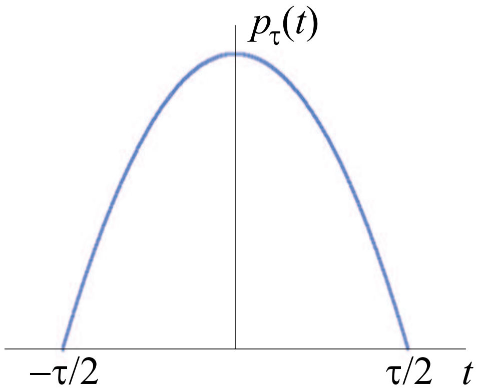


Gaussian assumption not justified, due to low number of clips.

Bit-error rate forecasts are far to optimistic!

Bahai's assumptions and shortcomings

Shape of the excursions assumed to be parabolic



$$p_\tau(t) = \left(\frac{1}{2}x''t^2 - \frac{1}{8}x''\tau^2\right) \cdot \mathrm{rect}\left(\frac{t}{\tau}\right)$$

From a Taylor expansion where constant term ensures zeros at $t = \pm\tau/2$

$$x'' \approx R''_{xx}(0) \cdot x$$

$x \approx l$???

$$p_\tau(t) = \left(-\frac{1}{2}lm_2t^2 + \frac{1}{8}lm_2\tau^2\right) \cdot \mathrm{rect}\left(\frac{t}{\tau}\right) ,$$

with

$$m_i = \begin{cases} \frac{1}{2\pi}\int \omega^i S_x(\omega)d\omega , & i = 2u \\ 0 , & i = 2u+1 \end{cases}$$

Bahai's assumptions and shortcomings

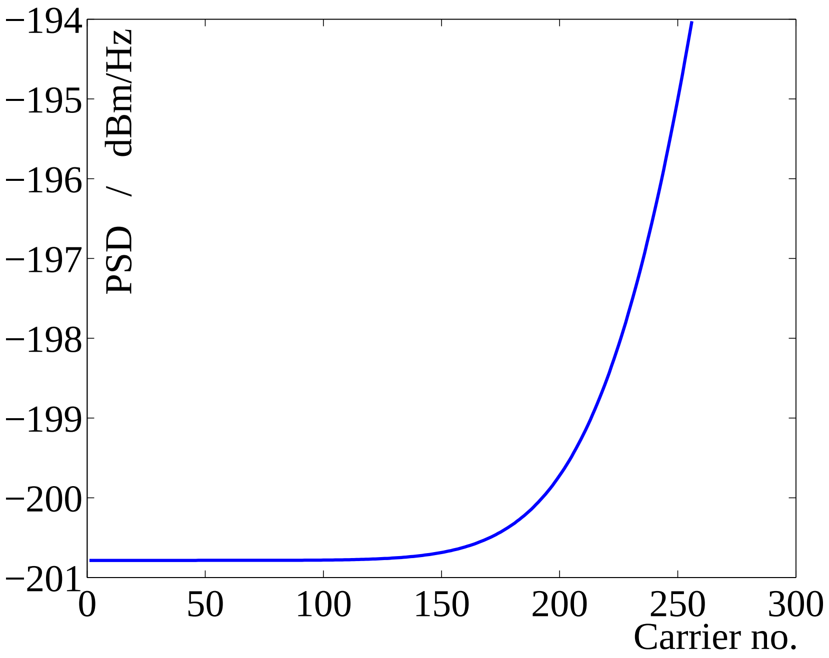
Discrete disturbance spectrum corresponding to parabolic excursion:

$$F_k = \frac{\sqrt{N_f} m_2 T l \tau}{4\pi^2 k^2} e^{-(j2\pi k(t_0+\frac{\tau}{2})/T)} \cdot \quad (1)$$

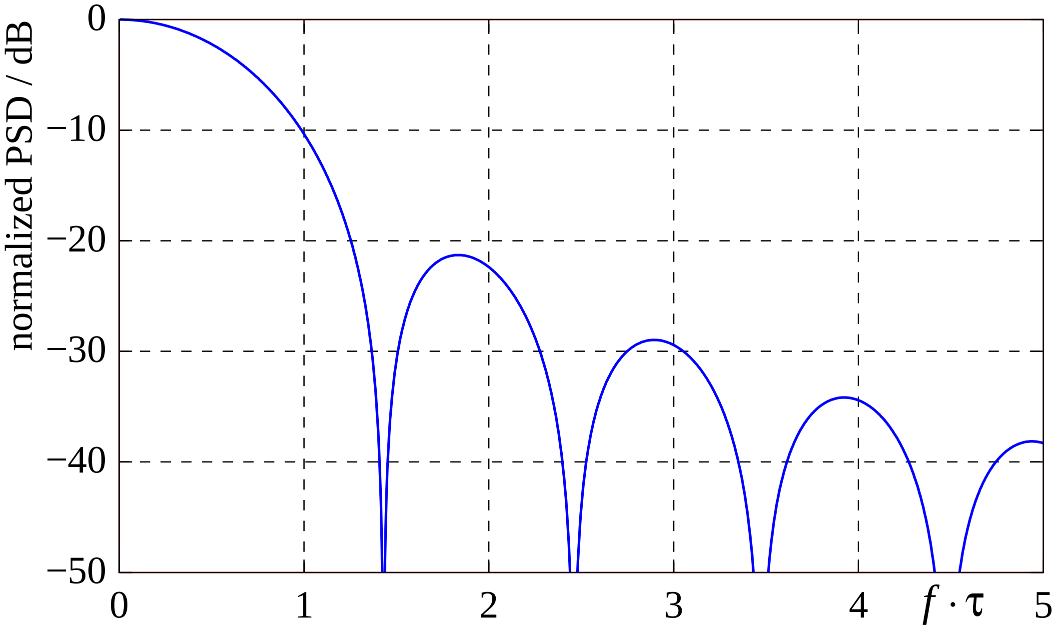
$$\cdot \left(\mathrm{sinc}\left(\frac{\pi k \tau}{T}\right) - \cos\left(\frac{\pi k \tau}{T}\right)\right) , \quad (2)$$

Unfortunately, this spectral shape does not materialize!

