Time–Frequency Analysis

Werner Kozek and
Götz Pfander

JACOBS UNIVERSITY

Progress in Mathematics for Communication Systems, 3.7.07

Tools in time–frequency analysis

Tools in time–frequency analysis

- Fouriertransform $\mathcal{F} : L^2(\mathbf{R}) \mapsto L^2(\widehat{\mathbf{R}})$, $\mathcal{F}f(\gamma) = \widehat{f}(\gamma) = \int f(x)e^{-2\pi i x\gamma}\,dx$ (a.e.).

Tools in time–frequency analysis

- Fouriertransform $\mathcal{F} : L^2(\mathbf{R}) \mapsto L^2(\widehat{\mathbf{R}})$, $\mathcal{F}f(\gamma) = \widehat{f}(\gamma) = \int f(x)e^{-2\pi i x\gamma}\,dx$ (a.e.).

  Note $f(x) = \int \widehat{f}(\gamma)e^{2\pi i x\gamma}\,d\gamma$

Tools in time–frequency analysis

- Fouriertransform $\mathcal{F} : L^2(\mathbf{R}) \mapsto L^2(\widehat{\mathbf{R}})$, $\mathcal{F}f(\gamma) = \widehat{f}(\gamma) = \int f(x)e^{-2\pi i x\gamma}\,dx$ (a.e.).

  Note $f(x) = \int \widehat{f}(\gamma)e^{2\pi i x\gamma}\,d\gamma$ (FT on abelian groups ...)

Tools in time–frequency analysis

- Fouriertransform $\mathcal{F} : L^2(\mathbf{R}) \mapsto L^2(\widehat{\mathbf{R}})$, $\mathcal{F}f(\gamma) = \widehat{f}(\gamma) = \int f(x)e^{-2\pi i x\gamma}\,dx$ (a.e.).

  Note $f(x) = \int \widehat{f}(\gamma)e^{2\pi i x\gamma}\,d\gamma$ (FT on abelian groups ...)

- Translation (time–shift) $T_t f(x) = f(x-t)$

- Modulation (frequency–shift) $M_\nu f(x) = e^{2\pi i \nu x} f(x)$ with $\widehat{M_\nu f} = T_\nu \widehat{f}$.

Tools in time–frequency analysis

- Fouriertransform $\mathcal{F} : L^2(\mathbf{R}) \mapsto L^2(\widehat{\mathbf{R}})$, $\mathcal{F}f(\gamma) = \widehat{f}(\gamma) = \int f(x) e^{-2\pi i x\gamma}\, dx$ (a.e.).

  Note $f(x) = \int \widehat{f}(\gamma) e^{2\pi i x\gamma}\, d\gamma$ (FT on abelian groups ...)

- Translation (time–shift) $T_t f(x) = f(x-t)$

- Modulation (frequency–shift) $M_\nu f(x) = e^{2\pi i \nu x} f(x)$ with $\widehat{M_\nu f} = T_\nu \widehat{f}$.

- Windowed Fourier transform V_g with respect to the window $g \in L^2(\mathbf{R})$, $\|g\| = 1$:

$$V_g f(t,\nu) = \int f(x) \overline{g}(x-t) e^{-2\pi i x\nu}\, dx = \langle f, M_\nu T_t g\rangle = \langle \widehat{f}, T_\nu M_{-t} \widehat{g}\rangle\,, \qquad \|f\| = \|V_g f\|$$

Tools in time–frequency analysis

- Fouriertransform $\mathcal{F} : L^2(\mathbf{R}) \mapsto L^2(\widehat{\mathbf{R}})$, $\mathcal{F}f(\gamma) = \widehat{f}(\gamma) = \int f(x)e^{-2\pi i x\gamma}\, dx$ (a.e.).

  Note $f(x) = \int \widehat{f}(\gamma)e^{2\pi i x\gamma}\, d\gamma$ (FT on abelian groups ...)

- Translation (time–shift) $T_t f(x){=}f(x{-}t)$

- Modulation (frequency–shift) $M_\nu f(x){=}e^{2\pi i\nu x}f(x)$ with $\widehat{M_\nu f}{=}T_\nu \widehat{f}$.

- Windowed Fourier transform V_g with respect to the window $g \in L^2(\mathbf{R})$, $\|g\| = 1$:

$$V_g f(t,\nu) = \int f(x)\overline{g}(x-t)e^{-2\pi i x\nu}\, dx = \langle f, M_\nu T_t g\rangle = \langle \widehat{f}, T_\nu M_{-t}\widehat{g}\rangle\,, \quad \|f\| = \|V_g f\|$$

  – Inversion formula: $f(x) = \int\int V_g f(t,\nu) M_\nu T_t g(x)\, d\nu dt$

Tools in time–frequency analysis

- Fouriertransform $\mathcal{F} : L^2(\mathbf{R}) \mapsto L^2(\widehat{\mathbf{R}})$, $\mathcal{F}f(\gamma) = \widehat{f}(\gamma) = \int f(x)e^{-2\pi i x\gamma}\,dx$ (a.e.).

  Note $f(x) = \int \widehat{f}(\gamma)e^{2\pi i x\gamma}\,d\gamma$ (FT on abelian groups ...)

- Translation (time–shift) $T_t f(x)=f(x-t)$

- Modulation (frequency–shift) $M_\nu f(x)=e^{2\pi i\nu x}f(x)$ with $\widehat{M_\nu f}=T_\nu \widehat{f}$.

- Windowed Fourier transform V_g with respect to the window $g \in L^2(\mathbf{R})$, $\|g\| = 1$:

$$V_g f(t,\nu) = \int f(x)\overline{g}(x-t)e^{-2\pi i x\nu}\,dx = \langle f, M_\nu T_t g\rangle = \langle \widehat{f}, T_\nu M_{-t}\widehat{g}\rangle\,, \quad \|f\| = \|V_g f\|$$

  - Inversion formula: $f(x) = \int\int V_g f(t,\nu)M_\nu T_t g(x)\,d\nu dt$
  - Uncertainty: $\int x^2|g(x)|^2\,dx \cdot \int \gamma^2|\widehat{g}(\gamma)|^2\,d\gamma \geq \frac{1}{16\pi^2}$

Tools in time–frequency analysis

- Fouriertransform $\mathcal{F} : L^2(\mathbf{R}) \mapsto L^2(\widehat{\mathbf{R}})$, $\mathcal{F}f(\gamma) = \widehat{f}(\gamma) = \int f(x)e^{-2\pi i x\gamma}\, dx$ (a.e.).

  Note $f(x) = \int \widehat{f}(\gamma)e^{2\pi i x\gamma}\, d\gamma$ (FT on abelian groups ...)

- Translation (time–shift) $T_t f(x) = f(x-t)$

- Modulation (frequency–shift) $M_\nu f(x) = e^{2\pi i \nu x} f(x)$ with $\widehat{M_\nu f} = T_\nu \widehat{f}$.

- Windowed Fourier transform V_g with respect to the window $g \in L^2(\mathbf{R})$, $\|g\| = 1$:

$$V_g f(t,\nu) = \int f(x)\overline{g}(x-t)e^{-2\pi i x\nu}\, dx = \langle f, M_\nu T_t g\rangle = \langle \widehat{f}, T_\nu M_{-t}\widehat{g}\rangle\,, \qquad \|f\| = \|V_g f\|$$

  - Inversion formula: $f(x) = \int\int V_g f(t,\nu) M_\nu T_t g(x)\, d\nu dt$
  - Uncertainty: $\int x^2|g(x)|^2\, dx \cdot \int \gamma^2|\widehat{g}(\gamma)|^2\, d\gamma \geq \frac{1}{16\pi^2}$ (Wavelet transform ...)

Tools in time–frequency analysis

- Fouriertransform $\mathcal{F} : L^2(\mathbf{R}) \mapsto L^2(\widehat{\mathbf{R}})$, $\mathcal{F}f(\gamma) = \widehat{f}(\gamma) = \int f(x)e^{-2\pi i x\gamma}\,dx$ (a.e.).

  Note $f(x) = \int \widehat{f}(\gamma)e^{2\pi i x\gamma}\,d\gamma$ (FT on abelian groups ...)

- Translation (time–shift) $T_t f(x) = f(x-t)$

- Modulation (frequency–shift) $M_\nu f(x) = e^{2\pi i \nu x} f(x)$ with $\widehat{M_\nu f} = T_\nu \widehat{f}$.

- Windowed Fourier transform V_g with respect to the window $g \in L^2(\mathbf{R})$, $\|g\| = 1$:

$$V_g f(t,\nu) = \int f(x)\overline{g}(x-t)e^{-2\pi i x\nu}\,dx = \langle f, M_\nu T_t g\rangle = \langle \widehat{f}, T_\nu M_{-t}\widehat{g}\rangle\,, \quad \|f\| = \|V_g f\|$$

  - Inversion formula: $f(x) = \int\int V_g f(t,\nu) M_\nu T_t g(x)\,d\nu dt$
  - Uncertainty: $\int x^2|g(x)|^2\,dx \cdot \int \gamma^2|\widehat{g}(\gamma)|^2\,d\gamma \geq \frac{1}{16\pi^2}$ (Wavelet transform ...)

Some goals and results in time–frequency analysis

- Discretize $f = \int\int V_g f(t,\nu) M_\nu T_t g \, d\nu dt$, i.e., find g, sampling set $\{(t_j,\nu_j)\}_{j\in J}$ such that

$$f(x) = \sum_{j\in J} V_g f(t_j,\nu_j) M_{\nu_j} T_{t_j} g(x)$$

Some goals and results in time–frequency analysis

- Discretize $f = \int\int V_g f(t,\nu) M_\nu T_t g \, d\nu dt$, i.e., find g, sampling set $\{(t_j,\nu_j)\}_{j\in J}$ such that

$$f(x) = \sum_{j\in J} V_g f(t_j,\nu_j) M_{\nu_j} T_{t_j} g(x)$$

E.g., for $g = \frac{1}{a}\chi_{[0,a)}$, $\{(t_j,\nu_j)\}_{j\in J} = \{(na, \frac{n}{a})\}_{j\in J}$, $\{M_{\frac{n}{a}} T_{ma} g\}_{m,n\in\mathbf{Z}}$ is an ONB of L^2.

Some goals and results in time–frequency analysis

- Discretize $f = \int\int V_g f(t,\nu) M_\nu T_t g \, d\nu dt$, i.e., find g, sampling set $\{(t_j,\nu_j)\}_{j\in J}$ such that

$$f(x) = \sum_{j\in J} V_g f(t_j,\nu_j) M_{\nu_j} T_{t_j} g(x)$$

  E.g., for $g = \frac{1}{a}\chi_{[0,a)}$, $\{(t_j,\nu_j)\}_{j\in J} = \{(na,\frac{n}{a})\}_{j\in J}$, $\{M_{\frac{n}{a}} T_{ma} g\}_{m,n\in\mathbf{Z}}$ is an ONB of L^2.

- If $\{M_{nb}T_{ma}g\}$ is a frame, i.e., $f \mapsto \{V_g f(ma,nb)\}$ is bounded and bounded below, then $ab \leq 1$

Some goals and results in time–frequency analysis

- Discretize $f = \int\int V_g f(t,\nu) M_\nu T_t g \, d\nu dt$, i.e., find g, sampling set $\{(t_j,\nu_j)\}_{j\in J}$ such that

$$f(x) = \sum_{j\in J} V_g f(t_j,\nu_j) M_{\nu_j} T_{t_j} g(x)$$

  E.g., for $g = \frac{1}{a}\chi_{[0,a)}$, $\{(t_j,\nu_j)\}_{j\in J} = \{(na, \frac{n}{a})\}_{j\in J}$, $\{M_{\frac{n}{a}} T_{ma} g\}_{m,n\in\mathbf{Z}}$ is an ONB of L^2.

- If $\{M_{nb}T_{ma}g\}$ is a frame, i.e., $f \mapsto \{V_g f(ma,nb)\}$ is bounded and bounded below, then $ab \leq 1$

- If $\{M_{nb}T_{ma}g\}$ is a Riesz basis for $\overline{\text{span}\{M_{nb}T_{ma}g\}}$, i.e., $\{c_{m,n}\} \mapsto \sum_{m,n} c_{m,n} M_{nb} T_{ma} g$ is bounded and stable, then $ab \geq 1$.

Some goals and results in time–frequency analysis

- Discretize $f = \int\int V_g f(t,\nu) M_\nu T_t g \, d\nu dt$, i.e., find g, sampling set $\{(t_j,\nu_j)\}_{j\in J}$ such that

$$f(x) = \sum_{j\in J} V_g f(t_j,\nu_j) M_{\nu_j} T_{t_j} g(x)$$

  E.g., for $g = \frac{1}{a}\chi_{[0,a)}$, $\{(t_j,\nu_j)\}_{j\in J} = \{(na, \frac{n}{a})\}_{j\in J}$, $\{M_{\frac{n}{a}} T_{ma} g\}_{m,n\in\mathbf{Z}}$ is an ONB of L^2.

- If $\{M_{nb}T_{ma}g\}$ is a frame, i.e., $f \mapsto \{V_g f(ma,nb)\}$ is bounded and bounded below, then $ab \leq 1$

- If $\{M_{nb}T_{ma}g\}$ is a Riesz basis for $\overline{\text{span}\{M_{nb}T_{ma}g\}}$, i.e., $\{c_{m,n}\} \mapsto \sum_{m,n} c_{m,n} M_{nb} T_{ma} g$ is bounded and stable, then $ab \geq 1$.

