TIME-FREQUENCY ANALYSIS: TUTORIAL

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Overview

- TF-Analysis: Spectral Visualization of nonstationary signals (speech, audio, ...)
  - *Spectrogram* (time-varying spectrum estimation)

- TF-methods for signal processing:
  - *Ambiguity function* (range/Doppler estimation)
  - *Short-time Fourier transform* (LTV filter design)

- TF-representation of underspread linear operators:
  - *Spreading Function* (representation & classification)
  - *Kohn-Nirenberg symbol* (LTV transfer function)
  - Application: MIMO-based SAR radar problem

Mathematical Setup

- Classical Theory:
  - signals defined on the real line
  - Hilbert space setup usual (Math. Physics and EE)
  - Gelfand brackets (pure mathematics)

- Numerical Practice:
  - signals are vectors in $\mathbb{C}^N$
  - Fourier Transform = DFT = realized by FFT

- Open Problems:
  - Algebraic & Number theoretic methods
  - try to take finite alphabet effects in account

Time-Frequency Shift

- Unitary time-frequency shift operator

$$(U(\tau,\nu)x)(t) = x(t-\tau)\exp(2\pi i \nu t)$$

- Superposition Law (Schrödinger Repr. of WH-Group)

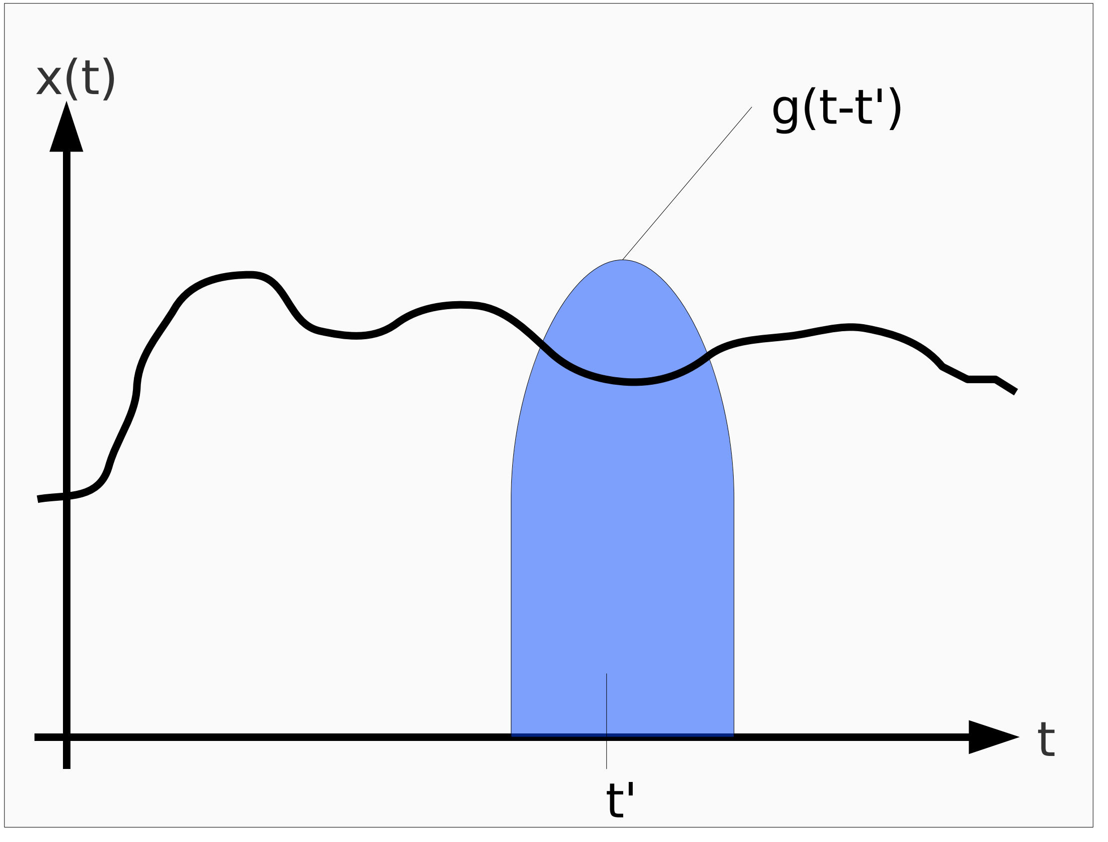
$$(U(\tau_{1,}\nu_1)U(\tau_{2,}\nu_2)x)(t) = x(t-(\tau_1+\tau_2))\exp(2\pi i((n_1+n_2)t-\nu_2\tau_1))$$

