Geometric Knot Theory

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Modern Mathematics
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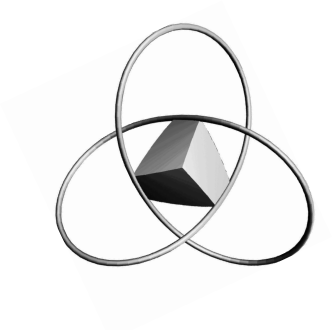


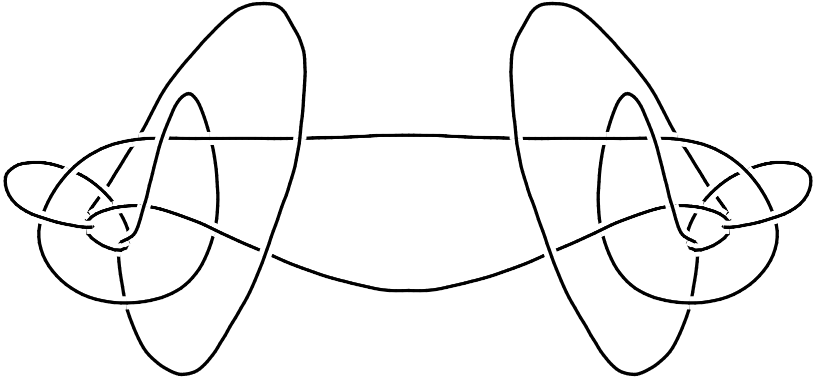
Berlin opportunities

Berlin Mathematical School

- International graduate school
- From Bachelor's to Doctorate
- Courses in English at three universities
- www.math-berlin.de

Knots and Links

- Closed curves embedded in space
- Classified topologically up to *isotopy*
- Two knotted curves are equivalent (same knot type) if one can be deformed into the other



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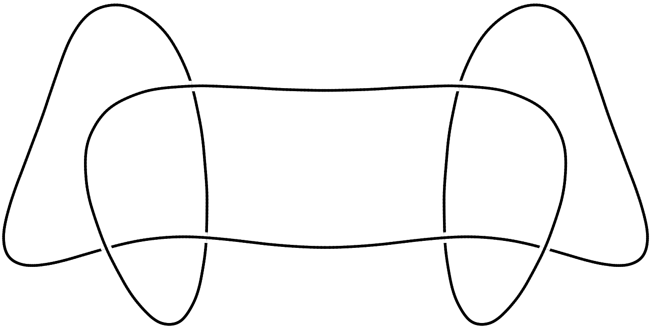


(Topological) Knot Theory

- Classify knot/link types
- Look for easily computed invariants
  to distinguish knots/links
- 3-manifold topology of complement

Geometric Knot Theory

Two threads:

Geometric properties of knotted space curve

determined by knot type or implied by knottedness
(e.g. Fáry/Milnor: $TC > 2\pi\mathrm{br} \geq 4\pi$)

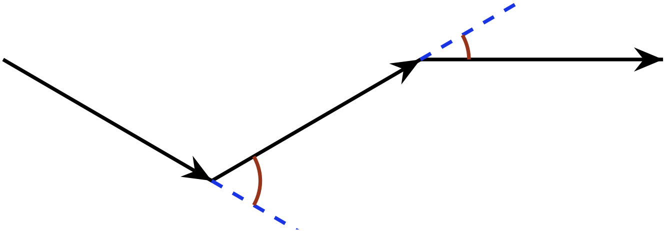
Optimal shape for a given knot

usually by minimizing geometric energy

Geometric optimization problems:
seek best geometric form for topological object

Total Curvature

- For K smooth, $\quad \mathrm{TC} := \int_K \kappa\, ds$
- For K polygonal, TC $:=$ sum of turning angles (exterior angles)

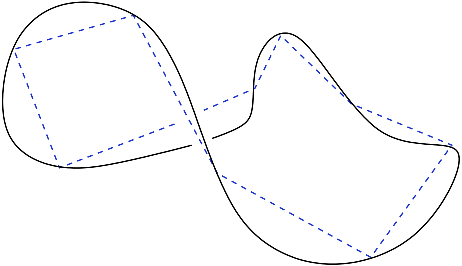


Total Curvature

Definition (Milnor)

For K arbitrary, $\mathrm{TC}(K) :=$ supremal TC of inscribed polygons

- Achieved by any limit of ever finer polygons.
- Analogous to Jordan's definition of length.



