Global Well-Posedness for the Averaged Euler Equations in Two Dimensions

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Abstract

We prove global well-posedness of the two-dimensional averaged Euler, or Euler-α equations for initial potential vorticity of class L^2. This model generalizes the one-dimensional Fokas–Fuchssteiner–Camassa–Holm equation which describes the propagation of unidirectional waves on the surface of shallow water. As such, it can be realized as a geodesic equation for the H^1 metric on the Lie algebra of vector fields. Moreover, in two dimensions the α-model obeys an advection equation for the so-called potential vorticity in close analogy to the vorticity form of the Euler equations.

We construct solutions to the weak form of the potential vorticity equation by taking the inviscid limit of solutions to a system regularized by artificial viscosity. Since the streamfunction–vorticity relation is of order four, we can show uniqueness even for potential vorticities in L^2.

Key words: Mean hydrodynamics, well-posedness, inviscid second grade fluids.

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1 Introduction

The averaged Euler, or Euler-α equations are a conservative regularization of the Euler equations and describe the motion of an ideal fluid averaged over a small spatial length scale α. In two dimensions they can be cast into the vorticity–streamfunction formulation

$$\partial_t q + \boldsymbol{u} \cdot \boldsymbol{\nabla} q = 0 \tag{1}$$

on a domain $\Omega \subset \mathbb{R}^2$, where $q = q(\boldsymbol{x}, t)$ is the so-called potential vorticity and $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ is the Eulerian fluid velocity. For incompressible flow, $\boldsymbol{u}$ is divergence free and can be written as $\boldsymbol{u} = \boldsymbol{\nabla}^\perp \psi = (-\partial_2 \psi, \partial_1 \psi)$, where the streamfunction ψ is the solution to the elliptic problem

$$q = -(1 - \alpha^2 \Delta)\Delta\psi \quad \text{in } \Omega\,, \tag{2a}$$
$$\psi = 0 \quad \text{on } \partial\Omega\,, \tag{2b}$$
$$\partial_{nn}\psi = 0 \quad \text{on } \partial\Omega\,. \tag{2c}$$

Here $\partial_{nn}\psi$ denotes the second normal derivative of ψ on the boundary. For the purposes of this paper we can, without loss of generality, set $\alpha = 1$.

Historically, the averaged Euler equations go back to at least three independent developments.

First, the Euler-α equations arise as a generalization into several spatial dimensions of the scalar Fokas–Fuchssteiner–Camassa–Holm (FFCH) equation,

$$u_t - u_{xxt} + 2\kappa u_x = -3uu_x + 2u_x u_{xx} + uu_{xxx}\,, \tag{3}$$

which models unidirectional surface waves in shallow water. Originally proposed by Fokas and Fuchssteiner [9] and independently re-derived by Camassa and Holm [3], the equation possesses a number of remarkable properties. It has a bi-Hamiltonian structure, and thus possesses an infinite number of conservation laws in involution. It has soliton solutions whose limiting form as the parameter $\kappa \to 0$ has peaks, i.e., whose first derivative has a jump discontinuity. When $\kappa = 0$, these "peakons" dominate the solution of the initial value problem: Any initial datum which has points of inflection with a sufficiently large negative slope develops singularities in the first derivative in finite time [4]. On the other hand, when the initial "momentum" $m = (1 - \partial_{xx})u$ is of a definite sign, the solution remains smooth for all times [7,8].

Moreover, the FFCH equation can be realized as a geodesic equation on the diffeomorphism group of the circle equipped with a metric that arises from the H^1 metric on its Lie algebra of vector fields [16]. It is this last property that can be used to obtain a multi-dimensional version of the equation—an idea introduced by Holm, Marsden, and Ratiu [14,15] (also see Kouranbaeva [16,17], Marsden, Ratiu, and Shkoller [23], and Shkoller [29]).

Second, the Euler-α equations are the equations of motion of an inviscid non-Newtonian fluid of second grade. In this interpretation, α^2 is a material constant expressing the elastic response of the fluid. In the literature on second grade fluids it is generally assumed that the viscosity does not vanish, as visco-elastic effects such as "swelling" are most important in engineering applications. However, the derivation of Markovitz and Coleman [22] and Noll and Truesdell [25], which essentially imposes material frame indifference and observer objectivity on a Rivlin–Ericksen constitutive law [28], does not depend on viscosity.

Existence, uniqueness and regularity of solutions to the equations of viscous second grade fluids was proved by Cioranescu and El Hacène [5], Cioranescu and Girault [6], and Galdi, Grobbelaar–van Dalsen, and Sauer [11]. Their work differs from ours in that they do not discuss the limit of vanishing viscosity—although a careful study of [5,6] shows that some of the underlying *a priori* estimates do not depend on viscosity—and work with the velocity rather than the vorticity form of the equation. [4]

Third, the Euler-α model can be seen as the continuous analog of a computational vortex blob method for two-dimensional hydrodynamics, in which the motion of a finite number of point vortices in a smoothed velocity field is computed by numerically solving a finite system of ordinary differential equations (see, e.g., the review by Leonard [18]). The operator $(1-\alpha^2\Delta)$ in (2a) corresponds to a particular choice of the smoothing kernel for the vortex blob algorithm. This connection will be explored in a forthcoming paper [27]; see also the related work by Nadiga and Shkoller [24].

The purpose of this paper is to prove that the α-model in two dimensions has global weak solutions, and that these solutions are unique and depend continuously on the initial data. The paper is organized as follows. We first sketch the derivation of the α-model. In Section 3 we recast the equations into their vorticity formulation. The advantage is that the vorticity equation can be analyzed within a framework of generalized Euler equations as was used by Levermore *et al.* [20,21]. The method is based on earlier work of Yudovitch [34] and Bardos [1].

In Section 4 we define the notion of weak solutions to the vorticity equations with initial data in L^2. To prove existence, we first introduce an artificial viscosity term into the system, thereby obtaining a sequence of approximate solutions parameterized by the viscosity ν. Second, we show that the approx-

[4] *Note added in proof:* A recent article by Busuioc [2] discusses the inviscid case in the framework of [5,6].

imating sequence converges to a solution of the inviscid equations as $\nu \to 0$. Uniqueness is proved by estimating the time evolution of the difference of two solutions in the H^{-1} Sobolev norm. The same estimate also establishes continuous dependence of the solution on the data.

