On the Role of the Heisenberg Group in Communication Systems

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Challenges for future communication systems

Users and providers want from communication systems:

- Higher data rates
- Reliable service
- Inexpensive
- Mobility, connectivity
- Power/battery efficient
- High spectral efficiency
- Convergence of platforms
- Scalability

These demands put enormous pressure on communication systems and engineers who design them.

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Mathematics could help ...

“The Hinduistic Diversity of the Unity of the Heisenberg Group”

[Roger Howe]

Agenda

- The Heisenberg group
- Spreading sequence design for CDMA
- Code design for MIMO
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The Heisenberg group (1)

For $x, \omega \in \mathbb{R}$ we define the translation operator by

$$T_x f(t) = f(t - x)$$

and the modulation operator by

$$M_\omega f(t) = e^{2\pi i \omega t} f(t).$$

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T_x and M_ω "almost" commute

$$T_x M_\omega = e^{-2\pi i x \omega} M_\omega T_x,$$

thus they commute if and only if $x \cdot \omega \in \mathbb{Z}$.
Operators of the form $T_x M_\omega$ or $M_\omega T_x$ are called time-frequency shift operators. We obtain for the composition of two such time-frequency shift operators

$$(T_x M_\omega)(T_{x'} M_{\omega'}) = e^{2\pi i x' \omega} T_{x+x'} M_{\omega+\omega'}.$$

The Heisenberg group (2)

To make the operators $T_x M_\omega, (x, \omega) \in \mathbb{R} \times \mathbb{R}$, into a group we "add" the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z} = \{e^{2\pi i \tau} : \tau \in \mathbb{R}\}$ by introducing the scalar operators

$$S_\tau f(x) = e^{2\pi i \tau} f(x).$$

Now the set

$$\{T_x M_\omega S_\tau, (x, y) \in \mathbb{R} \times \mathbb{R}, e^{2\pi i \tau} \in \mathbb{T}\}$$

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becomes a group of unitary operators.
More abstractly, we consider $\mathbb{R} \times \mathbb{R} \times \mathbb{T}$ and define on it a multiplication that is consistent with the composition before:

The Heisenberg group

$$(x, \omega, e^{2\pi i\tau}) \cdot (x', \omega', e^{2\pi i\tau'}) = (x + x', \omega + \omega', e^{2\pi i(\tau+\tau'+x'\omega)}).$$

With this multiplication, $\mathbb{R} \times \mathbb{R} \times \mathbb{T}$ becomes the *(reduced polarized)* Heisenberg group $\mathbb{H}(\mathbb{R})$.

The Heisenberg group (3)

Instead of defining T_x and M_ω over $\mathbb{R}$, we could use another field, such as $\mathbb{Z}$, $\mathbb{C}^N$ or $\mathbb{Z}_2^m$.
On $\mathbb{C}^N$: Instead of $\mathbb{R} \times \mathbb{R} \times \mathbb{T}$ we consider $\mathbb{C}^N \times \mathbb{C}^N \times \mathbb{T}_N$ where $\mathbb{T}_N$ is the sampled torus $\{e^{2\pi ik/N}, k = 0, \dots, N-1\}$.

$$T = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \ddots & & 0 \\ 0 & 1 & & & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & & 1 & 0 \end{bmatrix}, \quad \begin{aligned} M &= \operatorname{diag}[e^{2\pi i 0/N}, \dots, e^{2\pi i (N-1)/N}], \\ S &= e^{2\pi i/N} \cdot I, \end{aligned}$$

and

$$\mathbb{H}(\mathbb{C}^N) = \left\{ T^a M^b S^c, \;\; a, b, c \in \{0, \dots, N-1\} \right\}$$

The finite-field Heisenberg group $\mathbb{H}(\mathbb{Z}_2^m)$

We start with $\mathbb{Z}_2$: T_2 and M_2 are given by

$$T_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \text{we have } \; iT_2M_2 =: R_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

$\mathbb{H}(\mathbb{Z}_2)$ is generated by T_2, M_2, extended by $i \cdot I_2$.
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Note: could use $(-1) \cdot I_2$ instead, but $i \cdot I_2$ gives richer structure.
The elements of $\mathbb{H}(\mathbb{Z}_2^m)$ are unitary $2^m \times 2^m$ matrices of the form

$$X_0 \otimes \cdots \otimes X_{m-1}, \quad \text{where } X_k \in \{T_2, M_2, I_2, R_2\}.$$

$\mathbb{H}(Z_2^m)$ has 2^{2m+2} elements, each represented by a pair of binary m-tuples. For instance:

$$T_2 \otimes M_2 \otimes R_2 \otimes T_2 \otimes I_2 \leftrightarrow (10110|01100)$$

$\mathbb{H}(\mathbb{Z}_2^m)$ and mutually unbiased bases

Some important properties of $\mathbb{H}(\mathbb{Z}_2^m)$ (we denote $N := 2^m$):

- $\mathbb{H}(\mathbb{Z}_2^m)$ can be split into $N+1$ commutative subgroups $S_1, \ldots, S_{N+1}$.
- These subgroups are disjoint, except for the identity element.
- Each subgroup S_k has N elements, i.e., each S_k is a maximal commutative subgroup.