- NO unitary group representation of $\mathbb{R} x \mathbb{R}$

Short-Time Fourier Transform

- Sliding a window g(t) along the signal followed by Fourier transform of the windowed partial signal

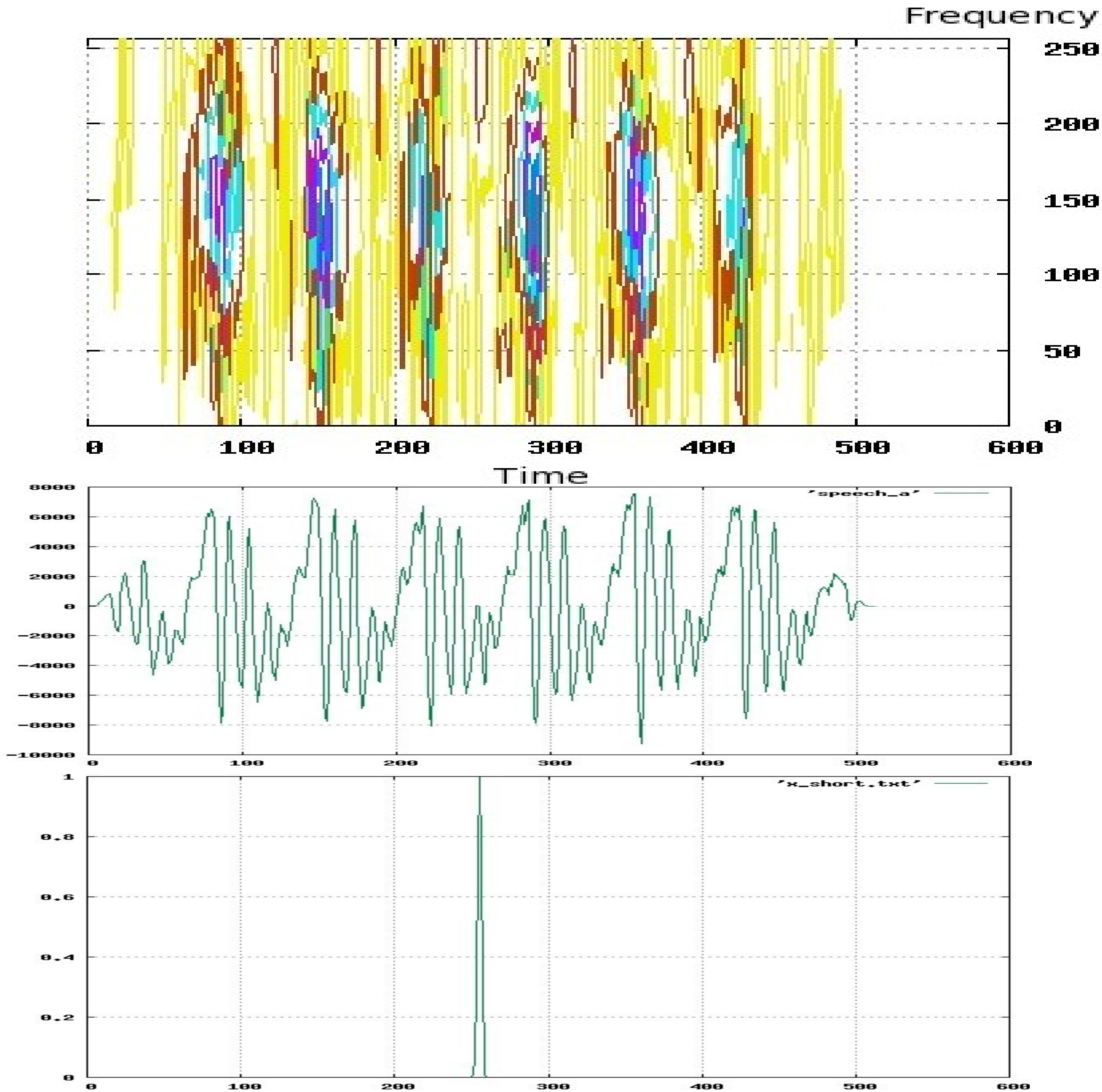
$$(V_g x)(t,f) = \int x(t')\overline{g(t-t')}\exp(-2\pi i f t')$$



