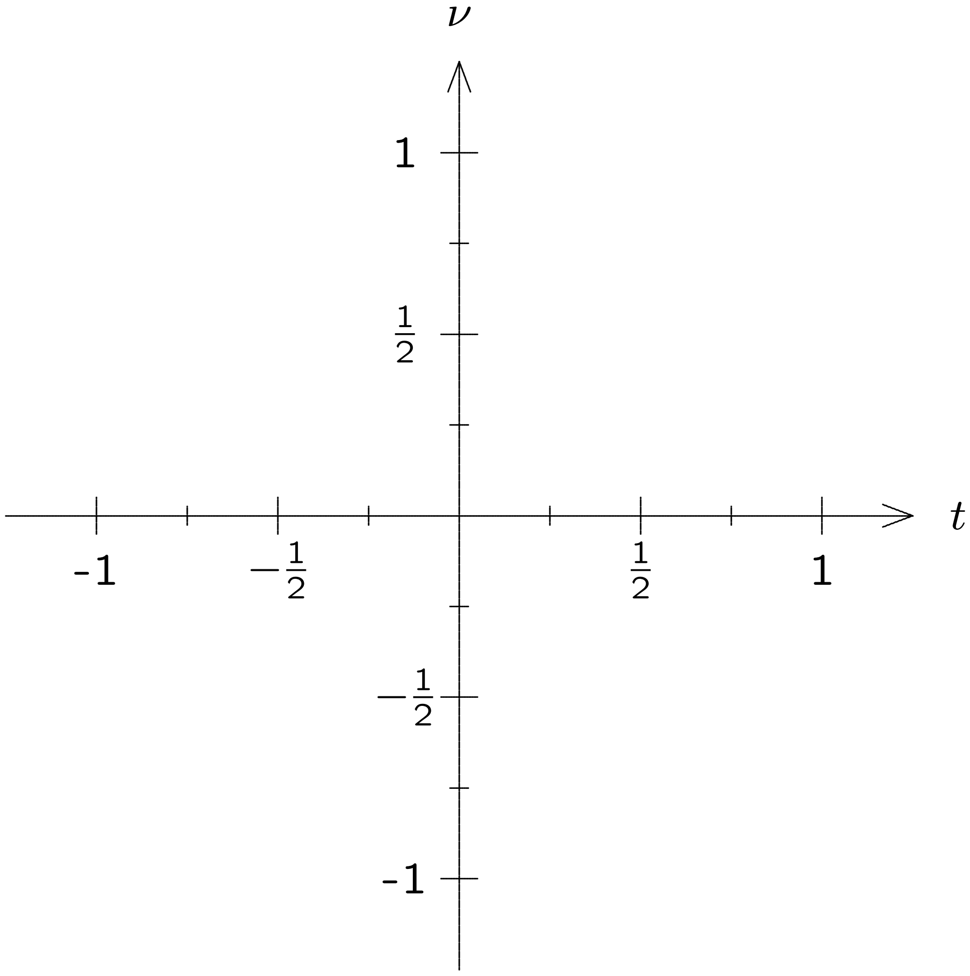
Operator Sampling

Götz Pfander

JACOBS UNIVERSITY

Summer 07, 12.7.07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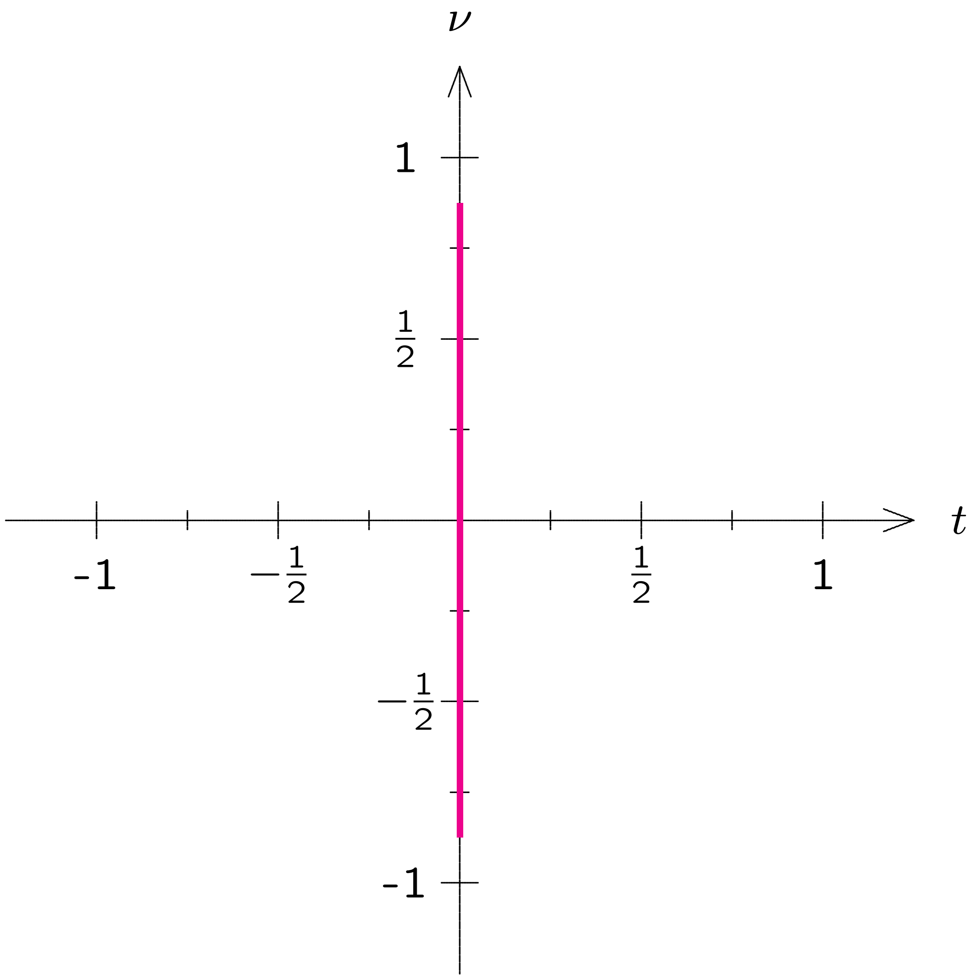
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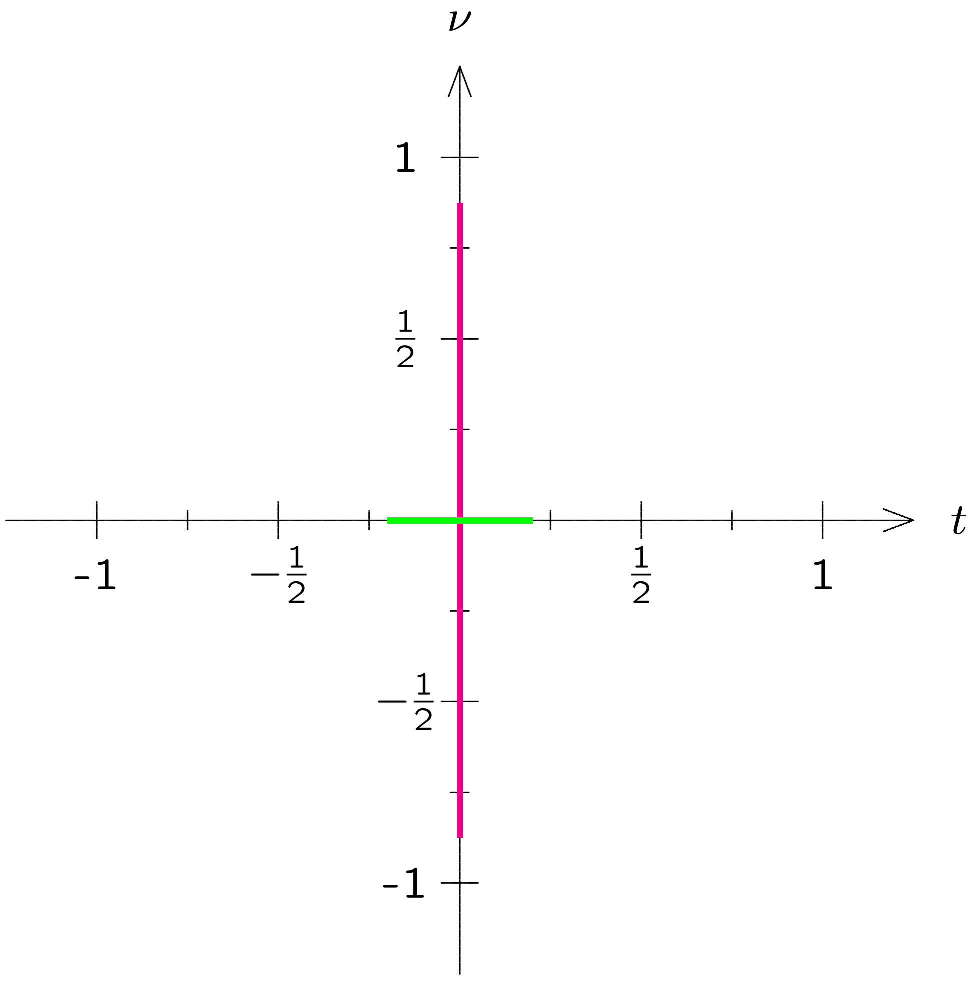
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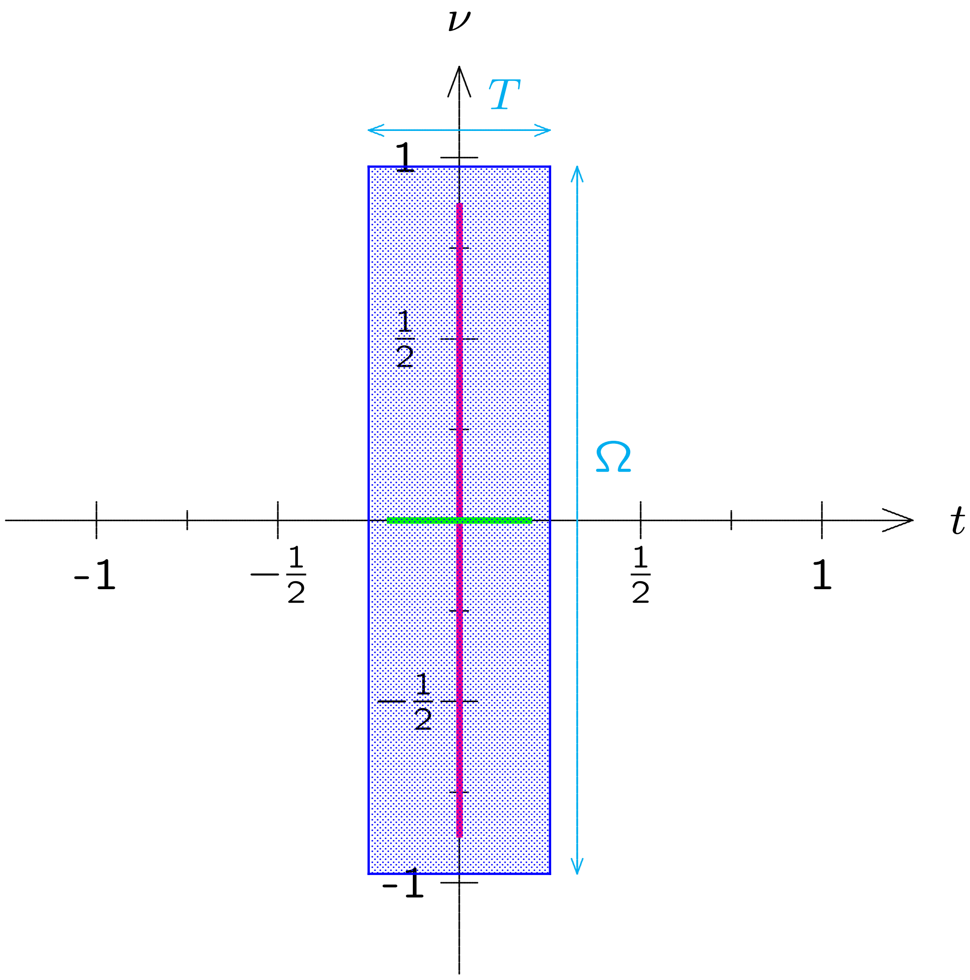
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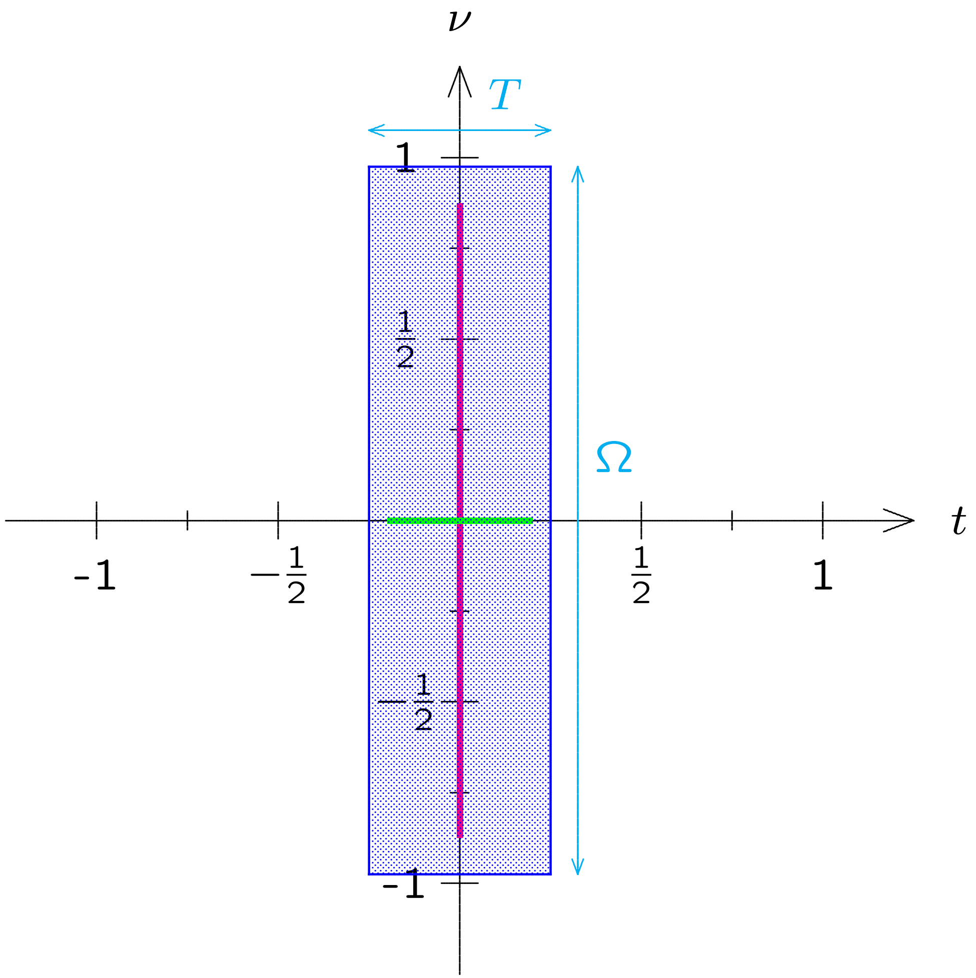
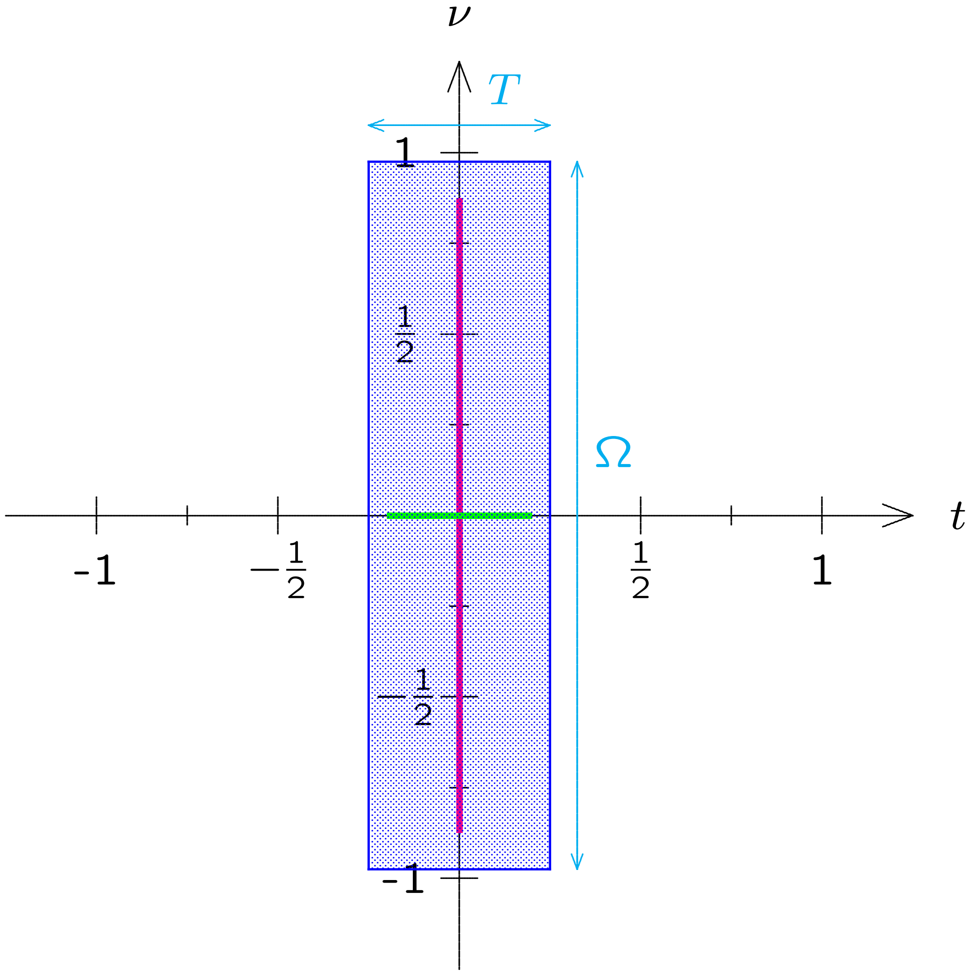