The case of L^2 potential vorticity for the α-model is technically easier than the corresponding situation for the Euler equations where uniqueness of such solutions is not known, because the vorticity–streamfunction relationship is of order four, thereby providing better *a priori* control on the velocity. In fact, L^1 potential vorticity is the Sobolev-critical case for uniqueness, which we discuss in forthcoming work [13,27].

For our purposes, the viscosity approach is mainly a matter of convenience. For the Euler equations one needs *a priori* estimates in L^∞, so that one cannot use Galerkin approximating sequences. Since we only require L^2 control, a direct Galerkin approximation of the inviscid problem would work here as well, as has been used by Cioranescu and El Hacène [5] for the equations of viscous second grade fluids.

The existence of classical solutions for the Euler-α model was proved by Shkoller [29] for domains without boundaries, and by Marsden, Ratiu, and Shkoller [23] for bounded domains, by realizing the equation as a geodesic on a subgroup of the volume-preserving diffeomorphism group. Equivalent results as well as analyticity of solutions when the equations are posed on a periodic domain follow from theorems in [19,26] once the regularity of the vorticity–streamfunction relation is established.

2 The Velocity Formulation

The Eulerian equation in velocity-momentum variables can be written

$$\partial_t \boldsymbol{m} - \boldsymbol{u}^\perp \boldsymbol{\nabla} \wedge \boldsymbol{m} + \boldsymbol{\nabla} \left(\boldsymbol{u} \cdot \boldsymbol{m} - \frac{1}{2} |\boldsymbol{u}|^2 - \frac{1}{2} \operatorname{tr} \left(\boldsymbol{\nabla} \boldsymbol{u} (\boldsymbol{\nabla} \boldsymbol{u})^{\mathrm{T}} \right) + p \right) = 0 \quad \text{in } \Omega \,, \tag{4a}$$

$$\boldsymbol{m} = (1 - \Delta) \boldsymbol{u} \quad \text{in } \Omega \,, \tag{4b}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega \,, \tag{4c}$$

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega \,, \tag{4d}$$

$$\boldsymbol{s} \cdot (\boldsymbol{n} \cdot \boldsymbol{\nabla}) \boldsymbol{u} = 0 \quad \text{on } \partial\Omega \,, \tag{4e}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}^{\mathrm{in}} \,, \tag{4f}$$

where $\boldsymbol{\nabla} \wedge \boldsymbol{m} = \partial_2 m_1 - \partial_1 m_2$, $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{x})$ is a unit vector on the boundary $\partial\Omega$, and $\boldsymbol{s} = \boldsymbol{n}^\perp$.

This system of equations arises formally as the Euler–Lagrange equations for the action

$$\mathcal{L} = \int L_\alpha \, \mathrm{d}t \tag{5}$$

under the constraint that the flow is volume preserving, where the Lagrangian L_α is given by

$$L_\alpha = \frac{1}{2} \int_\Omega \left(|\boldsymbol{u}|^2 + \alpha^2 \, \partial_j u^i \, \partial^j u_i \right) \, \mathrm{d}\boldsymbol{x} \, . \tag{6}$$

As pointed out by Shkoller [30] and Holm *et al.* [14], the boundary condition (4e) is inconsistent with the existence of a Lagrangian flow—the family of time advance maps for the particle labels does not form a group. This can be remedied by introducing a modified Lagrangian

$$L_{\text{mod}} = \frac{1}{2} \int_\Omega \left(|\boldsymbol{u}|^2 + 2\alpha^2 \, |\mathrm{Def}\, \boldsymbol{u}|^2 \right) \, \mathrm{d}\boldsymbol{x} \, , \tag{7}$$

where $\mathrm{Def}\, \boldsymbol{u} = (\boldsymbol{\nabla u} + \boldsymbol{\nabla u}^T)/2$ is the rate of deformation matrix. This choice leads to the natural boundary condition

$$\boldsymbol{s} \cdot (\boldsymbol{n} \cdot \boldsymbol{\nabla})\boldsymbol{u} + S_n(\boldsymbol{u}) = 0 \, , \tag{8}$$

where S_n is the second fundamental form of the boundary $\partial\Omega$.

However, our analysis is substantially independent of this issue, so that we stick to the somewhat simpler, but unphysical boundary condition (4e). In fact, it is interesting that the Eulerian formulation is well posed even in cases when the Lagrangian flow is not.

3 The Vorticity Formulation

Taking the curl of the momentum equation (4a), we obtain the advection equation (1) for the potential vorticity $q \equiv \boldsymbol{\nabla} \wedge \boldsymbol{m}$. Setting $\boldsymbol{u} \equiv \boldsymbol{\nabla}^\perp \psi$, where ψ is given by (2), relation (4b), the divergence condition (4c) and the essential boundary condition (4d) are immediately satisfied. Moreover, on $\partial\Omega$ provided that $\boldsymbol{\nabla}\psi \neq 0$,

$$\boldsymbol{s} \cdot (\boldsymbol{n} \cdot \boldsymbol{\nabla})\boldsymbol{u} = \boldsymbol{s} \cdot \left(\frac{\boldsymbol{\nabla}\psi}{|\boldsymbol{\nabla}\psi|} \cdot \boldsymbol{\nabla} \right) \boldsymbol{\nabla}^\perp \psi = \boldsymbol{n} \cdot \boldsymbol{\nabla} \, |\boldsymbol{\nabla}\psi| = \boldsymbol{n} \cdot \boldsymbol{\nabla}(\boldsymbol{n} \cdot \boldsymbol{\nabla}\psi) = \partial_{nn}\psi \, . \tag{9}$$

This establishes the formal equivalence of (2c) and (4e).

In the following we will prove that (2) has a unique solution ψ for a given potential vorticity q, and determine the regularity of that solution.

Let us introduce some assumptions and notation. The domain Ω is bounded and, for simplicity, assumed to be simply connected with $\partial\Omega \in C^4$. (The multiply connected case for the Euler equations was treated in [34] and also discussed in [21]; similar arguments apply here). The Sobolev space of order $n \in \mathbb{N}$ is denoted $H^n(\Omega)$, and has the inner product

$$(\varphi, \vartheta)_{H^n} = \sum_{0 \le |\beta| \le n} \int_\Omega \partial^\beta \varphi(\boldsymbol{x}) \, \partial^\beta \vartheta(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \,, \tag{10}$$

where $\beta = (i_1, i_2, \ldots, i_k)$, $|\beta| = k$, and $\partial^\beta = \partial^{i_1} \partial^{i_2} \cdots \partial^{i_k}$.