- Balian–Low. If $\{M_{nb}T_{ma}g\}$ is a frame or a Riesz basis and $ab = 1$, then

$$\int x^2 |g(x)|^2 \, dx \cdot \int \gamma^2 |\widehat{f}(\gamma)|^2 \, d\gamma = \infty$$

- HRT conjecture: For any J finite, $g \in L^2$, the family $\{M_{\nu_j} T_{t_j} g\}_{j \in J}$ is linear independent.

- HRT conjecture: For any J finite, $g \in L^2$, the family $\{M_{\nu_j} T_{t_j} g\}_{j \in J}$ is linear independent.

- For any $g \in L^2$, is $\{g(x),\ g(x-1),\ e^{2\pi i x} g(x),\ e^{2\pi i \sqrt{2} x} g(x-\sqrt{2})\}$ linear independent?

Useful operator representations

Useful operator representations

operator H $\qquad$ $Hf(x)$

Useful operator representations

operator H	$Hf(x)$
$\updownarrow$	$=$
kernel κ_H	$\int \kappa_H(x,s)f(s)\,ds$

Useful operator representations

operator H	$Hf(x)$
$\updownarrow$	$=$
kernel κ_H	$\int \kappa_H(x,s) f(s)\, ds$
$\updownarrow$	$=$
Kohn–Nirenberg symbol σ_H	$\int \sigma_H(x,\omega) \widehat{f}(\omega) e^{2\pi i x\cdot\omega}\, d\omega$

Useful operator representations

operator H	$Hf(x)$
$\updownarrow$	$=$
kernel κ_H	$\int \kappa_H(x,s)f(s)\,ds$
$\updownarrow$	$=$
Kohn–Nirenberg symbol σ_H	$\int \sigma_H(x,\omega)\widehat{f}(\omega)e^{2\pi i x\cdot\omega}\,d\omega$
$\updownarrow$	$=$
time–varying impulse response h_H	$\int h_H(t,x)f(x-t)\,dt$

Useful operator representations

operator H	$Hf(x)$
$\updownarrow$	$=$
kernel κ_H	$\int \kappa_H(x,s)f(s)\,ds$
$\updownarrow$	$=$
Kohn–Nirenberg symbol σ_H	$\int \sigma_H(x,\omega)\widehat{f}(\omega)e^{2\pi i x\cdot\omega}\,d\omega$
$\updownarrow$	$=$
time–varying impulse response h_H	$\int h_H(t,x)f(x-t)\,dt$
$\updownarrow$	$=$
spreading function η_H	$\int\int \eta_H(t,\nu)f(x-t)e^{2\pi i x\cdot\nu}\,dt\,d\nu$

Useful operator representations

operator H	$Hf(x)$
$\updownarrow$	$=$
kernel κ_H	$\int \kappa_H(x,s) f(s)\, ds$
$\updownarrow$	$=$
Kohn–Nirenberg symbol σ_H	$\int \sigma_H(x,\omega) \widehat{f}(\omega) e^{2\pi i x\cdot\omega}\, d\omega$
$\updownarrow$	$=$
time–varying impulse response h_H	$\int h_H(t,x) f(x-t)\, dt$
$\updownarrow$	$=$
spreading function η_H	$\int\int \eta_H(t,\nu) f(x-t) e^{2\pi i x\cdot\nu}\, dt\, d\nu$
	$=$
	$\int\int \eta_H(t,\nu) M_\nu T_t f(x), dt\, d\nu\,,$

Useful operator representations

operator H	$Hf(x)$
$\updownarrow$	$=$
kernel κ_H	$\int \kappa_H(x,s)f(s)\,ds$
$\updownarrow$	$=$
Kohn–Nirenberg symbol σ_H	$\int \sigma_H(x,\omega)\widehat{f}(\omega)e^{2\pi i x\cdot\omega}\,d\omega$
$\updownarrow$	$=$
time–varying impulse response h_H	$\int h_H(t,x)f(x-t)\,dt$
$\updownarrow$	$=$
spreading function η_H	$\int\int \eta_H(t,\nu)f(x-t)e^{2\pi i x\cdot\nu}\,dt\,d\nu$
	$=$
	$\int\int \eta_H(t,\nu)M_\nu T_t f(x),dt\,d\nu\,,$

(Translation (time–shift) $T_t f(x){=}f(x{-}t)$, Modulation (frequency–shift) $M_\nu f(x){=}e^{2\pi i\nu x}f(x)$)

Useful operator representations

operator H	$Hf(x)$
$\updownarrow$	$=$
kernel κ_H	$\int \kappa_H(x,s)f(s)\,ds$
$\updownarrow$	$=$
Kohn–Nirenberg symbol σ_H	$\int \sigma_H(x,\omega)\widehat{f}(\omega)e^{2\pi i x\cdot\omega}\,d\omega$
$\updownarrow$	$=$
time–varying impulse response h_H	$\int h_H(t,x)f(x-t)\,dt$
$\updownarrow$	$=$
spreading function η_H	$\int\int \eta_H(t,\nu)f(x-t)e^{2\pi i x\cdot\nu}\,dt\,d\nu$
	$=$
	$\int\int \eta_H(t,\nu)M_\nu T_t f(x),\,dt\,d\nu\,,$

(Translation (time–shift) $T_tf(x){=}f(x{-}t)$, Modulation (frequency–shift) $M_\nu f(x){=}e^{2\pi i\nu x}f(x)$)

Note $\displaystyle\int\int \sigma_H(x,\xi)e^{-2\pi i x\eta}e^{2\pi i t\xi} = \eta_H(t,\nu)$

Useful operator representations

operator H	$Hf(x)$
$\updownarrow$	$=$
kernel κ_H	$\int \kappa_H(x,s) f(s)\, ds$
$\updownarrow$	$=$
Kohn–Nirenberg symbol σ_H	$\int \sigma_H(x,\omega)\widehat{f}(\omega) e^{2\pi i x\cdot\omega}\, d\omega$
$\updownarrow$	$=$
time–varying impulse response h_H	$\int h_H(t,x) f(x-t)\, dt$
$\updownarrow$	$=$
spreading function η_H	$\int\int \eta_H(t,\nu) f(x-t) e^{2\pi i x\cdot\nu}\, dt\, d\nu$
	$=$
	$\int\int \eta_H(t,\nu) M_\nu T_t f(x),\, dt\, d\nu$,

(Translation (time–shift) $T_t f(x) = f(x-t)$, Modulation (frequency–shift) $M_\nu f(x) = e^{2\pi i \nu x} f(x)$)

Note $\int\int \sigma_H(x,\xi) e^{-2\pi i x\eta} e^{2\pi i t\xi} = \eta_H(t,\nu)$ $\quad \kappa_H(x,s) = \int \eta(x-s,\nu) e^{2\pi i x\nu}\ d\nu$

Useful operator representations

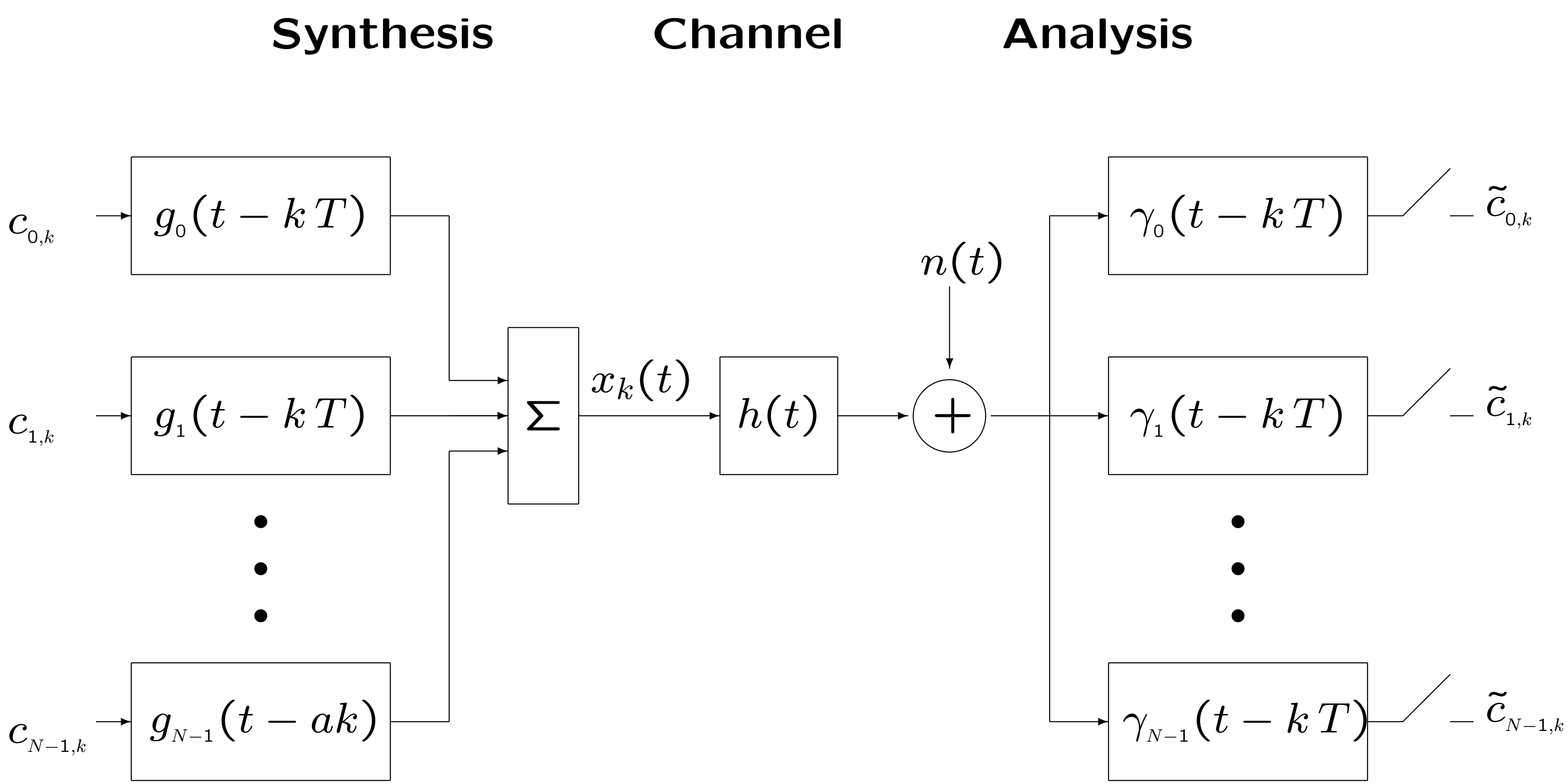
operator H	$Hf(x)$
$\updownarrow$	$=$
kernel κ_H	$\int \kappa_H(x,s)f(s)\,ds$
$\updownarrow$	$=$
Kohn–Nirenberg symbol σ_H	$\int \sigma_H(x,\omega)\widehat{f}(\omega)e^{2\pi i x\cdot\omega}\,d\omega$
$\updownarrow$	$=$
time–varying impulse response h_H	$\int h_H(t,x)f(x-t)\,dt$
$\updownarrow$	$=$
spreading function η_H	$\int\int \eta_H(t,\nu)f(x-t)e^{2\pi i x\cdot\nu}\,dt\,d\nu$
	$=$
	$\int\int \eta_H(t,\nu)M_\nu T_t f(x),dt\,d\nu$,

(Translation (time–shift) $T_t f(x)=f(x-t)$, Modulation (frequency–shift) $M_\nu f(x)=e^{2\pi i\nu x}f(x)$)

Note $\displaystyle\int\int \sigma_H(x,\xi)e^{-2\pi i x\eta}e^{2\pi i t\xi} = \eta_H(t,\nu) \quad \kappa_H(x,s) = \int \eta(x-s,\nu)e^{2\pi i x\nu}\ d\nu$

Pseudodifferential operators

1. Shift invariant multicarrier modulation



Principles of shift invariant multicarrier modulation

- Carrier functions/pulses: $\{g_l\}_{l=0,\dots,N-1}$.
- Transmission symbol: $x_0(t)=c_0 g_0(t)+\dots+c_{N-1}g_{N-1}(t)=\sum_{l=0}^{N-1} c_l\, g_l(t)$
- Transmission signal: $x(t)=\sum_{k=-\infty}^{\infty} x_k(t)=\sum_{k=-\infty}^{\infty}\sum_{l=0}^{N-1} c_{l,k}\,\underbrace{g_l(t-k\,T)}_{g_{l,k}(t)}$
- Channel distortion: $(K_h\, x)(t)=(h{*}x)(t)=\int_{\mathbb{R}} h(t{-}t')\, x(t')\, dt$
- Receiver strategy $\tilde{c}_{l,k}=\langle h*x,\gamma_{l,k}\rangle=\int_{\mathbb{R}}(h*x)\,(t)\,\overline{\gamma_l(t-k\,T)}\,dt$

JACOBS UNIVERSITY

Requirements on $\{g_{l,k}(t)\}$ and $\{\gamma_{l,k}(t)\}$ for stable and effective transmission:

- Stability and continuity of synthesis and analysis.
- If $K = \mathbf{Id}$entity then $c_{l,k} = \tilde{c}_{l,k}$.
- $\{g_l(t)\}$ and $\{\gamma_l(t)\}$ structured to allow for fast synthesis and analysis algorithms.
- Uniform compact support, i.e., $\operatorname{supp} g_l \subset [0, T]$, to restrict time delay.
- Efficient use of an assigned bandwidth.