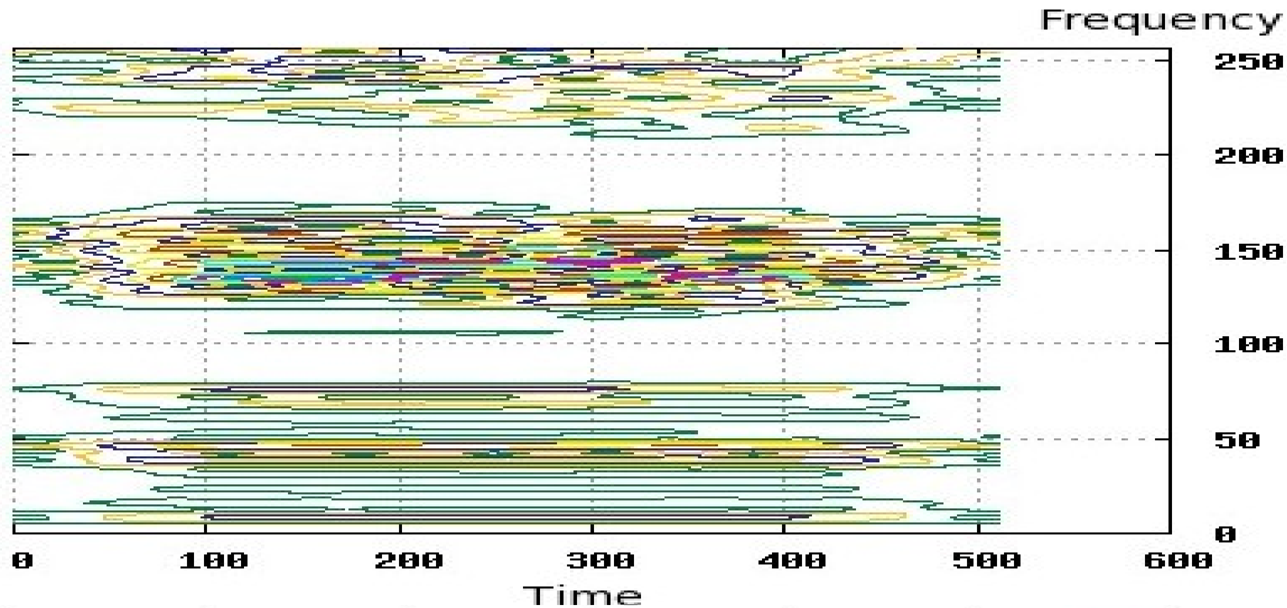
Spectrogram

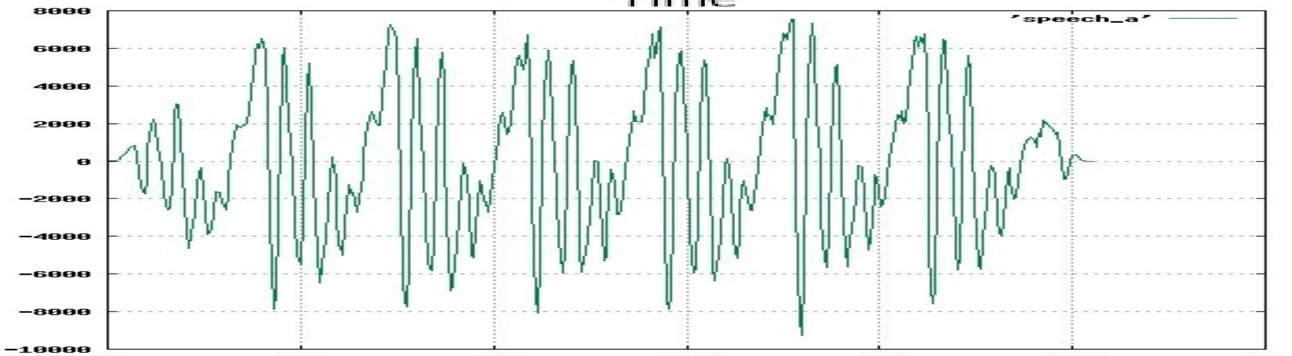
- The Short-time Fourier transform is complex valued and its real part and imaginary part are highly oscillatory
- adequate visualization is given by the squared magnitude => Spectrogram
- $$(S_g x)(t,f) = |(V_g x)(t,f)|^2$$
- The spectrogram can be interpreted as a smoothed Wigner distribution