The completion of the infinitely differentiable and compactly supported functions $C_0^\infty(\Omega)$ in the H^n norm is denoted $H_0^n(\Omega)$ with dual $H_0^n(\Omega)' \equiv H^{-n}(\Omega)$. The integral over the domain Ω is abbreviated by $\langle \cdot \rangle$, i.e.

$$\langle \varphi \rangle = \int_\Omega \varphi(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \,. \tag{11}$$

Hence, the $L^2(\Omega) \equiv H^0(\Omega)$ scalar product can be written $(\varphi, \vartheta)_{L^2} \equiv \langle \varphi \, \vartheta \rangle$. We introduce the space of infinitely differentiable and compactly supported functions which satisfy an incompressibility condition,

$$\mathbb{D} = \{ \boldsymbol{u} \in C_0^\infty(\Omega, \mathbb{R}^2) \colon \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \quad \text{in } \Omega \} \,,$$

and denote by $\mathbb{H}$ and $\mathbb{V}$ its completion in the L^2 and H^1 norms respectively. Finally, the relationship between $\boldsymbol{u}$ and q given in (2) is denoted by an abstract operator K, so that the full set of vorticity equations reads

$$\partial_t q + \boldsymbol{u} \cdot \boldsymbol{\nabla} q = 0 \,, \tag{12a}$$

$$\boldsymbol{u} = K q \,, \tag{12b}$$

$$q(0) = q^{\mathrm{in}} \,. \tag{12c}$$

Next, we formulate a lemma that assures K is invertible and has the smoothing properties we need in the next section to prove the existence theorem. For simplicity we set $\alpha = 1$.

Lemma 1 *For every $q \in H^{-2}(\Omega)$ there exists a unique function $\psi \in H^2(\Omega) \cap H_0^1(\Omega)$ and hence a unique function $\boldsymbol{u} = Kq \equiv \boldsymbol{\nabla}^\perp \psi \in \mathbb{H}$. Moreover, K is continuous as a mapping between the spaces $H^{-2}(\Omega) \to \mathbb{H}$, $H^{-1}(\Omega) \to \mathbb{H} \cap H^2(\Omega, \mathbb{R}^2)$ and $L^2(\Omega) \to \mathbb{H} \cap H^3(\Omega, \mathbb{R}^2)$.*

PROOF. We solve (2) by the variational method. Let φ, $\psi \in C_0^\infty(\Omega)$ and denote $\boldsymbol{v} \equiv \boldsymbol{\nabla}^\perp \varphi$ and $\boldsymbol{u} \equiv \boldsymbol{\nabla}^\perp \psi$. It can be verified directly that $\boldsymbol{u}$, $\boldsymbol{v} \in \mathbb{H}$. The geometric property of the two-dimensional Euler-α equation of being a geodesic equation for the H^1 metric suggests a natural bilinear form,

$$b(\varphi, \psi) \equiv (\boldsymbol{\nabla}^\perp \varphi, \boldsymbol{\nabla}^\perp \psi)_{H^1} = (\boldsymbol{v}, \boldsymbol{u})_{H^1} . \tag{13}$$

Due to the Poincaré inequality, $\|\boldsymbol{\nabla}^\perp \cdot\|_{L^2}$ is an equivalent norm on $H_0^1(\Omega)$, so that the following estimates hold:

$$b(\varphi, \psi) = (\boldsymbol{v}, \boldsymbol{u})_{H^1} \leq \|\boldsymbol{v}\|_{H^1} \|\boldsymbol{u}\|_{H^1} \leq c_1 \|\varphi\|_{H^2} \|\psi\|_{H^2} , \tag{14}$$

and

$$b(\varphi, \varphi) = \|\boldsymbol{\nabla}^\perp \varphi\|_{H^1}^2 \geq c_2 \|\varphi\|_{H^2}^2 . \tag{15}$$

This proves that b is continuous and coercive on $V \equiv H_0^1(\Omega) \cap H^2(\Omega)$. Then the Lax–Milgram Theorem guarantees that there is a unique operator $B \colon V \to V'$, such that $b(\varphi, \psi) = (\varphi, B\psi)_{L^2}$ for arbitrary $\varphi, \psi \in V$. Moreover, B is a continuous isomorphism. It follows that for given $q \in H^{-2}(\Omega)$ there exists a unique $\psi \in V$ such that

$$b(\varphi, \psi) = (\varphi, q)_{L^2} \tag{16}$$

for every $\varphi \in V$.

The classical regularity theory for elliptic equations (see, e.g., [12,33]) guarantees that, provided the domain is sufficiently regular, the inverse of B will gain four derivatives in Sobolev spaces. In particular, if $q \in L^2(\Omega)$ then $\psi \in H_0^1(\Omega) \cap H^4(\Omega)$.

It remains to be shown that the solution ψ to the weak equation (16) solves (2). First, consider only $\varphi, \psi \in C_0^\infty(\Omega)$, so that all integrations by parts are justified.

$$\begin{aligned}
b(\varphi, \psi) &\equiv (\boldsymbol{v}, \boldsymbol{u})_{L^2} + \int_\Omega \partial_j v^i \, \partial^j u_i \, \mathrm{d}\boldsymbol{x} \\
&= \int_\Omega \boldsymbol{v} \cdot (1 - \Delta) \boldsymbol{u} \, \mathrm{d}\boldsymbol{x} + \oint_{\partial\Omega} n^j \, \partial_j \, u^i v_i \, \mathrm{d}l \\
&= \int_\Omega \boldsymbol{\nabla}^\perp \varphi \cdot (1 - \Delta) \boldsymbol{\nabla}^\perp \psi \, \mathrm{d}\boldsymbol{x} + \oint_{\partial\Omega} \boldsymbol{v} \cdot (\boldsymbol{n} \cdot \boldsymbol{\nabla}) \boldsymbol{u} \, \mathrm{d}l \, .
\end{aligned} \tag{17}$$