... last but NOT least:

- Low ISI/ICI for an ensemble of convolution operators $\mathcal{H} \subset \mathcal{L}(L^2(\mathbb{R}))$, i.e., for all $h(t) \in \mathcal{H}$, the **channel matrix** G is diagonally dominant:

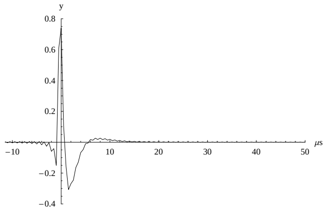
$$G^h_{l,k,l',k'} := \langle h * g_{l,k}, \gamma_{l',k'} \rangle \approx d_{l,k}\ \delta_{l,l'}\delta_{k,k'} \qquad \left(G\overrightarrow{c} = \overrightarrow{\widetilde{c}} \right)$$

ISI = inter symbol interference: effect of $c_{l,k}$ on $c_{l',k'}$ for $k \neq k'$

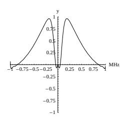
ICI = inter channel interference: effect of $c_{l,k}$ on $c_{l',k'}$ for $l \neq l'$

2. Numerical comparison of MCM schemes

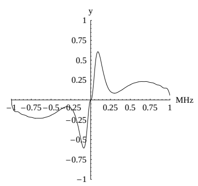
Simulations in a ADSL setting, using a impulse response which models a $2km$, $0,4mm\ PE$ twisted wire copper kabel. (ADSL ETSI ETR 328)



Convolution kernel h



$\mathrm{Re}\,\widehat{h}$

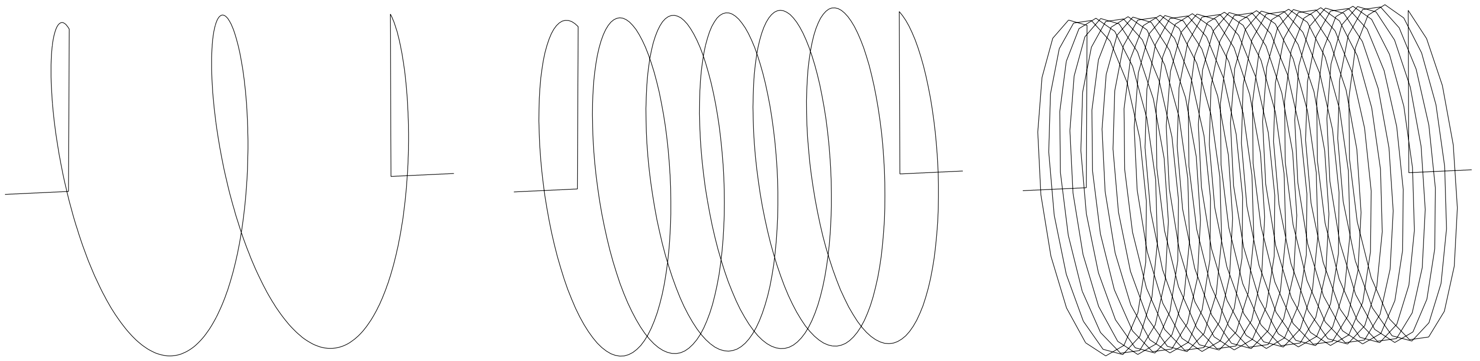


$\mathrm{Im}\,\widehat{h}$

i. Weyl–Heisenberg systems (DMT)

For g_0, γ_0, $b \geq \frac{1}{T}$ chosen appropriately, let

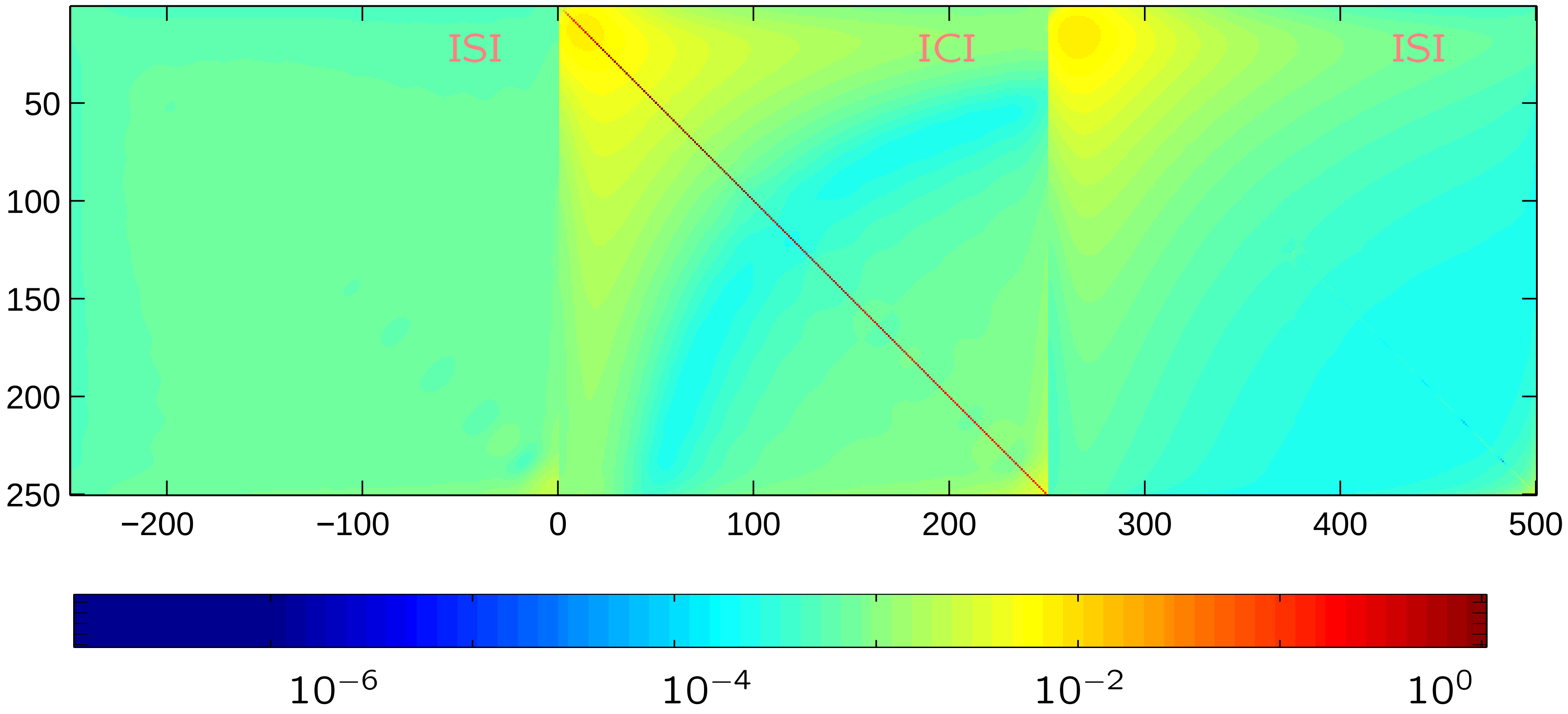
$$g_l(t) = g_0(t)e^{2\pi iblx} \quad \text{and} \quad \gamma_l(t) = \gamma_0(t)e^{2\pi iblx}.$$



Weyl–Heisenberg system channel matrix (no cyclic prefix)

- $g_0(t[\mu s]) = \gamma_0(t[\mu s]) = \frac{1}{250}\chi_{[0,250)}(t[\mu s])$
- $g_{l+250k} = g_{l,k}$, $l = 0, \ldots, 249$, $k \in \mathbf{Z}$

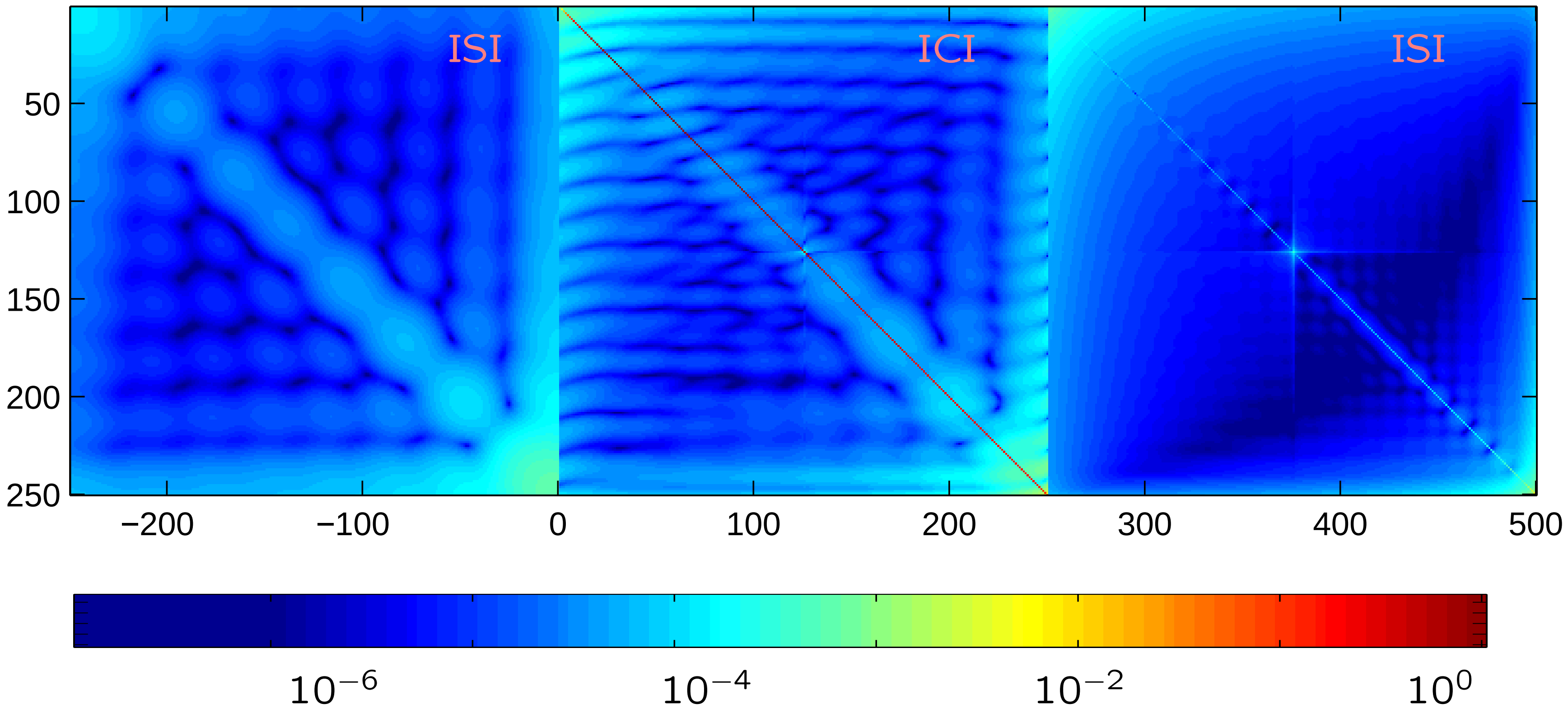
- $T = 250\mu s$,
- $b = \frac{1}{250} MHz$
- $N = 250$



Weyl–Heisenberg system channel matrix with 15μs cyclic prefix

- $g_0(t[\mu s]) = \frac{1}{250}\chi_{[-15,250)}(t[\mu s])$
- $\gamma_0(t[\mu s]) = \frac{1}{250}\chi_{[0,250)}(t[\mu s])$
- $g_{l+250k} = g_{l,k},\ l = 0, \dots, 249,\ k \in \mathbf{Z}$

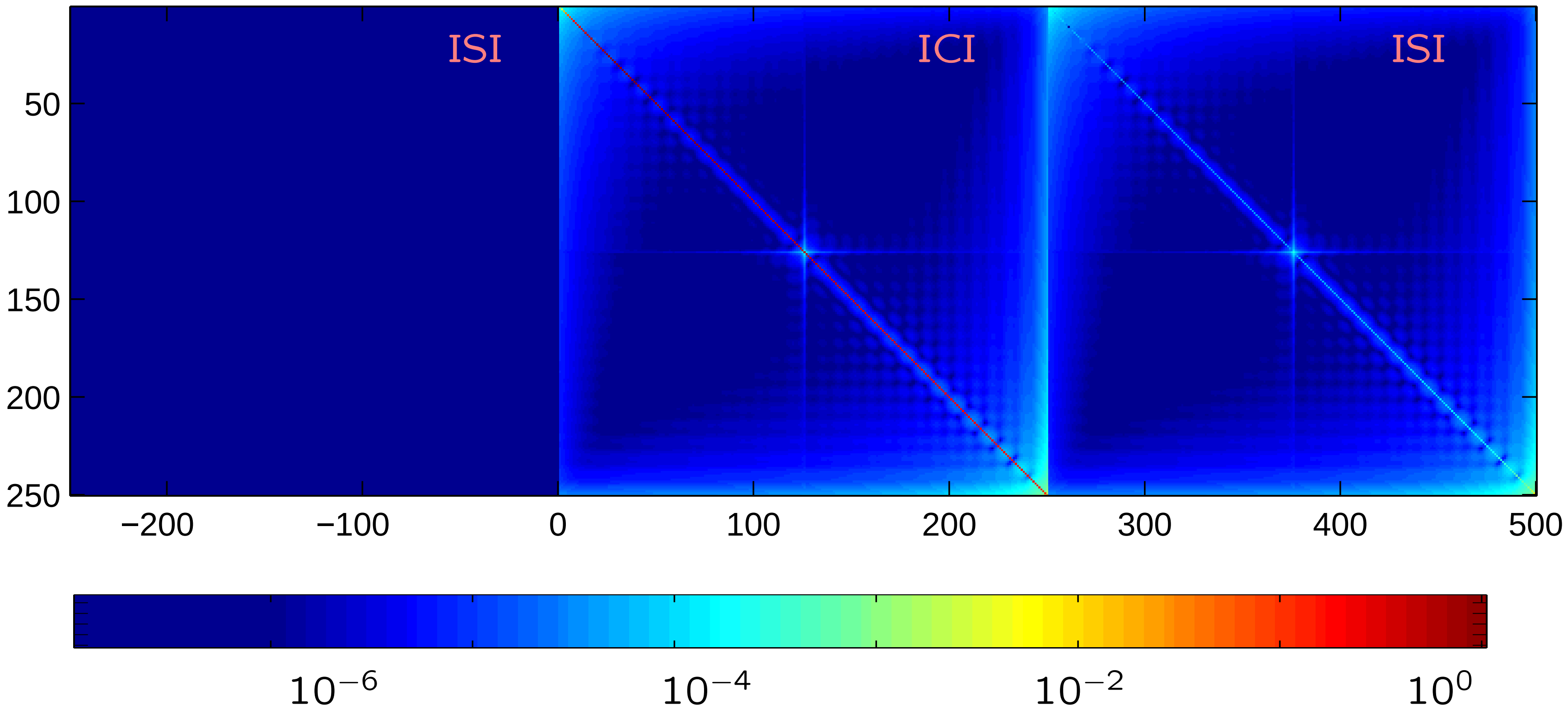
- $T = 265\mu s$
- $b = \frac{1}{250} MHz$
- $N = 250$