Classical sampling theorem $\Leftrightarrow$ *identifiability of operators* with distributional support of $\widehat{\sigma}$ (σ= Kohn–Nirenberg symbol) lying on ν axis.



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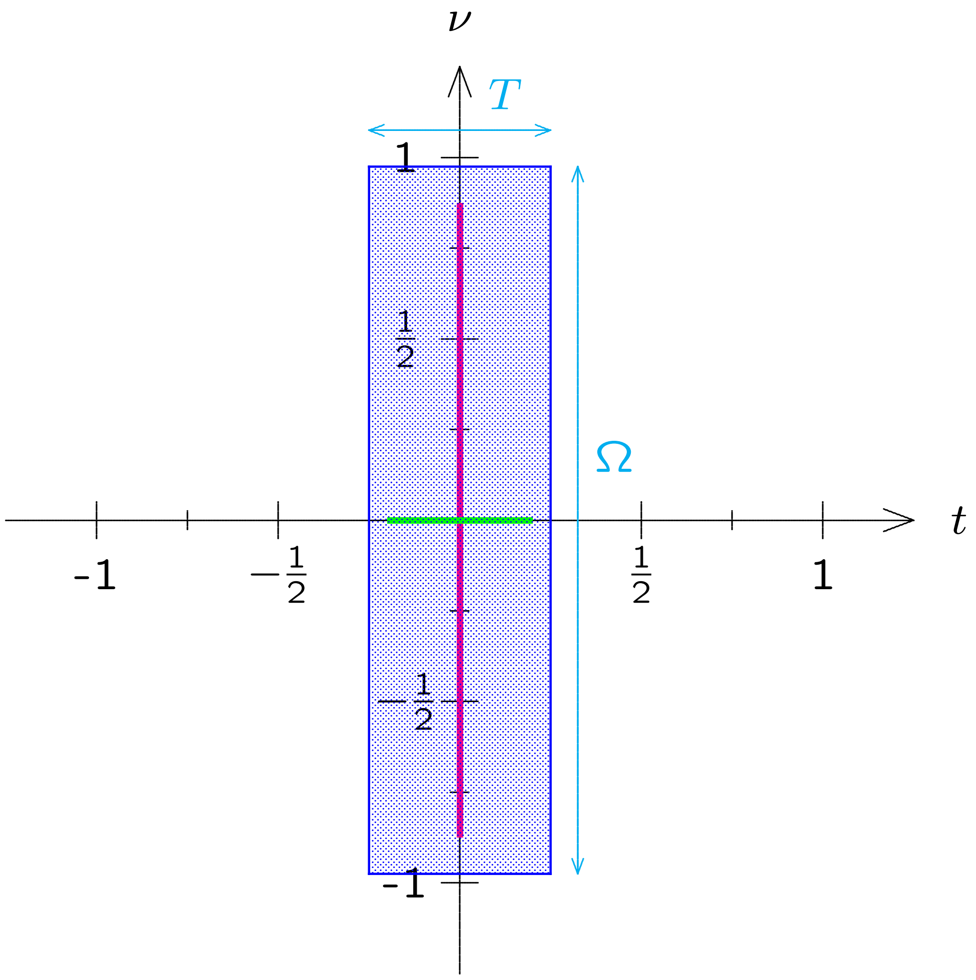

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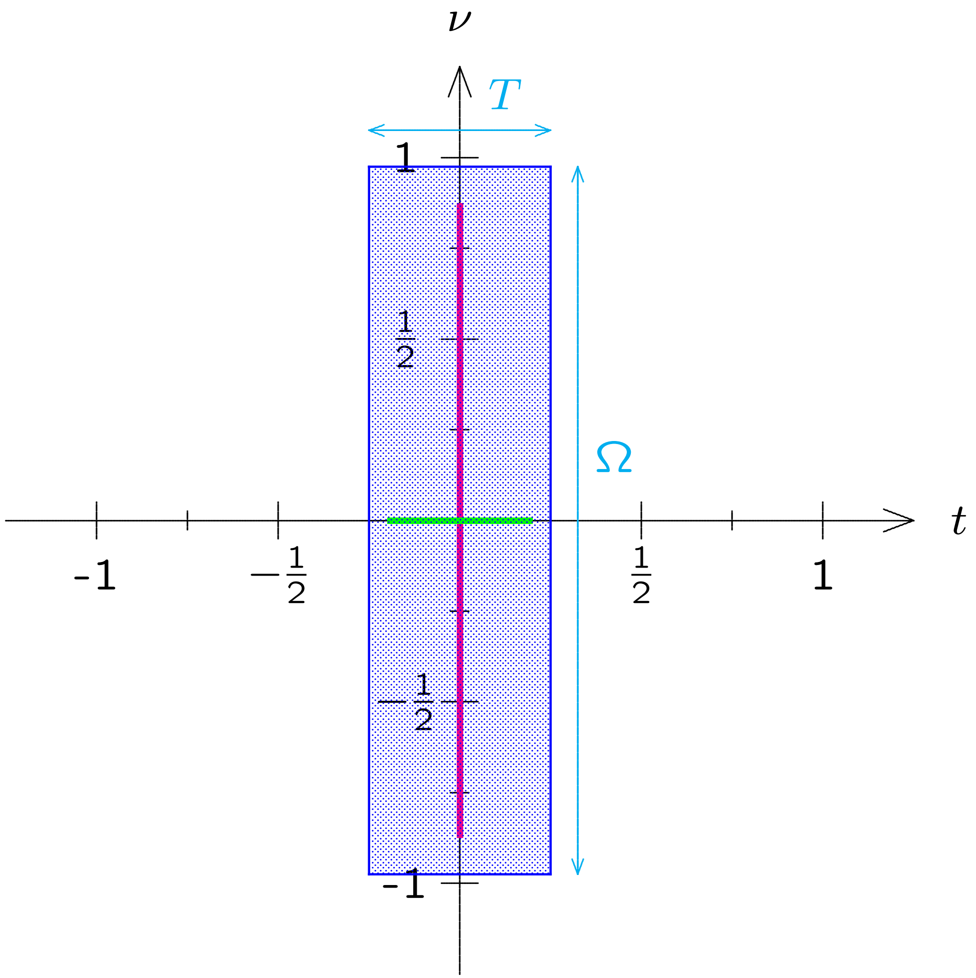
Our operator sampling theorem extends these results to operators with distributional support of $\widehat{\sigma}$ contained in sets of area less than one.



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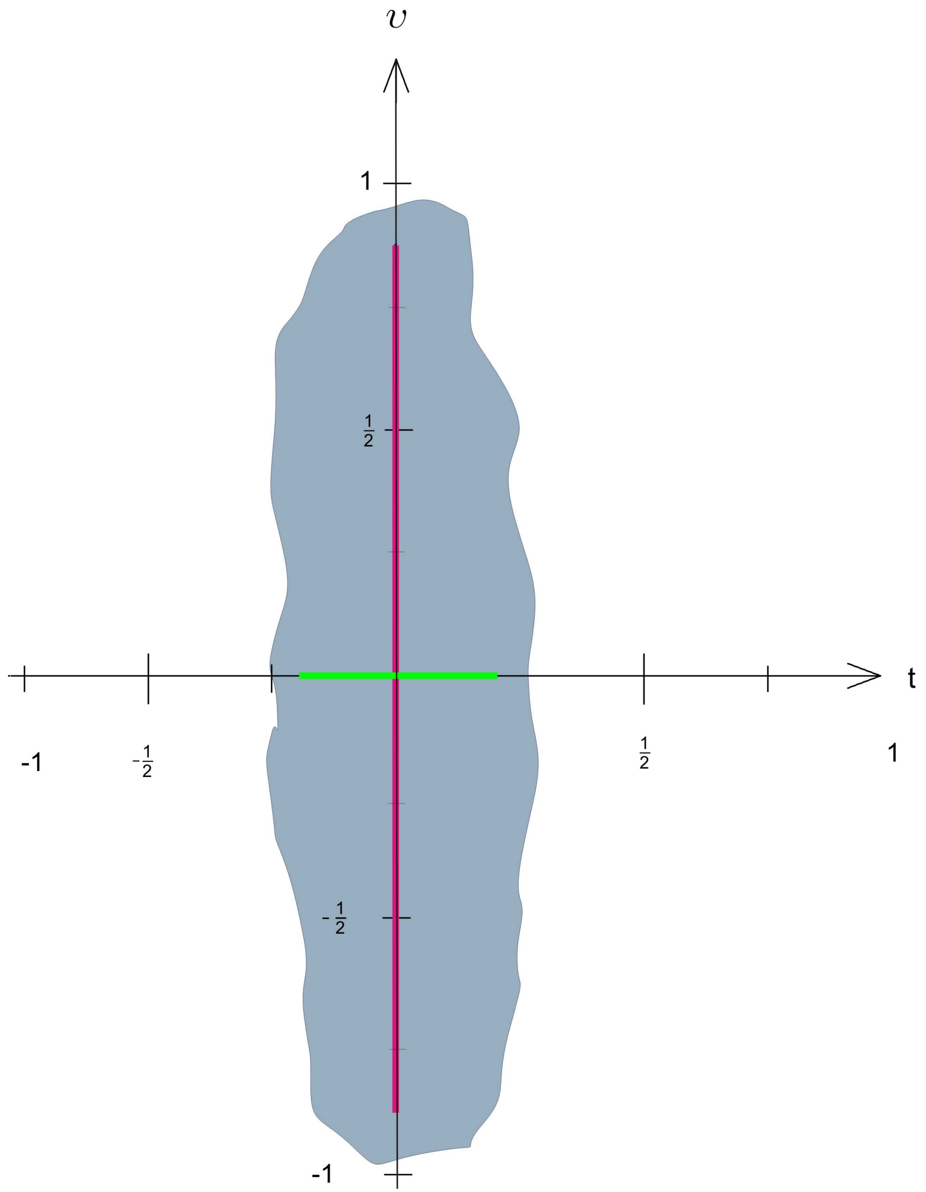

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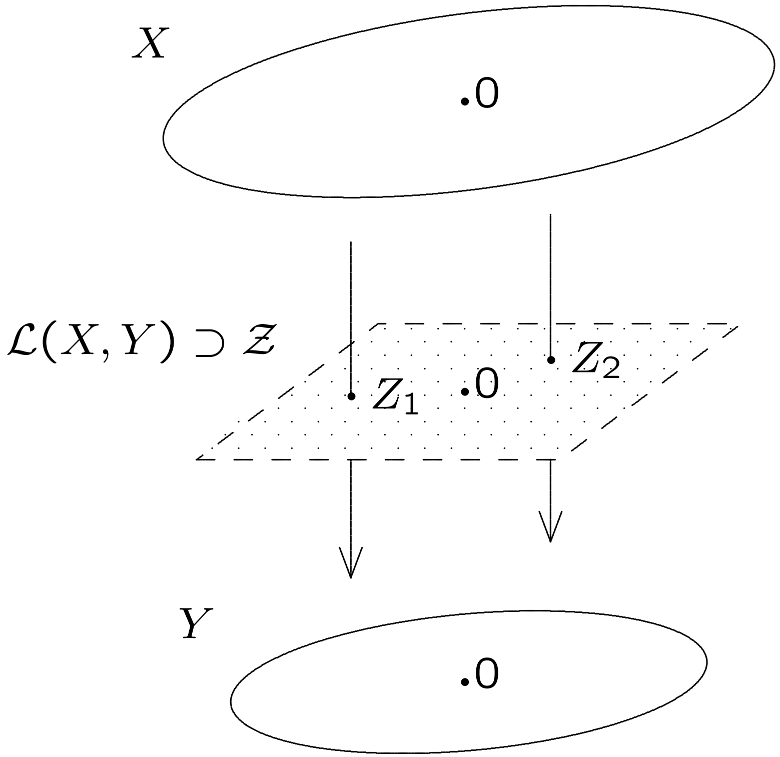
Theory I. The operator identification problem

Exists $g \in X$**, with** $\|Z\|_{\mathcal{Z}} \asymp \|Zg\|_Y$ **for all** $Z \in \mathcal{Z} \subseteq \mathcal{L}(X,Y)$ **?**