Using the Green formula

$$\int_\Omega \boldsymbol{\nabla}^\perp \varphi \cdot \boldsymbol{m} \, \mathrm{d}\boldsymbol{x} = - \int_\Omega \varphi \, \boldsymbol{\nabla} \wedge \boldsymbol{m} \, \mathrm{d}\boldsymbol{x} + \oint_{\partial\Omega} \varphi \, \boldsymbol{n}^\perp \cdot \boldsymbol{m} \, \mathrm{d}l \, , \tag{18}$$

and the hypothesis $\varphi \equiv 0$ on $\partial\Omega$, one obtains

$$b(\varphi, \psi) = -\int_\Omega \varphi \, \boldsymbol{\nabla}^\perp \cdot (1-\Delta)\boldsymbol{\nabla}^\perp \psi \; \mathrm{d}\boldsymbol{x} + \oint_{\partial\Omega} \boldsymbol{v} \cdot (\boldsymbol{n} \cdot \boldsymbol{\nabla})\boldsymbol{u} \; \mathrm{d}l \,. \tag{19}$$

As we assumed that $\varphi \in C_0^\infty(\Omega)$, the boundary integral vanishes and equation (16) becomes

$$-\int_\Omega \varphi \, \boldsymbol{\nabla}^\perp \cdot (1-\Delta)\boldsymbol{\nabla}^\perp \psi \; \mathrm{d}\boldsymbol{x} = \int_\Omega \varphi q \; \mathrm{d}\boldsymbol{x} \,. \tag{20}$$

By continuity, this relation is satisfied for solutions of the variational problem with $q \in L^2$ and $\psi \in H^4$, so that q and ψ satisfy (2a) in the L^2 sense. Further, this forces the integral along the boundary in (19) to vanish, i.e.,

$$\oint_{\partial\Omega} \boldsymbol{v} \cdot (\boldsymbol{n} \cdot \boldsymbol{\nabla})\boldsymbol{u} \; \mathrm{d}l = 0 \,. \tag{21}$$

Since $\boldsymbol{v}$ is tangent to the boundary, one can show as in (9) that locally on $\partial\Omega$

$$\boldsymbol{v} \cdot (\boldsymbol{n} \cdot \boldsymbol{\nabla})\boldsymbol{u} = \partial_n \psi \, \partial_{nn} \psi \,. \tag{22}$$

In other words, (21) implies that

$$\oint_{\partial\Omega} \partial_n \varphi \, \partial_{nn} \psi \; \mathrm{d}l = 0 \,, \tag{23}$$

for every $\varphi \in V$. It follows that $\partial_{nn}\psi = 0$ on the boundary. The regularity of $\boldsymbol{u}$ as stated in the lemma is a direct consequence of the above.

4 Weak Solutions

In this section we introduce a weak vorticity formulation, and prove existence and uniqueness of weak solutions. The proof is self-contained in the sense that we assume K is a linear operator satisfying all the properties as stated in Lemma 1; we will not refer explicitly to the elliptic problem (2) again. Our main result is the following.

Theorem 2 *Given $q^{\mathrm{in}} \in L^2(\Omega)$, there exists a unique function*

$$q \in C([0,\infty); \text{w-}L^2(\Omega)) \cap L^2_{\mathrm{loc}}([0,\infty); L^2(\Omega)) \,, \tag{24}$$

that satisfies the weak form of the vorticity equation (12)*, given by*

$$\langle \psi q(t_2) \rangle - \langle \psi q(t_1) \rangle - \int_{t_1}^{t_2} \big\langle (\boldsymbol{u} \cdot \boldsymbol{\nabla} \psi) \, q \big\rangle \, \mathrm{d}t = 0 \,, \tag{25a}$$

$$\boldsymbol{u} = Kq \,, \tag{25b}$$

$$q(0) = q^{\mathrm{in}} \,, \tag{25c}$$

for every $[t_1, t_2] \subset [0, \infty)$ *and every test function* $\psi \in H_0^1(\Omega)$*. It satisfies the "energy relation"*

$$\|q(t)\|_{L^2} \leq \|q^{\mathrm{in}}\|_{L^2} \,, \tag{26}$$

for every $t \in [0, \infty)$*. Moreover, for every fixed time* T *the mapping* $q^{\mathrm{in}} \mapsto q(T)$ *is continuous on* $H^{-1}(\Omega) \to H^{-1}(\Omega)$*.*

The method of proof is classical, our presentation will mostly follow [20]. In order to set up the viscous regularization, we introduce the operator A by

$$A\psi \equiv -\Delta \psi = q \quad \text{on } \Omega \,, \tag{27a}$$

$$\psi = 0 \quad \text{on } \partial\Omega \,. \tag{27b}$$

Due to the Poincaré inequality on $C_0^\infty(\Omega)$, one has

$$\langle A\psi \, \varphi \rangle = \big\langle \boldsymbol{\nabla} \psi \cdot \boldsymbol{\nabla} \varphi \big\rangle \leq c_1 \|\psi\|_{H^1} \, \|\varphi\|_{H^1} \,, \tag{28}$$

which shows that $\langle \boldsymbol{\nabla} \psi \cdot \boldsymbol{\nabla} \varphi \rangle$ can be extended to a symmetric, continuous and coercive bilinear form on $H_0^1(\Omega)$. Again by the Lax–Milgram theorem, $A \colon H_0^1(\Omega) \to H_0^1(\Omega)' \equiv H^{-1}(\Omega)$ is an isomorphism. By the Sobolev embedding theorem $H_0^1(\Omega) \hookrightarrow H^{-1}(\Omega)$ is compact, hence $\mathrm{id} \circ A^{-1}$ is also compact and A possesses countably many positive eigenvalues. This allows us to define arbitrary powers of the operator, whence the canonical H^1 norm on $H_0^1(\Omega)$ is equivalent to $\|A^{1/2} \cdot\|_{L^2}$, and the H^2 norm on $H^2(\Omega) \cap H_0^1(\Omega)$ is equivalent to $\|A \cdot\|_{L^2}$. Moreover, $H^{-1}(\Omega) = H_0^1(\Omega)'$ and $H^{-2}(\Omega) = H_0^2(\Omega)'$ can be endowed with the norms $\|A^{-1/2} \cdot\|_{L^2}$ and $\|A^{-1} \cdot\|_{L^2}$, respectively.

