ANALYTICITY OF SOLUTIONS FOR A GENERALIZED EULER EQUATION

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Abstract. We consider the so-called lake and great lake equations, which are shallow water equations that describe the long-time motion of an inviscid, incompressible fluid contained in a shallow basin with a slowly spatially varying bottom, a free upper surface and vertical side walls, under the influence of gravity and in the limit of small characteristic velocities and very small surface amplitude. If these equations are posed on a space-periodic domain and the initial data are real analytic, the solution remains real analytic for all times. The proof is based on a characterization of Gevrey classes in terms of decay of Fourier coefficients. In particular, our result recovers known results for the Euler equations in two and three spatial dimensions. We believe the proof is new.

1. Introduction

We investigate the analyticity of solutions to a class of shallow water equations that includes the so-called lake and great lake equations [12, 13] as well as the two-dimensional incompressible Euler equations. These equations describe the motion of an inviscid, incompressible fluid in a basin with a free upper surface in a regime where the aspect ratio δ (the ratio of typical horizontal to typical vertical length scales) is small, the Froude number ε (the ratio of typical horizontal speeds to the speed of gravity waves) is low, and the ratio of the surface amplitude to typical vertical length scales is of order ε^2. Moreover, the initial configuration to leading order in δ is required to be columnar, i.e. independent of the vertical coordinate.

The derivation of the shallow water equations from the three dimensional Euler equations employs a formal asymptotic expansion in ε and δ. Thus it is *a priori* not clear in what sense they approximate the full three dimensional flow, and such justification remains an open problem. A necessary first step is to find a setting in which both the great lake equations and the Euler equations on a domain with a free upper surface are well posed. The Euler problem is only known to be locally well-posed in a class of analytic functions [32, 33]. The purpose of this paper is to prove similar results for the great lake equations with the ultimate goal of justifying the small Froude number expansion. Partial results concerning the small aspect ratio expansion have been obtained in [28, 30].

For technical reasons that will become apparent later, we will pose the problem over $\boldsymbol{x} = (x_1, x_2)$ in a periodic horizontal domain. While such a domain is certainly unphysical, it will allow us to approach the question of validity of the asymptotic expansions. The evolution of the vertically averaged horizontal velocity field $\boldsymbol{u}(\boldsymbol{x}, t)$ is then governed to leading order in ε and first order in δ^2 by the set

Appeared in: J. Differential Equations **133** (1997), 321–339.

of nondimensional equations, the *great lake equations*

$$\partial_t \boldsymbol{v} - \boldsymbol{u}^{\perp}\, \boldsymbol{\nabla} \wedge \boldsymbol{v} + \boldsymbol{\nabla}(h - \tfrac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{v}) = 0\,, \tag{1a}$$

$$\boldsymbol{v} = \mathcal{L}\boldsymbol{u}\,, \tag{1b}$$

$$\boldsymbol{\nabla}\cdot(b\boldsymbol{u}) = 0\,, \tag{1c}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}^{\mathrm{in}}\,. \tag{1d}$$

We use the notation $\boldsymbol{u}^{\perp} = (u_2, -u_1)$ and $\boldsymbol{\nabla}\wedge\boldsymbol{v} = \partial_1 v_2 - \partial_2 v_1$; $h(\boldsymbol{x},t)$ is the surface height variation and $b(\boldsymbol{x}) \geq b_{\min} > 0$ the depth of the basin. The linear operator $\mathcal{L}$ in (1b) that relates the auxiliary field $\boldsymbol{v}$ to $\boldsymbol{u}$ is given by

$$\begin{aligned}\boldsymbol{v} = \mathcal{L}\boldsymbol{u} \equiv \boldsymbol{u} + \delta^2\big((\boldsymbol{u}\cdot\boldsymbol{\nabla} b)\boldsymbol{\nabla} b + \tfrac{1}{2} b(\boldsymbol{\nabla}\cdot\boldsymbol{u})\,\boldsymbol{\nabla} b \\ - \tfrac{1}{2} b^{-1}\boldsymbol{\nabla}(b^2\,\boldsymbol{u}\cdot\boldsymbol{\nabla} b) - \tfrac{1}{3} b^{-1}\boldsymbol{\nabla}(b^3\boldsymbol{\nabla}\cdot\boldsymbol{u})\big)\,.\end{aligned} \tag{2}$$

The leading order equations in δ are called the *lake equations.* It is easily verified that for a flat bottom both the lake and the great lake equations formally reduce to the two-dimensional Euler equations. In reference [12] it is shown that these shallow water models have a Hamiltonian structure, convect potential vorticity, and enjoy a Kelvin circulation theorem in complete analogy with the pure Euler case.

For a mathematical investigation, it is useful to note that the equations can be written in terms of an abstract initial value problem for the *potential vorticity* $\omega = b^{-1}\boldsymbol{\nabla}\wedge\boldsymbol{v}$, namely

$$\partial_t\omega + \boldsymbol{u}\cdot\boldsymbol{\nabla}\omega = 0\,, \tag{3a}$$

$$\boldsymbol{u} = K\omega\,, \tag{3b}$$

$$\omega(0) = \omega^{\mathrm{in}}\,. \tag{3c}$$

System (3) generalizes the two dimensional Euler equations in the following sense: Whereas $\omega = \boldsymbol{\nabla}\wedge\boldsymbol{u} \equiv \partial_1 u_2 - \partial_2 u_1$ for the Euler system, the relation between the velocity and vorticity fields is now prescribed by a linear operator K, which can be viewed as a perturbation of the inverse of the curl operator. The specifics of the model are all incorporated into K, which is analyzed in detail in Section 5.

By using the vorticity formulation, the great lake equations have been shown to be globally well-posed [25, 26, 28] and to possess global classical solutions as stated in the following theorem.

Theorem 1. *Assume that $b \in \mathrm{C}^{m+2}(\mathbb{T}^2)$ for some $m \geq 2$. Then for $\omega^{\mathrm{in}} \in \mathrm{H}^m(\mathbb{T}^2)$, there exists a unique global classical solution ω to the vorticity equation* (3) *such that*

$$\omega \in \bigcap_{k=0}^{m-2} \mathrm{C}^k([0,\infty); \mathrm{H}^{m-k}(\mathbb{T}^2))\,. \tag{4}$$

A proof for the case of a bounded domain can be found in [28, 29]. It applies to periodic domains with straightforward modifications.