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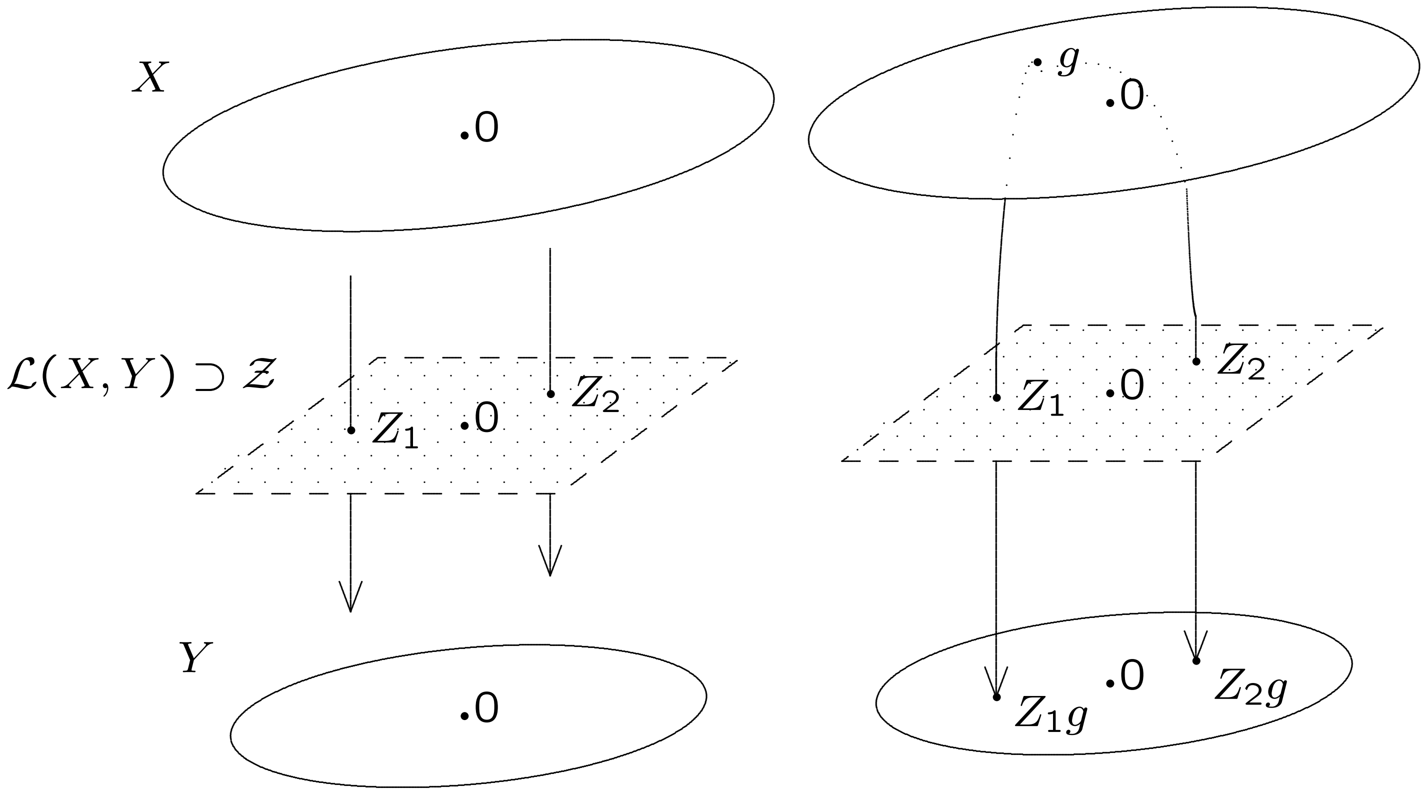

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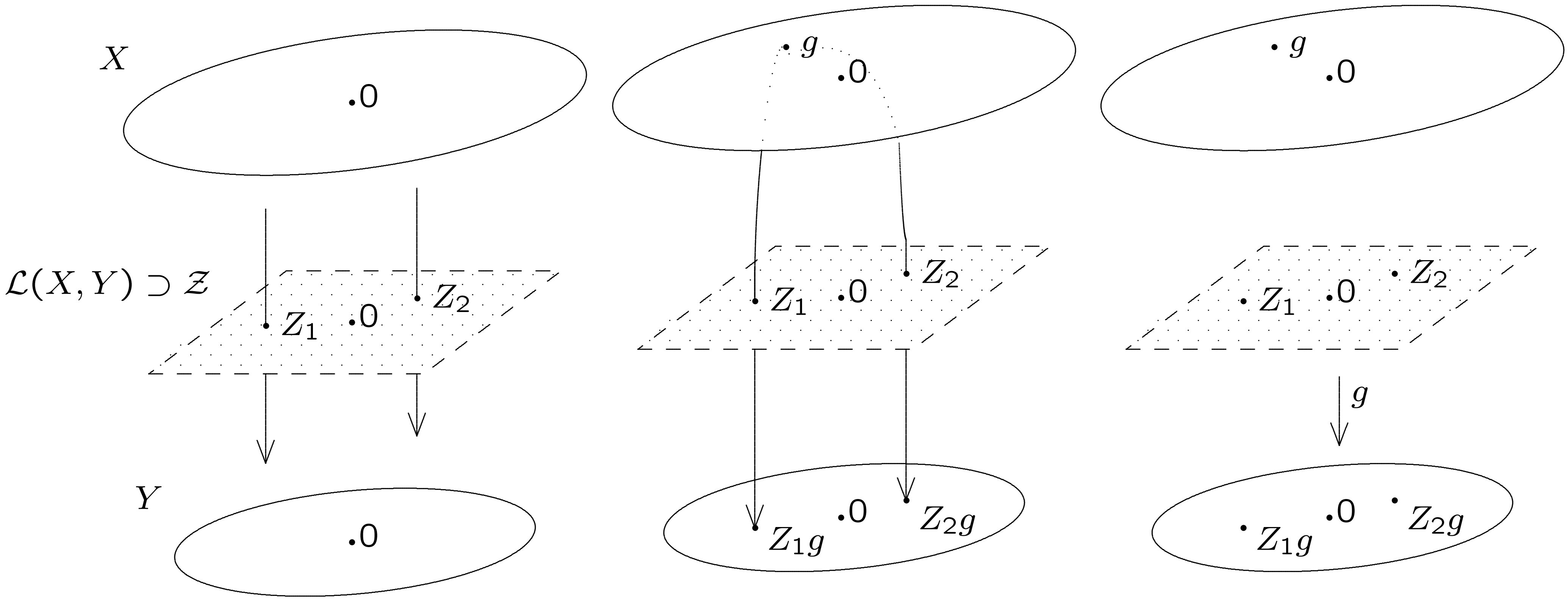

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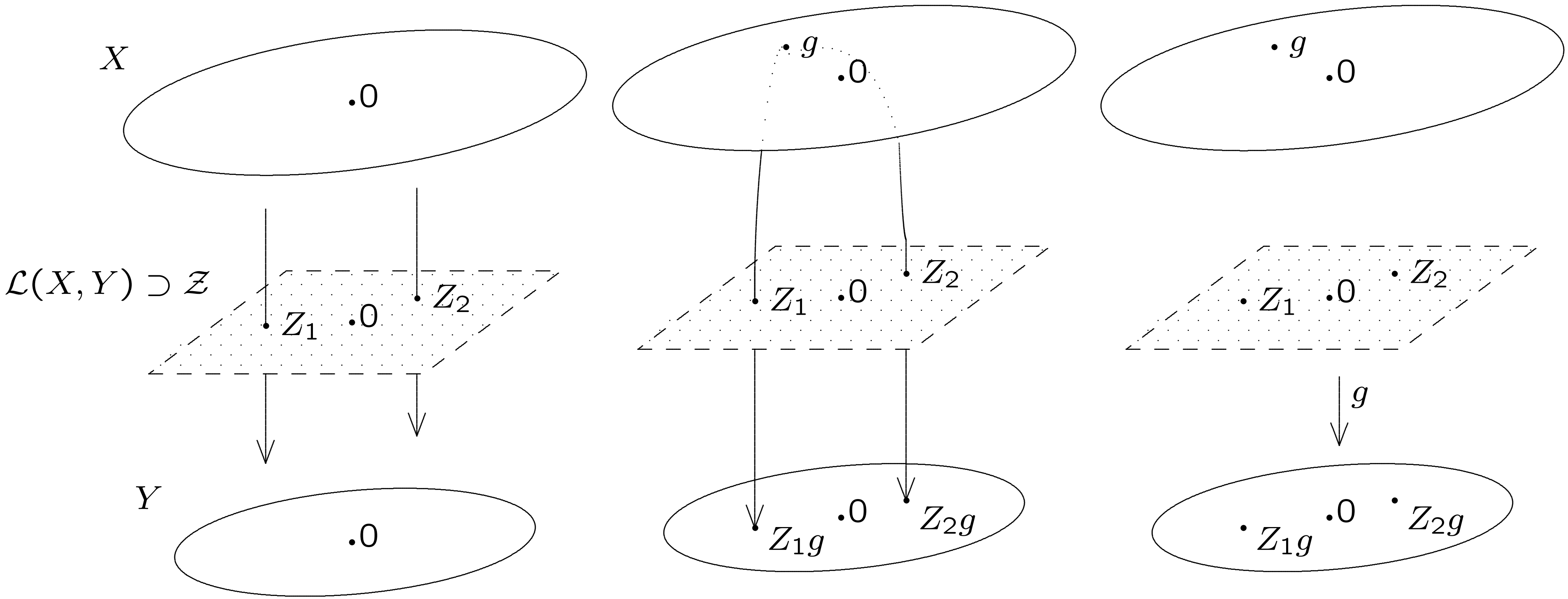
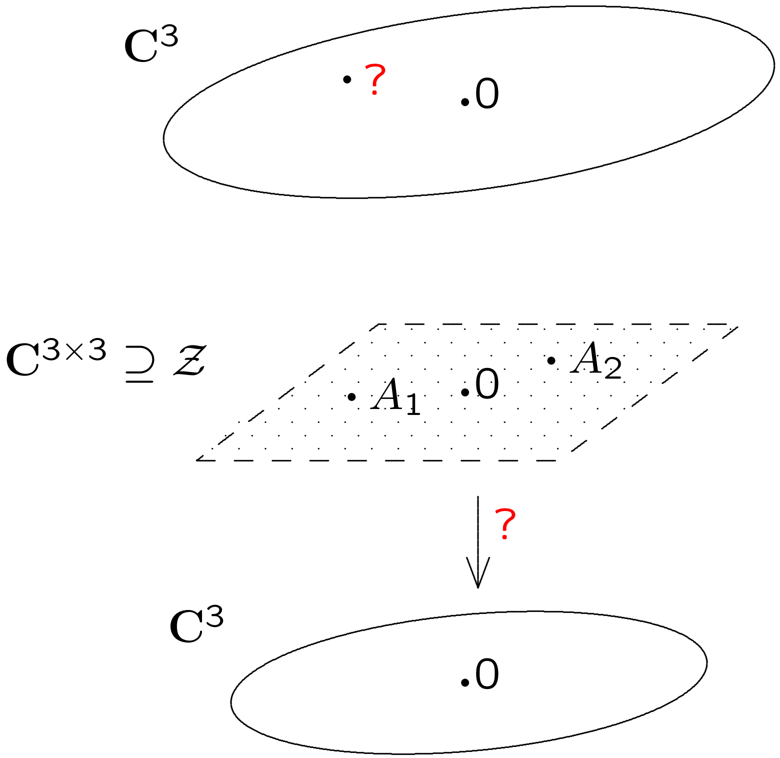


If $g = \sum c_j \delta_{s_j}$ then identification is referred to as operator sampling.

Examples



($\|\|_{\mathcal{Z}}$ arbitrary vector-space norm)

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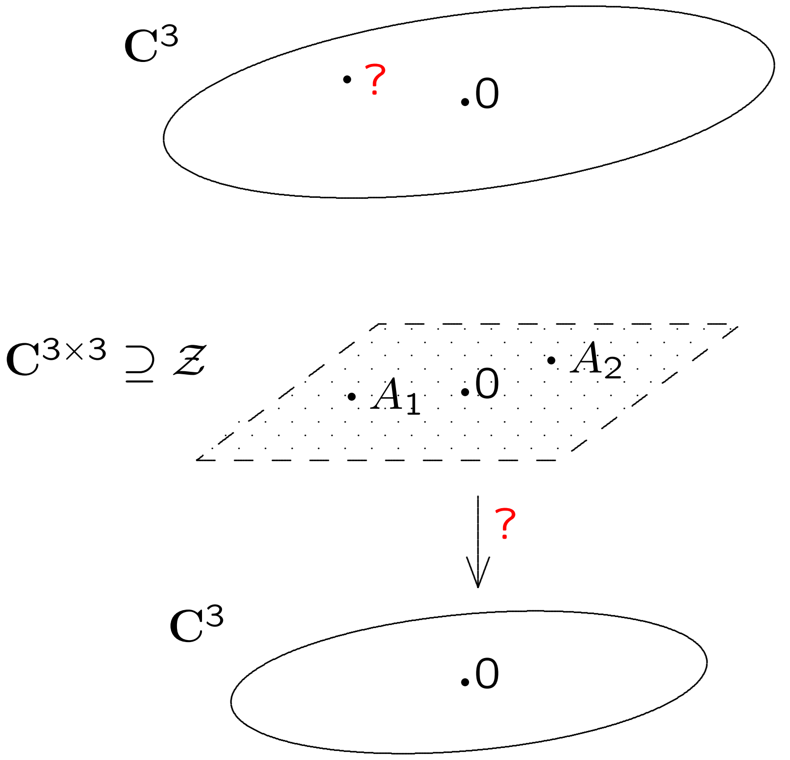

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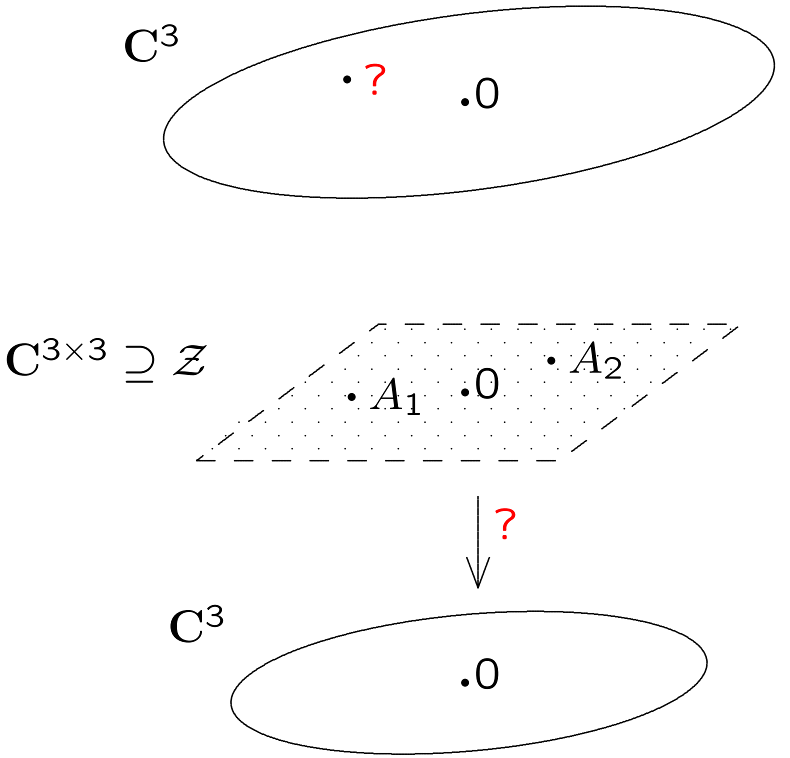