For $\boldsymbol{u} \in \mathbb{D}$ and ψ, $\varphi \in C^\infty(\overline{\Omega})$, using the Gauss integral formula, one can prove that

$$\big\langle (\boldsymbol{u} \cdot \boldsymbol{\nabla} \psi) \, \varphi \big\rangle = -\big\langle (\boldsymbol{u} \cdot \boldsymbol{\nabla} \varphi) \, \psi \big\rangle \,. \tag{29}$$

By continuity this relation extends to all $\boldsymbol{u} \in \mathbb{H}$ and ψ, $\varphi \in H^1(\Omega)$.

The proof now proceeds in three steps.

5 Existence

We construct solutions to (25) as the inviscid limit of the following viscous regularization of the α-model:

$$\partial_t q + \nu\, Aq + \boldsymbol{u} \cdot \boldsymbol{\nabla} q = 0 \quad \text{in } \Omega\,, \tag{30a}$$

$$\boldsymbol{u} = Kq \quad \text{in } \Omega\,, \tag{30b}$$

$$q = 0 \quad \text{on } \partial\Omega\,, \tag{30c}$$

$$q(0) = q^{\text{in}}\,. \tag{30d}$$

This system resembles the Navier–Stokes equations in two dimensions, and its existence and uniqueness theory is therefore well known (see, e.g., [21,31,32]). Theorem 2 from [20] guarantees that for each $\nu > 0$ there exists a unique function

$$q_\nu \in C([0,\infty); \text{w-}L^2(\Omega)) \cap L^2_{\text{loc}}([0,\infty); H^1_0(\Omega)) \tag{31}$$

that satisfies the weak form of the viscous equations (30), given by

$$\langle \psi q_\nu(t_2) \rangle - \langle \psi q_\nu(t_1) \rangle + \nu \int_{t_1}^{t_2} \big\langle \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla} q_\nu \big\rangle \, \mathrm{d}t - \int_{t_1}^{t_2} \big\langle (\boldsymbol{u}_\nu \cdot \boldsymbol{\nabla}\psi)\, q_\nu \big\rangle \, \mathrm{d}t = 0\,, \tag{32a}$$

$$\boldsymbol{u}_\nu = K q_\nu\,, \tag{32b}$$

$$q_\nu(0) = q_\nu^{\text{in}}\,, \tag{32c}$$

for every $[t_1, t_2] \subset [0,\infty)$ and every test function $\psi \in H^1_0(\Omega)$, as well as the "energy relation"

$$\|q_\nu(t)\|_{L^2} \le \|q_\nu^{\text{in}}\|_{L^2}\,, \tag{33}$$

for every $t \in [0,\infty)$.

We will now show how to pass to the inviscid limit.

Lemma 3 *The family $\{q_\nu\}$ is a relatively compact set in*

$$C([0,\infty); \text{w-}L^2(\Omega)) \wedge \text{w-}L^2_{\text{loc}}([0,\infty); \text{w-}L^2(\Omega))\,, \tag{34}$$

as well as in $L^2_{\text{loc}}([0,\infty); H^{-1}(\Omega))$ endowed with its strong topology.

Remark 4 *"$\wedge$" denotes the intersection of topological spaces equipped with the relative topology induced by the inclusion maps. The spaces here are first countable topological vector spaces which implies that a sequence converges in the intersection if and only if it converges in each space separately. Given a topological vector space* X *with dual* X′*, a sequence of functions $\phi_\nu \colon [0,\infty) \to \mathrm{X}$*

converges to ϕ in $C([0,\infty);\text{w-X})$, where w-X *denotes* X *endowed with its weak topology, when for every $\psi \in \mathrm{X}'$*

$$\langle \psi, \phi_\nu(t)\rangle_{\mathrm{X}',\mathrm{X}} \to \langle \psi, \phi(t)\rangle_{\mathrm{X}',\mathrm{X}} \tag{35}$$

uniformly on compact subsets of $[0,\infty)$. We also say that

$$\phi_\nu \to \phi \text{ in } \text{w-}L^p_{\mathrm{loc}}([0,\infty);\text{w-X})\,, \tag{36}$$

when for every $T>0$ and $\psi \in L^{p'}_{\mathrm{loc}}([0,\infty);\mathrm{X}')$

$$\int_0^T \langle \psi(t), \phi_\nu(t)\rangle_{\mathrm{X}',\mathrm{X}}\,\mathrm{d}t \to \int_0^T \langle \psi(t), \phi(t)\rangle_{\mathrm{X}',\mathrm{X}}\,\mathrm{d}t\,, \tag{37}$$

where $p' = p/(p-1)$ is the Hölder conjugate of p, and $p \in [1,\infty)$.

Proof of Lemma 3 The estimates

$$\|q_\nu(t)\|_{L^2} \le \|q_\nu^{\mathrm{in}}\|_{L^2} \le \|q^{\mathrm{in}}\|_{L^2} \tag{38}$$

follow from (33) and properties of the regularized initial data. This shows that $\{q_\nu\}$ is equibounded in $C([0,\infty);\text{w-}L^2(\Omega))$. In order to use the Arzela–Ascoli theorem we need—in addition to equiboundedness—equicontinuity in $C([0,\infty);\text{w-}L^2(\Omega))$. I.e., we need to prove that for every $\psi \in L^2(\Omega)$ the sequence $\{\langle \psi q_\nu\rangle\}$ is equicontinuous in $C([0,\infty))$. Let us first consider the case when $\psi \in C_0^\infty(\Omega)$. We estimate

$$\begin{aligned}
&|\langle \psi q_\nu(t_2)\rangle - \langle \psi q_\nu(t_1)\rangle| \\
&\qquad = \left| -\nu \int_{t_1}^{t_2} \big\langle \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla} q_\nu \big\rangle\,\mathrm{d}t + \int_{t_1}^{t_2} \big\langle (\boldsymbol{u}_\nu \cdot \boldsymbol{\nabla}\psi)\, q_\nu \big\rangle\,\mathrm{d}t \right| \\
&\qquad \le \nu\, \|A\psi\|_{L^\infty} \|1\|_{L^2} \left(\int_{t_1}^{t_2} \|q_\nu\|_{L^2}\,\mathrm{d}t \right) \\
&\qquad\quad + \|\boldsymbol{\nabla}\psi\|_{L^\infty} \left(\int_{t_1}^{t_2} \|\boldsymbol{u}_\nu\|_{L^2} \|q_\nu\|_{L^2}\,\mathrm{d}t \right) \\
&\qquad \le \nu\, \|A\psi\|_{L^\infty} V(\Omega)^{\frac12} \|q^{\mathrm{in}}\|_{L^2} |t_2 - t_1| \\
&\qquad\quad + \|\boldsymbol{\nabla}\psi\|_{L^\infty} \left(\int_{t_1}^{t_2} \mathrm{d}t \right)^{\frac14} \left(\int_{t_1}^{t_2} \|q_\nu\|_{L^2}^4\,\mathrm{d}t \right)^{\frac14} \left(\int_{t_1}^{t_2} \|\boldsymbol{u}_\nu\|_{L^2}^2\,\mathrm{d}t \right)^{\frac12}.
\end{aligned} \tag{39}$$