Weyl–Heisenberg system channel matrix with 30μs cyclic prefix

- $g_0(t[\mu s]) = \frac{1}{250}\chi_{[-30,250)}(t[\mu s])$
- $\gamma_0(t[\mu s]) = \frac{1}{250}\chi_{[0,250)}(t[\mu s])$
- $g_{l+250k} = g_{l,k},\ l = 0, \ldots, 249,\ k \in \mathbf{Z}$

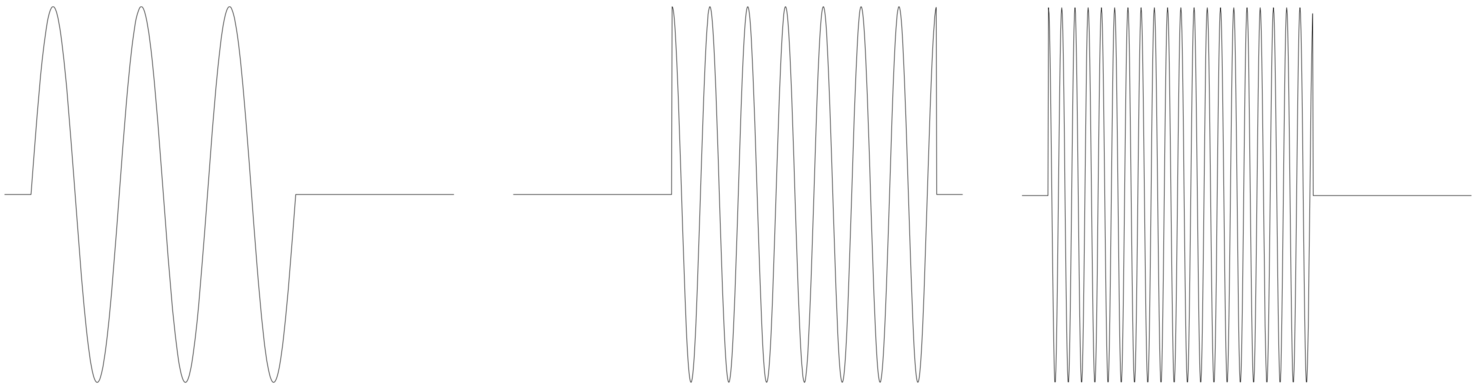
- $T = 280\mu s$
- $b = \frac{1}{250} MHz$
- $N = 250$



ii. Wilson bases (OQAM)

For an appropriate prototype function $g_0 = \gamma_0$ define

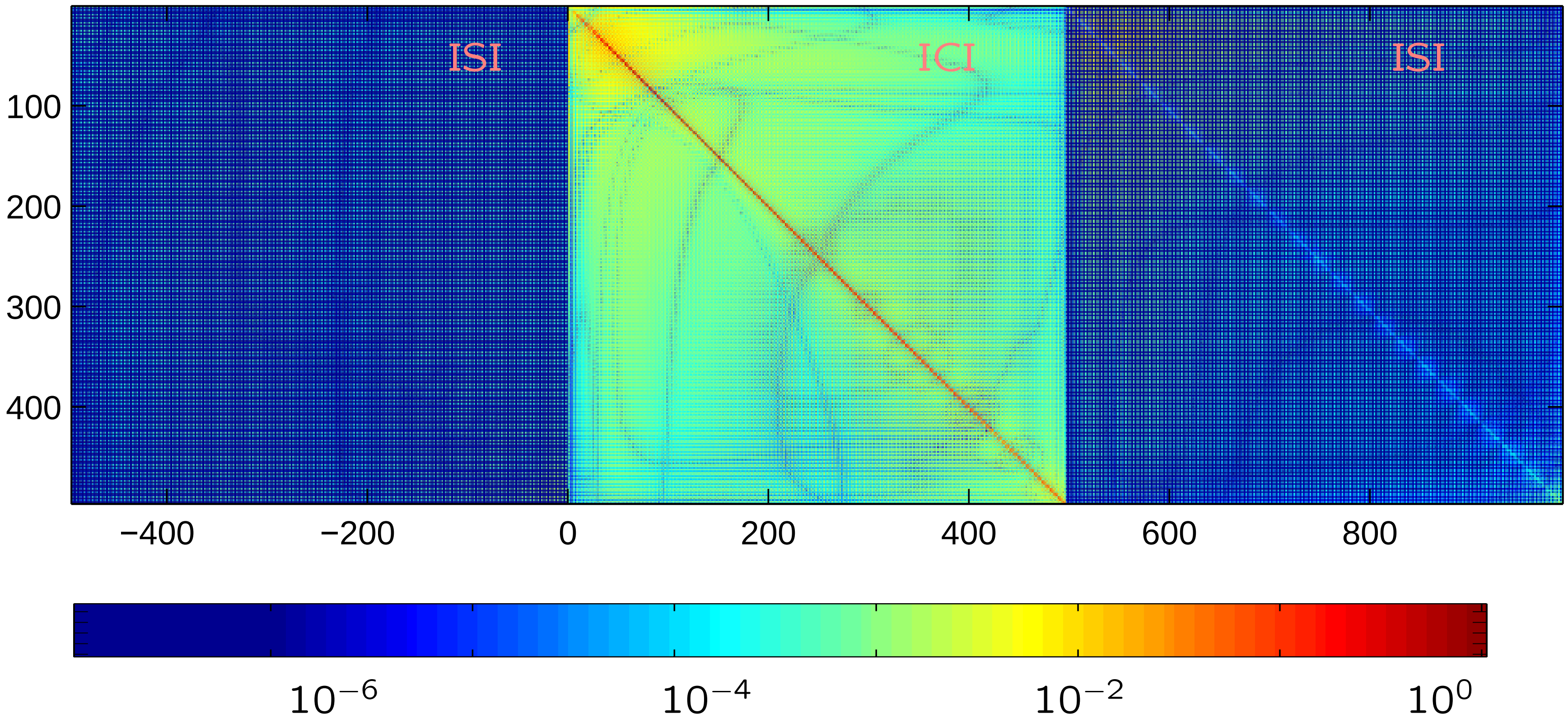
$$
\begin{aligned}
g_0(t) &= g(t)\,,\\
g_m^{(1)}(t) &= g(t)\,\sqrt{2}\cos(2\pi\tfrac{2m}{T}t)\,, \quad g_m^{(2)}(t) = g(t-\tfrac{T}{2})\,\sqrt{2}\cos(2\pi\tfrac{2m-1}{T}t)\,,\\
g_m^{(3)}(t) &= g(t)\,\sqrt{2}\sin(2\pi\tfrac{2m-1}{T}t)\,, \quad g_m^{(4)}(t) = g(t-\tfrac{T}{2})\,\sqrt{2}\sin(2\pi\tfrac{2m}{T}t)\,,\\
& \qquad m = 1,\ldots,M \ \ (\text{i.e., } N = 4M+1)\,.
\end{aligned}
$$



Wilson bases channel matrix

- $g_0(t[\mu s]) = \gamma_0(t[\mu s]) = \frac{1}{250}\chi_{[0,250)}(t[\mu s])$
- $g_{n+4l+497k} = g^{(n)}_{l,k}, \quad l = 1, \ldots, 124, k \in \mathbf{Z}$

- $T = 250\mu s$,
- $b = \frac{1}{250} MHz$
- $N = 497$



iii. Wavelet bases (DWMT)

For an appropriate prototype function g_0 define:

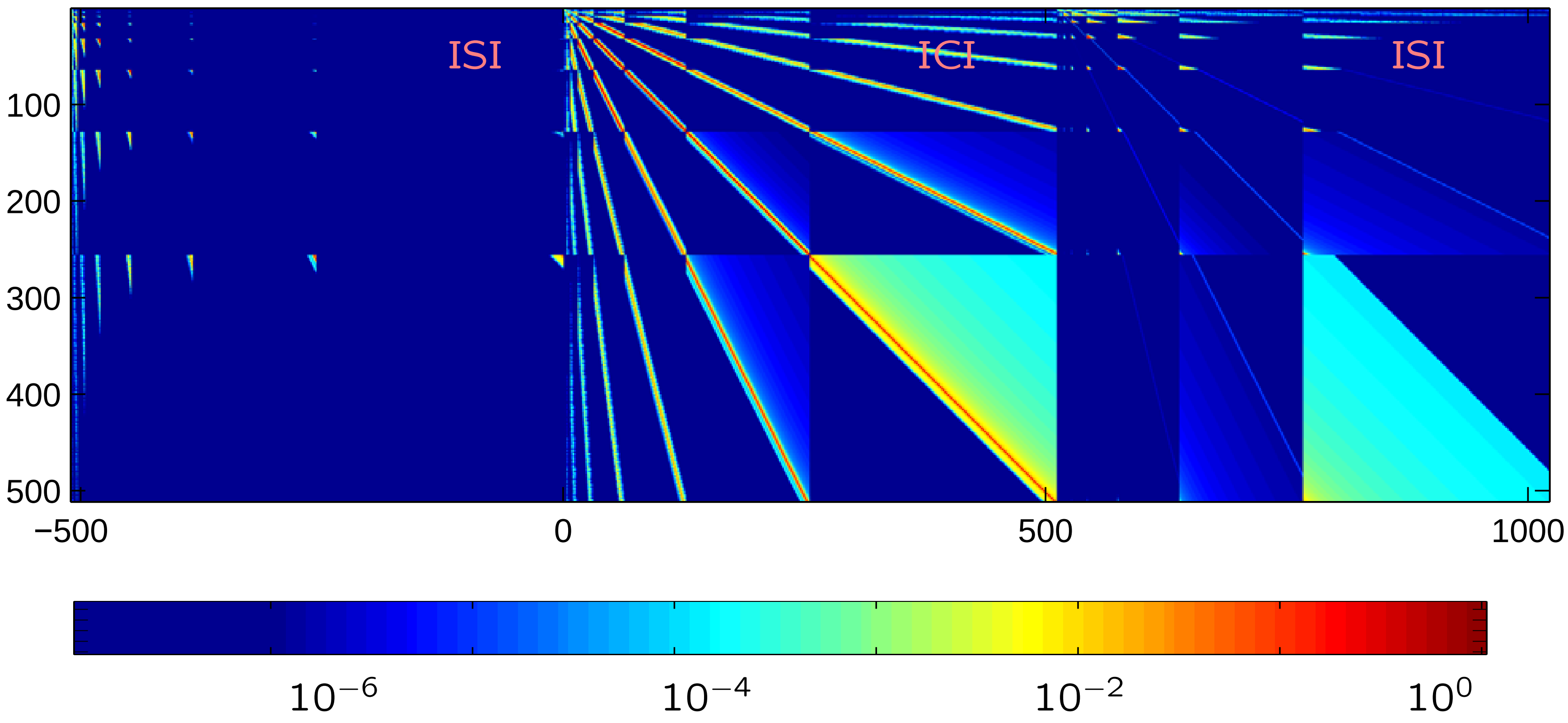
$$g_m^{(n)}(t) = 2^{m/2} g_0\big(2^m(t - n\frac{T}{2^m})\big)\,, m = 0, 1, \ldots, M\,, \quad n = 0, 1, \ldots, 2^m - 1$$