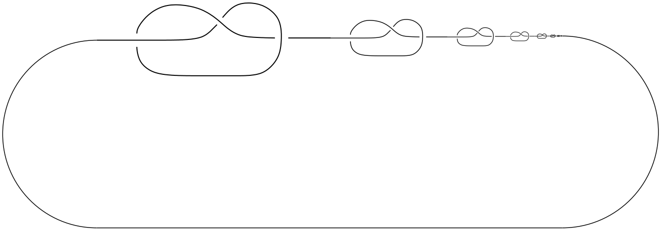
Curves of Finite Total Curvature

- FTC means TC $< \infty$
- Unit tangent vector
  Bounded variation (BV) function of arclength
- Curvature measure
  $dT = \kappa N\, ds$ as Radon measure
- Countably many corners
  where $T_+ \neq T_-$
  (curvature measure has atom)

See my survey in *Discrete Differential Geometry*, Birkäuser, 2008;
`arXiv:math.GT/0606007`

Approximation of FTC curves

- FTC knot has isotopic inscribed polygon [Milnor]
- Tame (not wild) knot type



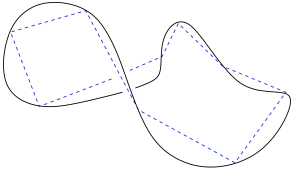
- K, K' each FTC and "C^1-close" $\implies$ isotopic [DS]
- FTC $\iff$ "geometrically tame"

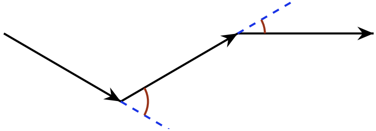
Projection of FTC curves

Theorem

Given an FTC curve $K \subset \mathbb{R}^n$ and some $k < n$, consider all projections of K to $\mathbb{R}^k$s. Their average TC equals TC(K).

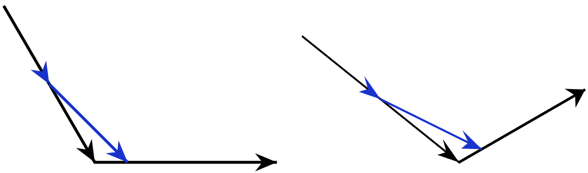
- Average is over Grassmannian
- Suffices to prove for polygons (dominated convergence) and thus for single corner





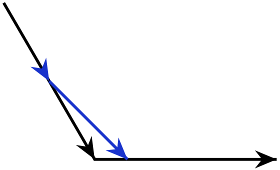
Projection of FTC curves (Proof)

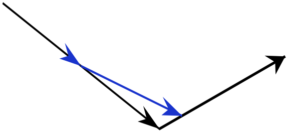
- Given angle θ, average turning angle of its projections is some function $f_k^n(\theta)$
- By cutting corner into two, f_k^n additive
  $f_k^n(\alpha+\beta) = f_k^n(\alpha) + f_k^n(\beta)$
- Continuous additive function is linear
  $f_k^n(\theta) = c_k^n\,\theta$
- What is the constant c_k^n? Should we try $\theta = \pi/2$?



Projection of FTC curves (Proof)

- Given angle θ, average turning angle of its projections is some function $f_k^n(\theta)$
- By cutting corner into two, f_k^n additive $f_k^n(\alpha+\beta) = f_k^n(\alpha) + f_k^n(\beta)$
- Continuous additive function is linear $f_k^n(\theta) = c_k^n\,\theta$
- Any projection of a cusp (angle π) is a cusp, so $f_k^n(\pi) = \pi$ Hence $c_k^n = 1$ as desired





Fenchel's Theorem

Corollary

$\gamma \subset \mathbb{R}^n$ *closed curve* $\implies TC(\gamma) \geq 2\pi$

Proof:

