Jacobs University February 12, 2018
Spring 2018

Analysis II

Homework 1

Due on February 19, 2018

Problem 1 [8 points]: The Stieltjes integral for a discontinuous α
Let $\alpha(x) = 0$ if $x \le 0$ and $\alpha(x) = 1$ if $x > 0$. Give a precise proof that $\int_{-1}^{1} f\, d\alpha = f(0)$ for every function $f\colon \mathbb{R} \to \mathbb{R}$ that is continuous at $x = 0$.

Problem 2 [4 points]: Explicit Stieltjes integrals
Let $a < b \in (0,4)$. Find a monotonically increasing bounded function $\alpha : \mathbb{R} \to \mathbb{R}$ such that

$$\int_0^4 f\, d\alpha = f(1) + 2f(2) + 3f(3) + \frac{1}{2}\int_a^b f(x)\, dx$$

for all f for which the integral exists.

Problem 3 [10 points]: Integrable and non-integrable functions
Define two functions $f, g : [0,1] \to \mathbb{R}$ via

$$\begin{aligned} f(x) &:= \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ 1/q & \text{if } x = p/q, \quad \text{with } p, q \text{ coprime}, \end{cases} \\ g(x) &:= \begin{cases} 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}, \\ 1 & \text{if } x \in \mathbb{Q}. \end{cases} \end{aligned}$$

(a) Show that $f \in \mathcal{R}[0,1]$ and $\int_0^1 f(x)dx = 0$.

(b) Show that g is not Riemann-integrable.

Problem 4 [10 points]: Riemann integrable or not?

(a) Show that $f(x) = e^x$ is Riemann integrable on $[a,b] \subset \mathbb{R}$. What is the value of $\int_a^b e^x\, dx$? (*Hint: Use an equidistant partition.*)

(b) Show that $f(x) = c$ ($c \in \mathbb{R}$) and $g(x) = x$ are Riemann integrable on $[a,b] \subset \mathbb{R}$. Find

$$\int_a^b c\, dx \quad \text{and} \quad \int_a^b x\, dx.$$

(c) Show that $f(x) = x^n$ is Riemann integrable on $[0, a] \subset \mathbb{R}$ for every $n \in \mathbb{N}$. What is the value of $\int_0^a x^n \, dx$? (*Hint: You may use the fact that* $\sum_{k=1}^N k^n$ *is a polynomial in* N *of degree* $n+1$ *and with leading coefficient* $1/(n+1)$*, or — if you know it — use the Stolz-Cesàro theorem.*)

Problem 5 [8 points]: Uniform continuity
Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is called *uniformly continuous* if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, x' \in I$ with $|x - x'| < \delta$ we have that $|f(x) - f(x')| < \varepsilon$ (in other words, the δ does not depend on x or x').

(a) Show that if $I = [a, b]$ is closed and bounded, then every continuous function $f : [a, b] \to \mathbb{R}$ is uniformly continuous. (*Hint: In general,* δ *may depend on* x *and thus defines a function* $\delta(x)$*; show that* $\delta(x)$ *is continuous.*)

(b) Does this answer change if I is no longer closed, and/or no longer bounded?

Bonus Problem 1 [3 points]: Uniform continuity continued
We continue Problem 5.

(a) Suppose $f, g : X \to \mathbb{R}$ are uniformly continuous on $X \subset \mathbb{R}$. Is it true that $f + g$ is uniformly continuous on X? How about $f \cdot g$?

(b) Does the answer to (c) change if f, g are bounded?

(c) Does the answer to (c) change if X is a closed interval?

Bonus Problem 2 [5 points]: Devil's staircase and Stieltjes integrals
Define a function $\alpha\colon [0, 1] \to \mathbb{R}$ as follows: Given $x \in [0, 1]$, write $x = \sum_{i \geq 1} b_i 3^{-i}$ with $b_i \in \{0, 1, 2\}$ (representation of x in base 3). Let n be minimal with $b_n = 1$ (or $n = \infty$ if all $b_i \neq 1$). Then $\alpha(x) := \sum_{i=1}^n a_i 2^{-i}$, where $a_i = 1$ if $b_i \in \{1, 2\}$ and $a_i = 0$ if $b_i = 0$. (For additional credit, you may check that the value of α is well defined even at points x that have two representations in base 3).

(a) Sketch the graph of α.

(b) Show that α is continuous and monotonically increasing.

(c) Show that for every $\varepsilon > 0$, there are finitely many intervals $I_{\varepsilon,1}, I_{\varepsilon,2}, \ldots I_{\varepsilon,n}$ with total length ε so that α is constant on $[0, 1] \setminus (I_{\varepsilon,1} \cup I_{\varepsilon,2} \cup \ldots I_{\varepsilon,n})$ (this means that α is constant except on a set of volume zero, but α is continuous and non-constant).

(d) Show that the integrals $\int_0^1 1 \, d\alpha$ and $\int_0^1 x \, d\alpha$ exist, and determine their values. (*Hint: Show that* $\int_0^1 (x - 1/2) \, d\alpha = 0$*.*)