In the following, we investigate the question of analyticity of solutions. Our method is based on the notion of Gevrey class regularity, which is a stronger concept than the C^∞ regularity which can be gained from Theorem 1. It not only asserts that all derivatives of the solution ω are bounded, but also that these bounds depend on the order of the derivatives in some prescribed way. Gevrey [18] used this notion as a setting in which to extend Cauchy-Kowalevski existence arguments to classes of functions that are not necessarily analytic (for a review of the analytic case see, e.g., [20]). In fact, they are special cases of the *quasianalytic* classes which had been introduced earlier by Hadamard [19]. La Vallée Poussin [23] showed that, among the quasianalytic functions, the Gevrey classes are characterized by an exponential decay of their Fourier coefficients (see also [22]). In turn, this characterization has recently proved useful for showing that the solutions of various nonlinear partial differential equations are analytic. Foias and Temam [17] developed this technique in the context of the two-dimensional incompressible Navier-Stokes equations, thereby simplifying earlier work of Kahane [21] that had been more in the style of Gevrey. Doelman and Titi [15] subsequently applied the technique to the cubic complex Ginzburg-Landau equation, and Ferrari and Titi [16] extended the results to a large class of analytic nonlinear parabolic equations. In particular, the authors of the last reference have exploited the fact that certain subclasses of a Gevrey class are normed algebras, which makes the estimation of polynomial nonlinearities a triviality. This strategy has been used, for example, to give a very compact proof of analyticity for the complex Ginzburg-Landau equation with a nonlinearity of arbitrary odd degree [24].

In Section 2, we will define Gevrey classes and prove the characterization theorem announced above. These methods will then be applied to our vorticity equation (3) in Section 3. The result is the following.

Theorem 2. *For real analytic ω^{in} and b there exists a unique global solution to the abstract vorticity equations* (3) *that is real analytic in both space and time.*

Details of the necessary technical estimates on the nonlinear term in (3) are given in Section 4, while details of the necessary regularity estimates for the linear operator K are given in Section 5.

A drawback of the Gevrey class approach to analyticity presented here is that it makes crucial use of the explicit representation of the solution ω in terms of its Fourier coefficients. As a consequence, we cannot treat the lake or great lake equations with prescribed lateral boundary conditions, and have to resort to a setting either on the whole space $\mathbb{R}^2$ or on the two-dimensional torus $\mathbb{T}^2$. Here, we restrict ourselves to the latter; the strategy however applies to the $\mathbb{R}^2$ case with only minor changes.

Lastly, we remark that our calculation readily yields a proof of analyticity for solutions of the Euler equations in any spatial dimension, for as long as a classical solution exists. Such a proof can be written in the velocity form of the equations; the adaptation requires only minor notational changes. Results of this type have first been derived by Bardos, Benachour and Zerner (see references [7, 8, 9, 10]) who use estimates on the Greens function of the Poisson kernel in compexified space to describe the region of analyticity. For results on the local propagation of analyticity see the works by Baouendi and Goulaouic [6], Alinhac and Metivier [4, 5], and references therein.

2. Preliminaries

A function $w \in \mathrm{C}^\infty$ is said to be of *Gevrey class* s for some $s > 0$ if there exist constants $\rho > 0$ and $M < \infty$ such that for every $x \in \mathbb{T}^d$ and every $\alpha \in \mathbb{N}^d$ one has

$$\left|\partial^\alpha w(x)\right| \le M \left(\frac{\alpha!}{\rho^{|\alpha|}}\right)^s . \tag{5}$$

Here we employ the usual multi-index notation in which

$$|\alpha| \equiv \sum_{j=1}^{d} \alpha_j \,, \qquad \alpha! \equiv \prod_{j=1}^{d} \alpha_j! \,, \qquad \partial^\alpha \equiv \prod_{j=1}^{d} \partial_j^{\alpha_j} \,. \tag{6}$$

The set of all functions of Gevrey class s is a vector space, denoted $\mathrm{G}^s(\mathbb{T}^d)$. It is closed under multiplication and differentiation. Moreover, the composition of two functions of Gevrey class s is again of class s.

It is classical that $\mathrm{G}^1(\mathbb{T}^d)$ is the space of real analytic functions $\mathrm{C}^\omega(\mathbb{T}^d)$; a proof can be found, for example, in John [20, page 65]. For $0 < s < 1$ the class $\mathrm{G}^s(\mathbb{T}^d)$ is a subclass of the analytic functions, while for $1 < s < \infty$ it contains the analytic functions. In fact, one has a hierarchy of spaces such that $0 < s_1 < s_2 < \infty$ implies the proper containments

$$\mathrm{G}^{s_1}(\mathbb{T}^d) \subset \mathrm{G}^{s_2}(\mathbb{T}^d) \subset \mathrm{C}^\infty(\mathbb{T}^d) \,. \tag{7}$$

Moreover, the union of the $\mathrm{G}^s(\mathbb{T}^d)$ does not exhaust $\mathrm{C}^\infty(\mathbb{T}^d)$ because there are quasianalytic functions that are not members of a Gevrey class [23].

In what follows it will be more convenient to characterize Gevrey classes in terms of the fractional Sobolev spaces $\mathrm{H}^r(\mathbb{T}^d)$ with $r \geq 0$, rather than uniform bounds such as (5).

To do so, we will use $\widehat{w}_{\boldsymbol{k}}$ to denote the Fourier coefficients of a function $w \in \mathrm{L}^2(\mathbb{T}^d)$, so that

$$w(x) = \sum_{\boldsymbol{k} \in 2\pi\mathbb{Z}^d} \widehat{w}_{\boldsymbol{k}} \, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \,, \qquad \widehat{w}_{\boldsymbol{k}} = \int_{\mathbb{T}^d} \mathrm{e}^{-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \, w(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \,. \tag{8}$$

The H^r norm of a function w can be defined as

$$\|w\|_{\mathrm{H}^r} \equiv \left(\sum_{\boldsymbol{k} \in 2\pi\mathbb{Z}^d} (1+k^2)^r \, |\widehat{w}_{\boldsymbol{k}}|^2 \right)^{\frac{1}{2}} \,, \tag{9}$$

where $k \equiv |\boldsymbol{k}|$. The space $\mathrm{H}^r(\mathbb{T}^d)$ is then the set of all $\mathrm{L}^2(\mathbb{T}^d)$ functions for which the norm (9) is finite.

Lemma 3. *Given $s > 0$ and $r \geq 0$. Then $w \in \mathrm{G}^s(\mathbb{T}^d)$ if and only if there are constants ρ, $M \in (0,\infty)$ that may depend on r, s and w such that for every $n \in \mathbb{N}$ one has*

$$\|\nabla^n w\|_{\mathrm{H}^r} = \left(\sum_{\boldsymbol{k} \in 2\pi\mathbb{Z}^d} (1+k^2)^r \, k^{2n} \, |\widehat{w}_{\boldsymbol{k}}|^2 \right)^{\frac{1}{2}} \leq M \left(\frac{n!}{\rho^n} \right)^s \,. \tag{10}$$

The proof is achieved by a direct application of the Sobolev embedding theorem, see Adams [2].

This lemma enables us to characterize functions in G^s in terms of the decay of their Fourier coefficients. A result of the type was first due to La Vallée Poussin [23, 27], although this attribution has been obscured in the recent literature. The construction here uses the operator $A = \sqrt{-\Delta}$ that, like $-\Delta$ itself, is nonnegative and self-adjoint so that arbitrary powers can be defined by spectral theory. For each $s \in (0,\infty)$ we define a family, parameterized by τ, of normed spaces

$$\mathcal{D}(\mathrm{e}^{\tau A^{1/s}} : \mathrm{H}^r(\mathbb{T}^d)) \equiv \left\{ w \in \mathrm{H}^r(\mathbb{T}^d) \,:\, \|\mathrm{e}^{\tau A^{1/s}} w\|_{\mathrm{H}^r} < \infty \right\}. \tag{11}$$

The functions in any such space have Fourier coefficients that decay faster than $\exp(-\tau k^{1/s})$. The next theorem recovers the Gevrey class $\mathrm{G}^s(\mathbb{T}^d)$ as the union of all such classes.