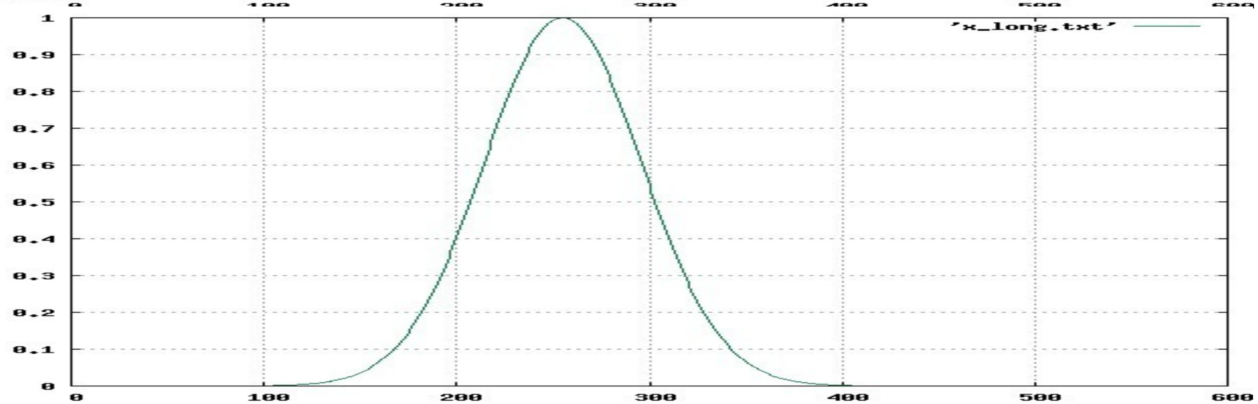
Spectrogram: „Short“ Window



Spectrogram: „Long“ Window







STFT-based Filtering

- Reconstruction of signal from STFT:

$$x(t) = \int \int V_g(t', f')(U(t', f')g)(t)\,dt'\,df'$$

- Reconstruction of signal from multiplicatively modified STFT:

$$(Hx)(t) = \int \int M(t', f')V_g(t', f')(U(t', f')g)(t)\,dt'\,df'$$

- this allows synthesis of HS operator (LTV filter) based on the time-frequency model M(t,f)

Radar Ambiguity Function

- How behaves the inner product of a signal and its TF-shifted version => time-frequency correlation function
- Well-known as Radar ambiguity function