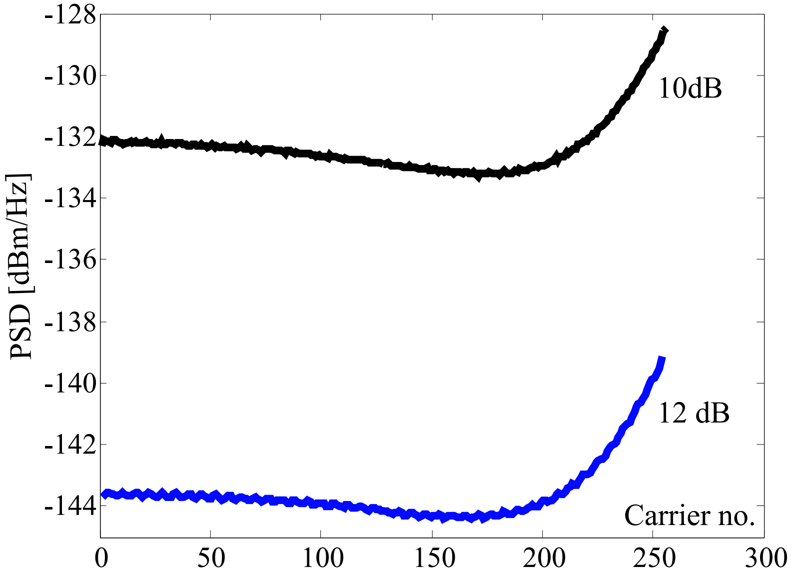
Simulated average single-clip PSD

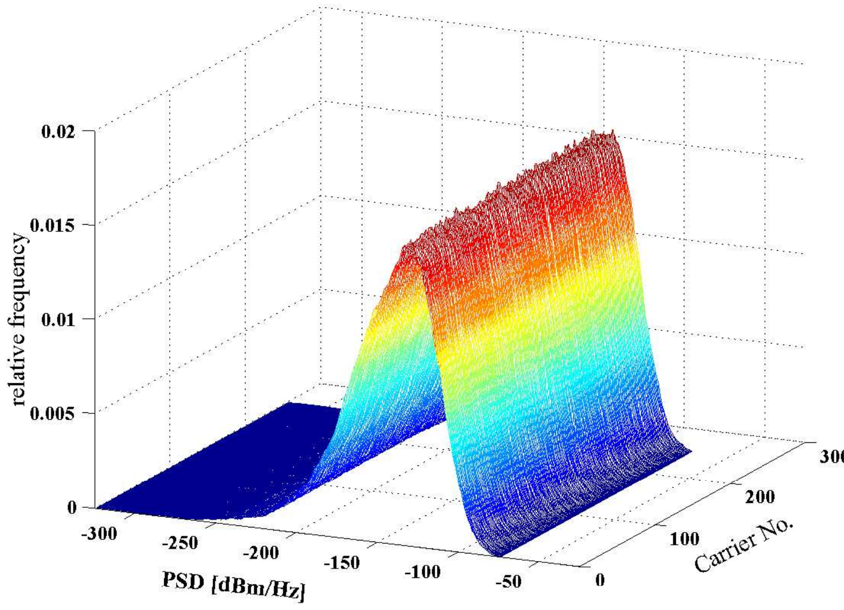


Bahai’s single-clip spectrum



Simulation results

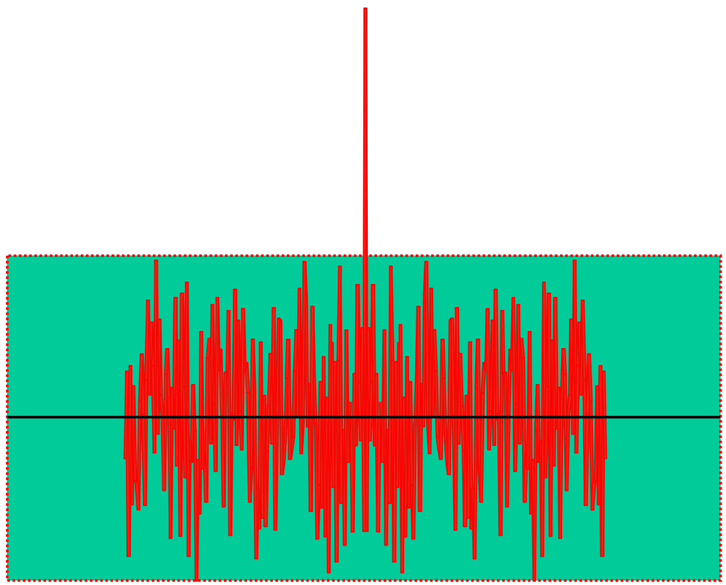




Tellado's Tone Reservation Algorithm

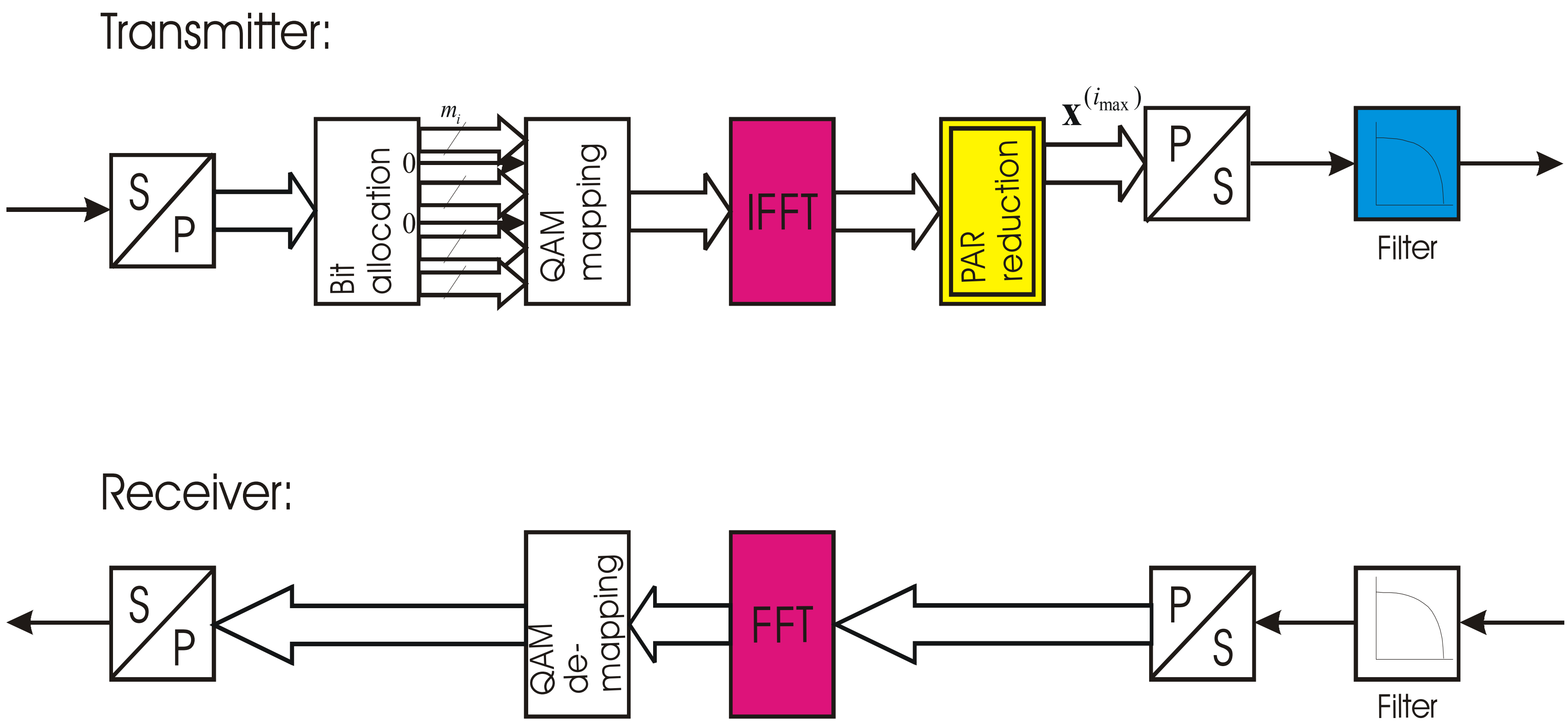
The idea

- Subtraction of Dirac-like functions in time domain



Tellado's Tone Reservation Algorithm

Location



Tellado's Tone-Reservation Algorithm

1. Initialize **X** to be the DFT-domain information vector with reserved carriers set to zero.

2. Initialize the time domain solution $\mathbf{x}^{(0)}$ to $\mathbf{x}$, which results from the IFFT(**X**).

3. Find the value $x_m^{(i)}$ and location m for which $|x_m^{(i)}| = \max_k |x_k^{(i)}|$

4. If $|x_m^{(i)}| < x_{target}$ of if $i > i_{max}$ then stop the iteration and transmit $\mathbf{x}^{(i)}$, otherwise

5. Update the time-domain vector according to