By the continuity of $K\colon L^2(\Omega) \to H^3(\Omega,\mathbb{R}^2) \cap \mathbb{H}$ and the fact that $H^1(\Omega) \subset L^p(\Omega)$ for $2 \le p < \infty$ we get for every $t \in [0,\infty)$

$$\|\boldsymbol{u}(t)\|_{L^2} \le c_1 \|\boldsymbol{u}(t)\|_{H^1} \le c_2 \|q\|_{L^2}. \tag{40}$$

Furthermore, $\nu < 1$, so that for every $\nu \in (0,1)$

$$|\langle \psi q_\nu(t_2)\rangle - \langle \psi q_\nu(t_1)\rangle| \le c(\psi,\Omega,q^{\mathrm{in}})\,|t_2 - t_1|\,. \tag{41}$$

Because the right side of this equation is independent of ν, the set $\{\langle \psi q_\nu\rangle\}$ is equicontinuous in $C([0,\infty))$.

Now we extend the argument to test functions in the space $L^2(\Omega)$. Let $\varepsilon > 0$ and choose ψ_ε to be a finite linear combination of eigenvectors of A such that

$$\|\psi - \psi_\varepsilon\|_{L^2} < \frac{\varepsilon}{3\,\|q^{\mathrm{in}}\|_{L^2} + 1}\,. \tag{42}$$

By the triangle inequality

$$\begin{aligned} |\langle \psi q_\nu(t_2)\rangle - \langle \psi q_\nu(t_1)\rangle| &= \left|\langle (\psi - \psi_\varepsilon)(q_\nu(t_2) - q_\nu(t_1))\rangle + \langle \psi_\varepsilon(q_\nu(t_2) - q_\nu(t_1))\rangle\right| \\ &\le 2\,\|\psi - \psi_\varepsilon\|_{L^2}\|q^{\mathrm{in}}\|_{L^2} + \left|\langle \psi_\varepsilon(q_\nu(t_2) - q_\nu(t_1))\rangle\right|\,. \end{aligned} \tag{43}$$

Choosing $|t_2 - t_1|$ small enough and applying (41) we can make the second term on the right smaller than $\frac{1}{3}\varepsilon$. By the choice of ψ_ε in (42) the first term on the right is smaller than $\frac{2}{3}\varepsilon$, so that the whole expression is less than ε. This proves that $\{q_\nu\}$ is relatively compact in $C([0,\infty);\text{w-}L^2(\Omega))$.

Note that the sequence $\{q_\nu\}$ is bounded in $L^\infty_{\mathrm{loc}}([0,\infty);L^2(\Omega))$, hence $\{q_\nu\}$ is relatively compact in $\text{w-}L^2_{\mathrm{loc}}([0,\infty);\text{w-}L^2(\Omega))$.

Finally, relative compactness in $L^2_{\mathrm{loc}}([0,\infty);H^{-1}(\Omega))$ is now a direct consequence of the continuity of the embedding

$$C([0,\infty);\text{w-}L^2(\Omega)) \hookrightarrow L^2_{\mathrm{loc}}([0,\infty);H^{-1}(\Omega)) \tag{44}$$

(see, e.g., [10]).

Lemma 3 implies existence of a weakly converging subsequence of $\{q_\nu\}$, which we still denote by $\{q_\nu\}$, to a limit function

$$q \in C([0,\infty);\text{w-}L^2(\Omega)) \cap L^2_{\mathrm{loc}}([0,\infty);L^2(\Omega)) \tag{45}$$

in the following sense:

(i) $q_\nu \to q$ in $C([0,\infty);\text{w-}L^2(\Omega))$;
(ii) $q_\nu \to q$ in $\text{w-}L^2_{\mathrm{loc}}([0,\infty);\text{w-}L^2(\Omega))$;
(iii) $q_\nu \to q$ in $L^2_{\mathrm{loc}}([0,\infty);H^{-1}(\Omega))$.

Next, let us show that q satisfies the weak form (25) and the energy relation (26). This will be done by showing that each of the terms in the weak form

(32) of the viscous equations converges to the corresponding term in (25). It is sufficient to assume a test function $\psi \in C_0^\infty(\Omega)$ and then proceed by applying a density argument.

The convergence (i) in $C([0,\infty);\text{w-}L^2(\Omega))$ implies that

$$\langle \psi q_\nu(t)\rangle \to \langle \psi q(t)\rangle \tag{46}$$

for every $t \geq 0$ as $\nu \to 0$. The viscosity term can be bounded as

$$\nu \int_{t_1}^{t_2} \big\langle \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla} q_\nu \big\rangle \,\mathrm{d}t \leq \nu \left\|\psi\right\|_{H^2} \int_{t1}^{t2} \left\|q_\nu\right\|_{L^2} \,\mathrm{d}t\,. \tag{47}$$

The last two factors on the right are bounded independent of ν, so that the left side must converge to zero as $\nu \to 0$. To show the convergence of the nonlinear term we use the estimate

$$\begin{aligned}
&\int_{t_1}^{t_2} \Big(\big\langle(\boldsymbol{u}_\nu \cdot \boldsymbol{\nabla}\psi)\, q_\nu\big\rangle - \big\langle(\boldsymbol{u} \cdot \boldsymbol{\nabla}\psi)\, q\big\rangle\Big)\,\mathrm{d}t \\
&= \int_{t_1}^{t_2} \Big(\big\langle((\boldsymbol{u}_\nu - \boldsymbol{u}) \cdot \boldsymbol{\nabla}\psi)\, q_\nu\big\rangle + \big\langle(\boldsymbol{u} \cdot \boldsymbol{\nabla}\psi)\,(q_\nu - q)\big\rangle\Big)\,\mathrm{d}t \\
&\leq \int_{t_1}^{t_2} \left\|\boldsymbol{u}_\nu - \boldsymbol{u}\right\|_{L^2} \left\|\boldsymbol{\nabla}\psi\right\|_{L^\infty} \left\|q_\nu\right\|_{L^2} \,\mathrm{d}t + \left|\int_{t_1}^{t_2} \big\langle(\boldsymbol{u} \cdot \boldsymbol{\nabla}\psi)\,(q_\nu - q)\big\rangle \,\mathrm{d}t\right|\,.
\end{aligned} \tag{48}$$