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$$|\langle u_m^{(k)}, u_n^{(l)} \rangle| = \frac{1}{\sqrt{N}} \quad \text{if } k \neq l;\ m, n = 1, \ldots, N.$$

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- $$u_m^{(k)}(j) \in \left\{ \frac{\pm 1}{\sqrt{N}}, \frac{\pm i}{\sqrt{N}} \right\}.$$

Example

$\mathbb{H}(\mathbb{Z}_2^2)$ contains 16 elements, it can be split into 5 maximal commutative subgroups, each containing 4 elements.
One such subgroup of $\mathbb{H}(\mathbb{Z}_2^2)$ is already diagonal, the four other ones are diagonalized by Hadamard matrices U_k of the form

$$U_k := D_k U D_k^*$$

where the D_k are diagonal matrices with entries $\pm 1, \pm i$ and

$$U = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

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Some classical mathematical questions

Definition: Given a set of vectors $V = \{v_1, \ldots, v_n\}$ in $\mathbb{C}^d$. The incoherence μ of V is defined as

$$\mu(V) := \max_{j \neq k} |\langle v_j, v_k \rangle|$$

- How can one arrange n lines through the origin in $\mathbb{R}^d$ (or $\mathbb{C}^d$), such that the incoherence between them is as large as possible?
- How many equiangular lines exist in $\mathbb{R}^d$? (i.e., $|\langle v_j, v_k \rangle| = c$ for all $j \neq k$)
- How many equiangular k-dimensional subspaces ($k \leq d$) exist in $\mathbb{R}^d$?
- How many ONBs can we construct in $\mathbb{C}^d$ such that abs. value of the inner product between vectors from different ONBs is $\frac{1}{\sqrt{d}}$? $\rightarrow$ Mutually unbiased bases (MUB).

The Heisenberg group and Grassmannian packings

The Heisenberg group $\mathbb{H}(\mathbb{Z}_2^m)$ provides some constructive answers to the questions above.
For instance there exist at most N^2 equiangular lines in $\mathbb{C}^N$ with incoherence $\mu = \frac{1}{\sqrt{N+1}}$.
Explicit constructions are known for $N \leq 12$, they are of the form

$$v_{j+k} := T^k M^j x, \quad j, k = 0, \dots, N-1$$

for a specifically designed vector x. Proofs (and thus constructions) rely on special properties of $\mathbb{H}(\mathbb{C}^N)$.

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Related to packings in Grassmannian spaces [Conway, Sloane, Seidel, Calderbank], as well as concept of Grassmannian frames [T.S., R.Heath].
Many problems in communications can be linked to these questions!

Agenda

- The Heisenberg group
- Spreading sequence design for CDMA
- Code design for MIMO
- Pulse shape design for OFDM
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Spreading sequence design for CDMA ($\mathbb{H}(\mathbb{Z}_2^m)$)

We consider the downlink of a direct sequence CDMA system.
We assume equal power transmission, chip rate sampling, i.i.d. Raleigh fading channel, spreading gain N and K users.
User n observes the $N \times 1$ vector

$$y = h_n\Big(\sum_{k=1}^{K} x_k u_k\Big) + w$$

where $x_k \in \{\pm 1\}$ is the data symbol transmitted to the k-th user, h_n is the channel gain for the n-th user, and $u_1, \dots, u_K$ are the spreading sequences.

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where $x_k \in \{\pm 1\}$ is the data symbol transmitted to the k-th user, h_n is the channel gain for the n-th user, and $u_1, \ldots, u_K$ are the spreading sequences. In matrix notation:

$$y = HUx + w$$

where U is an $N \times K$ spreading matrix, H is an $N \times N$ diagonal matrix with Raleigh fading coefficients as entries, w is AWGN.
Typical setup: $K = N$ and U is the Hadamard Walsh matrix.
Using a conventional matched filter, user k computes

$$\tilde{x}_k = \langle y, u_k \rangle = h_k x_k \langle u_k, u_k \rangle + h_k \sum_{k \neq l} x_l \langle u_l, u_k \rangle + \langle w, u_k \rangle.$$