$$\mathbf{x}^{(i+1)} = \mathbf{x}^{(i)} - \alpha \cdot (x_m^{(i)} - \mathrm{sign}(x_m^{(i)}) \cdot x_{target}) \cdot (\mathbf{p} \rightarrow m)$$

$i := i + 1;$

goto 3.

Low-complexity peak-reduction methods

Tone reservation (real case, DMT)

$$\mathbf{x}^{(i+1)} = \mathbf{x}^{(i)} - \alpha \cdot \left(x_m^{(i)} - \mathrm{sign}(x_m^{(i)}) \cdot x_{target}\right) \cdot (\mathbf{p} \to m)$$

Tone reservation (complex case, OFDM)

$$\mathbf{x}^{(i+1)} = \mathbf{x}^{(i)} - \alpha \cdot \left(x_m^{(i)} - e^{j\,\mathrm{arc}(x_m^{(i)})} \cdot x_{target}\right) \cdot (\mathbf{p} \to m)$$

The oversampled new Tellado-like proc.

The idea

- Oversampling by, *e.g.*, L=4
- Generating Dirac-like functions in the oversampled time domain after the filter response
- Processing non-oversampled and oversampled sequences in parallel
- Precomputing L pairs of non-oversampled and oversampled Dirac-like functions
- In the iteration: according to the maximum pos. m select one of the L pairs and shift it to the maximum pos.

The oversampled new Tellado-like proc.