In order to pass to the limit it is sufficient to have strong convergence of $\{\boldsymbol{u}_\nu\}$ in $L^2_{\text{loc}}([0,\infty);L^2(\Omega,\mathbb{R}^2))$ and weak convergence of $\{q_\nu\}$ in $L^2_{\text{loc}}([0,\infty);L^2(\Omega))$. The latter follows directly from (ii). Moreover, the strong convergence (iii) in $L^2_{\text{loc}}([0,\infty);H^{-1}(\Omega))$ and the continuity of $K\colon\ H^{-1}(\Omega) \to H^2(\Omega,\mathbb{R}^2)\cap\mathbb{H}$ imply strong convergence of $\{\boldsymbol{u}_\nu\}$ in $L^2_{\text{loc}}([0,\infty);L^2(\Omega,\mathbb{R}^2))$.

Finally, the convergence (i) implies

$$\left\|q(t)\right\|_{L^2} \leq \liminf_{\nu\to 0} \left\|q_\nu(t)\right\|_{L^2}\,, \tag{49}$$

for every $t \geq 0$. This together with the fact that $q_\nu^{\text{in}} \to q^{\text{in}}$ in L^2 guarantees the energy relation (26).

6 Uniqueness

We first demonstrate the existence of a distributional time derivative $\partial_t q \in L^2_{\mathrm{loc}}([0,\infty); H^{-1}(\Omega))$. From the general fact that $H^2 \subset L^\infty$ and the estimate

$$\int_0^t \left|\left\langle (\boldsymbol{u}(t') \cdot \boldsymbol{\nabla}\psi)\, q(t') \right\rangle\right|^2 \,\mathrm{d}t' \le \|\psi\|^2_{H^1} \int_0^t \|\boldsymbol{u}(t')\|^2_{L^\infty} \|q(t')\|^2_{L^2} \,\mathrm{d}t' \,, \tag{50}$$

we conclude that the integral on the right is finite, which means that the mapping

$$(t,\psi) \mapsto \left\langle (\boldsymbol{u}(t) \cdot \boldsymbol{\nabla}\psi)\, q(t) \right\rangle \tag{51}$$

belongs to $L^2_{\mathrm{loc}}([0,\infty); H^{-1}(\Omega))$.

Let therefore $\psi \in H^1_0(\Omega)$. The weak form (25) includes in particular

$$\langle \psi q(t) \rangle = \langle \psi q^{\mathrm{in}} \rangle + \int_0^t \left\langle (\boldsymbol{u} \cdot \boldsymbol{\nabla}\psi)\, q \right\rangle \,\mathrm{d}t' \,. \tag{52}$$

Take a test function $\theta \in C_0^\infty([0,\infty))$, multiply the previous line with $\mathrm{d}\theta/\,\mathrm{d}t$, integrate from 0 to ∞ and then integrate by parts to obtain

$$\int_0^\infty \langle \psi q(t) \rangle \,\frac{\mathrm{d}\theta}{\mathrm{d}t} \,\mathrm{d}t = -\int_0^\infty \left\langle (\boldsymbol{u}(t) \cdot \boldsymbol{\nabla}\psi)\, q(t) \right\rangle \theta(t) \,\mathrm{d}t \,. \tag{53}$$

Integration by parts is legitimate here because from the earlier estimate it follows that the integrand on the right side of (52) is a $\mathrm{L}^1_{\mathrm{loc}}([0,\infty))$ function. Relation (53) is equivalent to saying that

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \psi q(t) \rangle = \left\langle (\boldsymbol{u}(t) \cdot \boldsymbol{\nabla}\psi)\, q(t) \right\rangle \tag{54}$$

in the sense of distributions on $[0,\infty)$, so that we can write

$$\partial_t q = -\boldsymbol{\nabla} \cdot (\boldsymbol{u} q) \in L^2_{\mathrm{loc}}([0,\infty); H^{-1}(\Omega)) \,. \tag{55}$$

The following identity is useful in further calculations and it holds in the sense of distributions on $\mathbb{R}$

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|q(t)\|^2_{H^{-1}} = (\partial_t q, q)_{H^{-1}} \,. \tag{56}$$

This identity is obvious for functions $q \in C_0^\infty(\mathbb{R}; H^{-1}(\Omega))$ and can be extended to all $q \in L^2_{\mathrm{loc}}([0,\infty); H^{-1}(\Omega))$ with $\partial_t q \in L^2_{\mathrm{loc}}([0,\infty); H^{-1}(\Omega))$ by density. As proved in [33, Theorem 25.5] the embedding

$$\left\{ \varphi \in L^2_{\mathrm{loc}}([0,\infty); H^{-1}(\Omega)) \colon \partial_t \varphi \in L^2_{\mathrm{loc}}([0,\infty); H^{-1}(\Omega)) \right\} \hookrightarrow C([0,\infty); H^{-1}(\Omega)) \tag{57}$$

is continuous. Hence $q \in C([0,\infty); H^{-1}(\Omega))$.

Now assume there are two solutions $\hat\omega$ and $\bar\omega$ to equation (12) that correspond to identical initial data and set $\hat{\boldsymbol{u}} = K\hat\omega$, $\bar{\boldsymbol{u}} = K\bar\omega$ and $\varphi = \hat\omega - \bar\omega$. Our goal is to compute the time evolution of φ in the H^{-1} norm. If we can show that this norm remains zero for all times, the two solutions must be identical. Note that this does not mean that we allow general H^{-1} initial data for $\hat\omega$ or $\bar\omega$.