Theorem 4. *For any $s > 0$ and $r \geq 0$,*

$$\mathrm{G}^s(\mathbb{T}^d) = \bigcup_{\tau>0} \mathcal{D}(\mathrm{e}^{\tau A^{1/s}} : \mathrm{H}^r(\mathbb{T}^d)). \tag{12}$$

Remark 1. The use of the more general H^r rather than simply L^2 as the base space does not complicate the structure of the proof, but it is advantageous for the arguments that will follow.

Proof. Let $w \in \mathcal{D}(\mathrm{e}^{\tau A^{1/s}} : \mathrm{H}^r(\mathbb{T}^d))$ for some $\tau > 0$ and let $\rho = \tau/s$. Then

$$\begin{aligned} \|\nabla^n w\|_{\mathrm{H}^r}^2 &= \left(\frac{n!}{\rho^n}\right)^{2s} \sum_{\boldsymbol{k}} (1+k^2)^r \left(\frac{\rho^n\, k^{n/s}}{n!}\right)^{2s} |\widehat{w}_{\boldsymbol{k}}|^2 \\ &\leq \left(\frac{n!}{\rho^n}\right)^{2s} \sum_{\boldsymbol{k}} (1+k^2)^r\, \mathrm{e}^{2s\rho k^{1/s}}\, |\widehat{w}_{\boldsymbol{k}}|^2 \\ &= \left(\frac{n!}{\rho^n}\right)^{2s} \|\mathrm{e}^{\tau A^{1/s}} w\|_{\mathrm{H}^r}^2 . \end{aligned} \tag{13}$$

By setting $M = \|\mathrm{e}^{\tau A^{1/s}} w\|_{\mathrm{H}^r}$ we obtain (10), whereby $w \in \mathrm{G}^s(\mathbb{T}^d)$.

On the other hand, let $w \in \mathrm{G}^s(\mathbb{T}^d)$. For an arbitrary $\tau \geq 0$ one has

$$\begin{aligned} \|\mathrm{e}^{\tau A^{1/s}} w\|_{\mathrm{H}^r}^2 &\equiv \sum_{\boldsymbol{k}} (1+k^2)^r\, \mathrm{e}^{2\tau k^{1/s}}\, |\widehat{w}_{\boldsymbol{k}}|^2 \\ &= \sum_{m=0}^{\infty} \frac{(2\tau)^m}{m!} \sum_{\boldsymbol{k}} (1+k^2)^r\, k^{\frac{m}{s}}\, |\widehat{w}_{\boldsymbol{k}}|^2 . \end{aligned} \tag{14}$$

Now let ρ and M be such that (10) is satisfied. By interpolating (10) between $n = 0$ and any integer n such that $m/s \leq 2n$, the inner sum appearing in (14) can be bounded as

$$\sum_{\boldsymbol{k}} (1+k^2)^r\, k^{\frac{m}{s}}\, |\widehat{w}_{\boldsymbol{k}}|^2 \leq M^2 \frac{(n!)^{\frac{m}{n}}}{\rho^m}. \tag{15}$$

This bound is best if we choose $n = n_m \equiv [m/(2s)] + 1$, where $[\,\cdot\,]$ denotes the "greatest integer less than" function. Upon making this choice and applying the result in (14), one arrives at the bound

$$\|\mathrm{e}^{\tau A^{1/s}} w\|_{\mathrm{H}^r}^2 \leq \sum_{m=0}^{\infty} \frac{(2\tau)^m}{m!} M^2 \frac{(n_m!)^{\frac{m}{n_m}}}{\rho^m} = M^2 \sum_{m=0}^{\infty} \left(\frac{2\tau}{\rho}\right)^m \frac{(n_m!)^{\frac{m}{n_m}}}{m!}. \tag{16}$$

By making use of the Stirling formula [1] in the form

$$\lim_{n\to\infty} \frac{\mathrm{e}}{n} (n!)^{\frac{1}{n}} = 1, \tag{17}$$

the limit of the mth root of the mth term in the last series of (16) can be evaluated as

$$\lim_{m\to\infty} \frac{2\tau}{\rho} \frac{(n_m!)^{\frac{1}{n_m}}}{(m!)^{\frac{1}{m}}} = \frac{2\tau}{\rho} \lim_{m\to\infty} \frac{n_m}{m} = \frac{2\tau}{\rho} \frac{1}{2s} = \frac{\tau}{s\rho} \,. \tag{18}$$

Hence, by the Hadamard root test, the series in (16) converges for every $\tau < \rho s$, whence $w \in \mathcal{D}(\mathrm{e}^{\tau A^{1/s}} : \mathrm{H}^r(\mathbb{T}^d))$. $\square$

Remark 2. The above proof gives a sharp relationship between the τ of (11) and the ρ of (10). Had we been less careful in our choice of the n_m used in (16) then this relationship would have been missed.

Remark 3. The theorem also holds over $\mathbb{R}^d$. The proof can proceed in the same way with Fourier integrals in place of Fourier sums.

One reason Gevrey classes are useful in the context of nonlinear partial differential equations is that each G^s is closed under multiplication. It would be particularly useful if this property extended to each of the approximating (normed!) spaces $\mathcal{D}(\mathrm{e}^{\tau A^{1/s}} : \mathrm{H}^r(\mathbb{T}^d))$. An abstract statement of this fact is given in the following theorem, which we state for completeness. However, this result is not always the sharpest possible—in order to treat the vorticity system (3), we are forced to derive an estimate which is similar in spirit, but takes explicit advantage of the symmetry of the convective nonlinearity $\boldsymbol{u} \cdot \boldsymbol{\nabla}\omega$.

Theorem 5. *If $s \geq 1$, $\tau \geq 0$, and $r > d/2$ then $\mathcal{D}(\mathrm{e}^{\tau A^{1/s}} : \mathrm{H}^r(\mathbb{T}^d))$ is a Banach algebra. This means that it is closed under multiplication and that there exists a finite constant $C(r,d)$ such that any two functions v and w in $\mathcal{D}(\mathrm{e}^{\tau A^{1/s}} : \mathrm{H}^r(\mathbb{T}^d))$ satisfy the inequality*

$$\left\| \mathrm{e}^{\tau A^{1/s}}(vw) \right\|_{\mathrm{H}^r} \leq C(r,d) \left\| \mathrm{e}^{\tau A^{1/s}} v \right\|_{\mathrm{H}^r} \left\| \mathrm{e}^{\tau A^{1/s}} w \right\|_{\mathrm{H}^r} . \tag{19}$$

The proof is a direct extension of the usual proof that $H^r(\mathbb{T}^d)$ is a Banach algebra when $r > d/2$ [2], a result that is recovered above by setting $\tau = 0$. A proof of this theorem for the case when $s = 1$ is given in [16]; this proof is easily generalized for any $s > 1$, see [28].