The idea

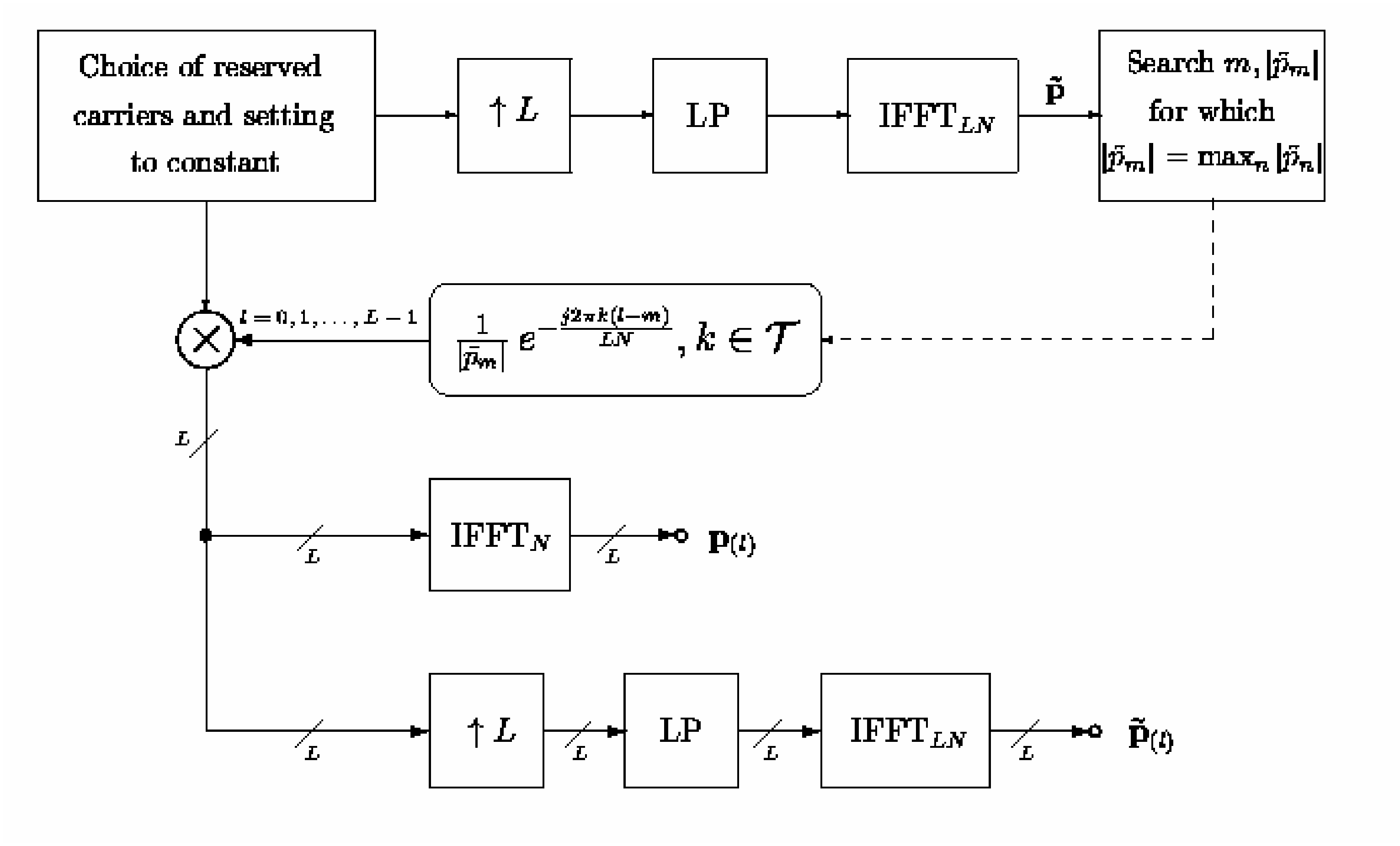
The shift property of the DFT is used

- in precomputing the *L* shifted versions of the Dirac-like pairs
- to shift the selected functions to the desired maximum position

$$x((n-l) \bmod (LN)) \quad \circ\!\!-\!\!\bullet \quad X(k)e^{-jkl\frac{2\pi}{LN}}$$

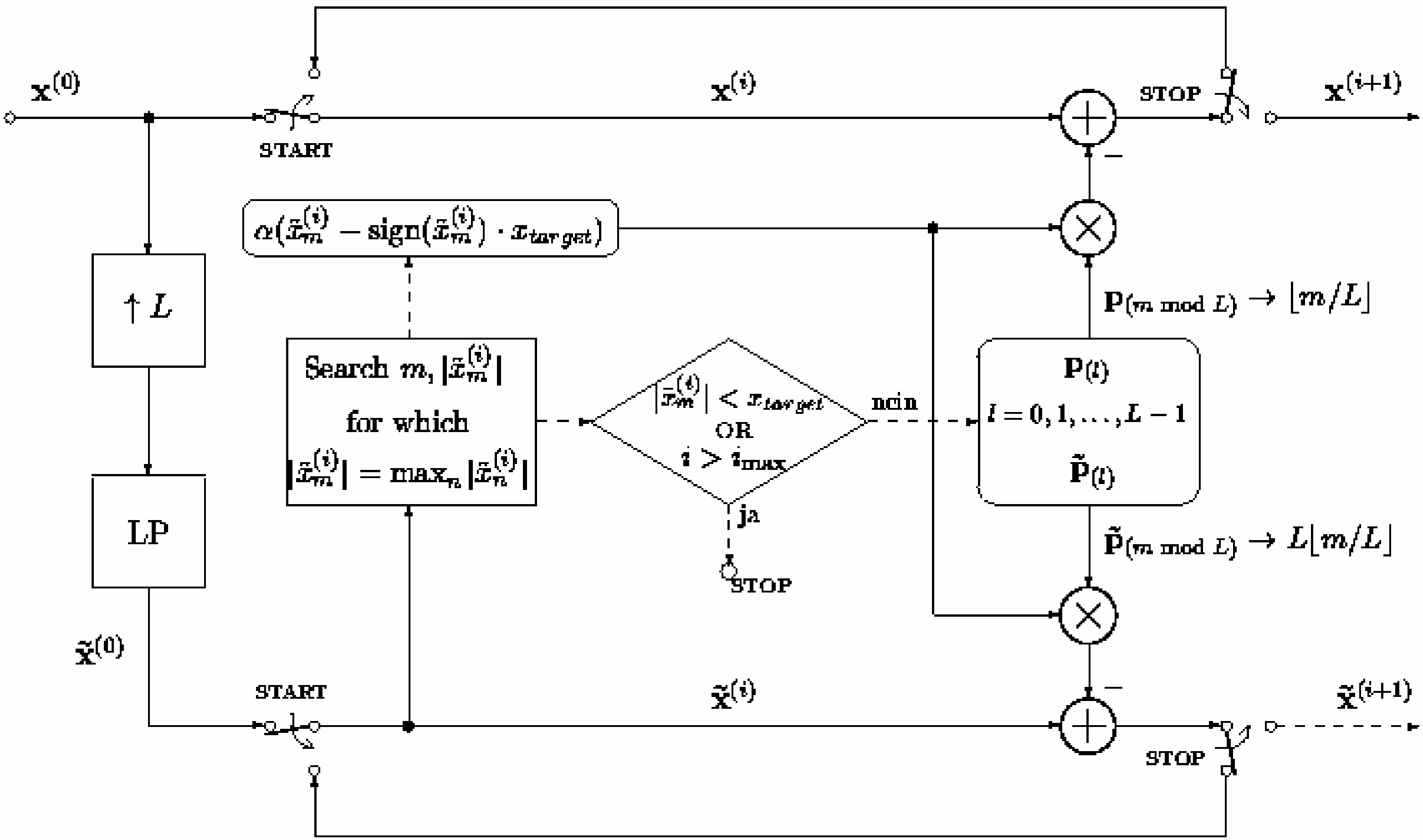
The oversampled new Tellado-like proc.

Generation of the Dirac-like pairs



The oversampled new Tellado-like proc.