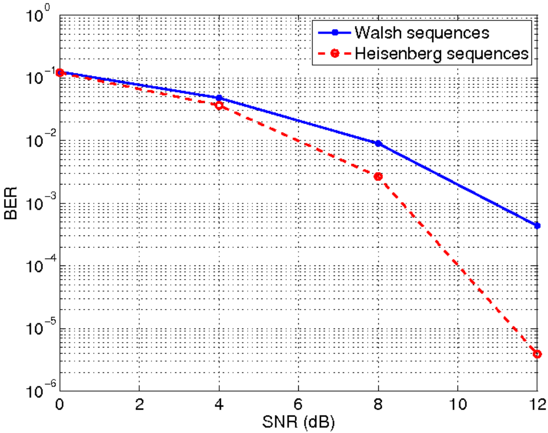
Overloaded CDMA

CDMA systems are interference limited.
Capacity translates to increasing the number of spreading sequences.
Question: How many spreading sequences can be added to the Walsh sequences of length N subject to the condition

$$|\langle u, v \rangle|^2 = 0 \text{ or } \frac{1}{N}, \quad \text{for all } u, v?$$

Answer: We can add N ONBs (thus N^2 spreading sequences) to the Walsh ONB, such that the above condition is satisfied. They are obtained from the maximal commuting subspace construction of the Heisenberg group $\mathbb{H}(\mathbb{Z}_2^m)$.
Designing spreading sequences for overloaded systems as equiangular sequences or as MUBs is optimal in several ways [R.Heath, A.Paulraj, T.S.]

Numerical comparison of spreading sequences



Other applications, further improvements

- LTE - single-carrier OFDM, spreading OFDM.
- Kashin representation should give even further improvements (relies on results by Vershynin, random matrix theory, dual to sparse representations a'la Candes-Tao).

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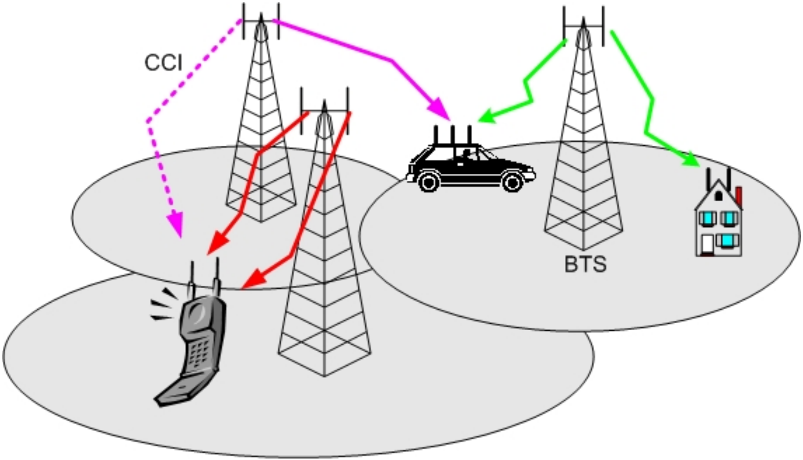
MIMO Wireless communication



MIMO Communication: Multiple-input multiple-output.
In certain radio environments we get channel variation in both space and time. If we use two (or more) antennas at the transmitter and receiver we can exploit this diversity.
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MIMO can provide significant increases in system performance and capacity.
MIMO can even simplify hardware design. Atheros' 802.11 chip utilizes MIMO and CMOS to integrate power amplifier into integrated circuit, which may allows them to reduce PAPR problem of OFDM.
[For more on PAPR see this workshop: Oswald and Henkel: *"PAPR Reduction - Mathematical and Realization Aspects"*]

Space-time coding

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Space-time coding takes advantage by correlating the data to be transmitted both in space and time.
Assume we have M transmit antennas and we encode information across T time intervals.

- Codewords become matrices:

Antenna

$$X = \begin{bmatrix} f_{11}(S) & f_{12}(S) & \dots & f_{1N}(S) \\ f_{21}(S) & f_{22}(S) & \dots & f_{2N}(S) \\ \vdots & & & \vdots \\ f_{T1}(S) & f_{T2}(S) & \dots & f_{TN}(S) \end{bmatrix} \quad \text{Time}$$

- S is a symbol (or bit) sequence
- T symbol periods, N transmit antennas
- $f_{i,j}$ are linear functions of S

Alamouti code

Example: Two transmit antennas, and we transmit two symbols ($S = [s_1, s_2]$) over two symbol periods

$$X(s_1, s_2) = \begin{bmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{bmatrix}$$

Note that

$$X = \Re[s_1] I_2 + \Im[S_1] M_2 + \Re[S_2] T_2 + \Im[S_2] R_2$$

thus underlying the Alamouti code are the orthogonal coefficient matrices for $\mathbb{H}(\mathbb{Z}_2)$ (Pauli matrices).
Alamouti code is an example of an orthogonal space-time block code.
Want to express codewords as linear combination of specific coefficient matrices

$$X = \sum_{k=1}^{N} a_k C_k$$

(Quasi)Orthogonal space time block codes

Calderbank et al. have developed a general theory for designing orthogonal space time block codes (OSTBC). The coefficient matrices C_k of such OSTBC satisfy (among others):