The spaces $\mathcal{D}(\mathrm{e}^{\tau A^{1/s}} : \mathrm{H}^r(\mathbb{T}^d))$ are well-suited for application to parabolic partial differential equations. One can identify τ with time so that $\mathcal{D}(\mathrm{e}^{tA^{1/s}} : \mathrm{H}^r(\mathbb{T}^d))$ evolves from being identical to H^r at $t = 0$ to being a subset of Gevrey class s in an arbitrarily short time. So if one can show that for some $T > 0$ the solutions to the equation with H^r initial data are in $\mathcal{D}(\mathrm{e}^{tA^{1/s}} : \mathrm{H}^r(\mathbb{T}^d))$ for every $t \in (0, T]$ then one has proved Gevrey regularity of class s over that interval.

Moreover, as we will show in the next section, by choosing a different time dependence of τ, one can prove Gevrey class 1 regularity for hyperbolic equations in a very similar way.

3. The Main Result

Let us now proceed to show that the solution ω of the abstract vorticity equation (3) remains of Gevrey class 1 for all times, if it is so initially. In other words, we will show that real analytic solutions propagate analytically. To do this, we employ the time dependent seminorm $\|A^r \mathrm{e}^{\tau(t)A}\omega\|$ where $r > \frac{5}{2}$ is fixed. The L^2 norm and inner product on $\mathbb{T}^2$ are denoted $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ respectively. It is easy to verify that the L^2 norm of ω is conserved; therefore bounds on the seminorm $\|A^r e^{\tau A}\omega\|$ already imply bounds on the norm of $\mathcal{D}(e^{\tau A}: \mathrm{H}^r(\mathbb{T}^d))$.

The most crucial ingredient is the following estimate on the convective nonlinearity, which combines two more general results—Lemma 8 and Lemma 12—which we prove in the subsequent two sections.

Lemma 6. *For $\omega \in \mathcal{D}(A^{r+1}e^{\tau A})$ and $b \in \mathcal{D}(A^{r+\frac{3}{2}}e^{\tau A})$, $r > \frac{5}{2}$ and $\boldsymbol{u} = K\omega$,*

$$\left|\left\langle A^r e^{\tau A}(\boldsymbol{u}\cdot\boldsymbol{\nabla}\omega), A^r e^{\tau A}\omega\right\rangle\right| \leq c_1 \|A^r\omega\|^3 + \tau\, c_2 \|A^r e^{\tau A}\omega\| \, \|A^{r+\frac{1}{2}}e^{\tau A}\omega\|^2 , \tag{20}$$

where c_1 and c_2 do not depend on τ.

Given this estimate, and ω^{in}, b and $\tau(0)$ such that the assumptions of Lemma 6 are satisfied at $t=0$, then by direct calculation

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|A^r e^{\tau A}\omega\|^2 &= \dot\tau \left\langle A^{r+1}e^{\tau A}\omega, A^r e^{\tau A}\omega\right\rangle + \left\langle A^r e^{\tau A}\partial_t\omega, A^r e^{\tau A}\omega\right\rangle \\ &= \dot\tau\, \|A^{r+\frac{1}{2}}e^{\tau A}\omega\|^2 - \left\langle A^r e^{\tau A}(\boldsymbol{u}\cdot\boldsymbol{\nabla}\omega), A^r e^{\tau A}\omega\right\rangle \\ &\leq \left(\dot\tau + \tau\, c_2 \|A^r e^{\tau A}\omega\|\right)\|A^{r+\frac{1}{2}}e^{\tau A}\omega\|^2 + c_1 \|A^r\omega\|^3 .\end{aligned} \tag{21}$$

Theorem 1 guarantees the existence of global classical solutions. In particular, we know that there exists a function $\theta(t)$ such that $\|A^r\omega(t)\| \leq \theta(t)$ for all $t \in [0,\infty)$. Provided the first term on the right of (21) is negligible, we would have the global bound

$$\|A^r e^{\tau(t)A}\omega(t)\|^2 \leq h^2(t) \equiv \|A^r e^{\tau(0)A}\omega^{\mathrm{in}}\|^2 + c_1 \int_0^t \theta^3(t')\,\mathrm{d}t' . \tag{22}$$

A necessary condition for this to happen is

$$\dot\tau(t) \leq -c_2\, h(t)\, \tau(t) , \tag{23}$$

so that we can set

$$\tau(t) = \tau(0)\, \exp\left(-c_2 \int_0^t h(t')\,\mathrm{d}t'\right) . \tag{24}$$

It is easily verified that condition (23) is also sufficient. In summary, we have derived global *a priori* bounds on ω in the time dependent space $\mathcal{D}(A^r e^{\tau(t)A})$, which is a subclass of the real analytic functions for any $t \geq 0$. Moreover, for every analytic ω^{in} and b there exists a $\tau(0)$ such that an estimate of this form is true.

The rigorous construction of global analytic solutions is now standard. For example, one can first construct local analytic solutions in $\mathcal{D}(A^r e^{\tau(t)A})$ as the limit of a Fourier-Galerkin approximating sequence, and then globalize the result by estimate (22). This shows that

$$\omega \in \mathrm{C}^\infty([0,\infty); \mathrm{C}^\omega(\mathbb{T}^d)) .$$

Moreover, the Cauchy-Kowalevski theorem asserts local analyticity in space and time. For fixed $t = t_0$, the radius of the space-time ball of analyticity depends only on the radius of the ball of spatial analyticity [20]. It is uniform in the spatial variables because $\omega(t_0,\cdot)$ is in a Gevrey class, and uniform for all $t_0 \in [0,T]$ through estimate (22). This implies global analyticity in time as well as space, whereby we have established the following theorem.

Theorem 7. *For real analytic ω^{in} and b there exists a unique global analytic solution to the abstract vorticity equations* (3)*, i.e.*

$$\omega \in \mathrm{C}^\omega([0,\infty)\times\mathbb{T}^2)\,. \tag{25}$$

Moreover, for every time $t \geq 0$ equation (24) *states in which of the subspaces $\mathcal{D}(\mathrm{e}^{\tau(t)A}\colon \mathrm{H}^r(\mathbb{T}^2))$ of Gevrey class G^1 the solution $\omega(t)$ lies.*

Remark 4. The function $\tau(t)$ is the radius of the strip of analyticity in the complex plane around the real axis. With our current estimates $\tau(t)$ decays at least as fast as $\exp(-\exp(\exp t))$, which is a faster decay than the $\exp(-\exp t)$ given by Bardos and Benachour [9, 10]. However, we believe it will eventually be possible to achieve the same rate of decay with our Gevrey class approach.

In the subsequent two sections we will prove the estimates on the convective nonlinearity and the operator K which have been summarized as Lemma 6 above.