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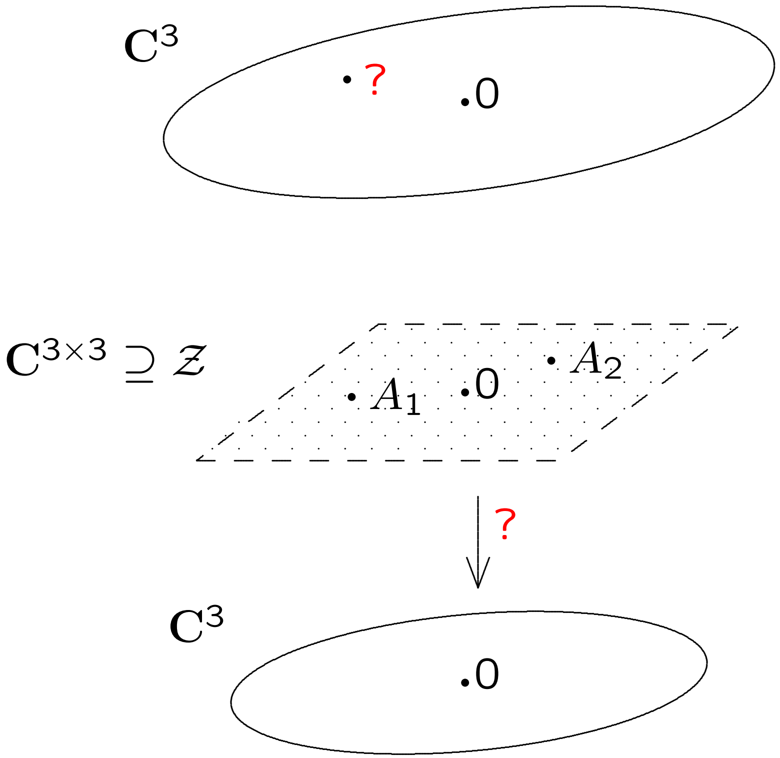

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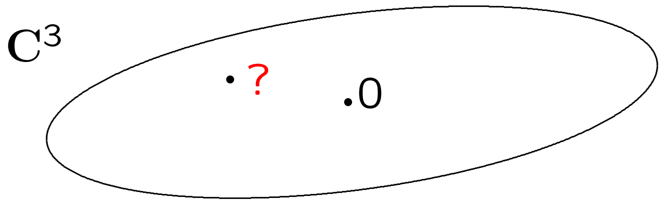

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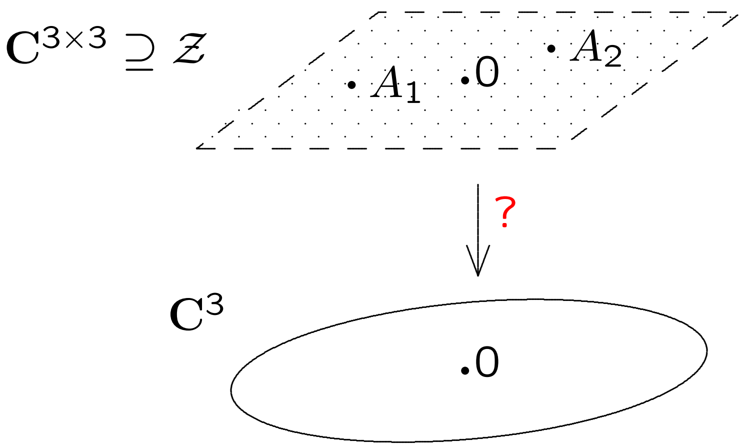

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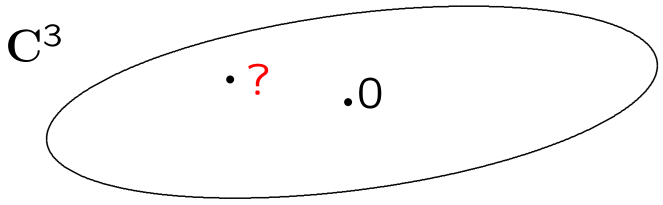

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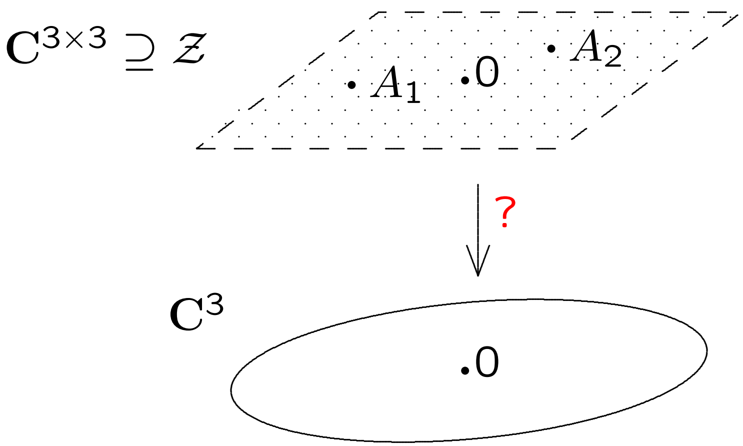

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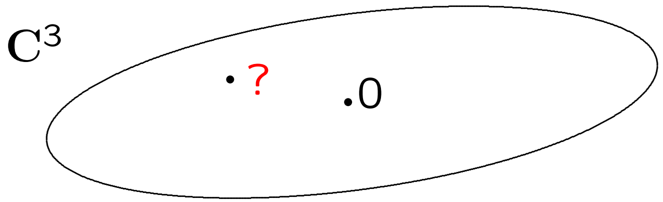

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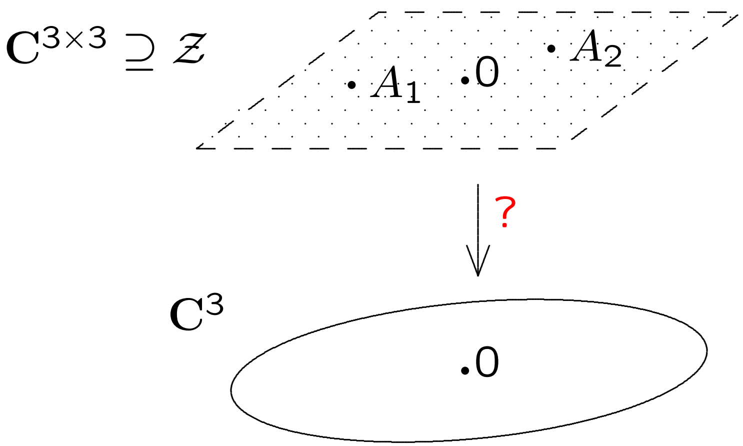

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$\mathcal{Z}$ identifiable since $\begin{pmatrix} a & 0 & 0 \\ b & 0 & 0 \\ c & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$

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Definition. For $1 \leq p,q \leq \infty$ and $\mathsf{s} \in \mathbf{R}$ set (with corresponding norm)

$$OPW_{\mathsf{s}}^{pq}(S) = \{H: \ \text{supp}\,\widehat{\sigma_H} \subseteq S \text{ and } V_{\varphi_0\otimes\varphi_0}\sigma_H(x,\xi,\nu,t)(1+\xi^2)^{\frac{\mathsf{s}}{2}} \in L^{pq,11}\}.$$

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Note. If $\mathsf{s} \leq -n$, $n \in \mathbf{N}$, then $OPW^{\infty,\infty}_{\mathsf{s}}(S)$ include • linear differential operators $\sum_{k=0}^n a_k(x)\frac{\partial^k}{\partial x^k}$ with bounded a_k and $\{0\} \times \bigcup_k \mathrm{supp}\,\widehat{a}_k \subset S$, • any pseudodifferential operators K of order n for which σ_K satisfies $\mathrm{supp}\,\widehat{\sigma_K} \subseteq S$, • finite delay convolution operators, • multiplication operators with bandlimited symbols.

Main Result I.

Classical Sampling Theorem. Given a function $m \in PW^2(\Omega)$ and T with $T\Omega < 1$. Choose $s \in PW^2\big(\frac{2}{T} - \Omega\big)$ with $\widehat{s} = 1$ on $[-\frac{\Omega}{2}, \frac{\Omega}{2}]$.

Then $\{m(nT)\}$ fully characterizes m, $\|\, \{m(kT)\}\, \|_{l^2} \asymp \|m\|_{L^2}$ and

$$m(x) = T \sum_{k \in \mathbf{Z}} m(kT)\, s(x - kT) = T \sum_{k \in \mathbf{Z}} m(kT)\; \frac{\sin 2\pi\Omega(x - kT)}{\pi(x - kT)}\,.$$

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Theorem (GP, D. Walnut) For $H \in OPW_s^{pq}([-\frac{\Omega}{2}, \frac{\Omega}{2}] \times [-\frac{T'}{2}, \frac{T'}{2}])$ and $\Omega T' < \Omega T < 1$, choose $s \in PW^p\big(\frac{2}{T} - \Omega\big)$ with $\widehat{s} = 1$ on $[-\frac{\Omega}{2}, \frac{\Omega}{2}]$ and $r \in \mathcal{S}$ with $\operatorname{supp} r \subset [-T + \frac{T'}{2}, T - \frac{T'}{2}]$ and $r = 1$ on $[-\frac{T'}{2}, \frac{T'}{2}]$.

Then $H Ш_T = H \sum_k \delta_{kT}$ fully determines H, in fact, we have $\|H\, Ш\,\|_{M_s^{pq}} \asymp \|\sigma_H\|_{PW_s^{pq}}$ and

$$h_H(t,x) = r(t) \sum_{k\in\mathbf{Z}} (H Ш_T)(t + kT)\, s(x - kT)\,. \qquad \left(Hf(x) = \int h_H(t,x) f(x-t)\, dt \right)$$

Corollary. Given $m \in PW(\Omega)$ and T with $T\Omega < 1$. Choose $s \in PW\big(\frac{2}{T} - \Omega\big) \supset PW(\frac{1}{T}) \supset PW(\Omega)$ with $\widehat{f}s = 1$ on $[-\frac{\Omega}{2}, \frac{\Omega}{2}]$.

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$$Mf(x) = \int \overbrace{m(x)}^{\sigma_M(x,\xi)}\, \widehat{f}(\xi) e^{2\pi i x\xi} d\xi = \int \overbrace{m(x)\delta_0(t)}^{h_M(t,x)}\, f(x-t)dt = \int\int \overbrace{\delta_0(t)\widehat{m}(\nu)}^{\eta_M(t,\nu)}\, e^{2\pi i x\nu} f(x-t)\, dt\, d\nu.$$

Since $M \in OPW^{2\infty}(\Omega, \frac{T}{2})$ we have

$$h_M(t,x) = r(t) \sum_{k \in \mathbf{Z}} (M \amalg_T)(t + kT)\, s(x - kT) = T \sum_{k \in \mathbf{Z}} m(kT)\, s(x - kT).$$

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$$Mf(x) = \int \overbrace{m(x)}^{\sigma_M(x,\xi)} \widehat{f}(\xi) e^{2\pi i x \xi} d\xi = \int \overbrace{m(x)\delta_0(t)}^{h_M(t,x)} f(x-t)dt = \int\int \overbrace{\delta_0(t)\widehat{m}(\nu)}^{\eta_M(t,\nu)} e^{2\pi i x \nu} f(x-t)\, dt\, d\nu.$$

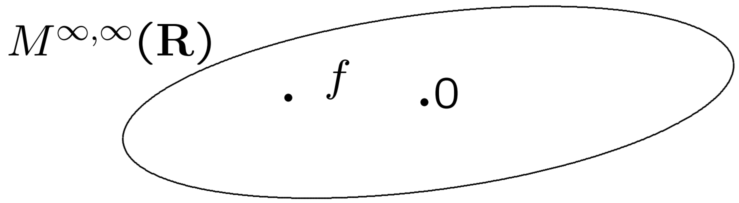
Since $M \in OPW^{2\infty}(\Omega, \frac{T}{2})$ we have

$$h_M(t,x) = r(t) \sum_{k \in \mathbf{Z}} (M Ш_T)(t + kT)\, s(x - kT) = T \sum_{k \in \mathbf{Z}} m(kT)\, s(x - kT).$$

Further $\|\sigma_M\|_{PW^{2\infty}} \asymp \|m\|_{M^{2,2}} \asymp \|m\|_{L^2}$ and $\|M\, Ш_T\, \|_{M^{2\infty}} \asymp \|\{m(nT)\}_n\|_2$.

Theorem(W. Kozek, GP) $OPW^{11}([-\frac{T}{2},\frac{T}{2}]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ identifiable $\Longrightarrow$ $T\Omega \leq 1$.

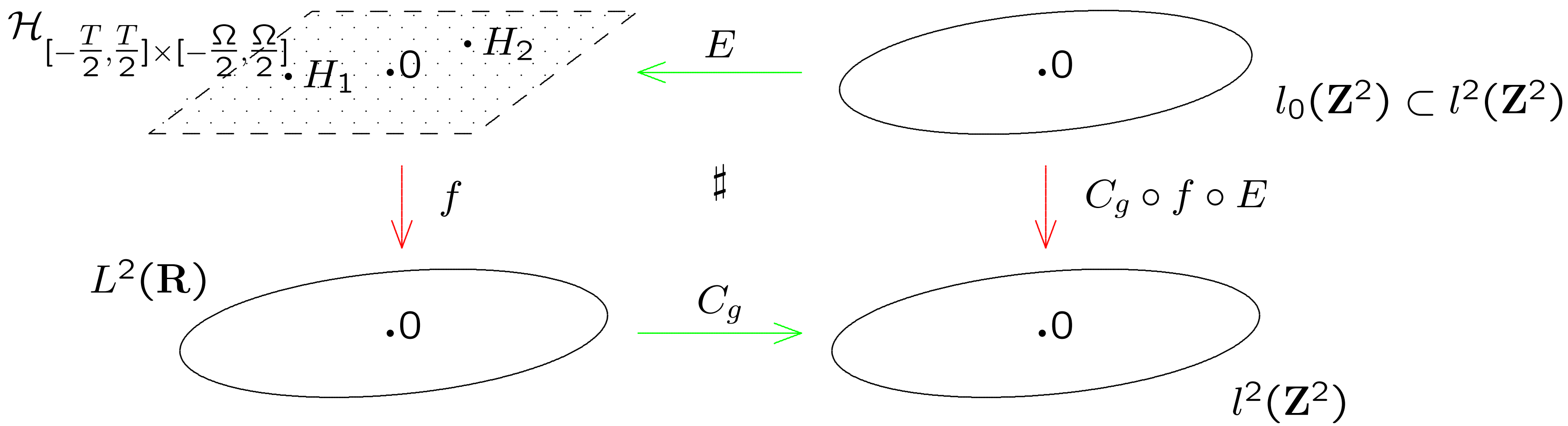
For $T\Omega > 1$, pick $\lambda > 1$ with $1 < \lambda^4 < T\Omega$, prototype operator $P_\lambda \in OPW^{\infty\infty}([-\frac{T}{2},\frac{T}{2}]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$.