- each C_k is unitary
- anti-Hermitian $C_k^* = -C_k$
- anti-commuting $C_k C_j + C_j C_k = 0, j \neq k$

OSTBCs are easy to decode.
Unfortunately they exist only for very specific dimensions, i.e., for dimensions 2, 4, 8. But code rate for dim. 4 and 8 is low.
Quasiorthogonal space time block codes:
Relax orthogonality requirement, decoding becomes more expensive, but higher rate codes become possible.
Many quasiorthogonal STBCs are based on $\mathbb{H}(\mathbb{Z}_2^m)$ (e.g. construction by A.Sezgin)

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In practice the feedback channel has often very low bandwidth. Thus instead of the exact CSI, the receiver will send a heavily quantized (and maybe very inaccurate) version of CSI.
Optimal low-rate quantization strategies for MIMO feedback channels are highly non-trivial.
Instead of sending a poorly quantized CSI, it is better to predesign a quantization codebook and then simply send the index of the nearest codeword.

Feedback codebook design for MIMO

Say, the codebook is $\mathcal{C} = \{C_1, \ldots, C_n\}$. If

$$\min_{k=1,\ldots,n} ||CSI - C_k||$$

is attained by C_j then we simply feed back index j.
E.g., if we have a 3-bit feedback channel we can use $n = 8$.
How shall optimal feedback codebook be designed for MIMO?

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- Algebraic constructions are based on $\mathbb{H}(\mathbb{Z}_2^m)$.
- Our codebook constructions are used in 802.16e standard (WIMAX).

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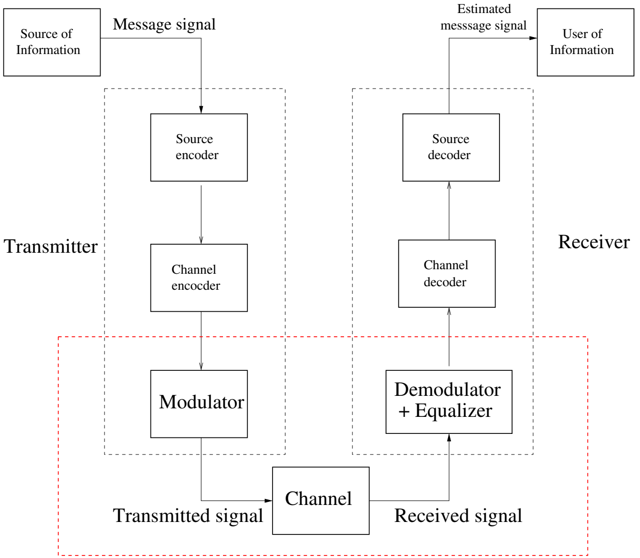
Basic communication scheme (1)



Basic communication scheme (2)

Transmitter: Let $x = \{x_k\}_{k\in\mathcal{I}}$ be the discrete data to be transmitted (for simplicity assume $x_k \in \{+1, -1\}$. Let $\{\varphi_k\}_{k\in\mathcal{I}}$ be a family of (bandlimited) functions – the transmission pulses. The (continuous-time) signal to be transmitted is

$$s(t) = \sum_{k\in\mathcal{I}} x_k \varphi_k(t)$$

Channel: **H** is the operator representing the radio channel. The received signal is

$$r = \mathbf{H}s + w$$

Receiver: The discrete data $y = \{y_k\}_{k\in\mathcal{I}}$ are extracted from r by computing

$$y_k = \langle r, \psi_k \rangle, \qquad k \in \mathcal{I}$$

where $\{\psi_k\}_{k\in\mathcal{I}}$ is a family of receiver functions. For simplicity we assume $\varphi_k = \psi_k$.

Basic communication scheme (3)

We introduce the coefficient operator C by

$$Cr = \{\langle r, \varphi_l \rangle\}_{l \in \mathcal{I}}$$

and note that $s = C^* x$. Then

$$y = C\mathbf{H}C^* x$$

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Defining the matrix $A = [A_{k,l}]_{k,l \in \mathcal{I}}$, where $A_{k,l} = \langle H\varphi_l, \varphi_k \rangle$ and using linearity of **H** we get

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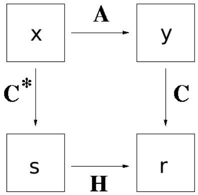

Equalization ...

We need to solve $Ax = y$.

