Jacobs University
Spring 2018

February 26, 2018

Analysis II

Homework 3

Due on March 5, 2018

Problem 1 [10 points]: Partial fractions
Let P and Q be two polynomials with real coefficients, and suppose that

$$Q(x) = (x - \alpha_1) \cdot (x - \alpha_2) \cdot \ldots \cdot (x - \alpha_n),$$

where all α_i are real and different. For this problem, you may use the fact $\int_a^b \frac{dx}{x} = \ln(x)\Big|_a^b$.

(a) Show that there is a unique polynomial R with real coefficients, and unique real numbers $A_1, \ldots, A_n$ such that

$$\frac{P(x)}{Q(x)} = R(x) + \frac{A_1}{x - \alpha_1} + \frac{A_2}{x - \alpha_2} + \cdots + \frac{A_n}{x - \alpha_n}.$$

(*Hint: Here you could use long division of polynomials or some basic results from linear algebra.*)

(b) Find a closed formula for $\int_a^b \frac{P(x)}{Q(x)}\, dx$ (say, provided Q has no zero on $[a, b]$).

(c) In particular, find a closed formula for $\int \frac{5x^4+4x^3+3x^2+2x+1}{x^3-2x^2-5x+6}\, dx$ (wherever the denominator is non-zero).

(d) Is there a similar formula if Q has multiple linear factors, or if Q does not split into real linear factors? Outline how every rational function can be integrated in closed form (no proof required).

Problem 2 [6 points]: Integration by substitution

(a) Compute $\int_0^{1/2} \frac{1}{1-x^2}\, dx$.

(b) Compute $\int_0^1 \frac{1}{\sqrt{1+x^2}}\, dx$ (think of hyperbolic trigonometric functions).

(c) For every odd $n \in \mathbb{N}$, show how $\int_0^1 x^n e^{-x^2}\, dx$ can be reduced to an integral of the form $\int_a^b t^k e^{-t}\, dt$.

Problem 3 [18 points]: Lots of integrals . . .

(1) Compute the following integrals using an appropriate substitution or the formula for the derivative of the inverse function (*hint: trigonometric substitutions*):

   (a) $\int \frac{1}{x}\, dx$,

   (b) $\int \frac{1}{\sqrt{1-x^2}}\, dx$,

   (c) $\int \frac{1}{1+x^2}\, dx$,

   (d) $\int \frac{1}{a^2+x^2}\, dx$,

   (e) $\int_0^1 \frac{x}{x^2+4}\, dx$,

   (f) $\int_0^1 \frac{1}{x^2\sqrt{x^2+1}}\, dx$.

(2) Compute the following integrals using integration by parts:

   (a) $\int_0^1 \arcsin(x)\, dx$,

   (b) $\int_0^1 \arccos(x)\, dx$, (compare this with (a))

   (c) $\int_0^1 e^x(x^2+1)\, dx$.

Problem 4 [6 points]: Uniform convergence of second derivatives
Suppose that $f_n\colon [a,b] \to \mathbb{R}$ is continuous and twice differentiable on (a,b) and so that $f_n''\colon \to g$ uniformly on $[a,b]$. Give sufficient conditions so that the f_n converge to a limit function as well.

Bonus Problem 1 [4 points]: Null sets
A set $X \subset \mathbb{R}$ is called a *set of volume* 0 ("Jordan measure zero") if for every $\varepsilon > 0$ there exists a *finite* family of open intervals $(U_n)_{n=1}^k$ with total length less than ε that covers X, i.e., such that

$$\sum_{n=1}^{k} \text{length}(U_n) < \varepsilon \quad \text{and} \quad X \subset \bigcup_{n=1}^{k} U_n.$$

The set X is called a *null set* (a set of "Lebesgue measure zero") if for every $\varepsilon > 0$ there exists a *countable* family of open intervals $(U_n)_{n\in\mathbb{N}}$ with total length less than ε that covers X, i.e., such that

$$\sum_{n=1}^{\infty} \text{length}(U_n) < \varepsilon \quad \text{and} \quad X \subset \bigcup_{n=1}^{\infty} U_n.$$

Obviously, any set of Jordan measure 0 is a null set. Prove that:

(a) A finite union of sets of Jordan measure 0 still has Jordan measure 0. Is this still true for countable unions?

(b) A countable union of null sets is a null set.

(c) $\mathbb{Q}$ is a null set, but does not have Jordan measure 0, and the same holds for $\mathbb{Q} \cap [0, 1]$.

(d) The standard Cantor middle-third set has Jordan measure 0.

Bonus Problem 2 [4 points]: A criterion for Riemann integrability
Let $f : [a, b] \to \mathbb{R}$ be a bounded function, and denote by D the set of its discontinuities. Prove that if D has Jordan measure 0 then f is Riemann integrable. Try to extend this for D being a null set and try to show the converse: any bounded function $f\colon [a, b] \to \mathbb{R}$ is Riemann integrable if and only if D is a null set.