4. Nonlinear Estimates in $\mathcal{D}(A^r\mathrm{e}^{\tau A})$

To prove the nonlinear estimate, we use the Fourier representation, so that the derivation is *de facto* independent of the spatial dimension d. We therefore formulate the lemma as an abstract result for functions on $\mathbb{T}^d$, which can also be used to prove (local) analyticity for solutions of the Euler or Navier-Stokes equations on $\mathbb{T}^3$.

Lemma 8. *For $\boldsymbol{u} \in \mathcal{D}(A^{r+\frac12}\mathrm{e}^{\tau A})$ and $\phi \in \mathcal{D}(A^{r+1}\mathrm{e}^{\tau A})$, where $r > \frac{d}{2}+\frac{3}{2}$, $\boldsymbol{u}$ has mean zero on $\mathbb{T}^d$, $\boldsymbol{\nabla}\cdot(b\boldsymbol{u}) = 0$ and $b \in \mathrm{W}^{1,\infty}(\mathbb{T}^d)$, one has the estimate*

$$\begin{aligned}&\bigl|\bigl\langle A^r\mathrm{e}^{\tau A}(\boldsymbol{u}\cdot\boldsymbol{\nabla}\phi), A^r\mathrm{e}^{\tau A}\phi\bigr\rangle\bigr| \leq c_1\,\|A^r\boldsymbol{u}\|\,\|A^r\phi\|^2\\ &+\tau\,c_2\left(\|A^r\mathrm{e}^{\tau A}\boldsymbol{u}\|\,\|A^{r+\frac12}\mathrm{e}^{\tau A}\phi\|^2 + \|A^{r+\frac12}\mathrm{e}^{\tau A}\boldsymbol{u}\|\,\|A^r\mathrm{e}^{\tau A}\phi\|\,\|A^{r+\frac12}\mathrm{e}^{\tau A}\phi\|\right)\,,\end{aligned} \tag{26}$$

where c_1 and c_2 do not depend on τ and $\|\cdot\| \equiv \|\cdot\|_{\mathrm{L}^2}$.

Remark 5. The restriction to $\boldsymbol{u}$ having zero mean is not necessary in the case where b is a constant, or can otherwise be removed by replacing $\|A^r\boldsymbol{u}\|$ with the full norm $\|\boldsymbol{u}\|_{\mathrm{H}^r}$ in estimate (26).

It is helpful to first derive the following technical lemma.

Lemma 9. *Let $r \geq 1$ and $\tau \geq 0$. Then for all positive ξ, $\eta \in \mathbb{R}$,*

$$\bigl|\xi^r\mathrm{e}^{\tau\xi} - \eta^r\mathrm{e}^{\tau\eta}\bigr| \leq c(r)\,|\xi-\eta|\Bigl(\bigl(|\xi-\eta|^{r-1}+\eta^{r-1}\bigr) + \tau\bigl(|\xi-\eta|^r+\eta^r\bigr)\mathrm{e}^{\tau|\xi-\eta|}\mathrm{e}^{\tau\eta}\Bigr)\,. \tag{27}$$

Proof. Without loss of generality, take $\xi > \eta$. Set $f(\theta) \equiv \theta^r\mathrm{e}^{\tau\theta}$. By the mean value theorem,

$$\xi^r\mathrm{e}^{\tau\xi} - \eta^r\mathrm{e}^{\tau\eta} \leq (\xi-\eta)\sup_{\theta\in[\eta,\xi]}|f'(\theta)|\,. \tag{28}$$

Compute f' and use the fact that $\mathrm{e}^y \leq 1 + y\mathrm{e}^y$ for $y \geq 0$ to obtain

$$\begin{aligned}f'(\theta) &= r\,\theta^{r-1}\mathrm{e}^{\tau\theta} + \tau\,\theta^r\mathrm{e}^{\tau\theta}\\ &\leq r\,\theta^{r-1}(1+\tau\theta\,\mathrm{e}^{\tau\theta}) + \tau\,\theta^r\mathrm{e}^{\tau\theta}\\ &= r\,\theta^{r-1} + \tau(1+r)\,\theta^r\mathrm{e}^{\tau\theta}\,.\end{aligned} \tag{29}$$

Since $f'(\theta)$ is a monotonically growing function for $r \geq 1$, the supremum in (28) is attained when $\theta = \xi$. For arbitrary positive ξ and η this implies

$$\left|\xi^r \mathrm{e}^{\tau\xi} - \eta^r \mathrm{e}^{\tau\eta}\right| \leq |\xi - \eta| \Big(r\big(\xi^{r-1} + \eta^{r-1}\big) + \tau(1+r)\big(\xi^r \mathrm{e}^{\tau\xi} + \eta^r \mathrm{e}^{\tau\eta}\big) \Big) . \tag{30}$$

Since for all positive ξ and η

$$\xi^\rho \leq \begin{cases} |\xi - \eta|^\rho + \eta^\rho & \text{when } \rho \in [0,1] \\ 2^{\rho-1}\big(|\xi-\eta|^\rho + \eta^\rho\big) & \text{when } \rho > 1 \,, \end{cases} \tag{31}$$

and $\mathrm{e}^{\tau\xi} \leq \mathrm{e}^{\tau|\xi-\eta|}\mathrm{e}^{\tau\eta}$, estimate (30) directly implies the statement of the lemma. □

Proof of Lemma 8. We generalize the proof for the case $\tau = 0$ which is given in Constantin and Foias [14, Lemma 10.4]. First, note that $\boldsymbol{\nabla} \cdot (b\boldsymbol{u}) = 0$ implies that

$$\begin{aligned} 2\big\langle \boldsymbol{u} \cdot \boldsymbol{\nabla}(A^r \mathrm{e}^{\tau A}\phi), A^r \mathrm{e}^{\tau A}\phi \big\rangle &= -\big\langle (A^r \mathrm{e}^{\tau A}\phi)^2, \boldsymbol{\nabla} \cdot \boldsymbol{u} \big\rangle \\ &= \big\langle (A^r \mathrm{e}^{\tau A}\phi)^2, b^{-1}\boldsymbol{u} \cdot \boldsymbol{\nabla} b \big\rangle \\ &\leq c(b)\, \|\boldsymbol{u}\|_{\mathrm{L}^\infty}\, \|A^r \mathrm{e}^{\tau A}\phi\|^2 \\ &\leq c\, \|A^r \boldsymbol{u}\|\, \|A^r \mathrm{e}^{\tau A}\phi\|^2 \,, \end{aligned} \tag{32}$$

the last inequality holding on the condition that $r > \frac{d}{2}$ and $\boldsymbol{u}$ being a mean-zero function. This expression can be bounded in terms of quantities on the right side of (26); therefore it is sufficient to find a bound of the form (26) for the expression

$$\begin{aligned} \Xi \equiv& \big\langle A^r \mathrm{e}^{\tau A}(\boldsymbol{u} \cdot \boldsymbol{\nabla}\phi), A^r \mathrm{e}^{\tau A}\phi \big\rangle - \big\langle \boldsymbol{u} \cdot \boldsymbol{\nabla}(A^r \mathrm{e}^{\tau A}\phi), A^r \mathrm{e}^{\tau A}\phi \big\rangle \\ =& \big\langle \boldsymbol{u} \cdot \boldsymbol{\nabla}\phi, A^r \mathrm{e}^{\tau A}\psi \big\rangle - \big\langle \boldsymbol{u} \cdot \boldsymbol{\nabla}(A^r \mathrm{e}^{\tau A}\phi), \psi \big\rangle \,, \end{aligned} \tag{33}$$