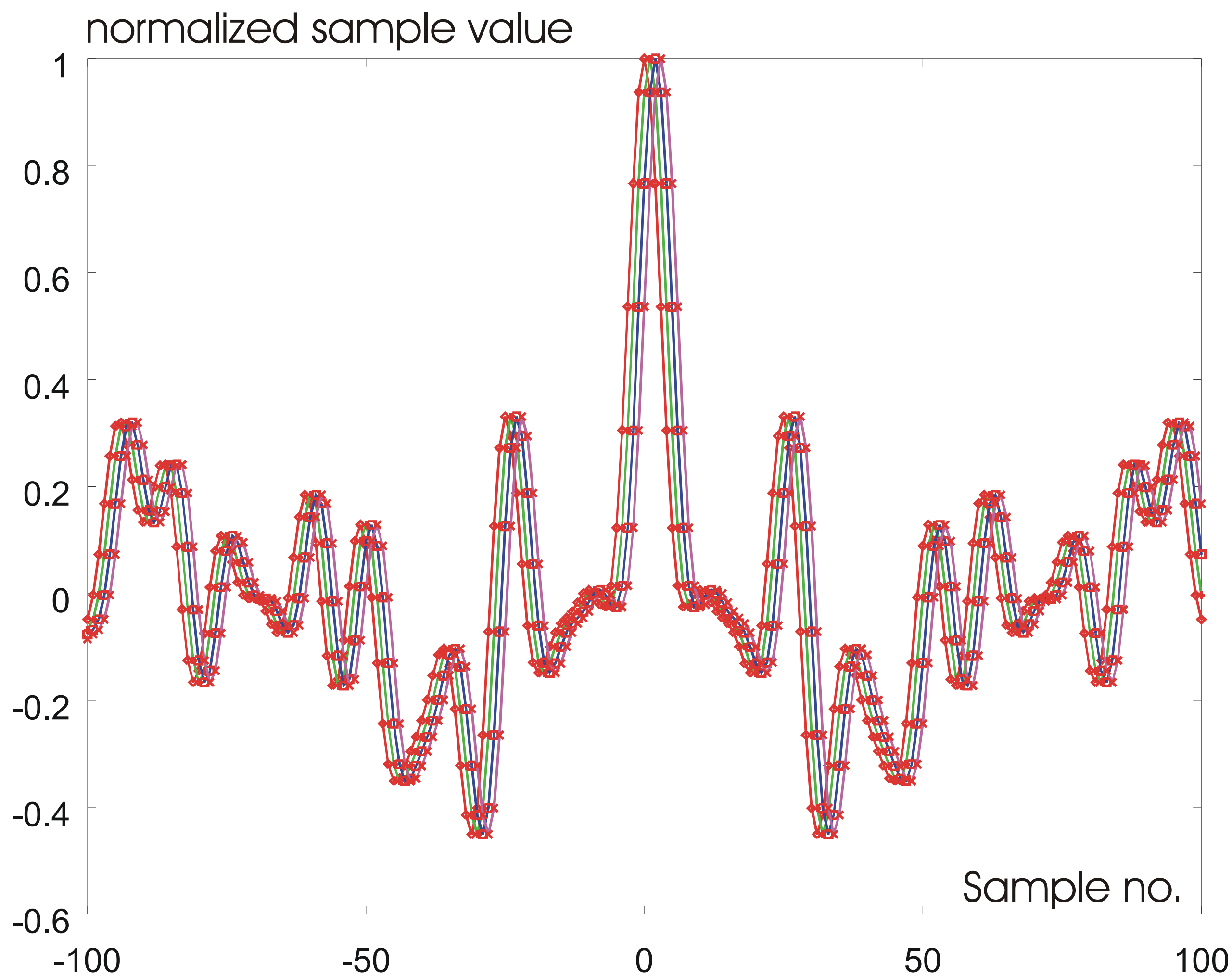
The iterations



The oversampled new Tellado-like proc.

The oversampled Dirac-like functions

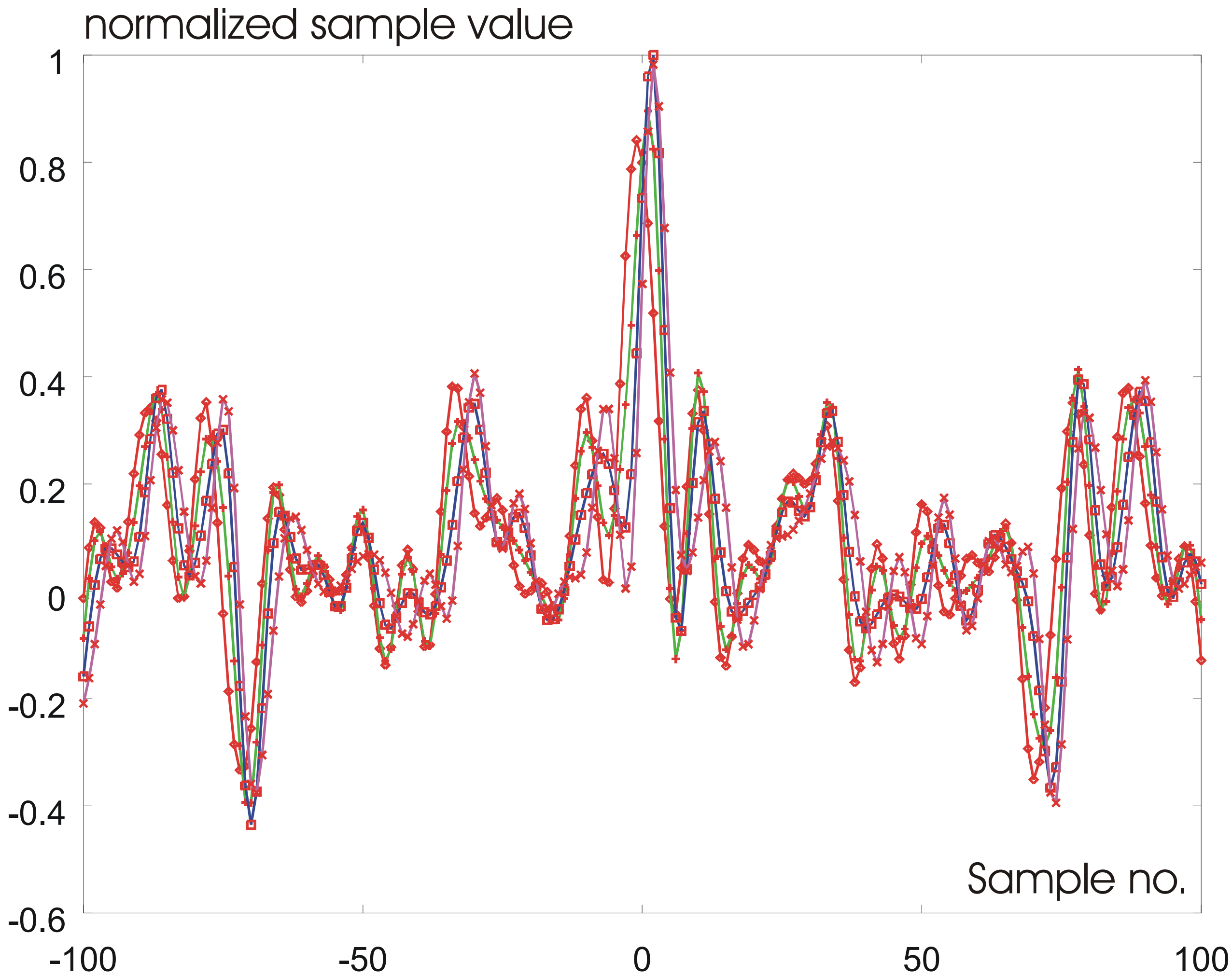
For brick-wall filter:



The oversampled new Tellado-like proc.