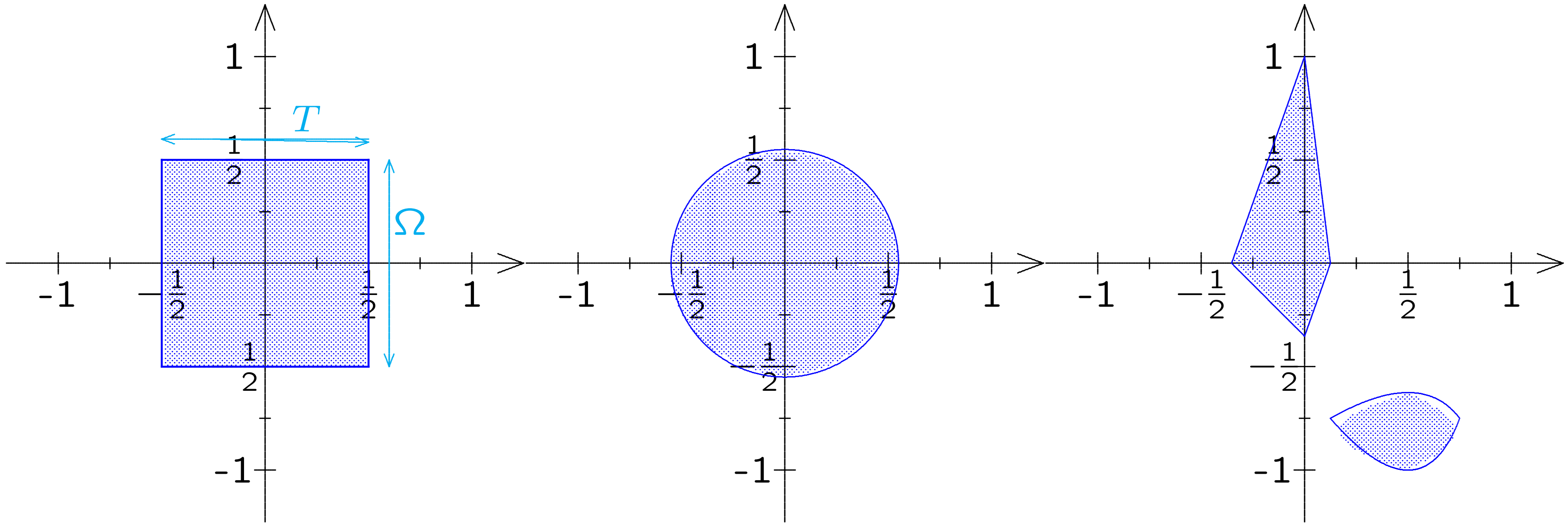
Define for $\alpha = T^{-1}$ and $\beta = \Omega^{-1}$:

$$E \;:\; l_0(\mathbf{Z}^2) \to \mathcal{H}_{[-\frac{T}{2},\frac{T}{2}]\times[-\frac{\Omega}{2},\frac{\Omega}{2}]}, \quad \{\sigma_{k,l}\} \mapsto \sum_{k,l} \sigma_{k,l}\, M_{k\lambda\alpha} T_{l\lambda\beta}\, P_\lambda\, T_{-l\lambda\beta} M_{-k\lambda\alpha}$$

$$C_g \;:\; L^2(\mathbf{R}) \to l^2(\mathbf{Z}^2), \quad h \mapsto \{\langle h, M_{k'\lambda^2\alpha} T_{l'\lambda^2\beta} f\rangle\}_{k',l'}$$



Götz Pfander



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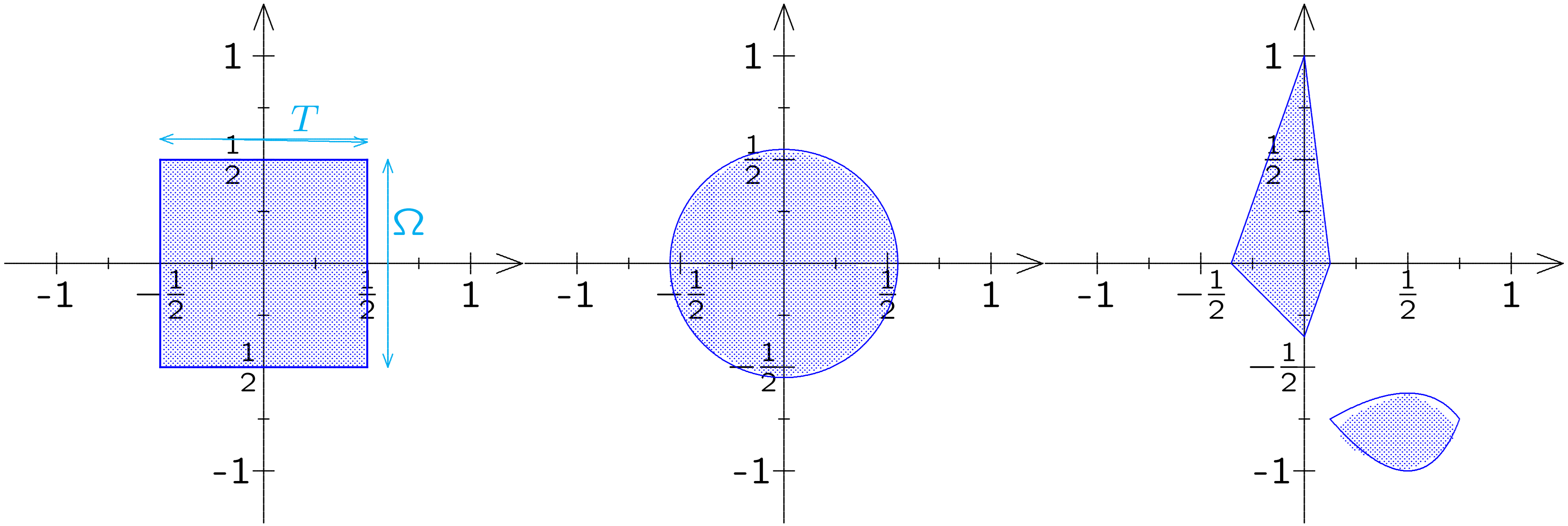
Extension II

Theorem (GP, D. Walnut) For $M \subset \mathbf{R} \times \widehat{\mathbf{R}}$ measurable and bounded, $\mu(\partial M) = 0$, we have

- If $\mu(M) < 1$ then $OPW^{11}(M)$ is identifiable.
- If $\mu(M) > 1$ then $OPW^{11}(M)$ is not identifiable.



Sparse signal recovery

For a prescribed dictionary $\mathcal{D} = \{\varphi_1, \varphi_2, \ldots, \varphi_N\} \subseteq \mathbf{C}^N$ set

$$\Sigma_k^{\mathcal{D}} = \{x = \sum c_j \varphi_j \in \mathbf{R}^N : \; \|c\|_0 = |\operatorname{supp} c| \leq k\}.$$

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Answer 1.

Necessary condition is $n \geq 2k$.

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to recover "sparse" quickly

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to recover "sparse" quickly from

Answer 2.

See sparse signal recovery, sparse approximations and compressed sensing literature.

Identification of matrices with sparse representations

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For a prescribed matrix dictionary $\mathcal{M} = \{M_1, M_2, \ldots, M_{Nn}\} \subseteq \mathbf{C}^{n\times N}$ set

$$M \in \mathbf{\Sigma}_{\mathsf{k}}^{\mathcal{M}} := \Big\{ M = \sum_r c_r M_r \text{ with } \|c\|_0 \leq k \Big\}.$$

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For $\mathcal{M}$, k, design $\rightarrow$ x

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Answer 1.

Necessary condition is $n \geq 2k$.

Identification of sparse matrices vs. sparse signal recovery

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Sparse signal recovery (sampling):

Design $\phi \in \mathbf{C}^{n\times N}$, $N > n$ so that all $x \in \Sigma_k^{\mathcal{D}} = \{x = \sum c_j \varphi_j \in \mathbf{R}^N : \ \|c\|_0 = |\operatorname{supp} c| \leq k\}$ can be recovered (quickly) from $y = \phi x$.

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Matrix identification: Let $\mathcal{M} = \{M_1, M_2, \ldots, M_N\} \subset \mathbf{C}^{n\times n}$ be a finite set of matrices.

Design g so that all $A \in \Sigma_{\mathsf{k}}^{\mathcal{M}} := \{A : \ A = \sum_r c_r M_r \text{ with } \|c\|_0 \leq k\}$ can be recovered (quickly) from $y = Ag$.

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Note. Recovery of $A = \sum_r x_r M_r \in \boldsymbol{\Sigma}_{\mathsf{k}}^{\mathcal{M}}$ from Ag

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Note.

Recovery of $A = \sum_r x_r M_r \in \boldsymbol{\Sigma}_{\mathsf{k}}^{\mathcal{M}}$ from Ag

$$\Leftrightarrow$$

Recovery of x from

$$y = Ag = \left(\sum_r x_r M_r\right) g = \sum_r x_r \left(M_r g\right) = \phi x$$

where $\phi = (\mathcal{M}g) = \left[M_1 g, M_2 g, \ldots, M_N g\right]$.

Identification of sparse matrices vs. sparse signal recovery

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Note.

$$\text{Recovery of } A = \textstyle\sum_r x_r M_r \in \Sigma_{\mathsf{k}}^{\mathcal{M}} \text{ from } Ag$$

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$$\text{where } \phi = (\mathcal{M}g) = \big[M_1 g, M_2 g, \ldots, M_N g\big].$$

We conclude that sparse matrix identification can be approached using sparse signal recovery methods. Note additional structure of $\Phi = (\mathcal{M}g)$.

Example. $\mathcal{G}$ = basis of time–frequency shifts

Definition. Let $\omega = e^{2\pi i/n}$ and $\mathcal{G} = \{T^k M^l\}$, where

$$Tx = T(x_0, \ldots, x_{n-1})^T = (x_1, x_2, \ldots, x_{n-1}, x_0)^T, \quad T = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Translation

$$Mx = (\omega^0 x_0, \omega^1 x_1, \ldots, \omega^{n-1} x_{n-1})^T, \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & 0 & \cdots & 0 \\ & & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \omega^{n-1} \end{pmatrix}$$

Modulation

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Remark. The columns of $\Phi = (\mathcal{G}g)$ form a tight Gabor frame with n^2 elements.

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$$Tx = T(x_0, \ldots, x_{n-1})^T = (x_1, x_2, \ldots, x_{n-1}, x_0)^T, \quad T = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Translation

$$Mx = (\omega^0 x_0, \omega^1 x_1, \ldots, \omega^{n-1} x_{n-1})^T, \quad M = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & 0 & \cdots & 0 \\ & & \vdots & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \omega^{n-1} \end{pmatrix}$$

Modulation

Remark. The columns of $\Phi = (\mathcal{G}g)$ form a tight Gabor frame with n^2 elements.

Theorem (J. Lawrence, GP, D. Walnut) For p **prime** exists $g \in \mathbb{C}^p$ which identifies $\Sigma_k^{\mathcal{G}}$ if $k \leq p/2$.

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Definition. The coefficient vector $x = \eta_A$ in $A = \sum_{l=0}^{n-1} \sum_{k=0}^{n-1} \eta_A(k,l) T^k M^l$ is called spreading function of A.

Theorem (GP, H. Rauhut).

- Let $\Lambda \subset \{0,\ldots,n-1\}^2$ with cardinality $|\Lambda| = k$.
- Let $A = \sum_{(k,l)\in\Lambda} \eta_A(k,l) T^k M^l$ and such that on Λ the random phases $\{\mathrm{sgn}(\eta_A(k,l))\}_{(k,l)\in\Lambda}$ are independent and uniformly distributed on the torus $\{z \in \mathbf{C}, |z| = 1\}$.
- Choose a random vector g with with entries $g_k = \frac{1}{\sqrt{n}} e_k$ with e_k being independent random variables with uniform distribution on the torus.
- Let $\sigma > 8$.

Then with probability at least

$$\exp\left(-\frac{n}{\sigma k} + \ln(2(n-k))\right) + \exp\left(-\frac{n}{16ek} + \ln(Ck)\right) + 4n^{-(\sigma/4-2)}$$

the algorithm Basis Pursuit (l^1–minimization) recovers η_A, and therefore A, from $Ag = (\mathcal{G}g)\eta_A$.

The constant $C \approx 1.075$ and the probability estimate above is effective once

$$k < C' \frac{n}{\ln(n)} .$$

Relevant Publications

W. Kozek and G.E. Pfander. Identification of operators with bandlimited symbols. *SIAM J. Math. Anal.*, 37(3):867–888, 2006.

F. Krahmer, G.E. Pfander, and P. Rashkov. Uncertainty principles for time–frequency representations on finite abelian groups. http://arxiv.org/abs/math.CA/0611493, preprint, 2006.

J. Lawrence, G.E. Pfander, and D. Walnut. Linear independence of Gabor systems in finite dimensional vector spaces. *J. Fourier Anal. Appl.* 11(6):715–726, 2005.

G.E. Pfander. Measurement of time–varying Multiple–Input Multiple–Output channels. Preprint, 2007.

G.E. Pfander, and H. Rauhut. Sparse representations in Gabor systems. In preparation, 2007.

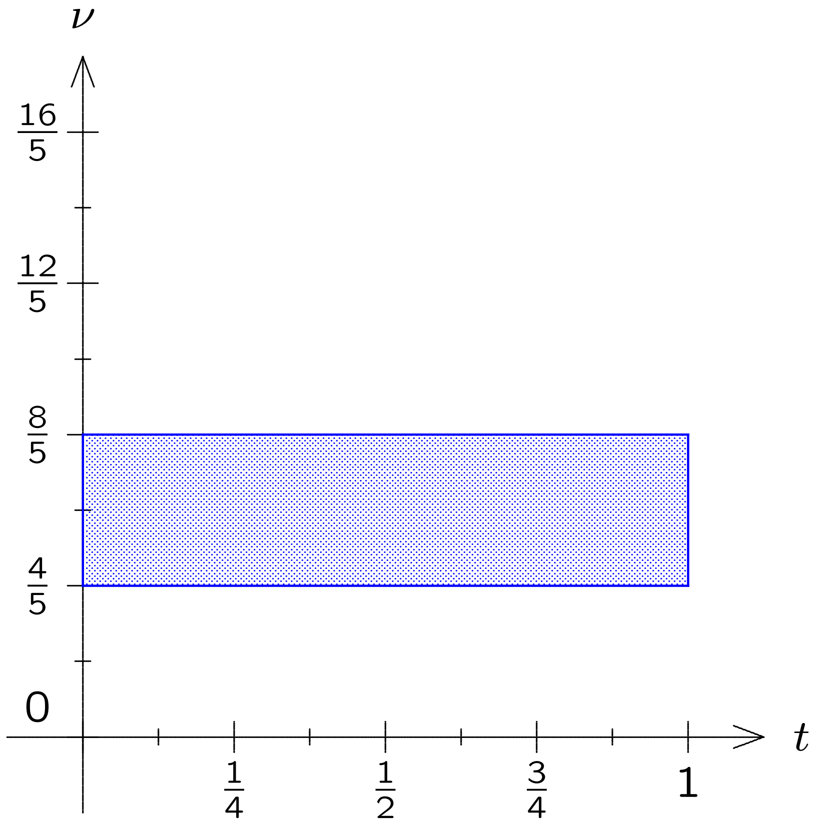
G.E. Pfander, H. Rauhut, and J. Tanner. Identification of matrices having a sparse representation. In preparation, 2007.

G.E. Pfander and D. Walnut. Measurement of time–variant channels. *IEEE Trans. Info. Theory*, 52(11):4808–4820, 2006.

G.E. Pfander and D. Walnut. Sampling of operators. In preparation, 2007.

Theorem (W. Kozek, GP) $OPW^{11}([-\frac{T}{2},\frac{T}{2}]\times[-\frac{\Omega}{2},\frac{\Omega}{2}])$ identifiable $\Longleftarrow$ $T\Omega \leq 1$.

Spreading support of H,
$\mu(\text{supp } \eta_H) = \frac{4}{5}$.