Problems:

- A is an infinite-dimensional, or very large, matrix
- Application of A or inversion of A may be very expensive
- The computation of C^*x and Cy may be expensive
- A and **H** are a priori not known at the receiver (requires channel estimation)

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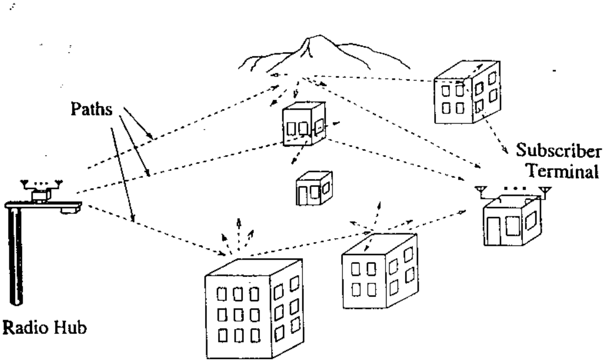
Ideal case: choose $\{\varphi_k\}$ such that A is diagonal and computation of $C^* x$ and Cr is cheap.
For diagonal A, MMSE equalization becomes identical to ML, since each y_k can be decoded individually by computing $\tilde{x}_k = y_k / A_{k,k}$ followed by solving $x_k = \text{argmin} |\tilde{x}_k - (\pm 1)|$.

Fixed wireless communication

Assume that transmitter and receiver are not moving.



Multipath propagation causes signal to arrive at receiver with different delays and different amplitudes: Transmission pulses are spread out in time ("delay spread").

Linear time-invariant channels

Let s denote the transmitted signal, the received signal r is given by (in absence of AWGN)

$$r(t) = (\mathbf{H}x)(t) = (h * s)(t) = \int_{-\infty}^{+\infty} h(t-\tau)s(\tau)d\tau$$

where **H** represents the linear operator modeling the wireless channel, h is the *impulse response*.
The emitted signal s is of the form

$$s(t) = \sum_k x_k \varphi_k$$

where φ_k are transmission pulses (yet to be determined!).

Equalization and channel diagonalization

We want a very simple equalizer at the receiver.

$$y_l = \langle r, \varphi_l \rangle = \sum_k x_k \langle H\varphi_k, \varphi_l \rangle = \sum_k A_{l,k} x_k$$

If the φ_k are eigenvectors of H with eigenvalues λ_k and if the φ_k are mutually orthogonal, then

$$\langle H\varphi_k, \varphi_l \rangle = \langle \lambda_k \varphi_k, \varphi_l \rangle = \lambda_l$$

thus A is diagonal with $A_{k,k} = \lambda_k$.
Eigenvectors of H are $e^{2\pi ik\cdot}$, eigenvalues are $\lambda_k = \hat{h}(k)$. Thus choose $\varphi_k(t) = e^{2\pi ikt}$, and we get the equalized data

$$\tilde{x}_l = \frac{y_l}{\hat{h}(l)}.$$

In presence of AWGN we simply round $\tilde{x}_l$ to ± 1, this is statistically optimal (maximum likelihood).

Mobile wireless communication

Moving transmitter and/or receiver cause the radio channel to be time-varying. Relative motion between transmitter and receiver results in Doppler effect. Thus in addition to delay spread caused by multipath propagation, signals are subject to Doppler spread.

We can no longer diagonalize operator **H** by Fourier transform.

No transform will simultaneously diagonalize all channels.

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Goal: We want to design an orthonormal system $\{\varphi_k\}_{k\in\mathcal{I}}$ such that the matrix A with entries

$$A_{k,l} = \langle H\varphi_l, \varphi_k \rangle$$

is "as diagonal as possible" for a large class of mobile channels.

Matrices with off-diagonal decay

How can we quantify “as diagonal as possible”?
Use off-diagonal decay of $A(\sigma)$ as measure for nearness of A to a diagonal matrix.
E.g., if

$$|A_{k,l}| \le c(1+|k-l|)^{-s}, \qquad s>0,$$

then larger s means closer to diagonal.

Even better would be if the off-diagonal decay of A satisfies

$$|A_{k,l}| \le c_1 e^{-c_2|k-l|}.$$

Representation of mobile channels

The mobile wireless channel can be written as

$$y(t) = \mathbf{H}s(t) = \int_{-\infty}^{+\infty} h_t(\tau)s(t-\tau)d\tau$$

where h_t is the impulse response at time t. By interpreting h_t as function of two variables, i.e., $h_t(\tau) = h(t,\tau)$ we can write

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Alternatively we can write

$$\mathbf{H}s(t) = \iint \hat{\sigma}(\eta,\tau)M_\eta T_{-\tau}s(t)d\tau d\eta$$

where

$$\hat{\sigma}(\cdot,\tau) = \mathcal{F}_1 h(-\cdot,\tau)$$

Mobile communication and pseudodifferential operators

We have the representation

$$\mathbf{H}s(t) = \iint \hat{\sigma}(\eta, \tau) M_\eta T_{-\tau} x(t) d\tau d\eta \qquad (1)$$

$\mathbf{H} = \mathbf{H}_\sigma$ is a pseudodifferential operator with (Kohn-Nirenberg) symbol σ. $\hat{\sigma}$ is called the spreading function.