$$\text{(i.e., } N = 2^{M+1} - 1\text{)}\,.$$

Wavelet bases channel matrix

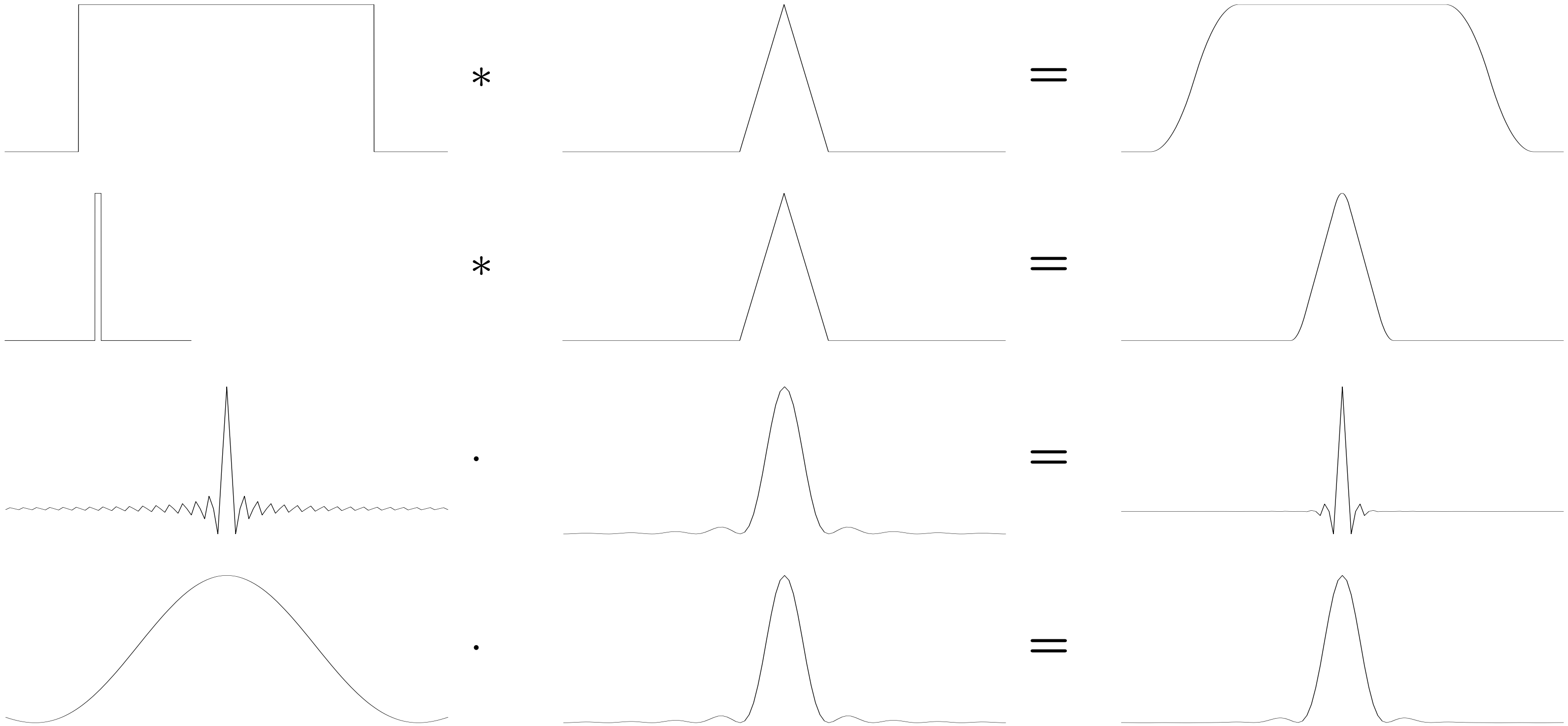
- $g_0(t[\mu s]) = \gamma_0(t[\mu s]) = \mathrm{db4}(t[\mu s])$
- $g_{511k+2^m-1+n} = g_{k,m}^{(n)}$.

- $T = 256\mu s$,
- $M = 8$
- $N = 511$



Why do wavelets fail?

Convolution of a pulse with the channel impulse response:

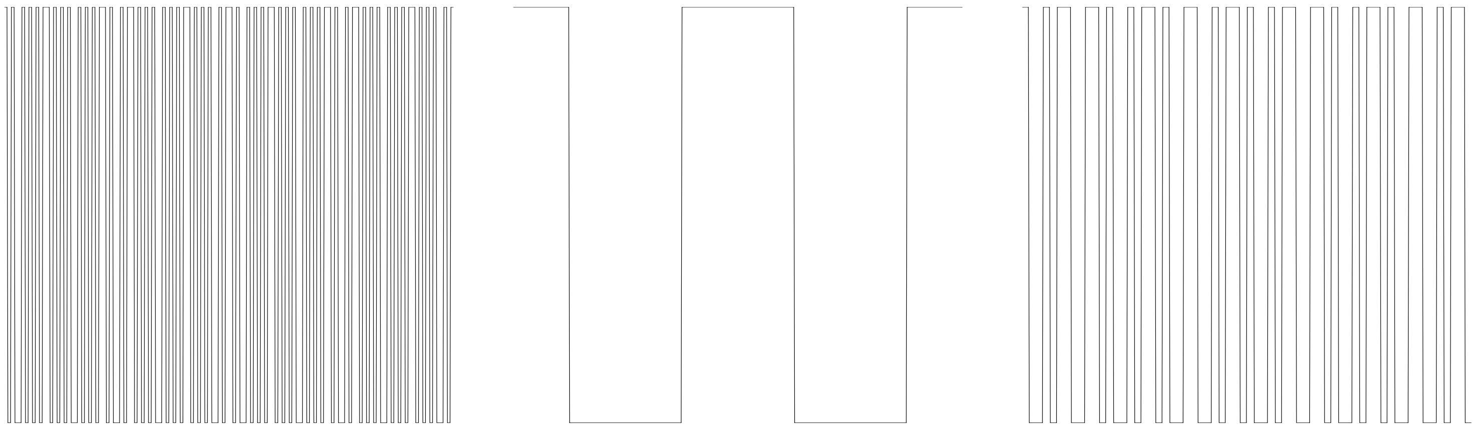


time
domain

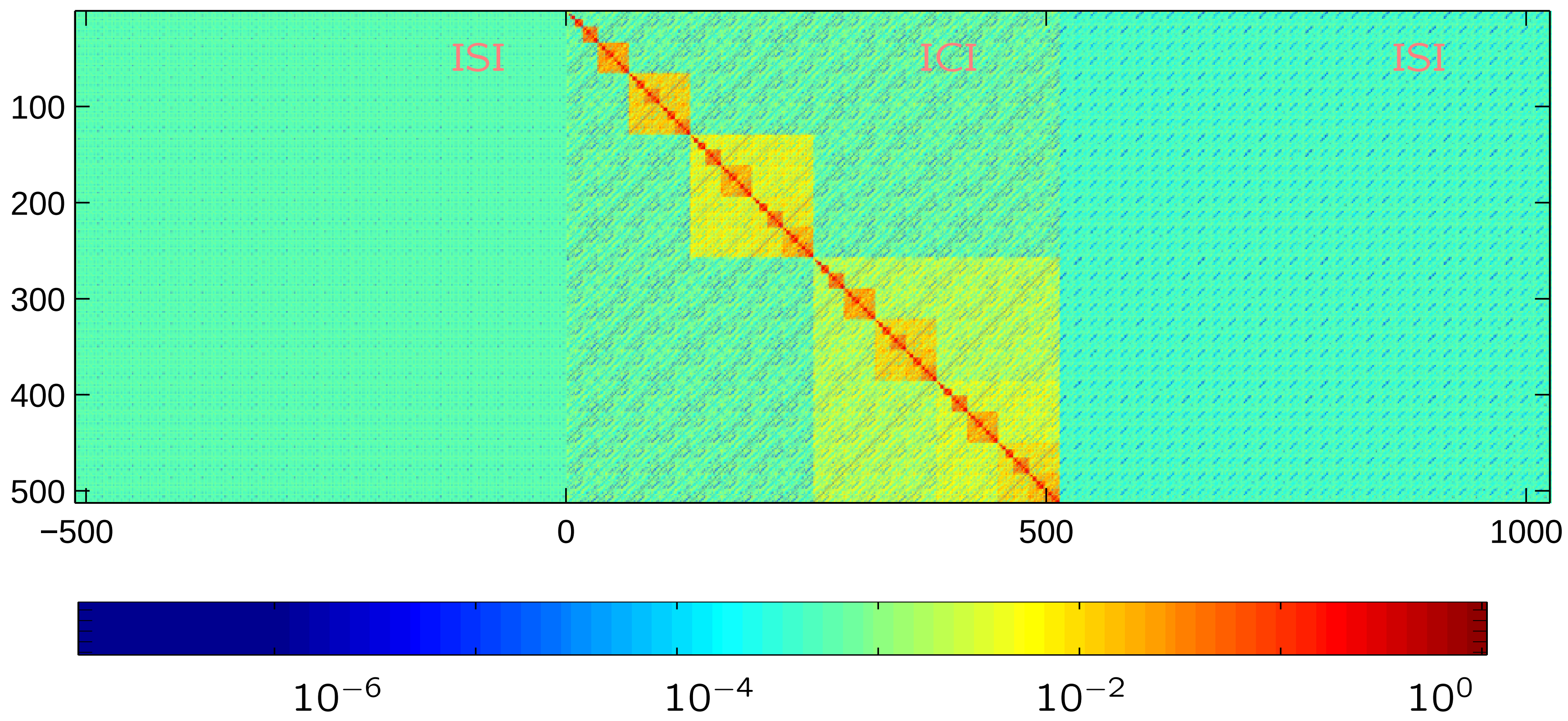
frequency
domain

iv. Walsh-Hadamard systems (CDMA)

$g^{(l)}(t)$ is the convolution of the binary-valued code sequence with index l and some bandlimited pulse-shaping signal.



Walsh-Hadamard system channel matrix

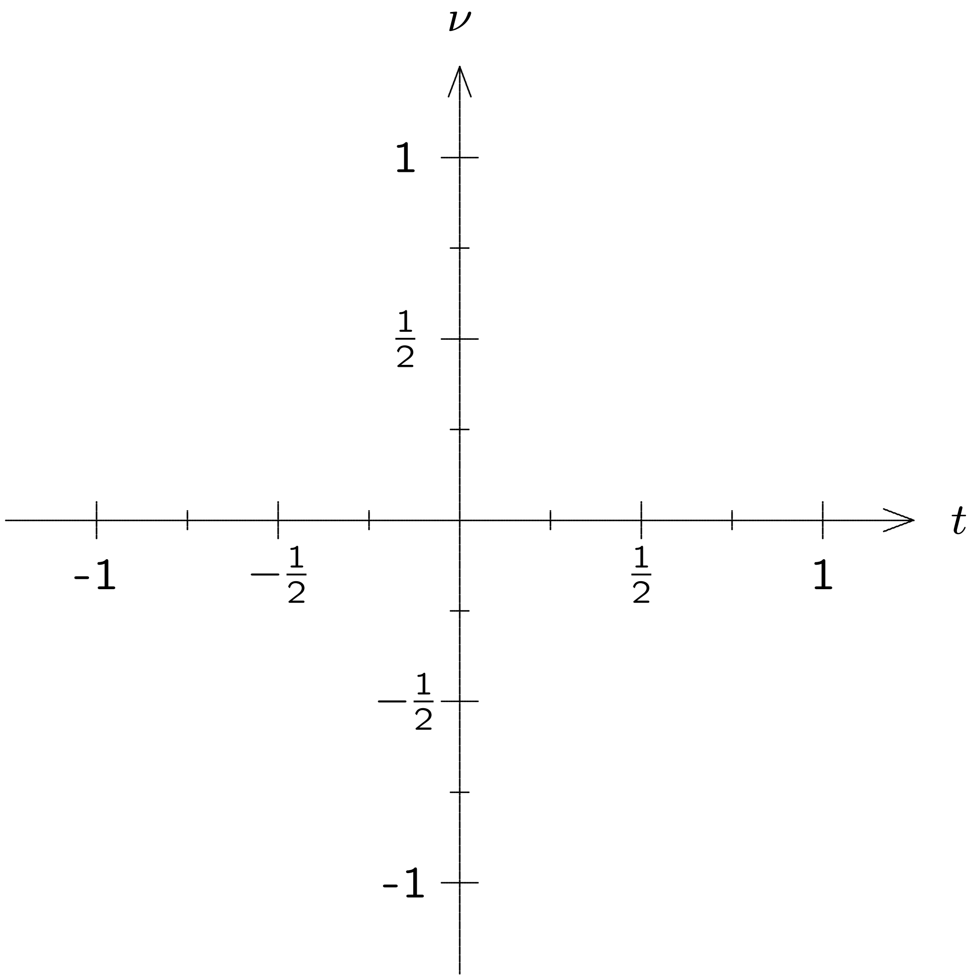


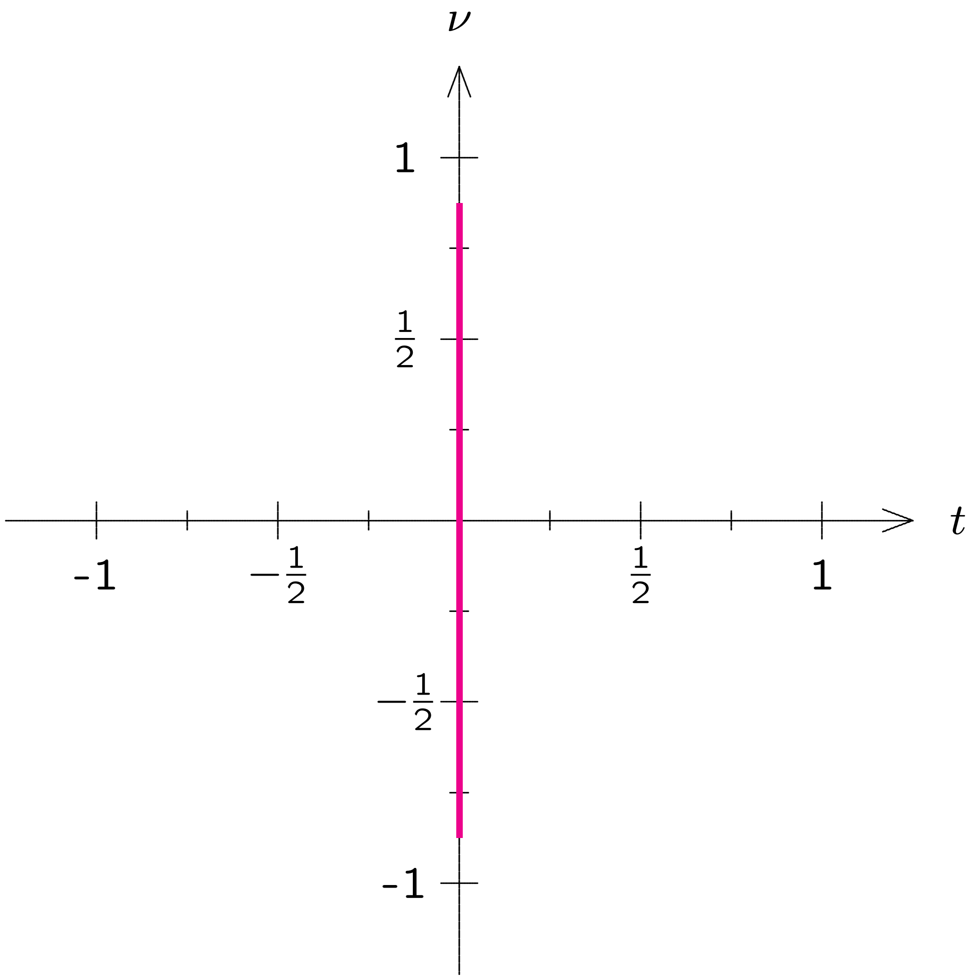
3. Mathematical contributions

- Upper and lower bounds on the orthogonal perturbation (ISI/ICI) of MCM systems. A magic superwavelet for digital transmission in time invariant environments does not exist.