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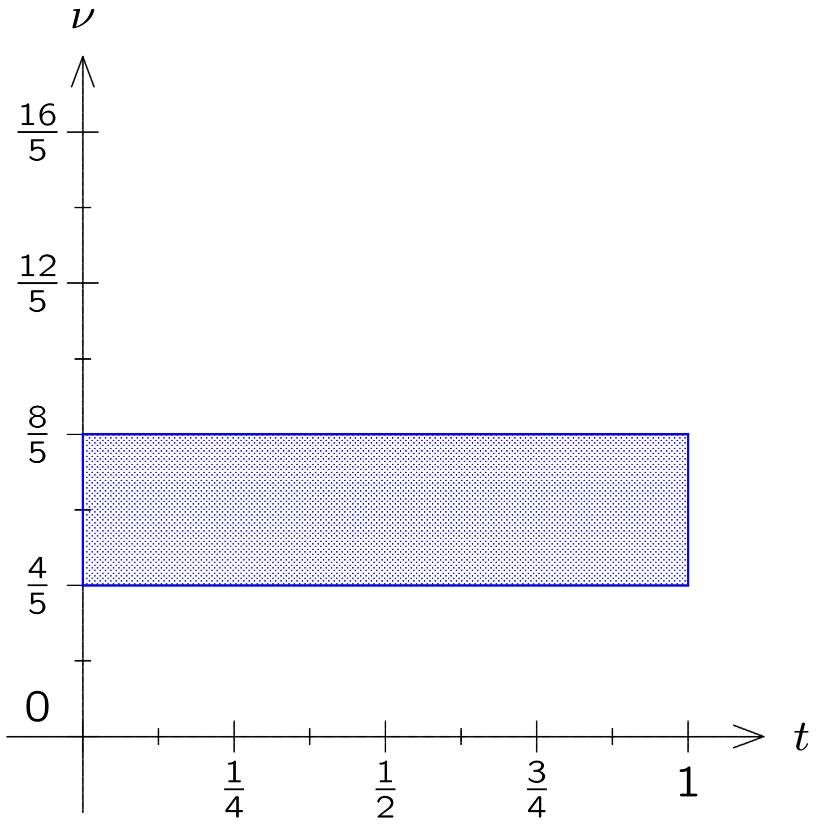
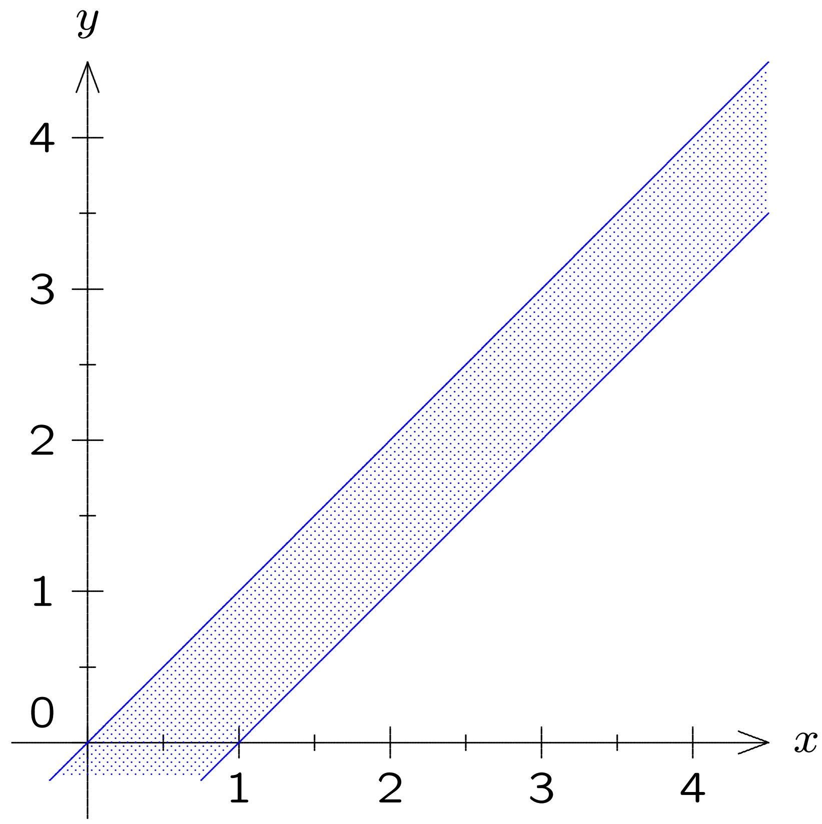


Support of κ_H,



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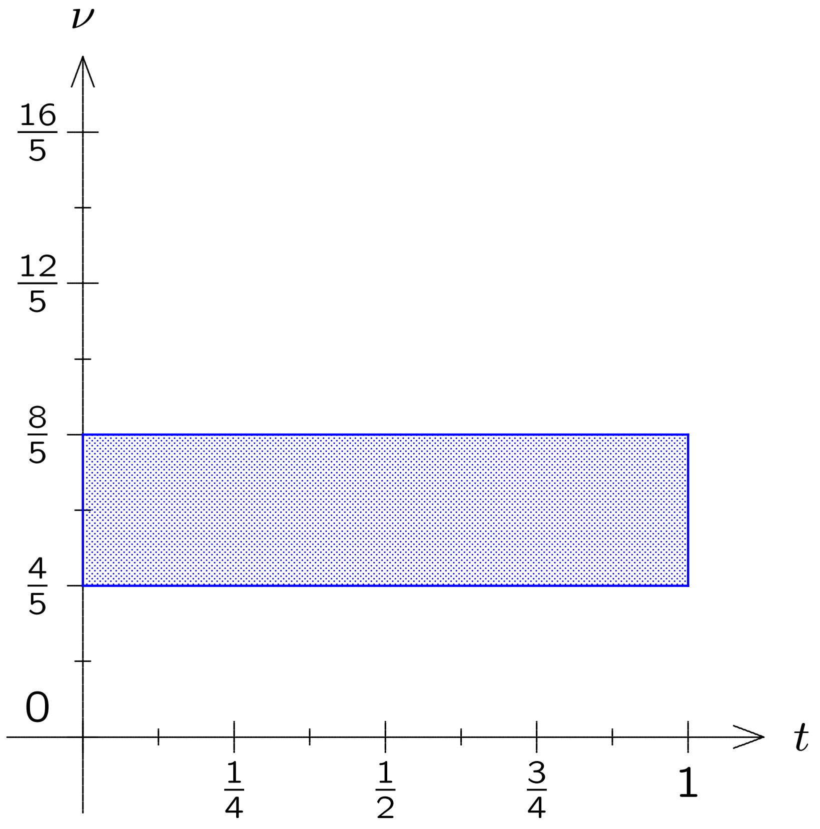

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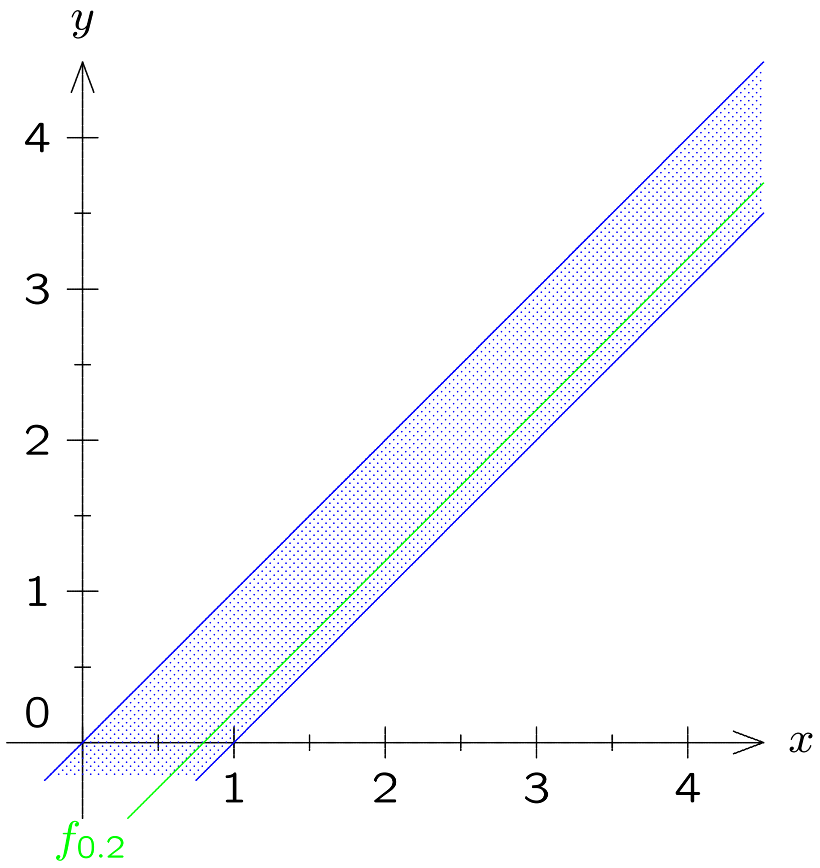

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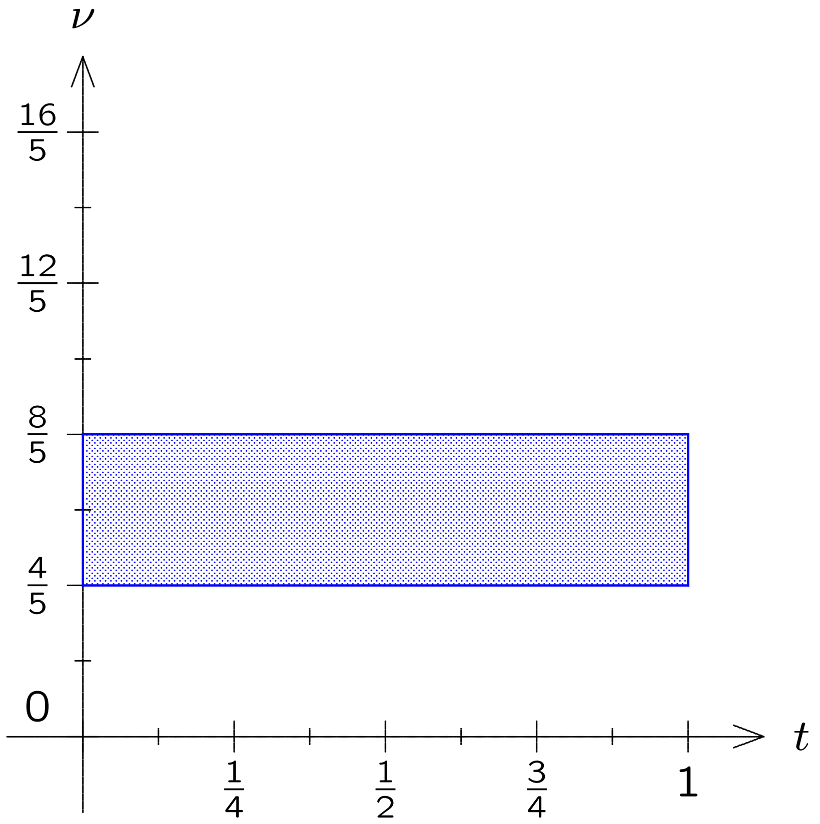

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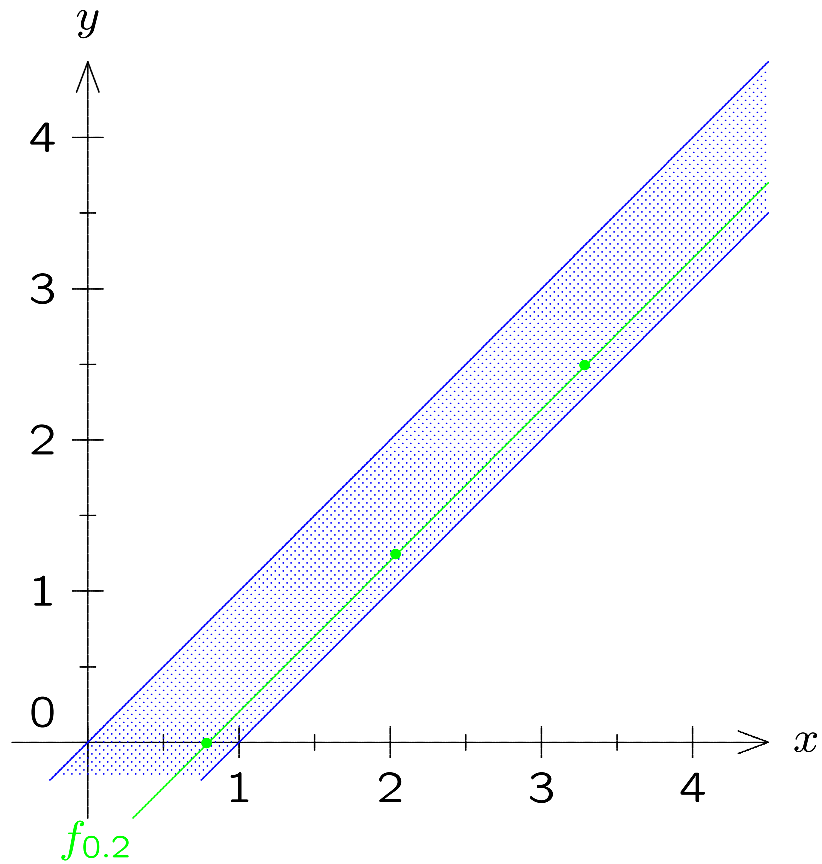

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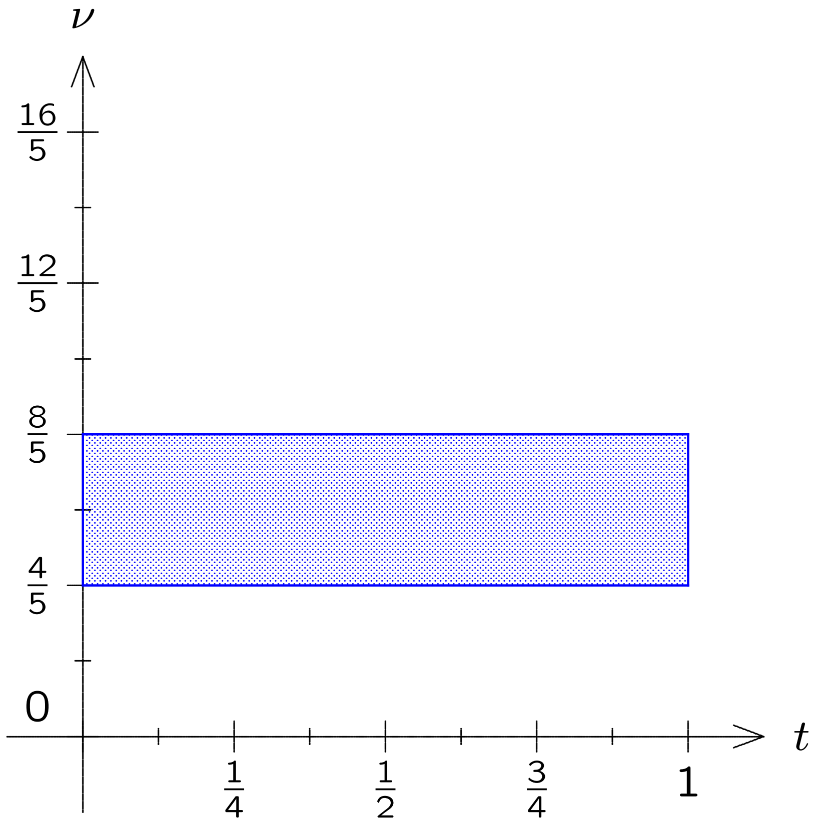