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Any reasonable linear operator can be written in the form (1). The different operators can be characterized in terms of properties of their symbol σ or their spreading function $\hat{\sigma}$. What properties of $\hat{\sigma}$ characterize operators representing mobile channels?

Mobile channels: quantitative analysis

Delay spread: energy of impulse response decays exponentially in time: for fixed τ there exist constants $a, c > 0$ such that

$$|h(t,\tau)|^2 \leq ce^{-a|t|},$$

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where v is velocity v, λ is wave length, θ is the angle between direction of moving object and direction of arrival of radio wave.

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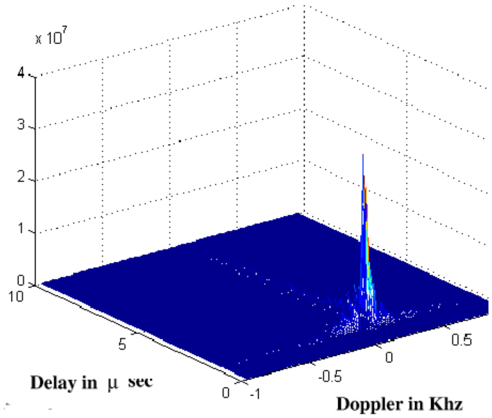
where v is velocity v, λ is wave length, θ is the angle between direction of moving object and direction of arrival of radio wave. Maximal Doppler shift $\nu_{max} = v/\lambda$. Hence Doppler shift of carrier frequency ω_c is confined to $[\omega_c - \nu_{max}, \omega_c + \nu_{max}]$ and $\hat{\sigma}(\eta, u)$ has compact support w.r.t. η for fixed u.
Properties of delay spread and Doppler spread imply that spreading function $\hat{\sigma}$ is localized.
(Extreme case: if $\hat{\sigma} = \delta$ then $\mathbf{H}_\sigma = I$).

Localization of spreading function



Spreading function $\hat{\sigma}(\eta, u)$

[Data: courtesy of Nokia]

Theorem: Let $\mathbf{H}_\sigma$ represent a narrowband mobile radio channel. Then σ belongs to the modulation space $\mathbf{M}^{\infty,1}_w(\mathbb{R}^2)$ where w is an exponentially increasing weight function.

$\mathbf{M}^{\infty,1}_w$ is also known as weighted Sjöstrand class.

Pseudodifferential operators

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Could also use model σ as bandlimited function, which leads to underspread operators. Pioneering work by W. Kozek on approximate diagonalization of underspread operators.
We use $\mathbf{M}_w^{\infty,1}$ as symbol space for σ and look for orthonormal functions $\{\varphi_k\}_{k\in\mathcal{I}}$ such that the matrix $A(\sigma)$ with entries

$$A(\sigma)_{l,k} = \langle \mathbf{H}_\sigma \varphi_k, \varphi_l \rangle, \quad k,l \in \mathcal{I}.$$

is as diagonal as possible for all $\sigma \in \mathbf{M}_w^{\infty,1}(\mathbb{R}^2)$.

Gabor systems as transmission functions

Definition: A Gabor system consists of functions $\{\varphi_{ma,nb}\}_{m,n\in\mathbb{Z}}$ of the form

$$\varphi_{ma,nb}(t) = M_{nb}T_{ma}\varphi(t), \quad m,n \in \mathbb{Z}$$

where $\varphi \in \boldsymbol{L}_2(\mathbb{R})$ is typically a function that is well localized in time and frequency (e.g., a Gaussian).
We denote such as system by (φ, a, b).

The coefficient operator $C : \boldsymbol{L}_2(\mathbb{R}) \mapsto \ell^2(a\mathbb{Z}\times b\mathbb{Z})$ is given by

$$Cf = \{\langle f, \varphi_{ma,nb}\rangle\}_{m,n\in\mathbb{Z}}.$$

and $C^* : \ell^2(a\mathbb{Z}\times b\mathbb{Z}) \mapsto \boldsymbol{L}_2(\mathbb{R})$ is given by

$$C^*\{x_{m,n}\} = \sum_{m,n} x_{m,n}\varphi_{ma,nb}.$$

Some properties of Gabor systems (1)

We have:

$$A(\sigma) = C^*\mathbf{H}_\sigma C = \{\langle \mathbf{H}_\sigma \varphi_{ma,nb}, \varphi_{m'a,n'b}\rangle\}_{m,n,m',n' \in \mathbb{Z}}.$$

- Necessary conditions for invertibility of $A(\sigma)$:
  Linear independence of (φ, a, b).
- To maximize data rate: range(C^*) should be as large as possible, ideally: whole $\boldsymbol{L}_2(\mathbb{R})$.
- Linear independence of (φ, a, b) requires $a \cdot b \geq 1$.
- For (φ, a, b) to span all of $\boldsymbol{L}_2(\mathbb{R})$ we need $a \cdot b \leq 1$ [Perelomov, Rieffel, Daubechies,...].