??? □

Fenchel's Theorem

Corollary

$\gamma \subset \mathbb{R}^n$ *closed curve* $\implies TC(\gamma) \geq 2\pi$

Proof 1:

Consider any inscribed 2-gon. □

Proof 2:

This is true in $\mathbb{R}^1$, where every angle is 0 or π □

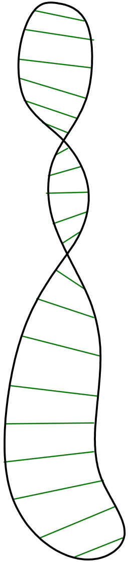
Fáry/Milnor Theorem

Theorem

$K \subset \mathbb{R}^3$ *knotted* $\implies TC(K) \geq 4\pi$

Proof [Milnor]:

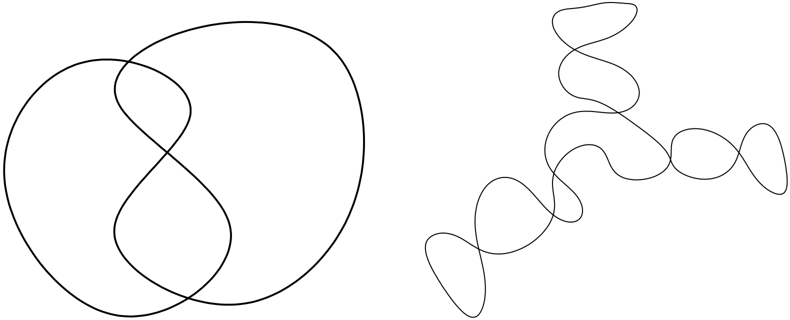
No projection to $\mathbb{R}^1$ can just go up & down, so true in $\mathbb{R}^1$ □



Fáry/Milnor Theorem: Fáry's Proof

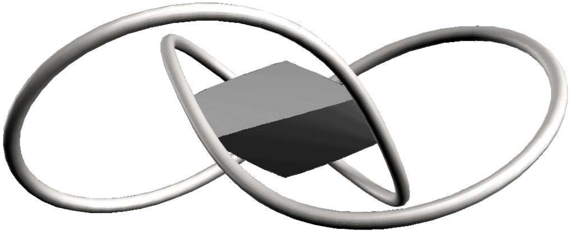
Proof [Fáry]:

True for knot diagrams in $\mathbb{R}^2$ because some region enclosed twice (perhaps not winding number two) □



Second Hull: Intiuition

- Fary/Milnor says knot K “wraps around” twice
- Intuition says K “wraps around some point” twice
- Some region (second hull) doubly enclosed by K
- How to make this precise?



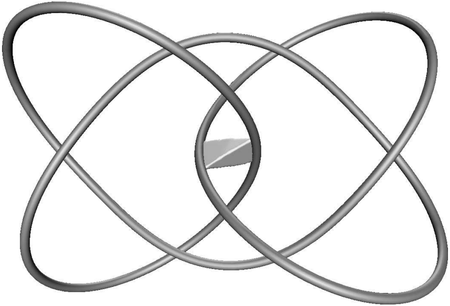
Second hull: Definition

Convex hull

$p \in \mathrm{cvx}(K) \iff$ every plane through p cuts K (at least twice)

Definition

$p \in n^{th}$ *hull* of $K \iff$ every plane through p cuts K at least $2n$ times

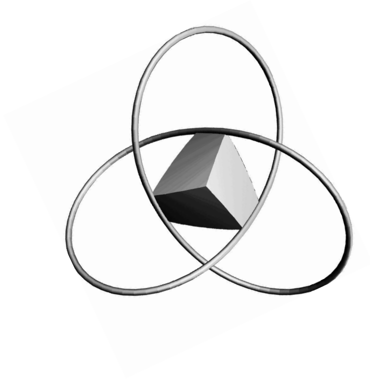


Second hull: Theorem

Amer. J. Math **125** (2003) pp 1335–1348, `arXiv:math.GT/0204106`
with Jason Cantarella, Greg Kuperberg, Rob Kusner