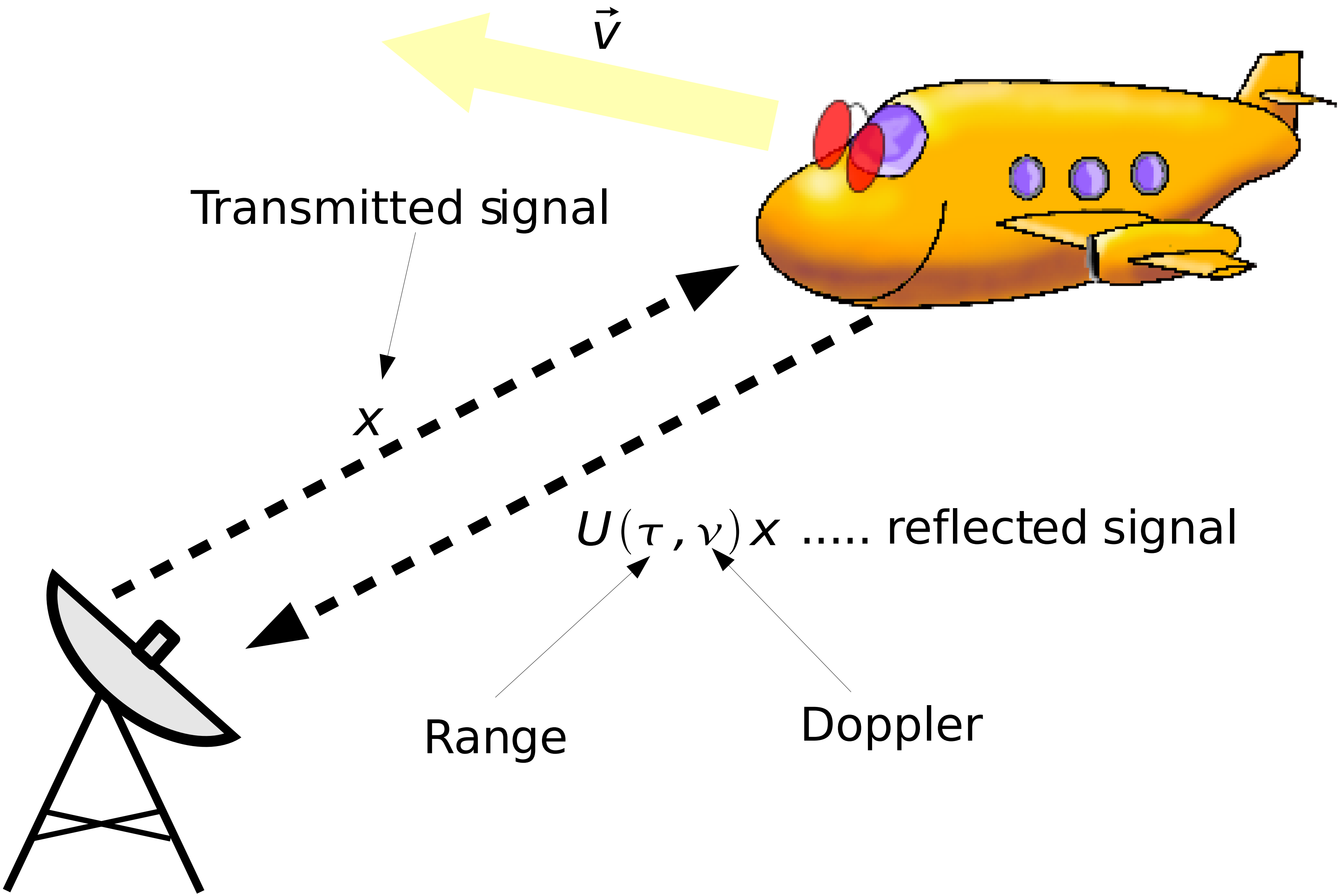
$$(A_x)(\tau,\nu)=\int x(t)\overline{x(t-\tau)}\exp(-2\pi i\nu t)dt$$

- Radar uncertainty principle:

$$\int\int |(A_x)(\tau,\nu)|^2 d\tau d\nu = \|x\|^4$$

$$|(A_x)(0,0)|^2 = \|x\|^4$$

Range-Doppler Radar



Range-Doppler Estimation

- The peak of the cross-ambiguity function is a ML-estimate for the Range-Doppler

$$(\tau,\nu)_{est}=argmax\left(A_{y,x}(\tau,\nu)\right)$$

- Curvature of Ambiguity function of x determines the Cramer-Rao bound for range-Doppler estimation => we want a peaky signal, however one has:

$$\frac{\partial^2 A_x}{\partial\nu^2}(0,0)=-4\pi^2\int t^2|x(t)|^2 dt$$

$$\frac{\partial^2 A_x}{\partial\tau^2}(0,0)=-4\pi^2\int f^2|X(f)|^2 df$$

Radar Synthesis Problem

- Ambiguity function is quadratic signal representation => inner symmetry, i.e. an arbitrary function is no valid ambiguity function
- Given a nonvalid time-frequency model how can we determine the closest valid ambiguity function

$$x_{opt} = arg\,min_x \left\| A_x - M \right\|^2$$

- Boils down to a partial eigenvalue problem of a self-adjoint matrix:

$$Q(M)\, x_{opt} = \lambda_{max}\, x_{opt}$$

Spreading Function

- Decomposition of linear operator into a superposition of time-frequency shift operators