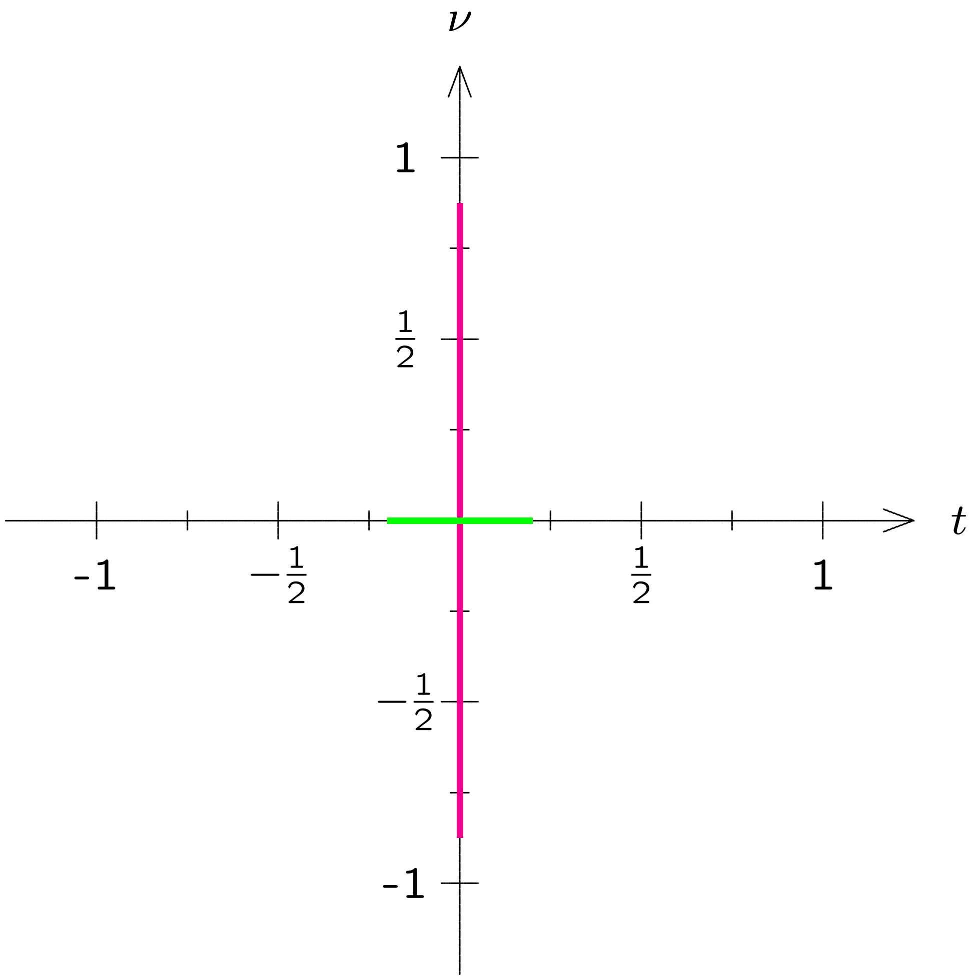
- Bounds on "oversampling" estimates of the crest factor in the *Weyl–Heisenberg* setup.

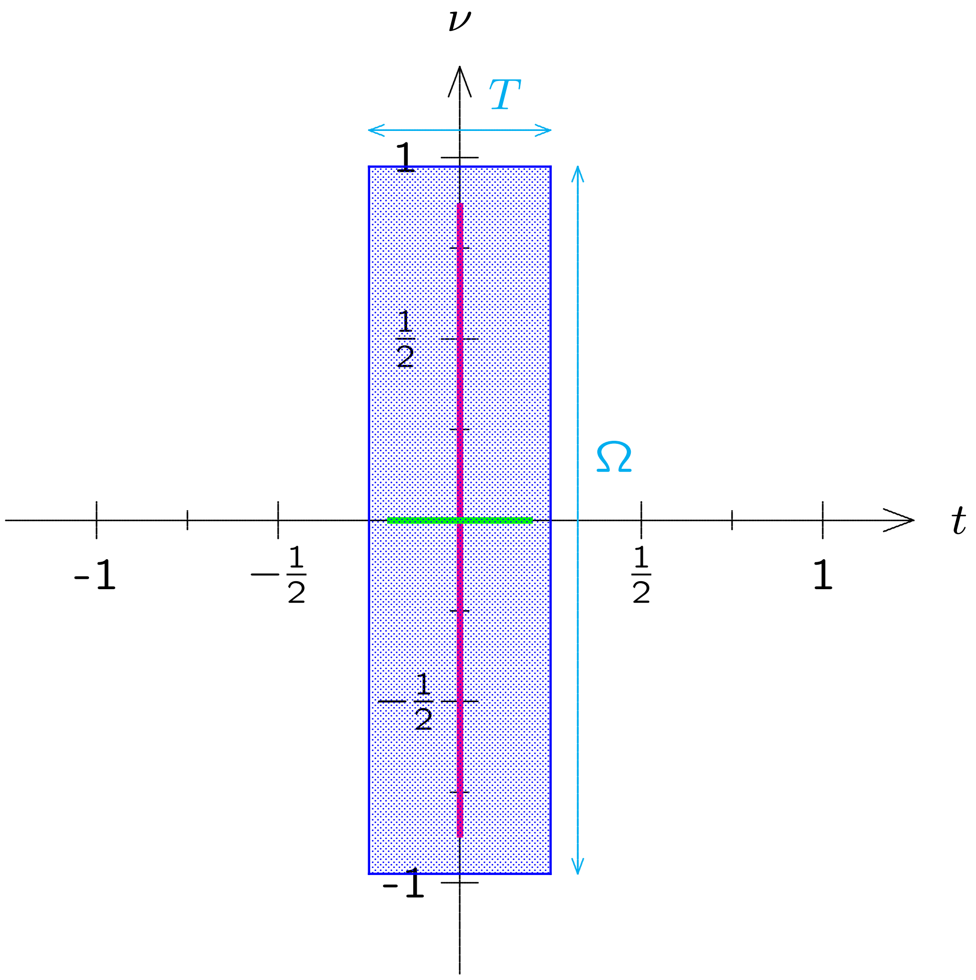
- Analysis of slowly time variant systems, solution of Kailath's underspread channel identification problem.