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For (φ, a, b) to be an ONB for $\boldsymbol{L}_2(\mathbb{R})$ one needs $a \cdot b = 1$.

Some properties of Gabor systems (2)

Balian-Low Theorem: If (φ, a, b) is an ONB for $\boldsymbol{L}_2(\mathbb{R})$, then φ cannot be well localized in time and frequency.

Hence having an orthonormal Gabor system (φ, a, b) with good time-frequency implies incompleteness and thus $a \cdot b > 1$!
Typical choice in practice $a \cdot b \in [\frac{6}{5}, 2]$. $a \cdot b = 2$ means we suffer 50% loss of spectral efficiency.
From now on we assume $a \cdot b > 1$.

Alternatively, we can use Wilson orthonormal bases (similar to Offset-QAM OFDM), which can be easily constructed from Gabor systems.

Optimal approximate diagonalization

Theorem [T.S.'04] The best approximate diagonalization of the mobile channel $\mathbf{H}_\sigma$ by an ONS is obtained by Gabor systems (φ, a, b) where φ decays exponentially in time and frequency. In that case $A(\sigma)$ has exponential off-diagonal decay.

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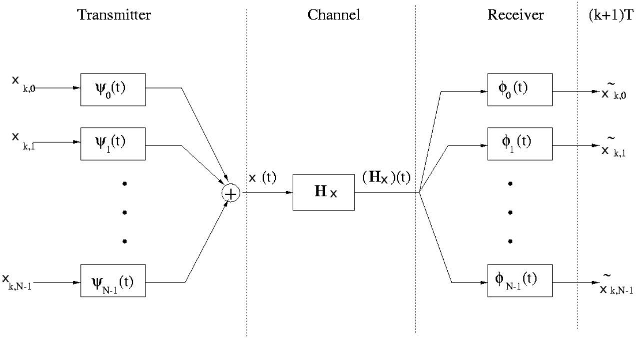
Proof-sketch: Results from pseudodifferential operator theory show that $\mathbf{H}_\sigma$ with $\sigma \in \mathbf{M}_w^{\infty,1}$, can be characterized by off-diagonal decay of the matrix $A(\sigma)$. This characterization is if-and-only-if!
Next use that such matrices form a Banach algebra.
Thus mobile wireless channels $\mathbf{H}_\sigma$ form a Banach algebra.
Now assume that some other system achieves better approximate diagonalization, i.e., $A(\sigma)$ would have faster than exponential decay.
But then the Banach algebra property would be violated. ❏
Still need to discuss design of φ.

Orthogonal Frequency Division Multiplexing

OFDM is a multicarrier system: bandwidth is split into subbands (subcarriers) and a block of data is transmitted simultaneously across subbands.



OFDM [contd.]

Assume we split bandwidth Ω into N subbands and let $F = \Omega/N$ (carrier separation). Let $\{x_{k,l}\}_{l=0}^{N-1}$ be the data to be transmitted at time slot k. Let a be the time interval (symbol period) between the transmission of two data blocks. The transmitted OFDM signal is then

$$s(t) = \sum_{k \in \mathbb{Z}} \sum_{l=0}^{N-1} x_{k,l} \varphi(t - kT) e^{2\pi i l F t},$$

where φ is a prototype transmission pulse.
Transmission pulses have Gabor-structure with $T = a, F = b$.

Pulse shape design for OFDM (1)

Usually φ, T, F are chosen such that $\{\varphi_{mT,nF}\}_{m,n\in\mathbb{Z}}$ is an ONS.
Simple choice that yields orthogonal system:

$$\varphi = \chi_{[0,T']}, \quad T' \leq T, T = 1/F.$$

This is used in most current OFDM systems.
By letting $T' < T$ we effectively insert a guard interval between two temporally adjacent pulses.

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Rectangular pulse plus guard interval is useful in case of multipath but it is a bad choice in presence of Doppler spread.
Reason: if $\varphi = \chi_{[0,T']}$ then $\hat{\varphi} = \mathrm{sinc}_{1/T'}$. While shifted sinc-functions $\{\mathrm{sinc}_{1/T'}(\cdot - mT)\}$ are orthogonal, a small perturbation caused by Doppler spread completely destroys orthogonality and we get large interference.

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Recall: want $A(\sigma) = \{\langle \mathbf{H}_\sigma \varphi_{mT,nF}, \varphi_{mT',nF'}\rangle\}_{m,n,m',n'\in\mathbb{Z}}$ to be (almost) diagonal. But $A(\sigma)$ is only (block-)diagonal if there is no Doppler spread, with Doppler spread we have $|A(\sigma)_{m,n,m',n'}| \approx \mathcal{O}(\|(m,n)-(m',n')\|^{-1})$.