First observe that for any two functions $\boldsymbol{v} \in L^2(\Omega, \mathbb{R}^2)$ and $\psi \in L^2(\Omega)$ one has

$$\big\langle A^{-1}(\boldsymbol{\nabla} \cdot \boldsymbol{v})\,\psi \big\rangle = -\big\langle \boldsymbol{v} \cdot \boldsymbol{\nabla} A^{-1}\psi \big\rangle\,. \tag{58}$$

This is easily verified for smooth functions and extends by continuity. Using (56) and then (58) one computes

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi(t)\|_{H^{-1}}^2 &= -\big\langle A^{-1}\big(\boldsymbol{\nabla}\cdot(\hat{\boldsymbol{u}}\hat\omega)\big)\,\varphi\big\rangle + \big\langle A^{-1}\big(\boldsymbol{\nabla}\cdot(\bar{\boldsymbol{u}}\bar\omega)\big)\,\varphi\big\rangle \\ &= \big\langle \hat\omega\, \hat{\boldsymbol{u}}\cdot \boldsymbol{\nabla} A^{-1}\varphi\big\rangle - \big\langle \bar\omega\, \bar{\boldsymbol{u}}\cdot \boldsymbol{\nabla} A^{-1}\varphi\big\rangle \\ &= \big\langle \varphi\, \hat{\boldsymbol{u}}\cdot \boldsymbol{\nabla} A^{-1}\varphi\big\rangle + \big\langle \bar\omega\, (\hat{\boldsymbol{u}}-\bar{\boldsymbol{u}})\cdot \boldsymbol{\nabla} A^{-1}\varphi\big\rangle\,. \end{aligned} \tag{59}$$

The second term on the right is estimated

$$\begin{aligned}\big|\big\langle \bar\omega\,(\hat{\boldsymbol{u}}-\bar{\boldsymbol{u}})\cdot\boldsymbol{\nabla}A^{-1}\varphi\big\rangle\big| &\le \|\bar\omega\|_{L^2}\,\|\hat{\boldsymbol{u}}-\bar{\boldsymbol{u}}\|_{L^\infty}\,\|\boldsymbol{\nabla}A^{-1}\varphi\|_{L^2} \\ &\le c_1\,\|q^{\mathrm{in}}\|_{L^2}\,\|\hat{\boldsymbol{u}}-\bar{\boldsymbol{u}}\|_{H^2}\,\|A^{-1}\varphi\|_{H^1} \\ &\le c_2\,\|\varphi\|_{H^{-1}}^2\,, \end{aligned} \tag{60}$$

where we used the global L^2 bound on $\bar\omega$ and the continuity of K and A^{-1}. The first term on the right of (59) is estimated by noting that for every $\boldsymbol{u} \in \mathbb{D}$ and $\psi \in C_0^\infty(\Omega)$,

$$\begin{aligned}\big\langle A(\boldsymbol{u}\cdot\boldsymbol{\nabla}\psi)\,\psi\big\rangle &= \big\langle \boldsymbol{\nabla}(\boldsymbol{u}\cdot\boldsymbol{\nabla}\psi)\cdot\boldsymbol{\nabla}\psi\big\rangle \\ &= \big\langle \boldsymbol{\nabla}\psi\cdot(\boldsymbol{\nabla}\boldsymbol{u})\cdot\boldsymbol{\nabla}\psi\big\rangle + \big\langle \boldsymbol{u}\cdot\boldsymbol{\nabla}(\boldsymbol{\nabla}\psi)\cdot\boldsymbol{\nabla}\psi\big\rangle\,. \end{aligned} \tag{61}$$

The second term on the right varnishes due to relation (29). Take $\psi = A^{-1}\varphi$ to obtain

$$\begin{aligned}\big|\big\langle \varphi\,\boldsymbol{u}\cdot\boldsymbol{\nabla}A^{-1}\varphi\big\rangle\big| = \big|\big\langle \psi\, A(\boldsymbol{u}\cdot\boldsymbol{\nabla}\psi)\big\rangle\big| &= \big|\big\langle \boldsymbol{\nabla}\psi\cdot(\boldsymbol{\nabla}\boldsymbol{u})\cdot\boldsymbol{\nabla}\psi\big\rangle\big| \\ &\le \|\boldsymbol{\nabla}\boldsymbol{u}\|_{L^\infty}\,\|\boldsymbol{\nabla}\psi\|_{L^2}^2 \\ &\le c\|\boldsymbol{u}\|_{H^3}\,\|\psi\|_{H^1}^2\,. \end{aligned} \tag{62}$$

This estimate can be extended by continuity to all $\boldsymbol{u} \in \mathbb{H}$, and all $\psi \in H_0^1(\Omega)$. Then

$$\big|\big\langle \varphi\,\hat{\boldsymbol{u}}\cdot\boldsymbol{\nabla}A^{-1}\varphi\big\rangle\big| \le c\|\hat{\boldsymbol{u}}\|_{H^3}\,\|A^{-1}\varphi\|_{H^1}^2 \le c_2\,\|\hat\omega\|_{L^2}\,\|\varphi\|_{H^{-1}}^2\,. \tag{63}$$

The last steps are valid because of the continuity of K and A^{-1}. Altogether, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi(t)\|_{H^{-1}}^2 \leq c_3 \, \|\varphi(t)\|_{H^{-1}}^2 \,, \tag{64}$$

where $c_3 = c_3(\|\hat\omega^{\rm in}\|_{L^2}, \|\bar\omega^{\rm in}\|_{L^2})$. This linear differential inequality can be integrated directly and yields

$$\|\varphi(t)\|_{H^{-1}} \leq \|\varphi(0)\|_{H^{-1}} \, \exp\bigl(c_3 \, t\bigr) \,. \tag{65}$$

Hence, the difference φ between the two solutions remains zero if it is so initially.

7 Continuous Dependence on the Data

Estimate (65) directly implies that the solution depends continuously on the data. Fix $T > 0$, and choose $\hat\omega^{\rm in}$ and $\bar\omega^{\rm in}$ from within a bounded open set $B \subset L^2$, so that for every $0 \leq t \leq T$,

$$\|\hat\omega(t) - \bar\omega(t)\|_{H^{-1}} \leq c_4(B,T) \, \|\hat\omega^{\rm in} - \bar\omega^{\rm in}\|_{H^{-1}} \,. \tag{66}$$

This completes the proof of Theorem 2.

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