Jacobs University April 25, 2018
Spring 2018

# Foundations of Mathematical Physics

## Homework 11

Due on May 2, 2018

**Problem 1 [3 points]: von Neumann theorem**

(a) Show that $-\Delta$ on the domain $C_0^\infty(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is measurable and open, satisfies the conditions of von Neumann's theorem, and thus has self-adjoint extensions.

(b) Find a conjugation that commutes with $-i\frac{d}{dx}$ on the domain $C_0^\infty(\mathbb{R})$, and leaves the domain invariant.

**Problem 2 [7 points]: Applying Kato-Rellich**

Let $V : \mathbb{R}^3 \to \mathbb{R}$ be such that $V = V_1 + V_2$ with $V_1 \in L^2(\mathbb{R}^3)$ and $V_2 \in L^\infty(\mathbb{R}^3)$ (one then writes $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$). Prove that $-\Delta + V$ is self-adjoint on $H^2(\mathbb{R}^3)$ by using Kato-Rellich: First, show that V is relatively $\Delta$-bounded. Then, show that it is actually infinitesimally $\Delta$-bounded. (*Hint: Sobolev Lemma, Fourier transform and Cauchy-Schwarz.*)

**Problem 3 [7 points]: Hardy's inequality**

(a) Prove that for all $\varepsilon > 0$ there is a $C_\varepsilon < \infty$ such that for all $\psi \in H^2(\mathbb{R}^d)$ and $j = 1, \dots, d$, we have

$$\|\partial_{x_j}\psi\|_{L^2(\mathbb{R}^d)} \leq \varepsilon\|\Delta\psi\|_{L^2(\mathbb{R}^d)} + C_\varepsilon\|\psi\|_{L^2(\mathbb{R}^d)}.$$

(b) Prove Hardy's inequality for $d = 3$, i.e., prove that there is a $C < \infty$ such that for all $\psi \in H^1(\mathbb{R}^3) \cap C_0^1(\mathbb{R}^3)$, we have

$$\||x|^{-1}\psi\|_{L^2(\mathbb{R}^3)} \leq C\|\nabla\psi\|_{L^2(\mathbb{R}^3)}.$$

Then show that this inequality actually holds for all $\psi \in H^1(\mathbb{R}^3)$, e.g., by using Fatou's lemma. If you can, show that the inequality holds for $C = 2$.

(c) Using (a) and (b), show that for all $\psi \in H^2(\mathbb{R}^3)$ there is an $\varepsilon > 0$ and $C_\varepsilon < \infty$ such that

$$\||x|^{-1}\psi\|_{L^2(\mathbb{R}^3)} \leq \varepsilon\|\Delta\psi\|_{L^2(\mathbb{R}^3)} + C_\varepsilon\|\psi\|_{L^2(\mathbb{R}^3)}.$$

**Problem 4 [3 points]: Dirac equation**

Consider the Dirac Hamiltonian with Coulomb interaction in three dimensions

$$H = H_0 - \frac{e}{|x|}$$

with

$$H_0 = -i \sum_{\mu=1}^{3} \gamma^0 \gamma^\mu \partial_{x_\mu} - \gamma^0 m,$$

where $m > 0$ and $\gamma^\mu$ are the $4 \times 4$ Dirac gamma matrices, i.e., they satisfy $\gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu$ for $\mu \neq \nu$ and $\gamma^0\gamma^0 = -\gamma^1\gamma^1 = -\gamma^2\gamma^2 = -\gamma^3\gamma^3 = 1$. Given that $(H_0, H^1(\mathbb{R}^3)^4)$ is self-adjoint, prove that also $H$ is self-adjoint for $e < \frac{1}{2}$. (Note: For $e > \frac{1}{2}$, $H$ is actually not essentially self-adjoint on $H^1(\mathbb{R}^3)^4$ anymore.)