ν
1
$\frac{1}{2}$
$-\frac{1}{2}$
-1
-1
$-\frac{1}{2}$
$\frac{1}{2}$
1
t

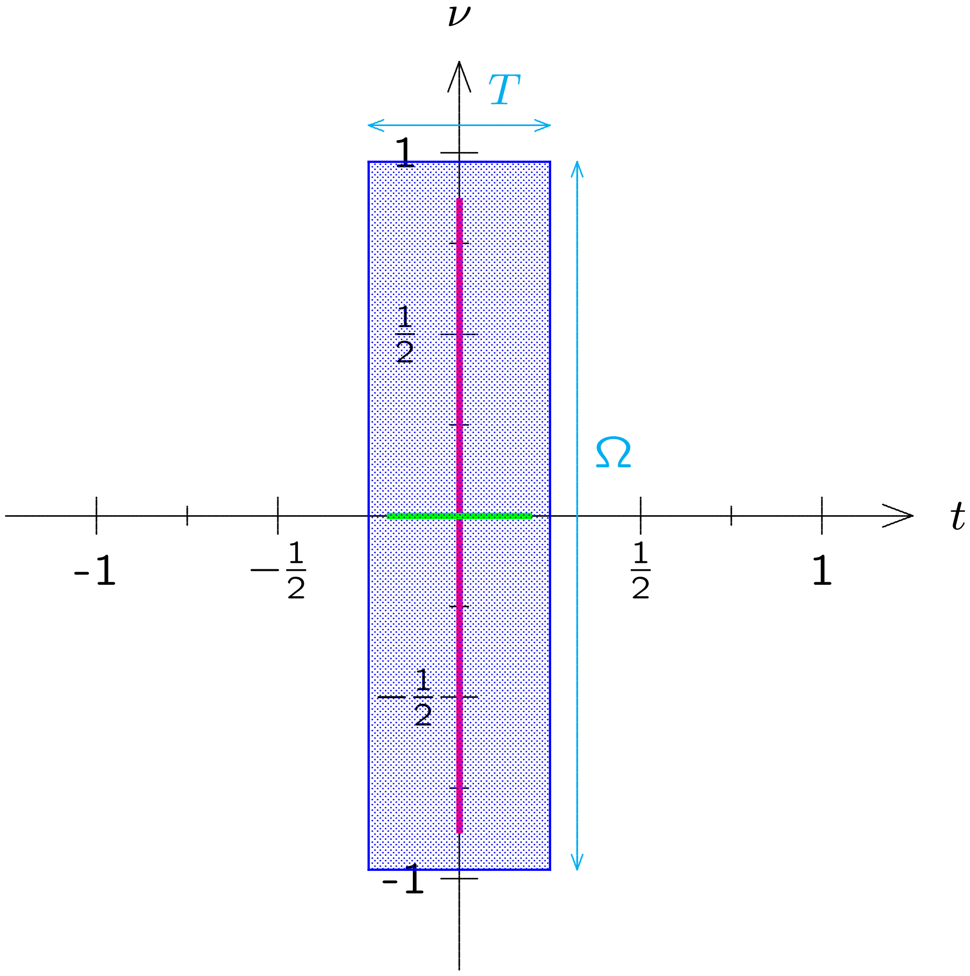


ν
1
$\frac{1}{2}$
$-\frac{1}{2}$
-1
t
-1
$-\frac{1}{2}$
$\frac{1}{2}$
1

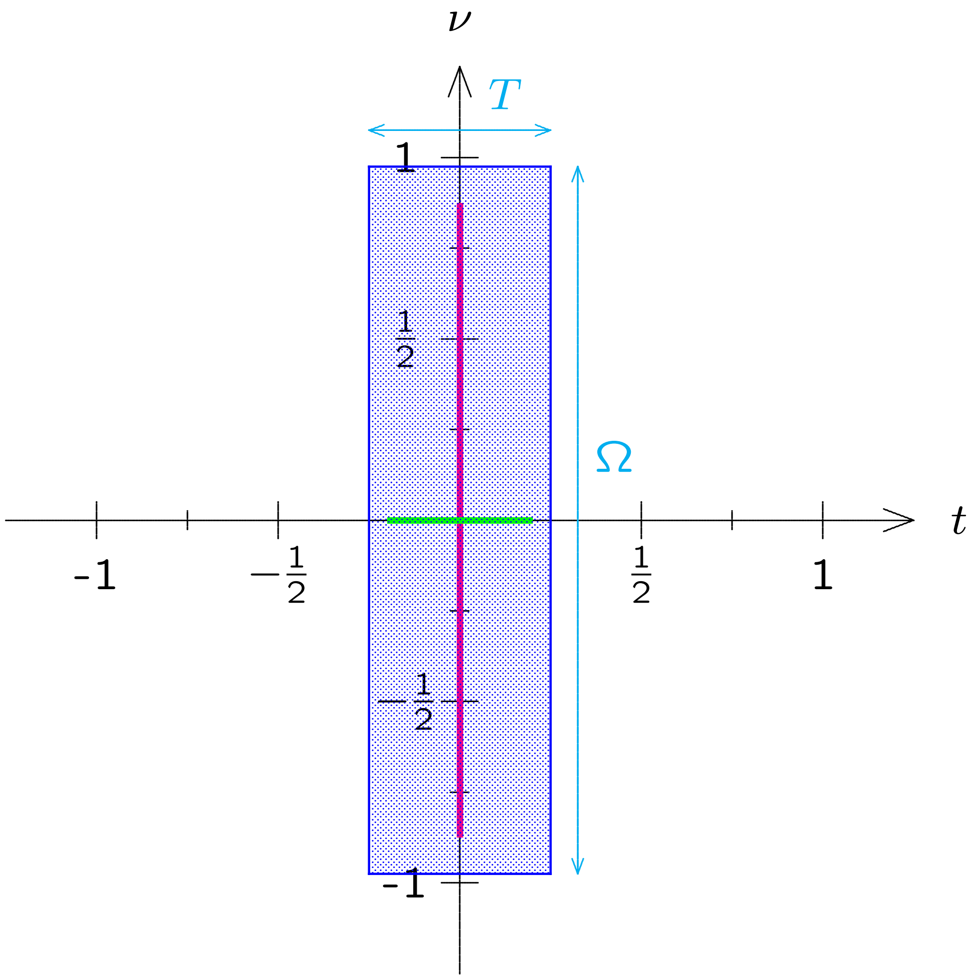

ν
1
$\frac{1}{2}$
$-\frac{1}{2}$
-1
-1
$-\frac{1}{2}$
$\frac{1}{2}$
1
t



ν
T
1
$\frac{1}{2}$
Ω
t
-1
$-\frac{1}{2}$
$\frac{1}{2}$
1
$-\frac{1}{2}$
-1

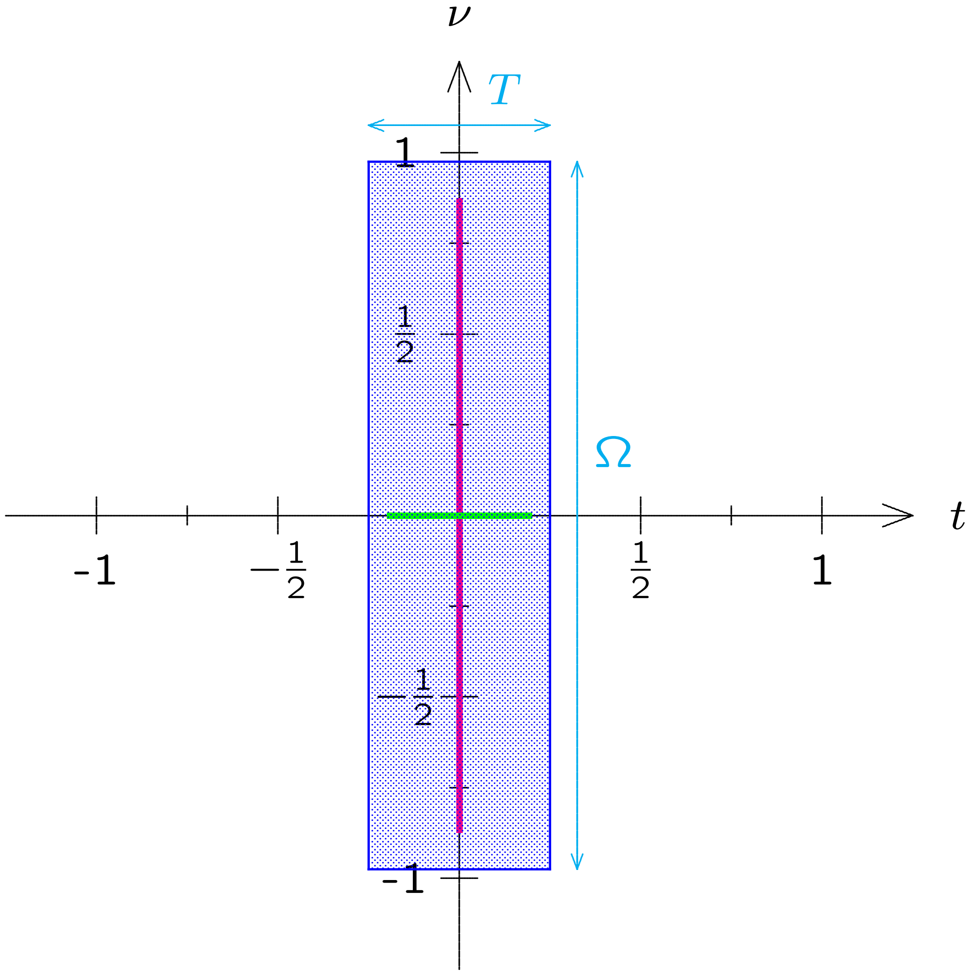


Classical sampling theorem ⇔ *identifiability of operators* with distributional support of $\widehat{\sigma}$ ($\sigma=$ Kohn–Nirenberg symbol) lying on ν axis.



Classical sampling theorem $\Leftrightarrow$ *identifiability of operators* with distributional support of $\widehat{\sigma}$ (σ= Kohn–Nirenberg symbol) lying on ν axis.

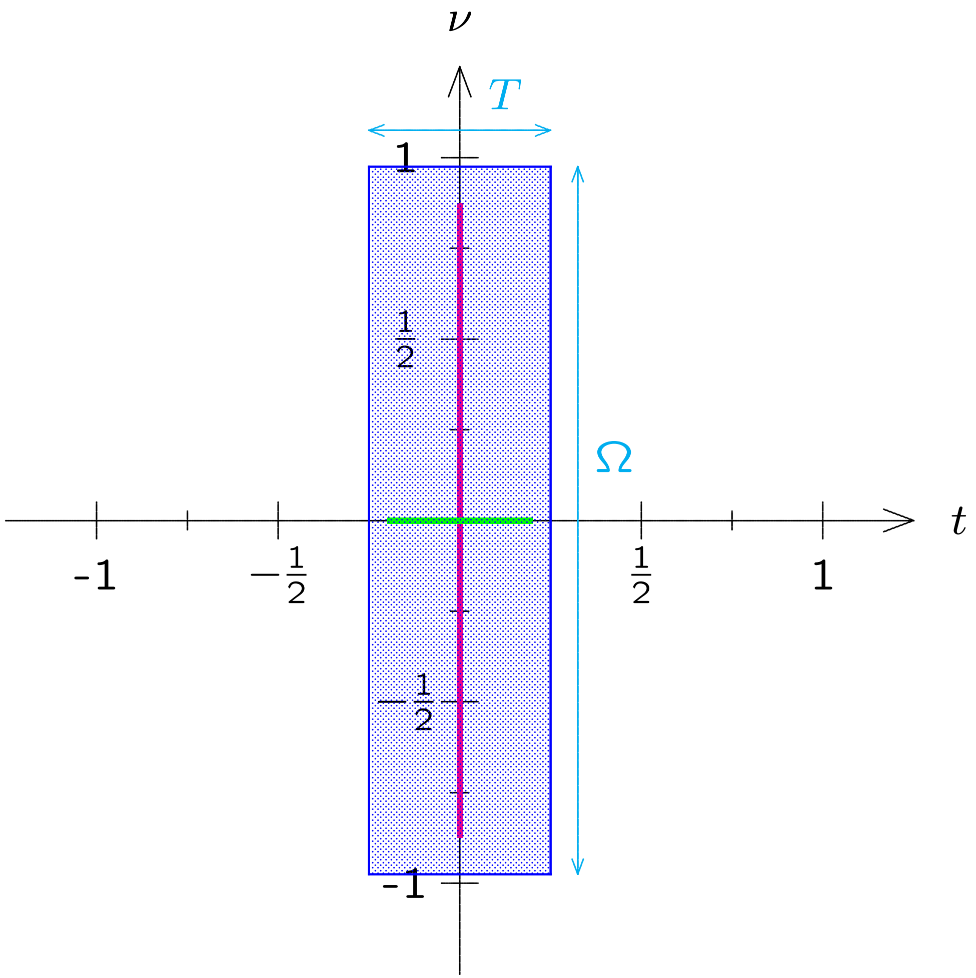
Time–invariant operators can be identified from their action on the δ $\Leftrightarrow$ *identifiability of operators* with distributional support of $\widehat{\sigma}$ on t axis.



Classical sampling theorem $\Leftrightarrow$ *identifiability of operators* with distributional support of $\widehat{\sigma}$ ($\sigma=$ Kohn–Nirenberg symbol) lying on ν axis.

Time–invariant operators can be identified from their action on the δ $\Leftrightarrow$ *identifiability of operators* with distributional support of $\widehat{\sigma}$ on t axis.

Our operator sampling theorem extends these results to operators with distributional support of $\widehat{\sigma}$ contained in sets of area less than one.

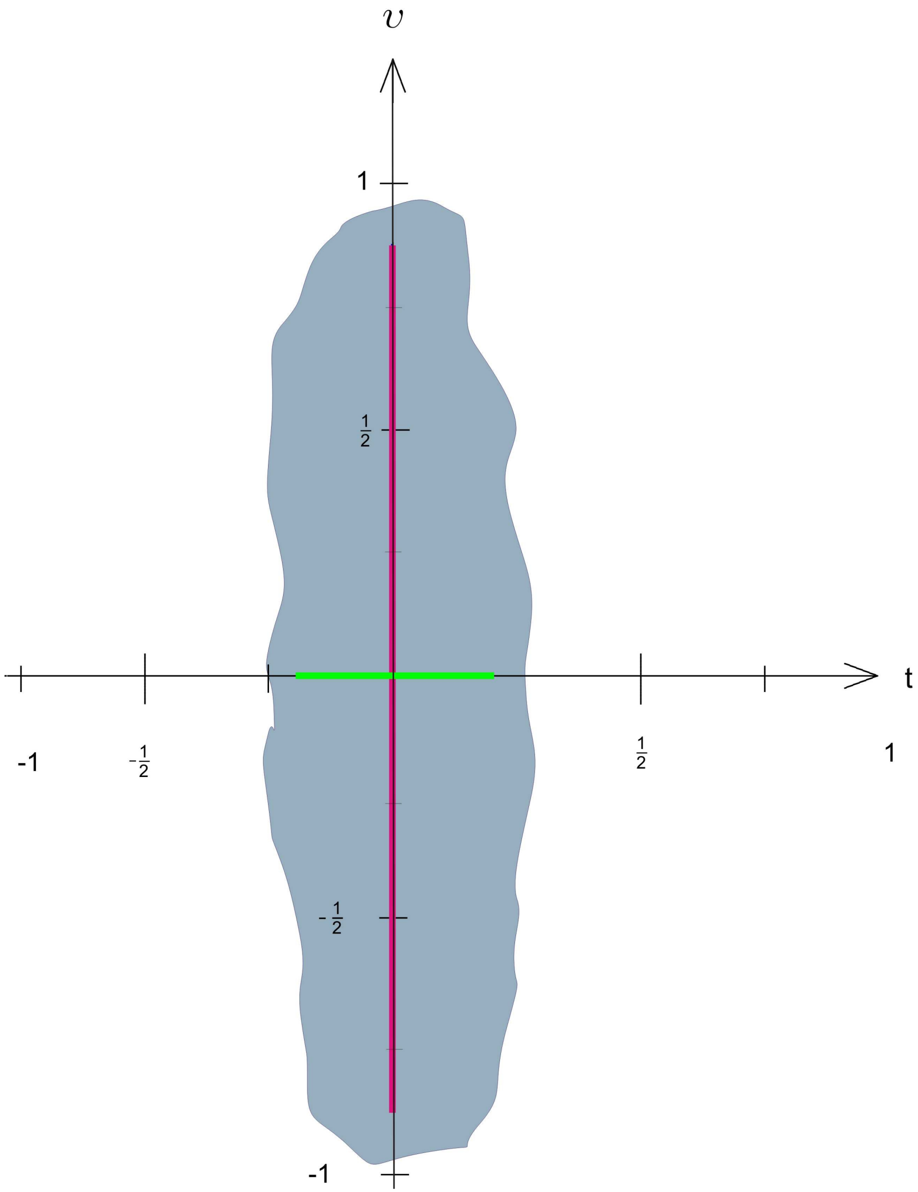


Classical sampling theorem $\Leftrightarrow$ *identifiability of operators* with distributional support of $\widehat{\sigma}$ (σ= Kohn–Nirenberg symbol) lying on ν axis.

Time–invariant operators can be identified from their action on the δ $\Leftrightarrow$ *identifiability of operators* with distributional support of $\widehat{\sigma}$ on t axis.

Our operator sampling theorem extends these results to operators with distributional support of $\widehat{\sigma}$ contained in sets of area less than one.

Results cover • linear differential operators with bandlimited coefficients • any pseudo-differential operators of arbitrary order and bandlimited Kohn–Nirenberg symbol • finite delay convolution operators, • multiplication operators with bandlimited symbols.



Classical sampling theorem $\Leftrightarrow$ *identifiability of operators* with distributional support of $\widehat{\sigma}$ on ν axis.

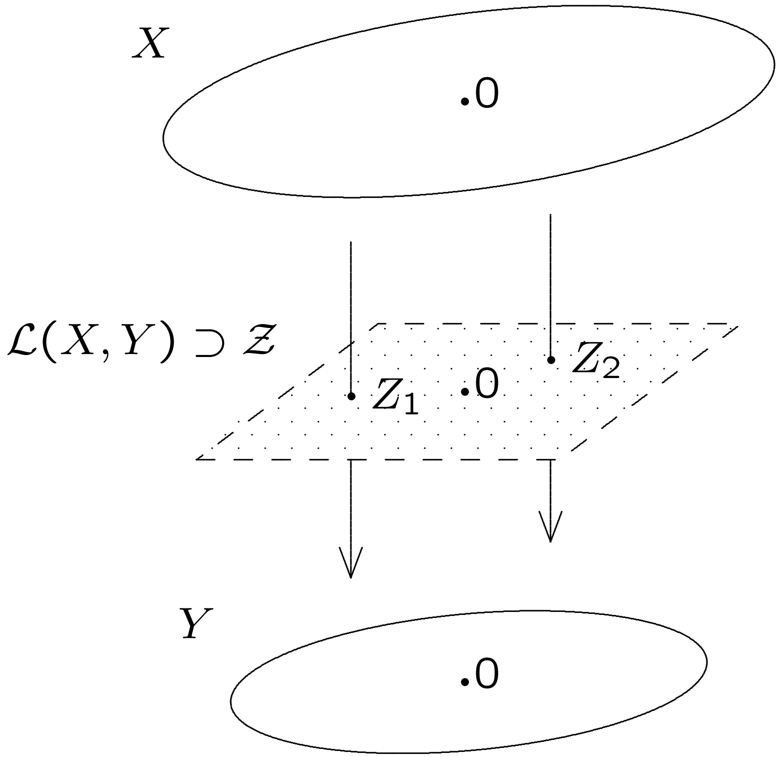
Time–invariant operators can be identified from their action on the δ $\Leftrightarrow$ *identifiability of operators* with distributional support of $\widehat{\sigma}$ on t axis.