Theorem

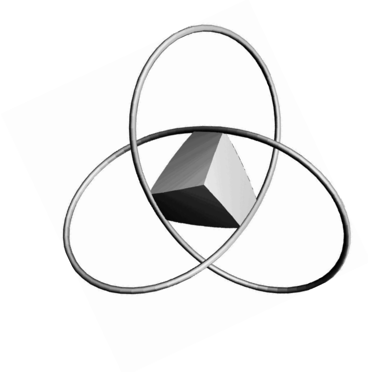
A knotted curve has nonempty second hull



Second hull: Proof

Proof for prime FTC knot:

An *essential halfspace* contains all of K except one unknotted arc.
Intersection of all essential halfspaces is (part of) second hull. □



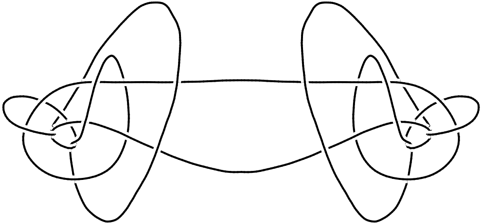
One notion of "where knotting happens"

Möbius energy

- Inspired by Coulomb energy (repelling electrical charges)

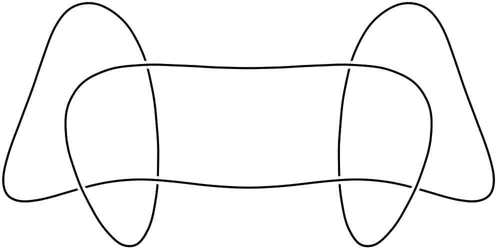
$$\iint_{K\times K} \frac{dx\,dy}{|x-y|^p}$$

- Renormalize to make this finite [O'Hara]
- Scale-invariant for $p = 2$
- Invariant under Möbius transformations [FHW]



Möbius energy

- Minimizers for prime knots [FHW]
- Probably no minimizers for composite knots
- Flow perhaps untangles all unknots



Ropelength

Definition

- *Thickness* of space curve = *reach*

  $=$ diameter of largest embedded normal tube
- *Ropelength* $=$ length / thickness

Positive thickness implies $C^{1,1}$

Definition

- *Gehring thickness* = minimum distance between components

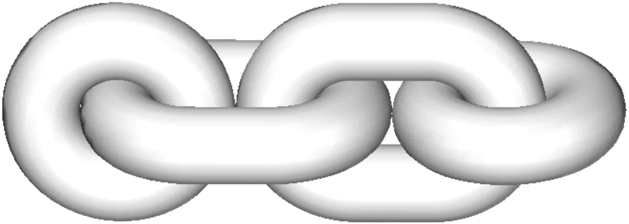
works with Milnor's *link homotopy*

Ropelength

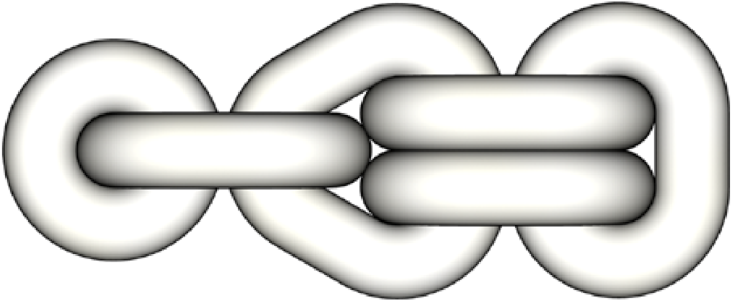
Inventiones **150** (2002) pp 257–286, `arXiv:math.GT/0103224`
with Jason Cantarella, Rob Kusner

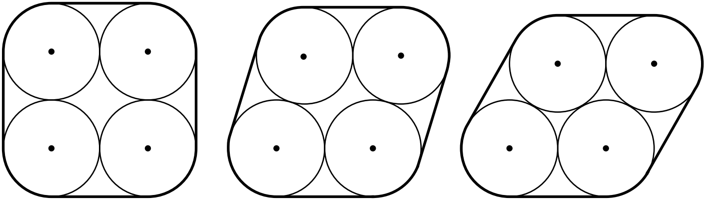
Results

- Minimizers exist for any link type
- Some known from sharp lower bounds
- Simple chain $=$ connect sum of Hopf links
  Middle components stadium curves: not C^2