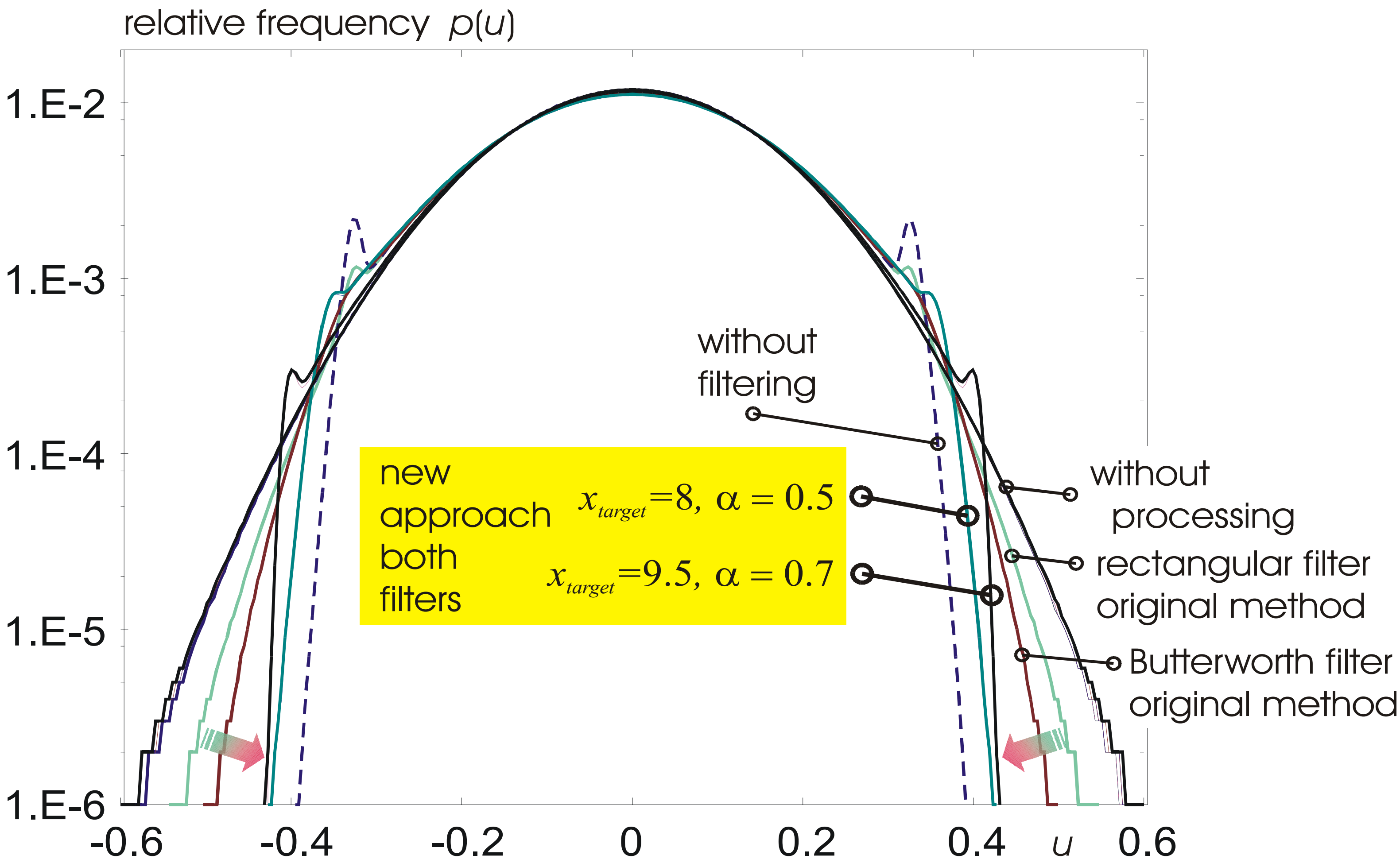
The oversampled Dirac-like functions

For Butterworth filter:

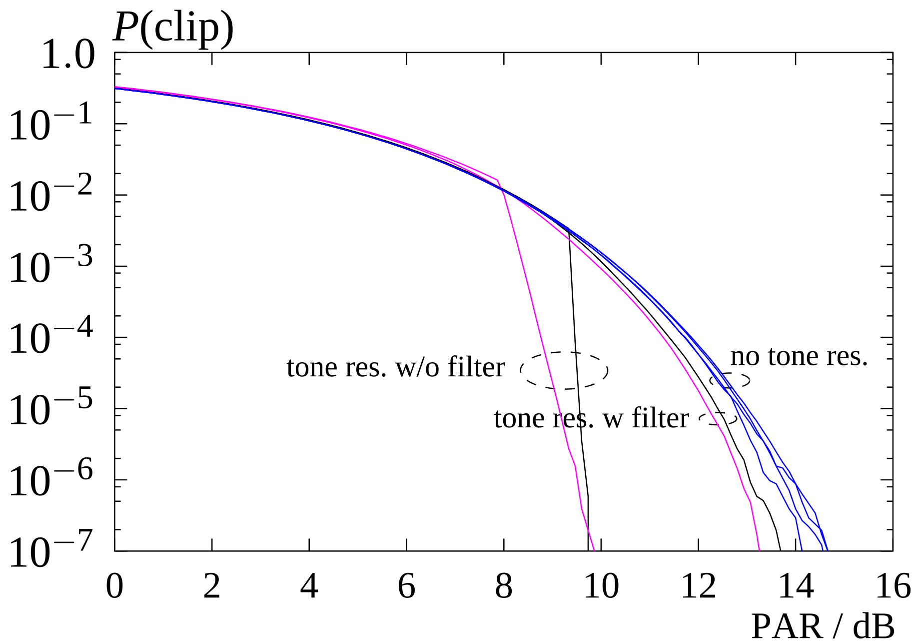


The oversampled new Tellado-like proc.