$$(S_H)(\tau, \nu) = \int H(t, t-\tau) \exp(-2\pi i \nu t) dt$$

- Inner product representation

$$(S_H)(\tau, \nu) = \langle H, U(\tau, \nu) \rangle$$

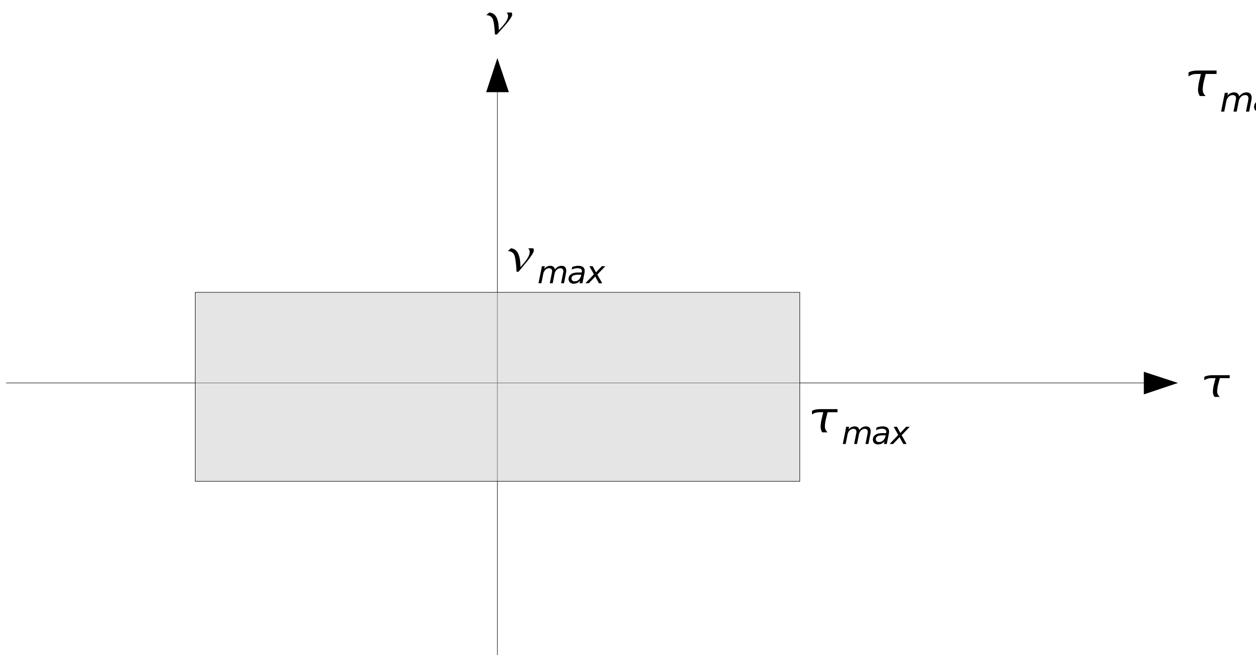
Kohn-Nirenberg Symbol

- Decomposition of linear operator into a superposition of time-frequency shift operators

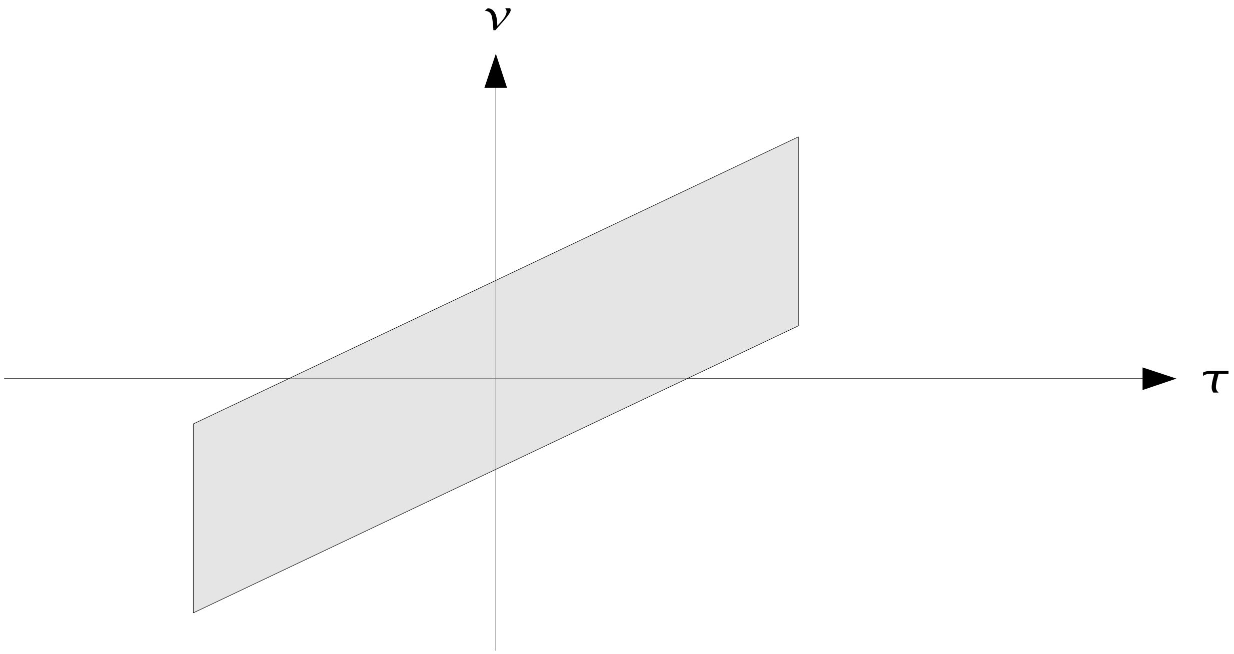
$$(K_H)(t,f)=\int H(t,t-\tau)\exp(-2\pi i f\tau)d\tau$$