Minimizers





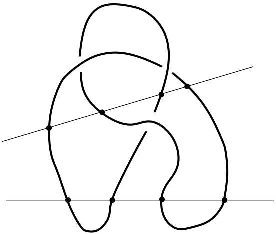
Lower bounds

Geom. & Topol. **10** (2006) pp 1–26,
arXiv:math.DG/0408026
with Elizabeth Denne and Yuanan Diao

Theorem

K *knotted* $\implies$ *ropelength* ≥ 15.66 *(within 5% for trefoil)*

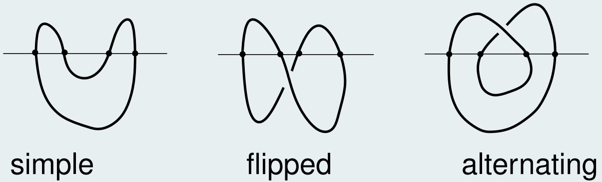
Proof uses *essential alternating quadrisecants*:



Quadrisecant

- Line intersecting a curve four times
- Every knot has one (Pannwitz – 1933 Berlin)

Three order types



Theorem (Denne thesis)

Every knot has an essential alternating quadrisecant

(Essential means no disk in $\mathbb{R}^3 \setminus K$ spans secant plus arc of K.)

Lower bound: Proof

Theorem

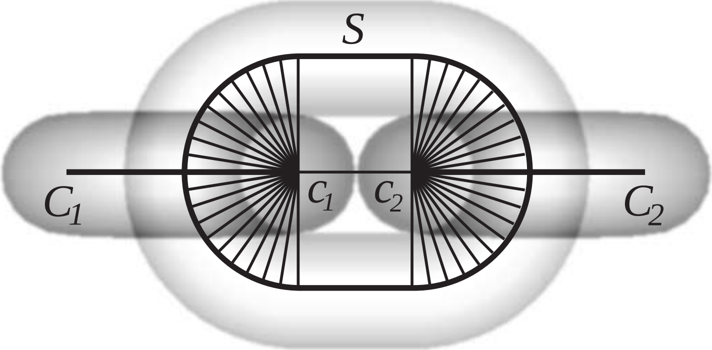
Ropelength > 15.66 *for any knotted curve*

- Denne gives essential alternating quadrisecant $abcd$
- Write lengths as $r := |a-b|$, $s := |b-c|$, $t := |c-d|$
- Scaling to thickness 1, we have $r, s, t \geq 1$
- Define $f(x) := \sqrt{x^2-1} + \arcsin(1/x)$
- $\ell_{ac} \geq f(r) + f(s)$, $\ell_{bd} \geq f(s) + f(t)$, $\ell_{da} \geq f(r) + s + f(t)$,
- $\ell_{cb} \geq \pi$ and $\ell_{cb} \geq 2\pi - 2\arcsin s/2$ if $s < 2$.
- Minimize sum separately in r, s, t.

Criticality

Balance Criterion: tension vs. contact force

Characterizes ropelength-critical links by force balance



Criticality papers

Gehring case – no curvature bound

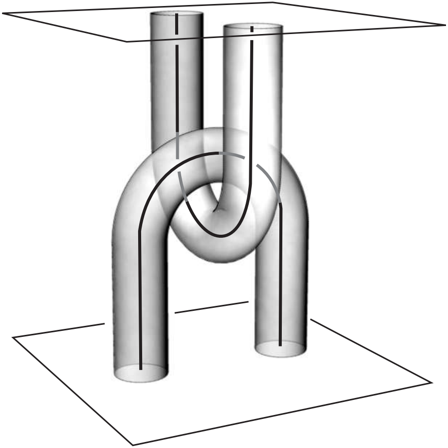
Geom. & Topol. **10** (2006) pp 2055–2115,
`arXiv:math.DG/0402212`
with Jason Cantarella, Joe Fu, Rob Kusner, Nancy Wrinkle

Ropelength case – with curvature bound

Geom. & Topol. **18** (2014) pp 1973–2043, `arXiv:1102.3234`
with Cantarella, Fu, Kusner