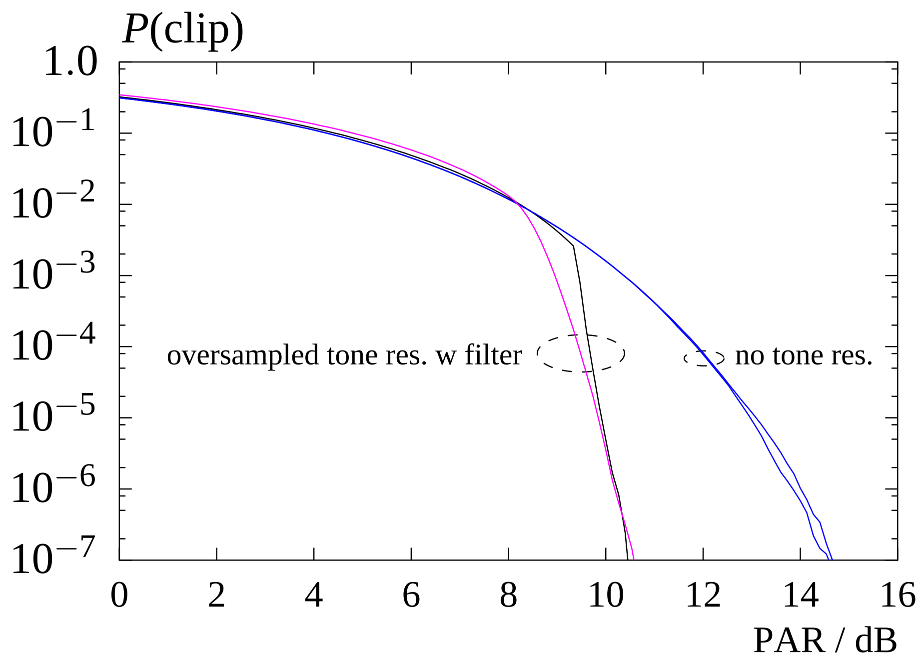
Results



Tone reservation



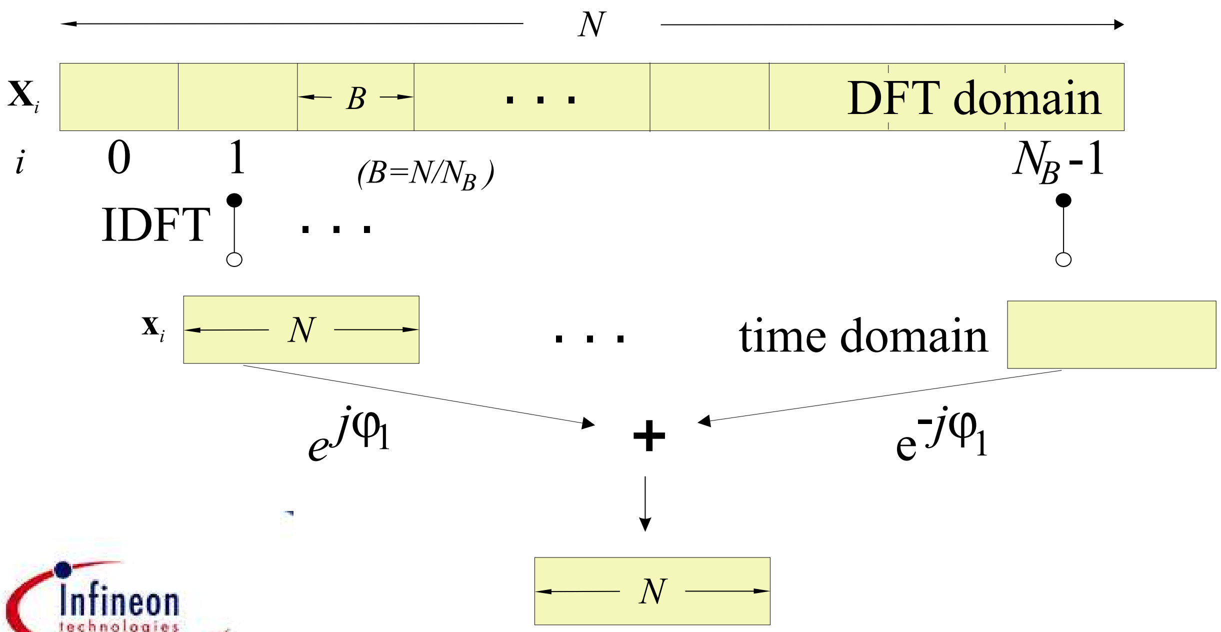
Non-oversampled

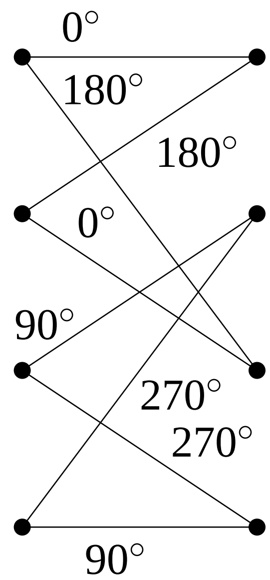


oversampled

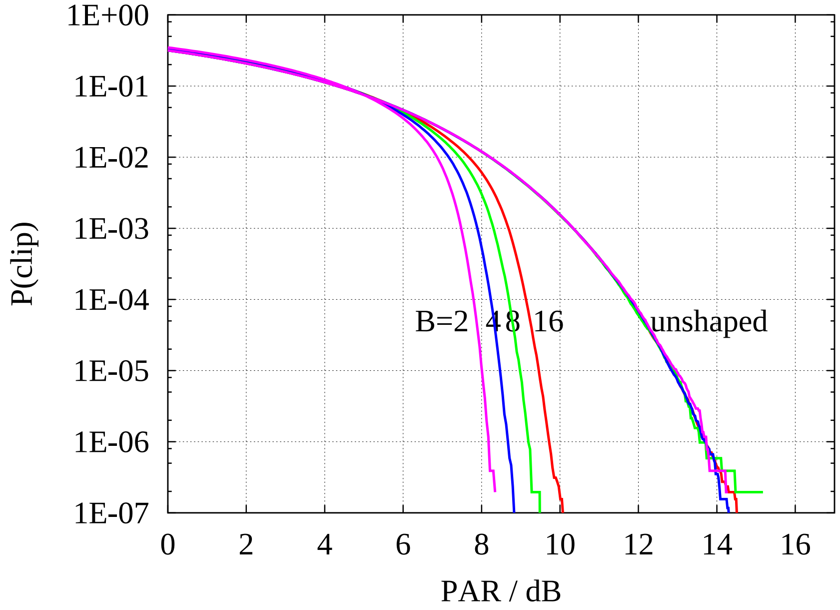
Low-complexity peak-reduction methods

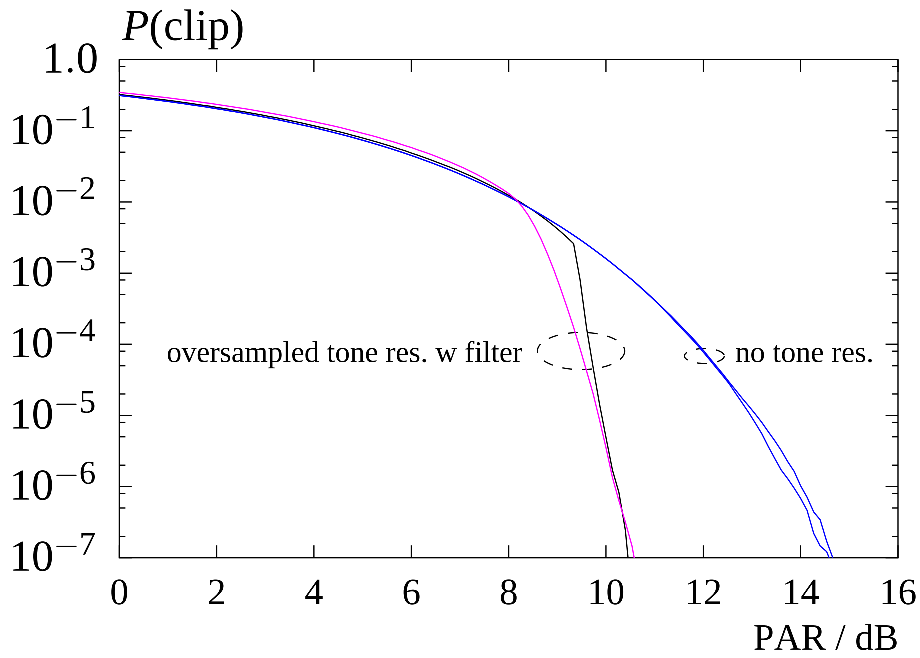
Trellis Partial Transmit Sequences





Low-complexity peak-reduction methods





Trellis PTS

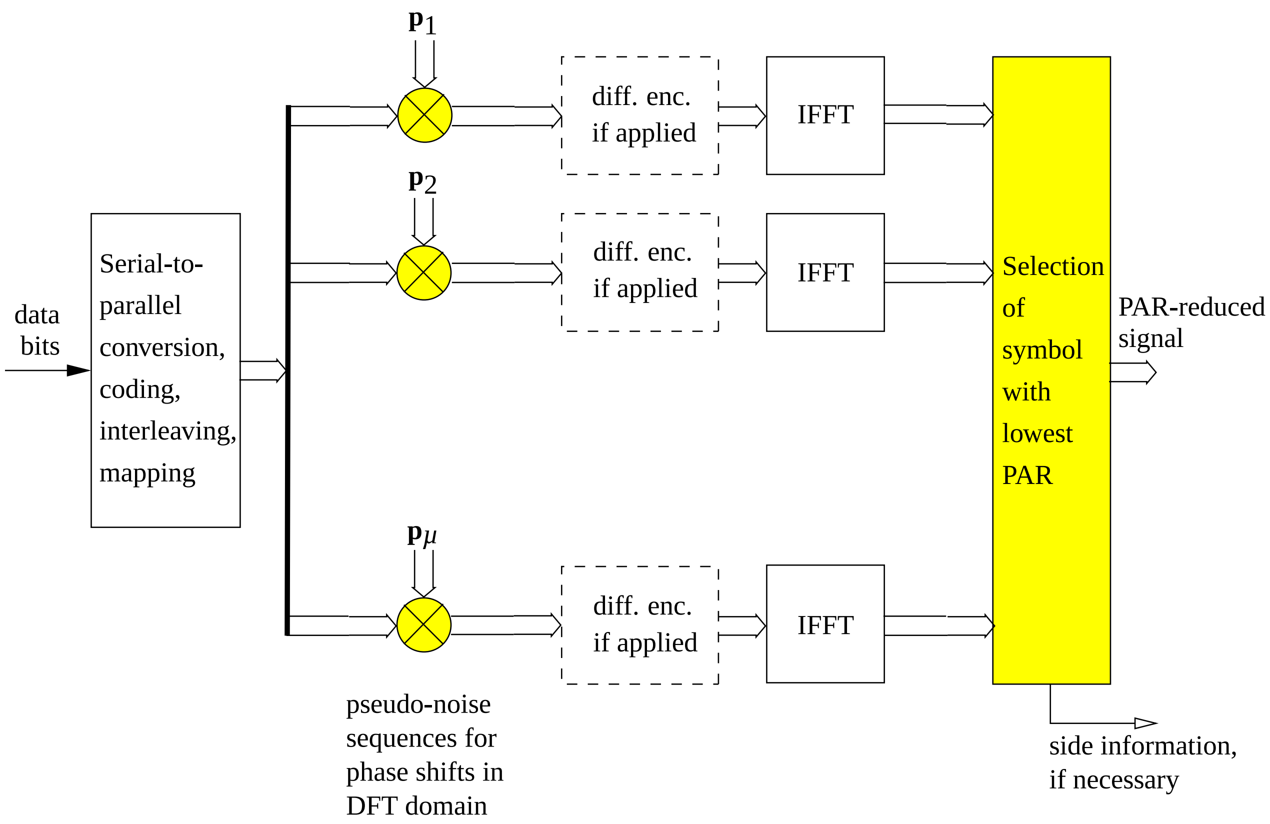
Tone reservation

⟶ Similar performance, but tone reservation has lower complexity.

Infineon technologies

IUB International University Bremen

The PAR-reduction method "**Selected Mapping (SLM)**"



The procedure multiplies the DFT-frame with phase-shift vectors, typically generated from *m*-sequences and after performing IDFTs for all resulting vectors, just selects the time-domain signal with the lowest PAR. Four possible phases addressed by the pseudo-noise sequences usually suffice. Side information needs to be transmitted with high reliability such that the receiver knows the chosen PN sequence. However, there are also realizations that do not require this side information.