Underspread Operators

$\tau_{max} \cdot \nu_{max} \leqslant 1$



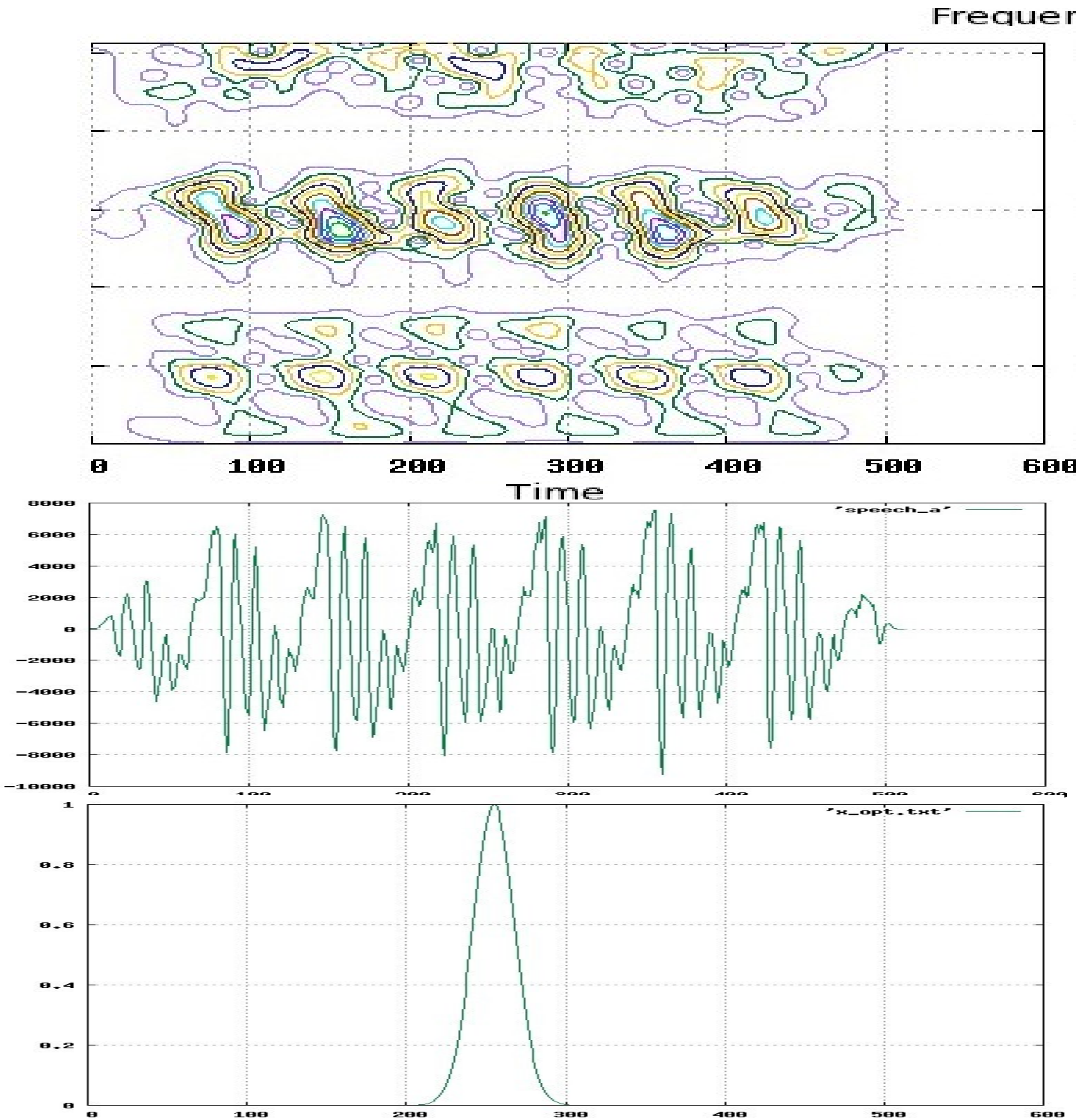
Underspread Operators



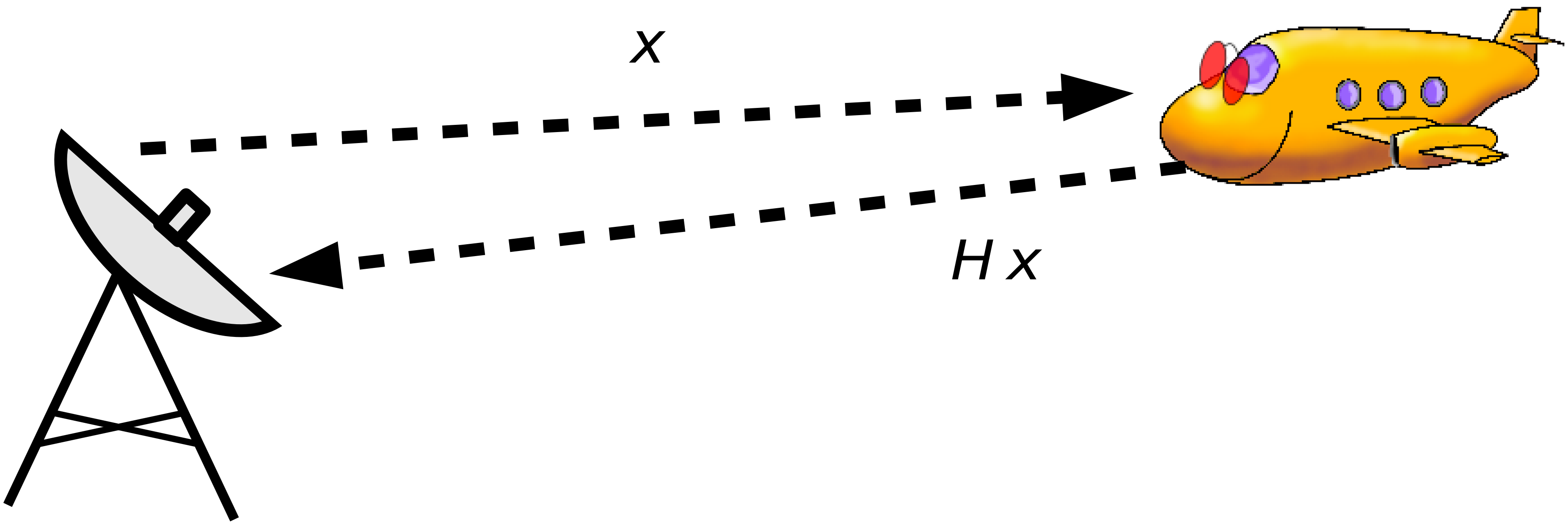
Underspread Asymptotics

- Underspread operators are approximately normal
- Underspread operators do approximately commute
- Underspread operators are approximately diagonalized by a properly adapted Gabor basis
- Underspread operators can be realized as STFT multipliers

Spectrogram: Adapted Window



SAR Radar



- Determine/Classify the whole object rather than its range and velocity from observation of reflected signal
- System identification problem: given x and Hx estimate H and then classify the object based on this estimate
- SAR = Synthetic Aperture Radar

Gabor/STFT based Source Coding