Our operator sampling theorem extends these results to operators with distributional support of $\widehat{\sigma}$ contained in sets of area less than one.

Results cover • linear differential operators with bandlimited coefficients • any pseudo-differential operators of arbitrary order and bandlimited Kohn–Nirenberg symbol • finite delay convolution operators, • multiplication operators with bandlimited symbols.

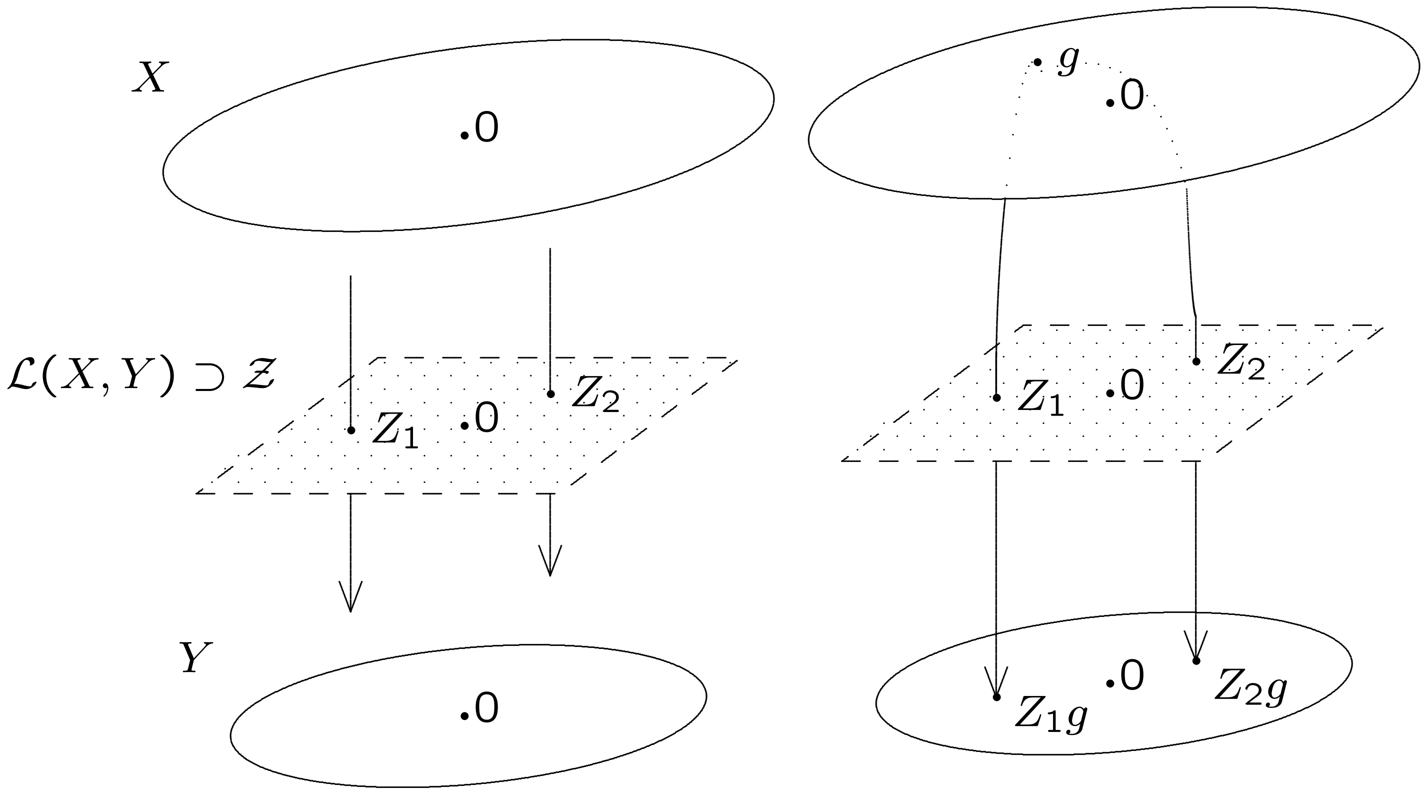
Theory I. The operator identification problem

Exists $g \in X$**, with** $\|Z\|_{\mathcal{Z}} \asymp \|Zg\|_Y$ **for all** $Z \in \mathcal{Z} \subseteq \mathcal{L}(X, Y)$ **?**



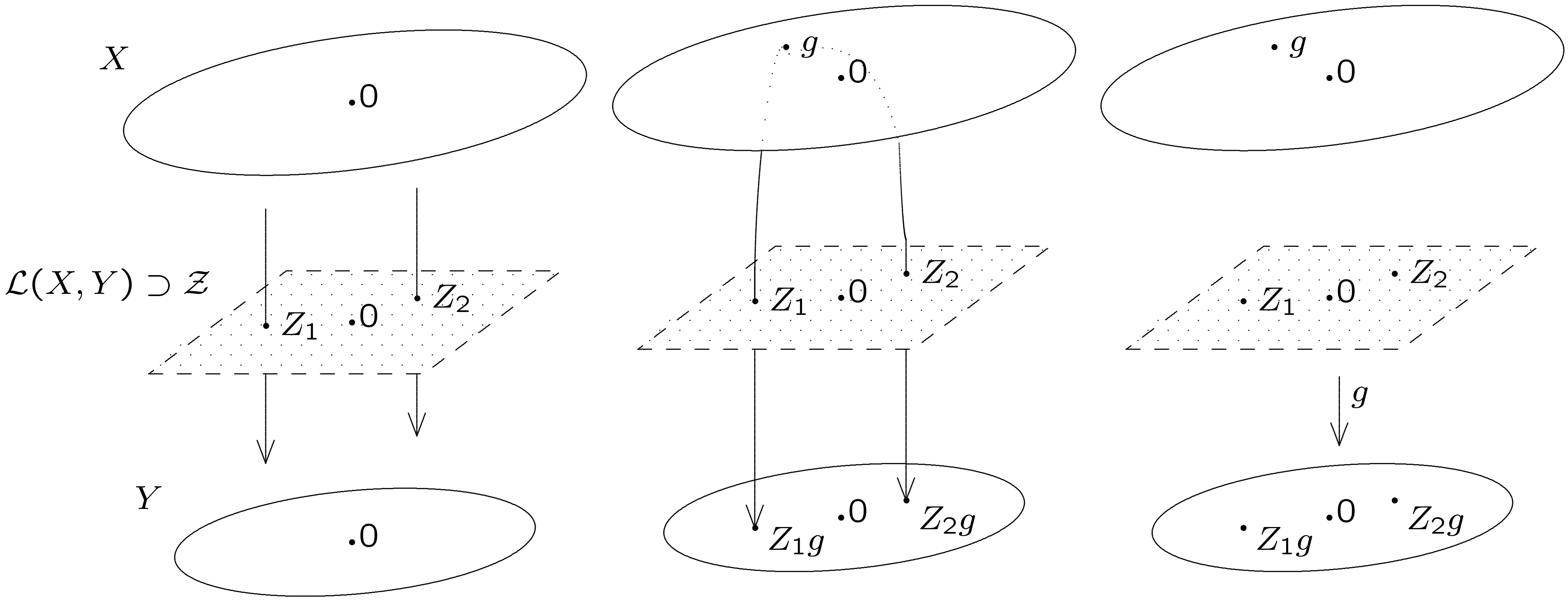
Theory I. The operator identification problem

Exists $g \in X$**, with** $\|Z\|_{\mathcal{Z}} \asymp \|Zg\|_Y$ **for all** $Z \in \mathcal{Z} \subseteq \mathcal{L}(X,Y)$ **?**



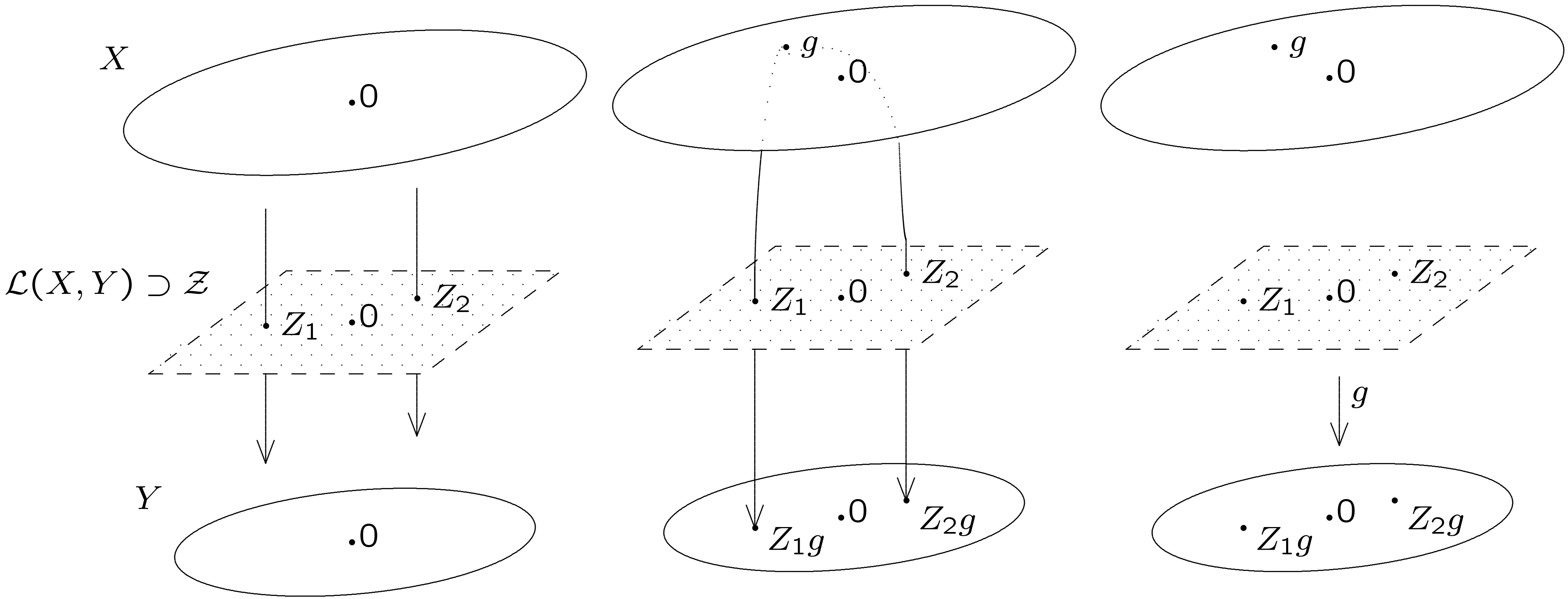
Theory I. The operator identification problem

Exists $g \in X$, with $\|Z\|_{\mathcal{Z}} \asymp \|Zg\|_Y$ for all $Z \in \mathcal{Z} \subseteq \mathcal{L}(X,Y)$?



Theory I. The operator identification problem

Exists $g \in X$, **with** $\|Z\|_{\mathcal{Z}} \asymp \|Zg\|_Y$ **for all** $Z \in \mathcal{Z} \subseteq \mathcal{L}(X,Y)$ **?**



If $g = \sum c_j \delta_{s_j}$ then identification is referred to as operator sampling.

Main Result I.

Classical Sampling Theorem. Given a function $m \in PW^2(\Omega)$ and T with $T\Omega < 1$. Choose $s \in PW^2\big(\frac{2}{T} - \Omega\big)$ with $\widehat{s} = 1$ on $[-\frac{\Omega}{2}, \frac{\Omega}{2}]$.

Then $\{m(nT)\}$ fully characterizes m, $\| \{m(kT)\} \|_{l^2} \asymp \|m\|_{L^2}$ and

$$m(x) = T \sum_{k \in \mathbf{Z}} m(kT)\, s(x - kT) = T \sum_{k \in \mathbf{Z}} m(kT)\, \frac{\sin 2\pi\Omega(x - kT)}{\pi(x - kT)}\,.$$

Main Result I.

Classical Sampling Theorem. Given a function $m \in PW^2(\Omega)$ and T with $T\Omega < 1$. Choose $s \in PW^2(\frac{2}{T} - \Omega)$ with $\widehat{s} = 1$ on $[-\frac{\Omega}{2}, \frac{\Omega}{2}]$.

Then $\{m(nT)\}$ fully characterizes m, $\| \{m(kT)\} \|_{l^2} \asymp \|m\|_{L^2}$ and

$$m(x) = T \sum_{k \in \mathbf{Z}} m(kT)\, s(x - kT) = T \sum_{k \in \mathbf{Z}} m(kT)\, \frac{\sin 2\pi\Omega(x - kT)}{\pi(x - kT)}\,.$$

Theorem (GP, D. Walnut) For $H \in OPW_s^{pq}([-\frac{\Omega}{2}, \frac{\Omega}{2}] \times [-\frac{T'}{2}, \frac{T'}{2}])$ and $\Omega T' < \Omega T < 1$, choose $s \in PW^p(\frac{2}{T} - \Omega)$ with $\widehat{s} = 1$ on $[-\frac{\Omega}{2}, \frac{\Omega}{2}]$ and $r \in \mathcal{S}$ with $\operatorname{supp} r \subset [-T + \frac{T'}{2}, T - \frac{T'}{2}]$ and $r = 1$ on $[-\frac{T'}{2}, \frac{T'}{2}]$.

Then $H Ш_T = H \sum_k \delta_{kT}$ fully determines H, in fact, we have $\|H\, Ш\,\|_{M_s^{pq}} \asymp \|\sigma_H\|_{PW_s^{pq}}$ and

$$h_H(t, x) = r(t) \sum_{k \in \mathbf{Z}} (H Ш_T)(t + kT)\, s(x - kT)\,. \qquad \left(Hf(x) = \int h_H(t, x) f(x - t)\, dt \right)$$