where $\psi \equiv A^r \mathrm{e}^{\tau A}\phi$. We now use the Fourier representations of $\boldsymbol{u}$, ϕ and ψ, in which we can write

$$\begin{aligned} \boldsymbol{u}(\boldsymbol{x}) &= \sum_{\boldsymbol{j} \in 2\pi\mathbb{Z}^d} \widehat{\boldsymbol{u}}_{\boldsymbol{j}}\, \mathrm{e}^{\mathrm{i}\boldsymbol{j}\cdot\boldsymbol{x}} \,, \\ \boldsymbol{\nabla}\phi &= \sum_{\boldsymbol{k} \in 2\pi\mathbb{Z}^d} \mathrm{i}\boldsymbol{k}\, \widehat{\phi}_{\boldsymbol{k}}\, \mathrm{e}^{\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}} \,, \\ A^r \mathrm{e}^{\tau A}\psi &= \sum_{\boldsymbol{l} \in 2\pi\mathbb{Z}^d} l^r \mathrm{e}^{\tau l}\, \widehat{\psi}_{\boldsymbol{l}} \mathrm{e}^{\mathrm{i}\boldsymbol{l}\cdot\boldsymbol{x}} \,, \end{aligned}$$

where $l \equiv |\boldsymbol{l}|$. Hence, the first term on the right of (33) can be written in terms of Fourier sums as

$$\begin{aligned} \big\langle \boldsymbol{u} \cdot \boldsymbol{\nabla}\phi, A^r \mathrm{e}^{\tau A}\psi \big\rangle &= \mathrm{i} \sum_{\boldsymbol{n}} \sum_{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=\boldsymbol{n}} \big\langle \widehat{\boldsymbol{u}}_{\boldsymbol{j}} \cdot \boldsymbol{k}\, \widehat{\phi}_{\boldsymbol{k}}, l^r \mathrm{e}^{\tau l}\, \widehat{\psi}_{\boldsymbol{l}}\, \mathrm{e}^{\mathrm{i}(\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l})\cdot\boldsymbol{x}} \big\rangle \\ &= \mathrm{i}\, (2\pi)^d \sum_{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0} \widehat{\boldsymbol{u}}_{\boldsymbol{j}} \cdot \boldsymbol{k}\, \widehat{\phi}_{\boldsymbol{k}}\, l^r \mathrm{e}^{\tau l}\, \widehat{\psi}_{\boldsymbol{l}} \,. \end{aligned} \tag{34}$$

In the second equality we have used the orthogonality relation for exponential functions on $\mathbb{T}^d$. Similarly, the second term on the right of (33) has the Fourier representation

$$\big\langle \boldsymbol{u} \cdot \boldsymbol{\nabla}(A^r \mathrm{e}^{\tau A}\phi), \psi \big\rangle = \mathrm{i}\, (2\pi)^d \sum_{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0} \widehat{\boldsymbol{u}}_{\boldsymbol{j}} \cdot \boldsymbol{k}\, k^r \mathrm{e}^{\tau k}\, \widehat{\phi}_{\boldsymbol{k}}\, \widehat{\psi}_{\boldsymbol{l}} \,. \tag{35}$$

By subtracting (35) from (34) and taking the absolute value, we obtain a bound on Ξ. Note in particular that $\widehat{\psi}_0 = 0$ by the definition of ψ, and that the contributions to Ξ from $\boldsymbol{k} = 0$ and $\boldsymbol{l} = 0$ cancel out. We have

$$\begin{aligned}\Xi &\le (2\pi)^d \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ \boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\neq 0}} \left| \widehat{\boldsymbol{u}}_{\boldsymbol{j}} \cdot \boldsymbol{k}\, \widehat{\phi}_{\boldsymbol{k}}\, l^r \mathrm{e}^{\tau l}\, \widehat{\psi}_{\boldsymbol{l}} - \widehat{\boldsymbol{u}}_{\boldsymbol{j}} \cdot \boldsymbol{k}\, k^r \mathrm{e}^{\tau k}\, \widehat{\phi}_{\boldsymbol{k}}\, \widehat{\psi}_{\boldsymbol{l}} \right| \\ &= (2\pi)^d \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ \boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\neq 0}} |\widehat{\boldsymbol{u}}_{\boldsymbol{j}}|\, |\widehat{\phi}_{\boldsymbol{k}}|\, |\widehat{\psi}_{\boldsymbol{l}}|\, k \left| l^r \mathrm{e}^{\tau l} - k^r \mathrm{e}^{\tau k} \right| .\end{aligned}$$

Now apply Lemma 9 with $\xi = l$ and $\eta = k$, so that $|\xi - \eta| = |l - k| \le |\boldsymbol{l} - \boldsymbol{k}| = j$, to obtain

$$\Xi \le c_1(r) \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ \boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\neq 0}} |\widehat{\boldsymbol{u}}_{\boldsymbol{j}}|\, |\widehat{\phi}_{\boldsymbol{k}}|\, |\widehat{\psi}_{\boldsymbol{l}}|\, k\, j \Big(\big(j^{r-1} + k^{r-1}\big) + \tau \big(j^r + k^r\big) \mathrm{e}^{\tau j} \mathrm{e}^{\tau k} \Big) . \tag{36}$$

By the definition of ψ,

$$|\widehat{\psi}_{\boldsymbol{l}}| = l^r \mathrm{e}^{\tau l}\, |\widehat{\phi}_{\boldsymbol{l}}| \le l^r (1 + \tau l\, \mathrm{e}^{\tau l})\, |\widehat{\phi}_{\boldsymbol{l}}| \le l^r\, |\widehat{\phi}_{\boldsymbol{l}}| + \tau (j + k)\, |\widehat{\psi}_{\boldsymbol{l}}| , \tag{37}$$

whence we can completely separate (36) into a sum with weights of the form k^r, and a sum with weights of the form $\tau k^{r+1} \mathrm{e}^{\tau k}$. More precisely, we have

$$\begin{aligned}\Xi &\le c_2(r) \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ \boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\neq 0}} |\widehat{\boldsymbol{u}}_{\boldsymbol{j}}|\, |\widehat{\phi}_{\boldsymbol{k}}|\, |\widehat{\phi}_{\boldsymbol{l}}|\, k\, j\, l^r \big(j^{r-1} + k^{r-1}\big) \\ &\quad + \tau\, c_3(r) \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ \boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\neq 0}} |\widehat{\boldsymbol{u}}_{\boldsymbol{j}}|\, |\widehat{\phi}_{\boldsymbol{k}}|\, |\widehat{\psi}_{\boldsymbol{l}}|\, k\, j \big(j^r + k^r\big) \mathrm{e}^{\tau j} \mathrm{e}^{\tau k} .\end{aligned} \tag{38}$$