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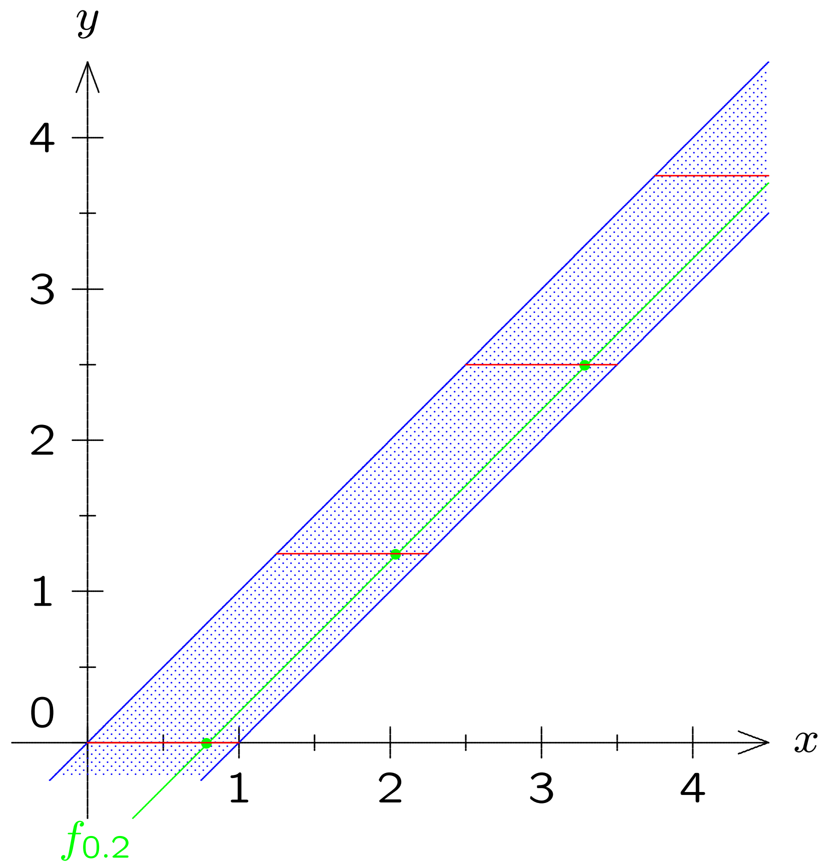

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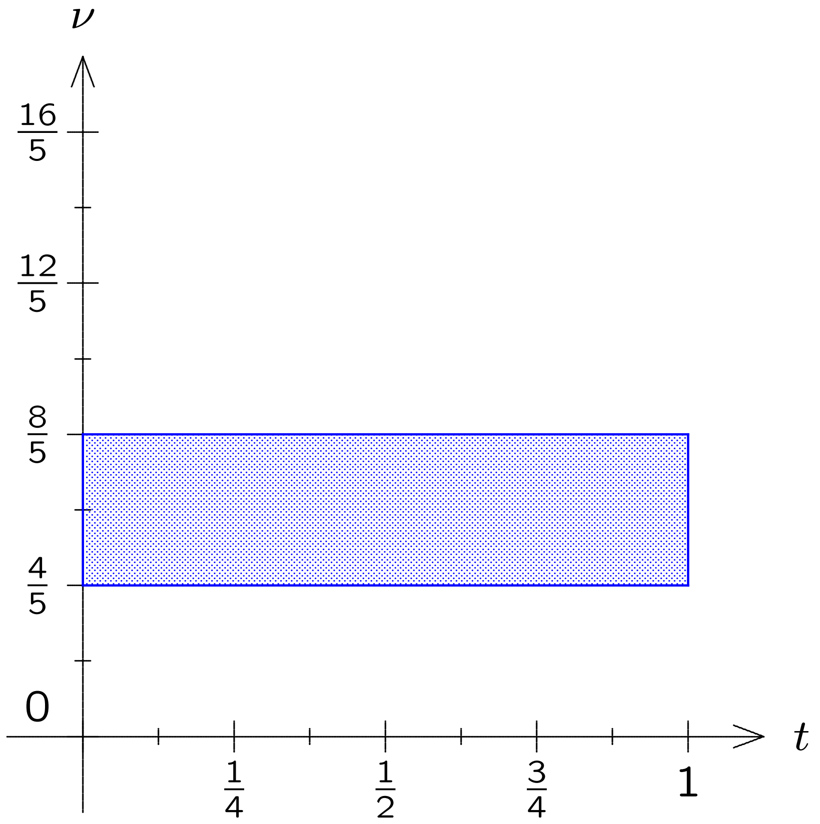

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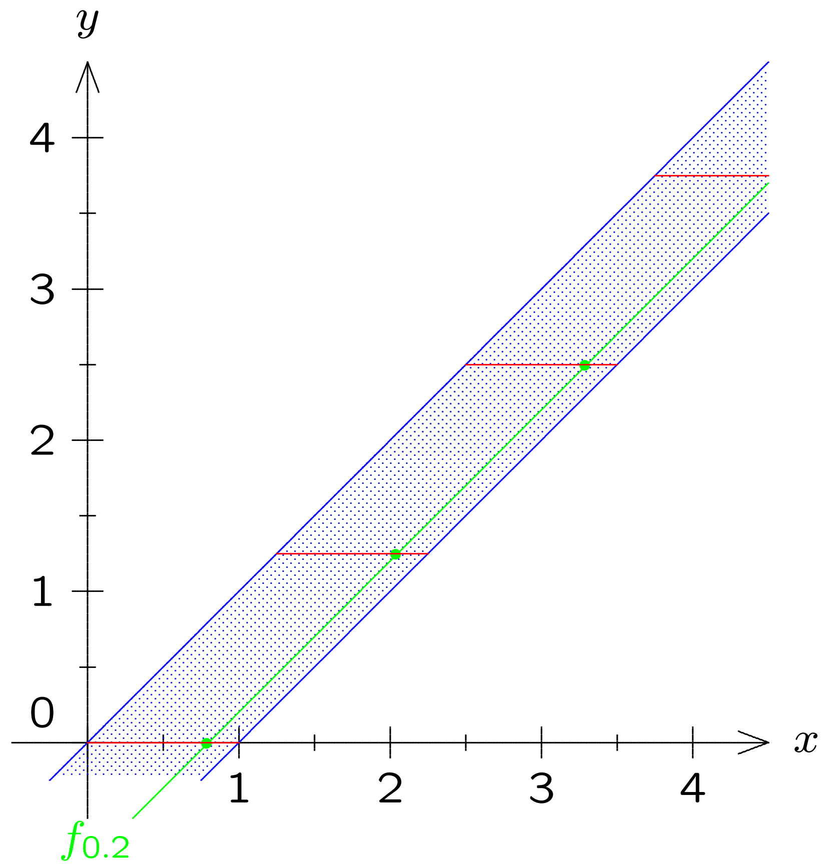

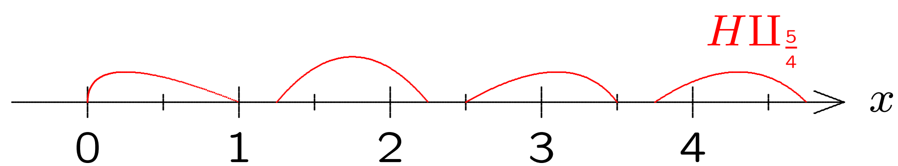

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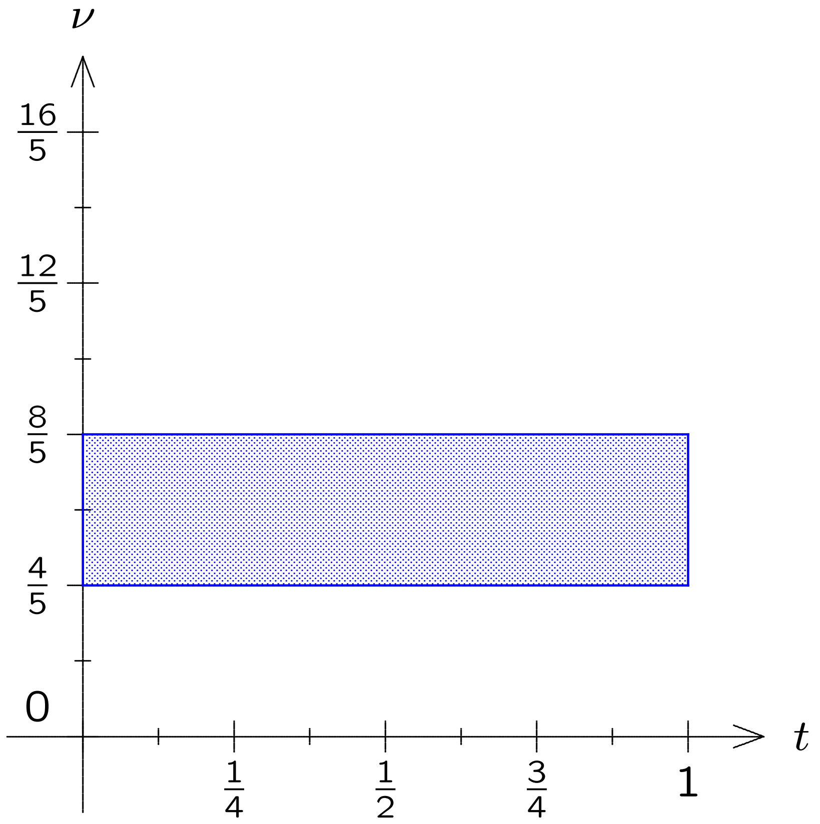

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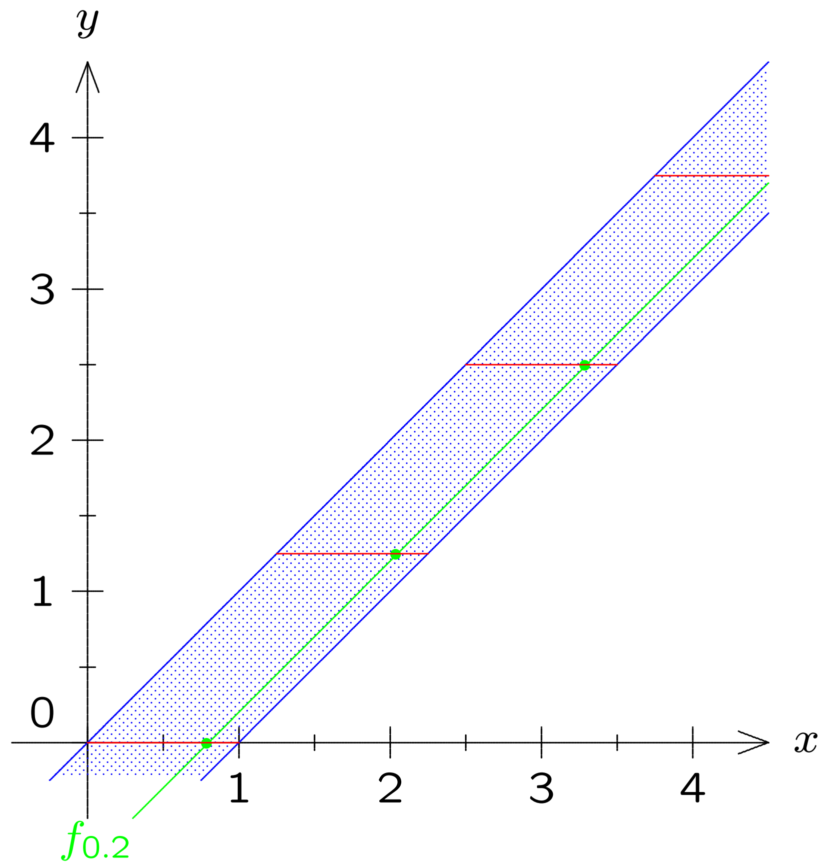
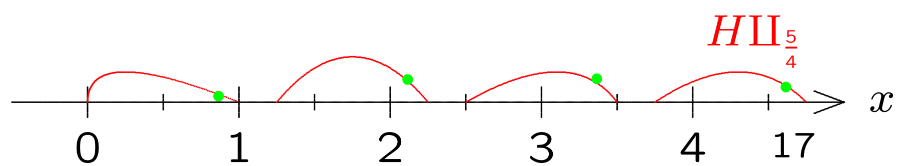
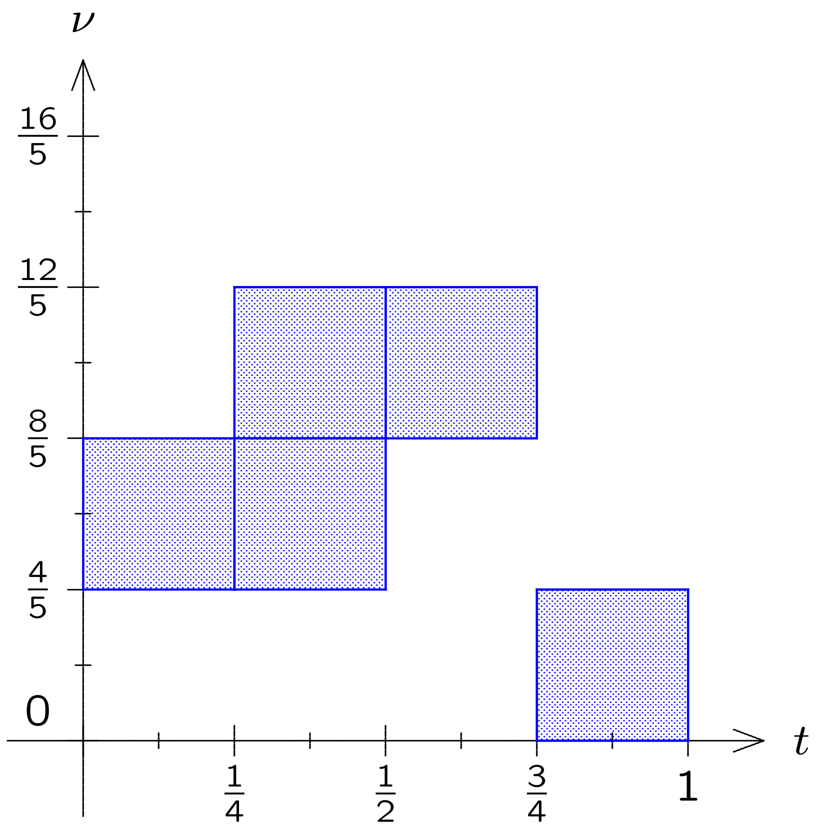




Spreading support of H,
$K = 4$, $L = 5$.