Pulse shape design for OFDM (2)

Our theory implies: In case of severe Doppler spread we should choose φ with exponential time-frequency decay, since this would result in matrix $A(\sigma)$ whose off-diagonal entries decay like $\mathcal{O}(e^{-\|(m,n)-(m',n')\|})$.

We also want $\varphi_{ma,nb}$ to be an orthonormal system!

But which φ shall we choose?

Let $g(t) := e^{-\pi t^2}$. We know that the Gaussian minimizes Heisenberg's Uncertainty Principle. However (g, T, F) is not an ONS.

Want ONS (φ, T, F) such that φ is "as close as possible" to g.

Transmission pulse design for OFDM (1)

Theorem:[Janssen-T.S.,'02]: Assume (ψ, T, F) is a Gabor Riesz basis for a subspace of $\boldsymbol{L}_2(\mathbb{R})$ and let R be the Gram matrix with entries

$$R_{m,n,m',n'} = \langle \psi_{m'T,n'F}, \psi_{mT,nF} \rangle, \qquad m, n, m', n' \in \mathbb{Z}.$$

Then the function φ_{opt} that minimizes

$$\|\psi - \varphi\|_2$$

among all functions φ such that (φ, T, F) is an ONS, is given by

$$\varphi_{\text{opt}} := \sum_{m,n} [R^{-\frac{1}{2}}]_{0,0,m,n} \psi_{mT,nF}$$

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Banach square root theorem implies that if ψ is time-frequency well-localized, then so is φ!
If ψ decays exponentially in time and frequency, then so does φ!

Transmission pulse design for OFDM (2)

Theoretical framework leads to the following practical algorithm for designing optimal OFDM transmission pulses [T.S.'02/'03]:

1. Start with Gabor system (ψ, T, F) with e.g., $\psi(t) = e^{-\alpha t^2}$.
2. Compute $\varphi = \sum_{m,n}[R^{-\frac{1}{2}}]_{0,0,m,n}\psi_{mT,nF}$
3. Include various important practical constraints by formulating them as projection onto convex sets.
4. Iterate between step 2 and step 3 until a prescribed tolerance is reached.

This algorithm is numerically extremely efficient (due to nice properties of $\mathbb{H}(\mathbb{C}^N)$).
Patent pending (jointly with A. Paulraj)

Practical application

Pulse-shaping method has been used in collaboration with Special Communication Systems to design new OFDM-based modem for short-radio-wave communications.

Pactor III

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Pactor III

Landfall Navigation says about Pactor III:

"*It is the most amazing thing we have seen in over 35 years of transmitting data over radio*" (www.landfallnavigation.com/pactor.html).

Agenda

- The Heisenberg group
- Spreading sequence design for CDMA
- Code design for MIMO
- Pulse shape design for OFDM
- High resolution radar via compressed sensing

High resolution radar and compressed sensing

Resolution in (monostatic) radar is limited by the Uncertainty Principle.

New theory of compressed sensing Donoho, Candes, Tao,... is tailored to recover sparse objects from few measurements.

Their framework can be extended to operator identification and thus to radar.

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Key fact: Let N be a power of a prime number ≥ 5 and $x(k) = e^{2\pi i k^3/N}$ for $k = 0, \dots, N-1$.

Then

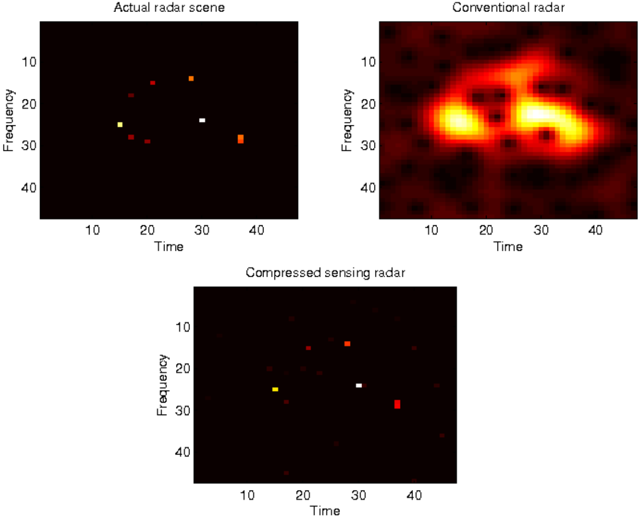
$$T^k M^n x, \qquad k, n = 0, \dots, N-1$$

generates mutually unbiased bases.

See also presentation by G. Pfander.

Compressed sensing radar

Comparison of conventional radar with compressed sensing radar at SNR of 10dB.



The Heisenberg group $\mathbb{H}(\mathbb{Z}_2^m)$ will be at the heart of next generation wireless communication systems.

In theory there is no difference between theory and practice,
in practice there is ....

[Yogi]