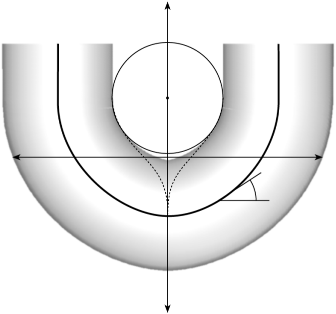
The clasp

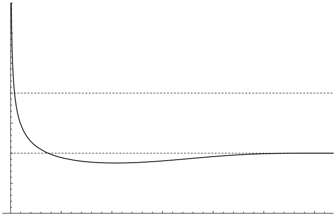
- Clasp: one rope attached to ceiling, one to floor
- Again with semicircles?



The Gehring clasp

- Gehring clasp has unbounded curvature (is $C^{1,2/3}$ and $W^{2,3-\varepsilon}$)
- Half a percent shorter than naive clasp





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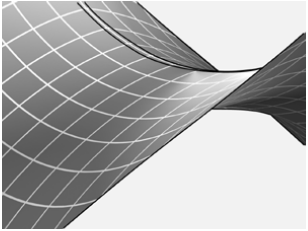
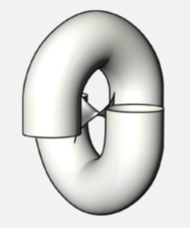
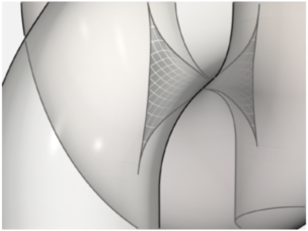
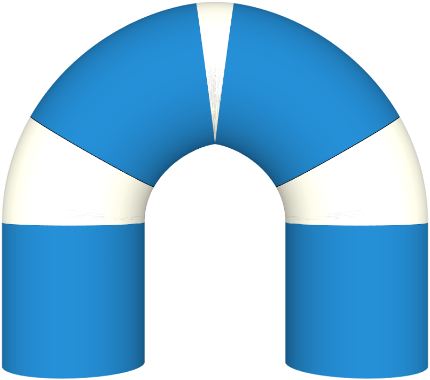
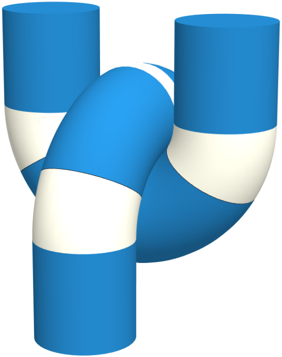




The tight clasp

- Tight clasp slightly longer
- Kink (arc of max curvature) at tip





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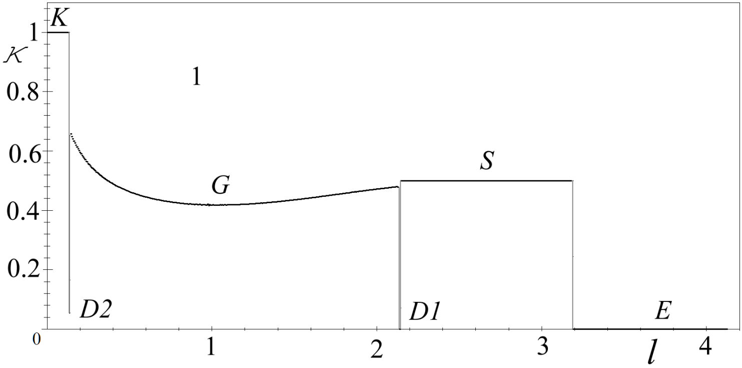
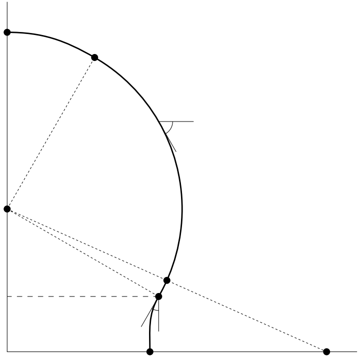
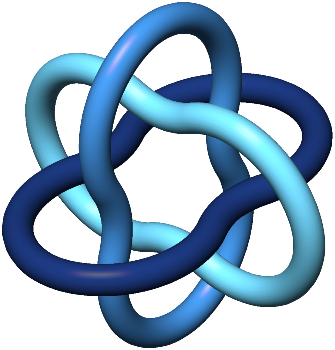