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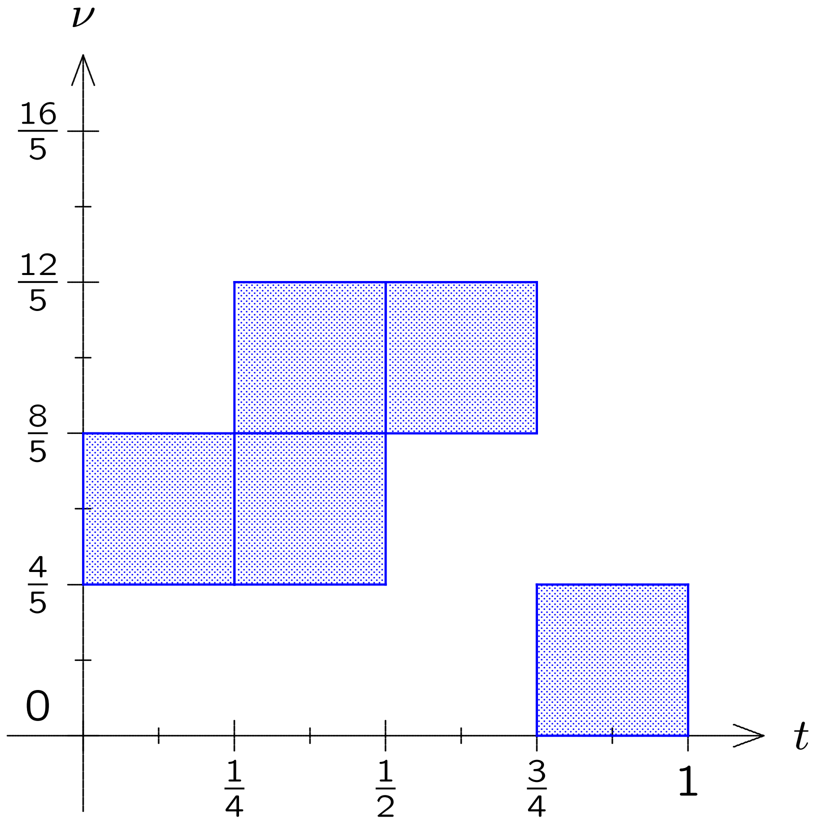

Support kernel κ_H,
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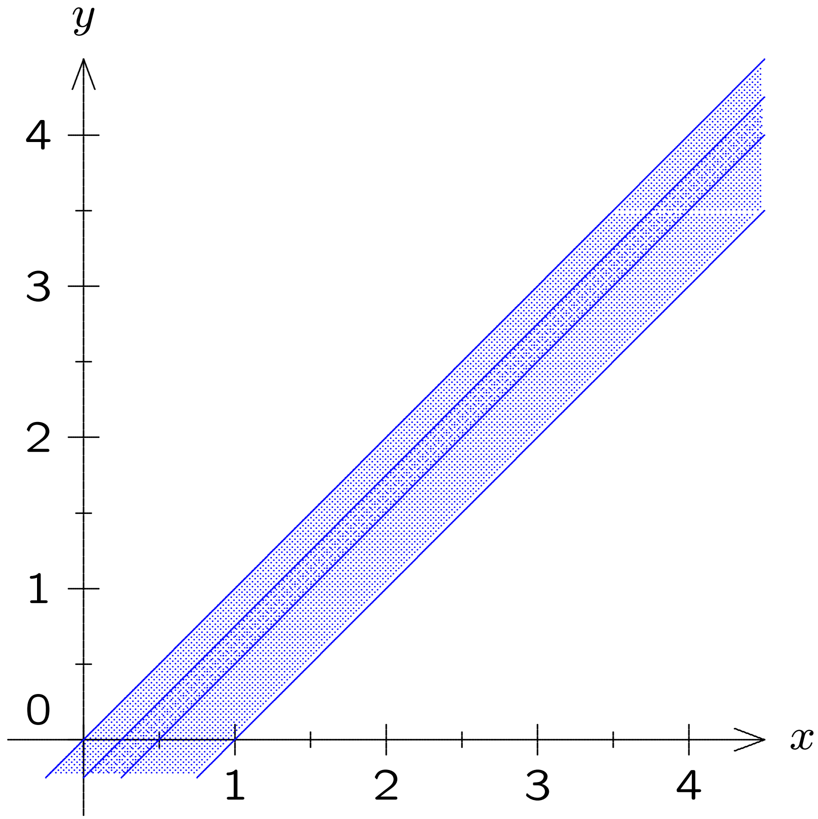

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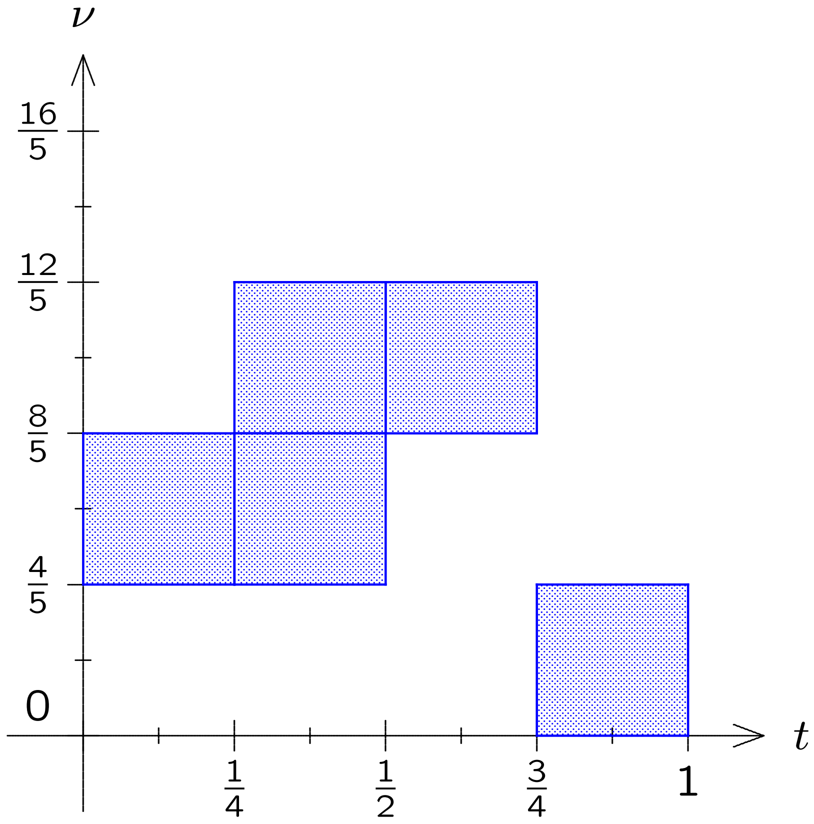

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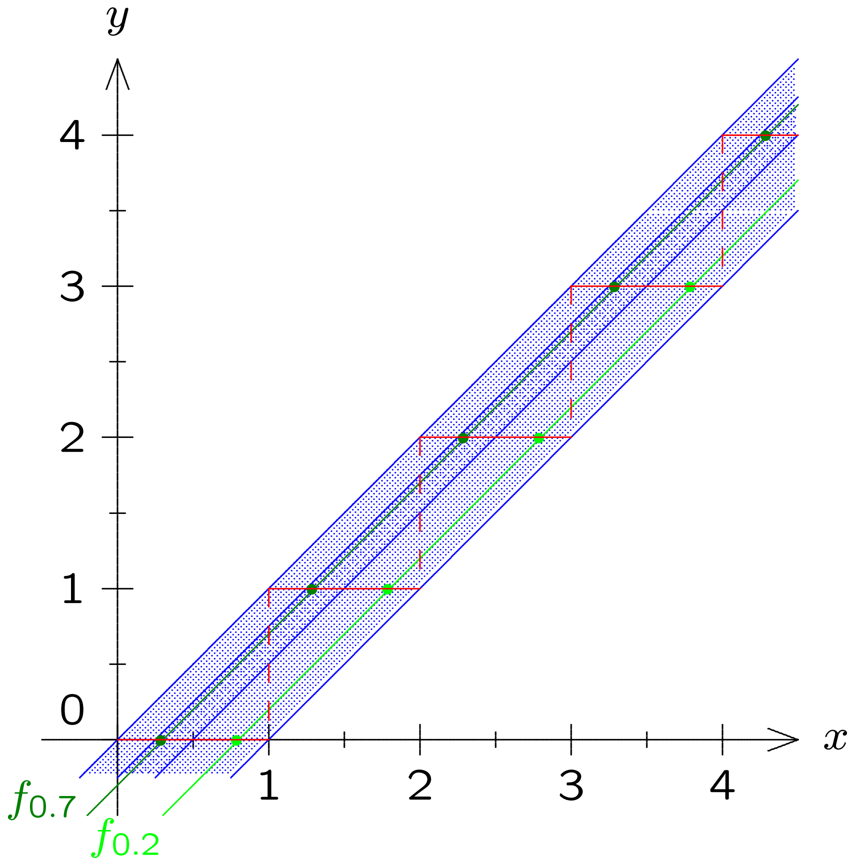

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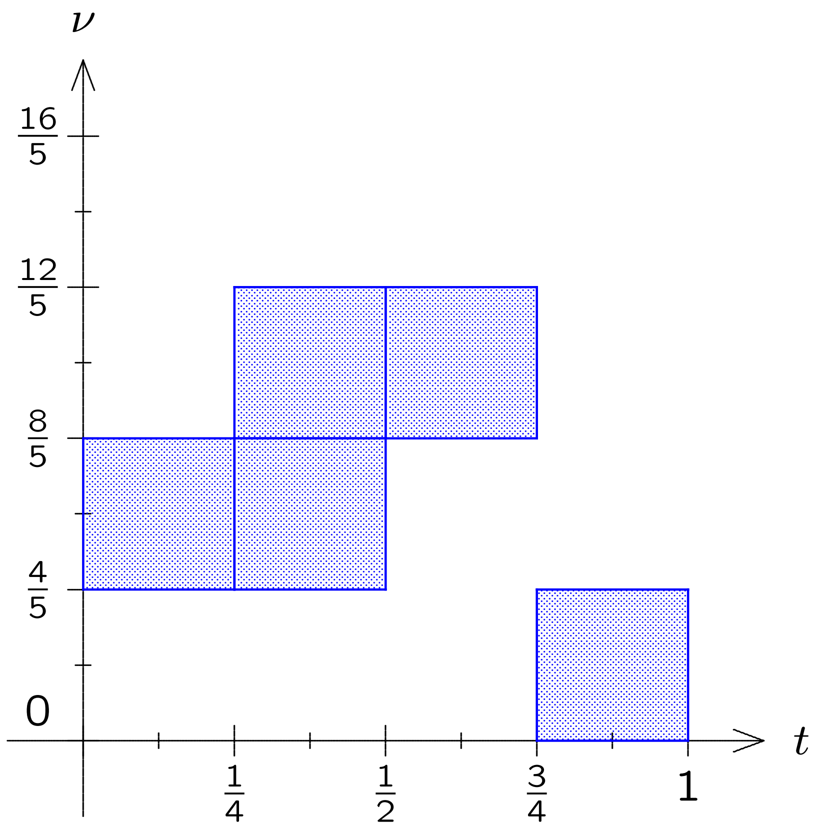

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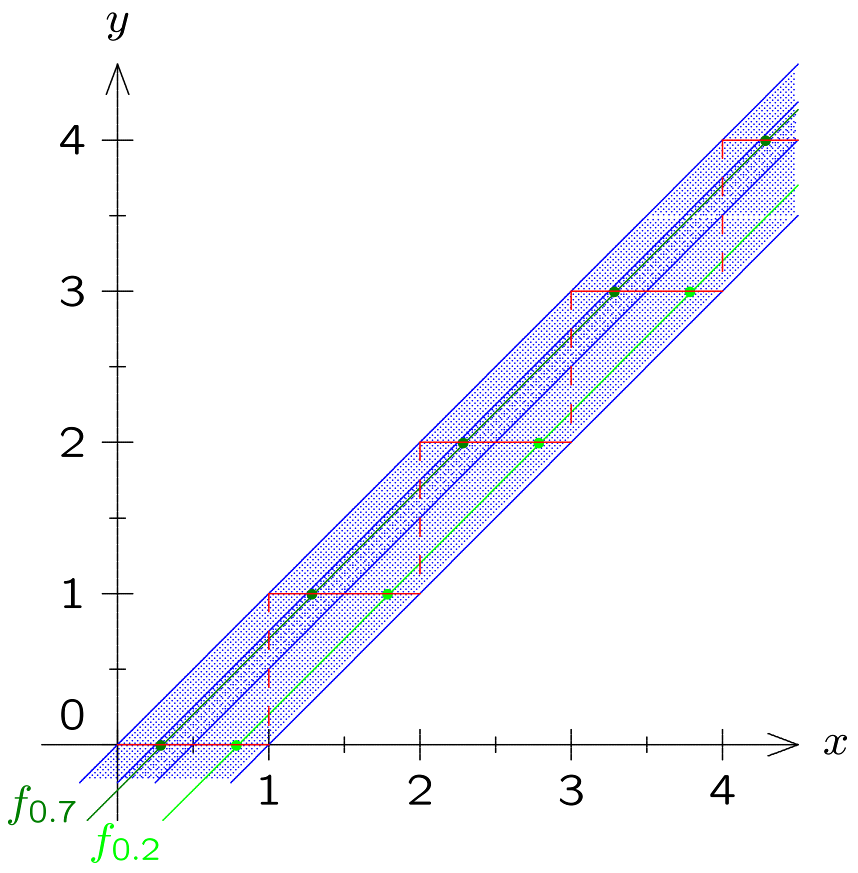

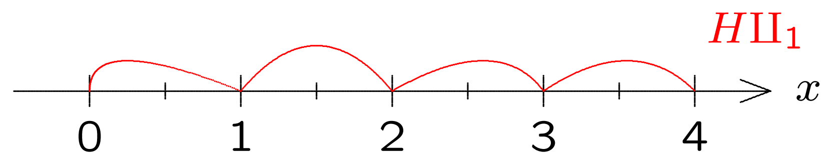

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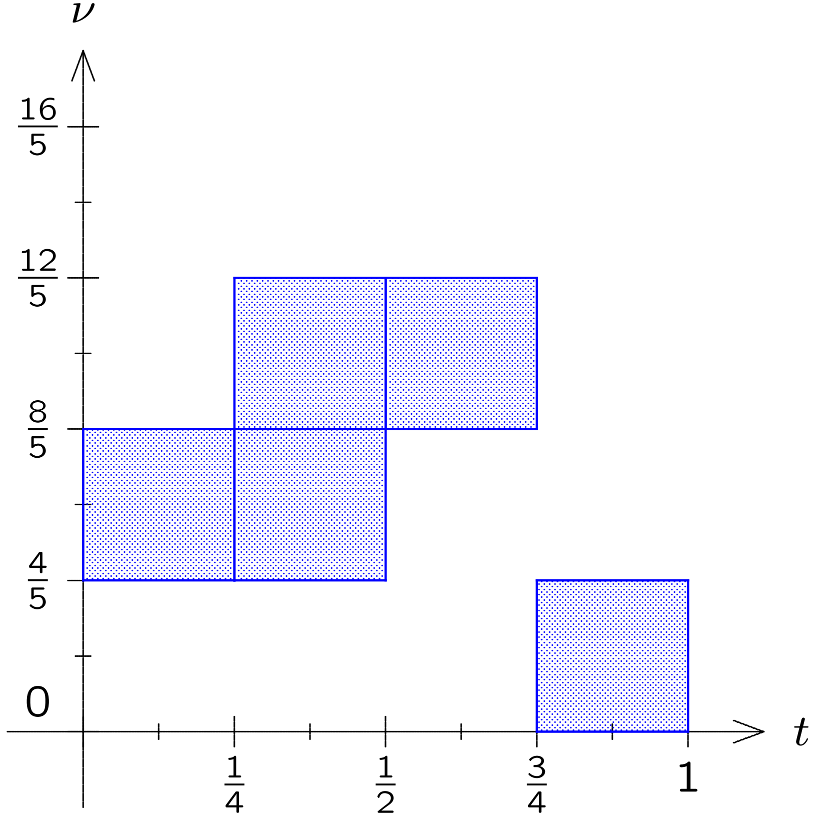

Support kernel $\widehat{\kappa_H}$,
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