Let us first consider the sum

$$\begin{aligned}\sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ \boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\neq 0}} |\widehat{\boldsymbol{u}}_{\boldsymbol{j}}|\, |\widehat{\phi}_{\boldsymbol{k}}|\, |\widehat{\phi}_{\boldsymbol{l}}|\, k\, j^r\, l^r &= \sum_{\substack{\boldsymbol{k}\in 2\pi\mathbb{Z}^d\\ \boldsymbol{k}\neq 0}} k\, |\widehat{\phi}_{\boldsymbol{k}}| \sum_{\substack{\boldsymbol{j}\in 2\pi\mathbb{Z}^d\\ \boldsymbol{j}\neq 0,-\boldsymbol{k}}} j^r\, |\widehat{\boldsymbol{u}}_{\boldsymbol{j}}|\, |\boldsymbol{j} + \boldsymbol{k}|^r\, |\widehat{\phi}_{-\boldsymbol{j}-\boldsymbol{k}}| \\ &\le \sum_{\substack{\boldsymbol{k}\in 2\pi\mathbb{Z}^d\\ \boldsymbol{k}\neq 0}} k\, |\widehat{\phi}_{\boldsymbol{k}}|\, \|A^r \boldsymbol{u}\|\, \|A^r \phi\| .\end{aligned} \tag{39}$$

The inequality on the second line is due to the Cauchy-Schwarz inequality. Moreover, if $r > \frac{d}{2} + 1$, we can apply the Cauchy-Schwarz inequality a second time as follows:

$$\begin{aligned}\sum_{\substack{\boldsymbol{k}\in 2\pi\mathbb{Z}^d\\ \boldsymbol{k}\neq 0}} k\, |\widehat{\phi}_{\boldsymbol{k}}| &= \sum_{\substack{\boldsymbol{k}\in 2\pi\mathbb{Z}^d\\ \boldsymbol{k}\neq 0}} k^{1-r}\, k^r\, |\widehat{\phi}_{\boldsymbol{k}}| \\ &\le \Bigg(\sum_{\substack{\boldsymbol{k}\in 2\pi\mathbb{Z}^d\\ \boldsymbol{k}\neq 0}} k^{2-2r} \Bigg)^{\frac{1}{2}} \Bigg(\sum_{\substack{\boldsymbol{k}\in 2\pi\mathbb{Z}^d\\ \boldsymbol{k}\neq 0}} k^{2r}\, |\widehat{\phi}_{\boldsymbol{k}}|^2 \Bigg)^{\frac{1}{2}} \\ &\le c_4(r)\, \|A^r \phi\| .\end{aligned} \tag{40}$$

Similarly,

$$\sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ \boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\neq 0}} |\widehat{\boldsymbol{u}}_{\boldsymbol{j}}|\,|\widehat{\phi}_{\boldsymbol{k}}|\,|\widehat{\phi}_{\boldsymbol{l}}|\,j\,k^r\,l^r \le c_4(r)\,\|A^r\boldsymbol{u}\|\,\|A^r\phi\|^2\,. \tag{41}$$

Next, we consider a sum from the second line of (38). To assure that the highest power on j, k and l is $r+\frac{1}{2}$, we use $j \le k+l \le kl$ (note that $k,l \ge 2$), so that

$$\begin{aligned}
\sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ \boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\neq 0}} |\widehat{\boldsymbol{u}}_{\boldsymbol{j}}|\,|\widehat{\phi}_{\boldsymbol{k}}|\,|\widehat{\psi}_{\boldsymbol{l}}|\,k\,j^{r+1}\,\mathrm{e}^{\tau j}\mathrm{e}^{\tau k} &\le \sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ \boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\neq 0}} |\widehat{\boldsymbol{u}}_{\boldsymbol{j}}|\,|\widehat{\phi}_{\boldsymbol{k}}|\,|\widehat{\psi}_{\boldsymbol{l}}|\,k^{\frac{3}{2}}\,l^{\frac{1}{2}}\,j^{r+\frac{1}{2}}\,\mathrm{e}^{\tau j}\mathrm{e}^{\tau k} \\
&= \sum_{\substack{\boldsymbol{k}\in 2\pi\mathbb{Z}^d\\ \boldsymbol{k}\neq 0}} k^{\frac{3}{2}}\,\mathrm{e}^{\tau k}\,|\widehat{\phi}_{\boldsymbol{k}}| \sum_{\substack{\boldsymbol{j}\in 2\pi\mathbb{Z}^d\\ \boldsymbol{j}\neq 0,-\boldsymbol{k}}} j^{r+\frac{1}{2}}\mathrm{e}^{\tau j}\,|\widehat{\boldsymbol{u}}_{\boldsymbol{j}}|\,|\boldsymbol{j}+\boldsymbol{k}|^{\frac{1}{2}}\,|\widehat{\psi}_{-\boldsymbol{j}-\boldsymbol{k}}| \\
&\le \sum_{\substack{\boldsymbol{k}\in 2\pi\mathbb{Z}^d\\ \boldsymbol{k}\neq 0}} k^{\frac{3}{2}}\,\mathrm{e}^{\tau k}\,|\widehat{\phi}_{\boldsymbol{k}}|\,\|A^{r+\frac{1}{2}}\mathrm{e}^{\tau A}\boldsymbol{u}\|\,\|A^{\frac{1}{2}}\psi\|\,.
\end{aligned} \tag{42}$$

The remaining sum on the right of this expression is estimated as in (40). If $r > \frac{d}{2}+\frac{3}{2}$,

$$\sum_{\substack{\boldsymbol{k}\in 2\pi\mathbb{Z}^d\\ \boldsymbol{k}\neq 0}} k^{\frac{3}{2}}\,\mathrm{e}^{\tau k}\,|\widehat{\phi}_{\boldsymbol{k}}| \le \Bigg(\sum_{\substack{\boldsymbol{k}\in 2\pi\mathbb{Z}^d\\ \boldsymbol{k}\neq 0}} k^{3-2r}\Bigg)^{\frac{1}{2}} \Bigg(\sum_{\substack{\boldsymbol{k}\in 2\pi\mathbb{Z}^d\\ \boldsymbol{k}\neq 0}} k^{2r}\,\mathrm{e}^{2\tau k}\,|\widehat{\phi}_{\boldsymbol{k}}|^2\Bigg)^{\frac{1}{2}} \le c_5(r)\,\|A^r\mathrm{e}^{\tau A}\phi\|\,. \tag{43}$$

Similarly,

$$\sum_{\substack{\boldsymbol{j}+\boldsymbol{k}+\boldsymbol{l}=0\\ \boldsymbol{j},\boldsymbol{k},\boldsymbol{l}\neq 0}} |\widehat{\boldsymbol{u}}_{\boldsymbol{j}}|\,|\widehat{\phi}_{\boldsymbol{k}}|\,|\widehat{\psi}_{\boldsymbol{l}}|\,j\,k^{r+1}\,\mathrm{e}^{\tau j}\mathrm{e}^{\tau k} \le c_5(r)\,\|A^r\mathrm{e}^{\tau A}\boldsymbol{u}\|\,\|A^{r+\frac{1}{2}}\mathrm{e}^{\tau A}\phi\|^2\,. \tag{44}$$