Example Tight Link

Critical Borromean rings – IMU logo

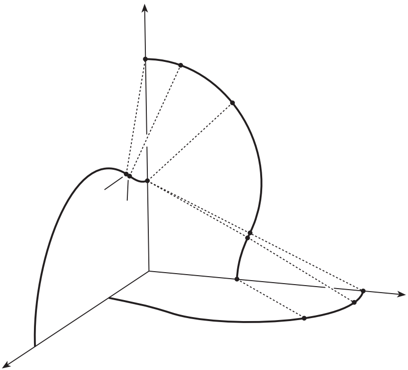
- maximal (pyritohedral) symmetry, each component planar
- piecewise smooth (42 pieces in total)
- some described by elliptic integrals

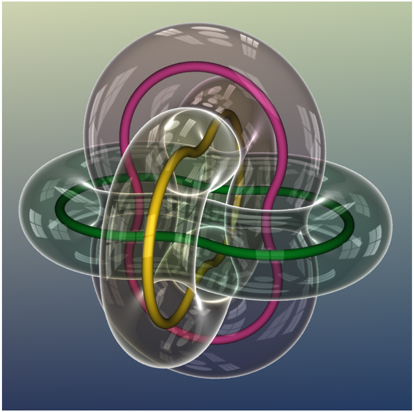




Borromean Rings

- Uses clasp arcs and circles; 0.08% shorter than circular
- Curvature < 2 everywhere $\implies$ also ropelength-critical





Linked table stands

From Africa, 3 components, Borromean rings

Linked table stands

From Ghana, 7 components

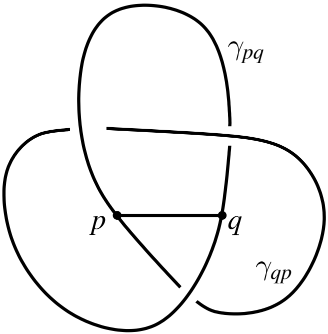
Linked table stands

From Turkey, 8 components!

Distortion

Notation

- Given $p, q \in K$, subarcs γ_{pq}, γ_{qp}
  have lengths ℓ_{pq}, ℓ_{qp}
- $d(p,q) := \min(\ell_{pq}, \ell_{qp})$
- $\delta(p,q) := d(p,q)/|p-q|$
  arc/chord ratio
- Distortion: $\delta(K) := \sup_{p,q} \delta(p,q)$.



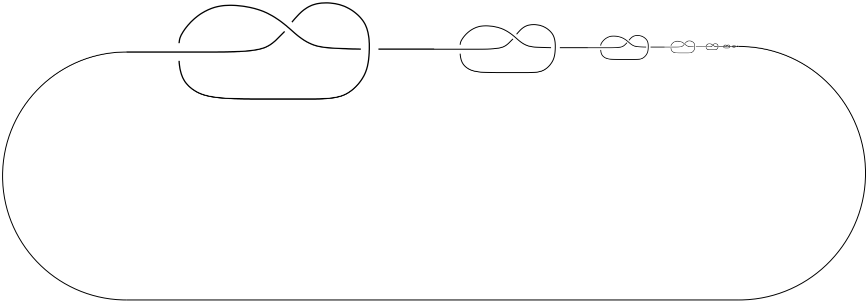
Gromov

- $\delta(K) \geq \pi/2$, equality only for round circle
- Can every knot be built with $\delta < 100$?

Distortion: Upper bounds

Computations

- Trefoil can be built with $\delta < 8.2$
- Open trefoil has more distortion, but still $\delta < 11$
- So infinitely many (even wild) knots with $\delta < 11$



Distortion: Lower bounds

Proc. AMS **137** (2009) pp 1139–1148, `arXiv:math.GT/0409438v2`
with Elizabeth Denne

Theorem

K knotted $\implies \delta > 5\pi/3$ *(within 30% for trefoil)*

Theorem (Pardon)

Torus knot $T_{p,q}$ has $\delta > \min(p,q)/160$

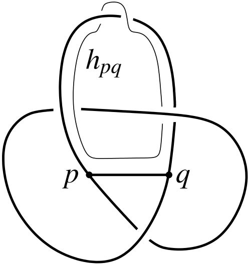
Theorem (Studer)

$\delta(T_{2,q}) \leq 7q/\log q$

Essential arcs

Given $p, q \in K$, when is γ_{pq} essential?

- Construct free homotopy class h_{pq} in $\mathbb{R}^3 \setminus K$
- h_{pq} parallel to $\gamma_{pq} \cup \overline{qp}$, zero linking with K
- γ_{pq} *essential* $\iff$ h_{pq} nontrivial
  $\iff$ $\gamma_{pq} \cup \overline{qp}$ spanned by no disk in $\mathbb{R}^3 \setminus K$

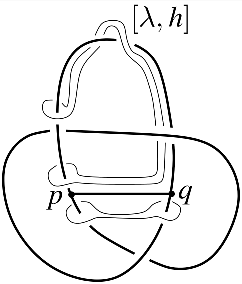


- K unknotted $\implies$
  all arcs inessential ($\pi_1 = H_1$)
- γ_{pq} and γ_{qp} inessential
  $\implies$ K unknotted (Dehn)

Essential secants

Definition

Secant $\overline{pq}$ *essential* if both γ_{pq} and γ_{qp} are

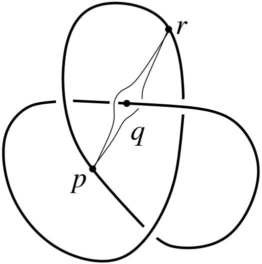


- $\lambda \in \pi_1$ is meridian
- Commutators $[\lambda, h_{pq}] = [\lambda, h_{qp}]$ nonzero only when $\overline{pq}$ essential

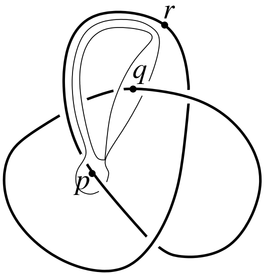
Arcs becoming essential

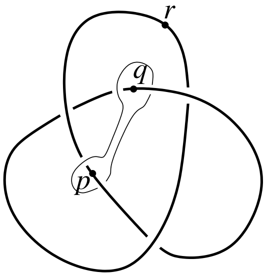
As r varies, when does γ_{pr} become essential?

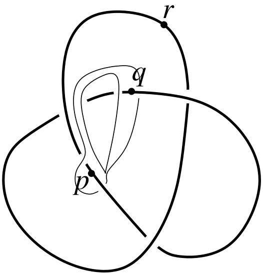
- Change in h_{pr} happens when $\overline{pr}$ crosses $q \in K$
- Change is $[\lambda, h_{pq}] = [\lambda, h_{qr}]$
- Both $\overline{pq}$ and $\overline{qr}$ must be essential



When pr becomes essential, pq is essential







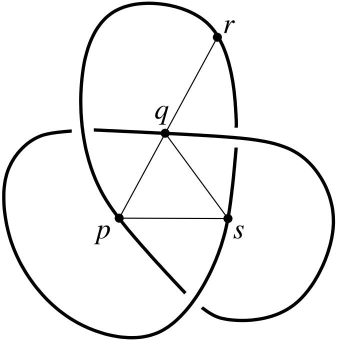
Distortion: Theorem

Theorem

$\delta \geq 5\pi/3$ *for any knot*

Proof:

- Find shortest essential secant $\overline{ps}$
- Scale so $|p - s| = 1$
- Find first $r \in \gamma_{ps}$ with γ_{pr} essential
- Get $q \in K \cap \overline{pr}$
- If $\overline{qx}$ essential $\forall x \in \gamma_{ps}$ then γ_{ps} stays outside $B_1(q)$, so $\ell_{ps} \geq (5/6)2\pi$



To become inessential, must go outside $B_2(q)$, thus even longer