By inserting estimates (39) to (44) into (38), one obtains the bound (26). □

5. Regularity Estimates for K

We now prove the necessary regularity properties of K. It is defined via the *stream function* ψ, which is the mean-zero solution of the elliptic problem

$$\omega = b^{-1}\,\boldsymbol{\nabla}\wedge\mathcal{L}(b^{-1}\,\boldsymbol{\nabla}^\perp\psi)\,, \tag{45}$$

where $\mathcal{L}$ is given by (2) and $\boldsymbol{\nabla}^\perp = (\partial_2, -\partial_1)$. Then $\boldsymbol{u} \equiv K\omega$ is given by

$$\boldsymbol{u} = b^{-1}\,\boldsymbol{\nabla}^\perp\psi\,. \tag{46}$$

It is easy to verify that $\boldsymbol{u}$ satisfies the weighted incompressibility condition

$$\boldsymbol{\nabla}\cdot(b\boldsymbol{u}) = 0. \tag{47}$$

Let $\mathbb{H}$ denote the subspace of $\mathrm{L}^2(\mathbb{T}^2,\mathbb{R}^2)$ functions that satisfy (47), and let P denote the orthogonal projector onto $\mathbb{H}$ with respect to a b-weighted L^2 scalar product. We quote two lemmas which are expressions of the invertibility of $\mathcal{L}$ and the ellipticity of (45). Their proofs follow those given in [25] of the corresponding results for bounded domains.

Lemma 10. *For $b \in \mathrm{C}^1(\mathbb{T}^2)$, the restriction of $\mathcal{L}$ to the space $\mathbb{H}$ is a continuous, invertible, positive and self-adjoint operator, explicitly given by*

$$P\mathcal{L}P\boldsymbol{u} = P\left(\boldsymbol{u} + \frac{\delta^2}{3}\,\boldsymbol{\nabla} b(\boldsymbol{\nabla} b \cdot P\boldsymbol{u})\right), \tag{48}$$

for every $\boldsymbol{u} \in \mathrm{L}^2(\mathbb{T}^2, \mathbb{R}^2)$.

Remark 6. This lemma is useful because the null space of the projector P is contained in the nullspace of the curl operator, and we therefore can write (45) as

$$\omega = b^{-1}\,\boldsymbol{\nabla} \wedge P\mathcal{L}P(b^{-1}\,\boldsymbol{\nabla}^\perp\psi)\,. \tag{49}$$

This can be seen, for example, by noting that

$$P(b^{-1}\,\boldsymbol{\nabla}^\perp\psi) = b^{-1}\,\boldsymbol{\nabla}^\perp\psi\,. \tag{50}$$

The adjoint relation then is

$$b^{-1}\,\boldsymbol{\nabla} \wedge P\boldsymbol{v} = b^{-1}\,\boldsymbol{\nabla} \wedge \boldsymbol{v}\,. \tag{51}$$

Lemma 11. *For $b \in \mathrm{C}^3(\mathbb{T}^2)$ and every $\omega \in \mathrm{H}^{-1}(\mathbb{T}^2)$ there exists a unique function $\boldsymbol{u} = K\omega \in \mathbb{H}$. Moreover, K is continuous as a mapping between the spaces $\mathrm{H}^{-1}(\mathbb{T}^2) \to \mathbb{H}$, $\mathrm{L}^2(\mathbb{T}^2) \to \mathbb{H} \cap \mathrm{H}^1(\mathbb{T}^2, \mathbb{R}^2)$ and $\mathrm{H}^1(\mathbb{T}^2) \to \mathbb{H} \cap \mathrm{H}^2(\mathbb{T}^2, \mathbb{R}^2)$, and for some $p_0 > 1$ satisfies the estimate*

$$\|K\omega\|_{\mathrm{W}^{1,p}} \le c\,p\,\|\omega\|_{\mathrm{L}^p}\,, \tag{52}$$

for all $p \ge p_0$ where the constant c depends only on p_0.

To prove analyticity of the solution to (3), we need to strengthen the statement of Lemma 11.

Lemma 12. *For $b \in \mathcal{D}(A^{r+\frac{3}{2}}\mathrm{e}^{\tau A})$ and $r > \frac{1}{2}$, the operator K is continuous from $\mathcal{D}(A^r\mathrm{e}^{\tau A})$ into $\mathcal{D}(A^{r+1}\mathrm{e}^{\tau A})$.*

The proof uses the following technical lemma.

Lemma 13. *For $\phi \in \mathcal{D}(A^{r+\frac{1}{2}}\mathrm{e}^{\tau A})$ and every $\varepsilon > 0$ one has the estimate*

$$\|A^r\mathrm{e}^{\tau A}\phi\|^2 \le \varepsilon\,\|A^{r+\frac{1}{2}}\mathrm{e}^{\tau A}\phi\|^2 + \varepsilon^{-2r}\mathrm{e}^{\frac{2\tau}{\varepsilon}}\,\|\phi\|^2\,. \tag{53}$$

Proof. Take the interpolation inequality

$$\|A^{r+\frac{i}{2}}\phi\| \le \|A^{r+\frac{1}{2}+\frac{i}{2}}\phi\|^\theta\,\|\phi\|^{1-\theta} \tag{54}$$

where $\theta = \frac{2r+i}{2r+1+i}$, square both sides, and apply the Young inequality on the right. Thus, for every $\varepsilon > 0$, one has

$$\begin{aligned} \|A^{r+\frac{i}{2}}\phi\|^2 &\le \varepsilon\,\|A^{r+\frac{1}{2}+\frac{i}{2}}\phi\|^2 + (1-\theta)\left(\frac{\theta}{\varepsilon}\right)^{\frac{\theta}{1-\theta}}\,\|\phi\|^2 \\ &\le \varepsilon\,\|A^{r+\frac{1}{2}+\frac{i}{2}}\phi\|^2 + \varepsilon^{-(2r+i)}\,\|\phi\|^2\,. \end{aligned} \tag{55}$$

Finally, multiply this inequality by $(2\tau)^i/i!$ and sum over i:

$$\sum_{i=0}^\infty \frac{(2\tau)^i}{i!}\|A^{r+\frac{i}{2}}\phi\|^2 \le \varepsilon\,\sum_{i=0}^\infty \frac{(2\tau)^i}{i!}\|A^{r+\frac{1}{2}+\frac{i}{2}}\phi\|^2 + \varepsilon^{-2r}\mathrm{e}^{\frac{2\tau}{\varepsilon}}\,\|\phi\|^2\,. \tag{56}$$

The Parseval identity then shows that this expression is equivalent to (53). $\square$

Proof of Lemma 12. We start with proving the lemma for K associated with the lake equations only. The key step is to show the invertibility of the operator $L\colon \mathcal{D}(A^{r+2}\mathrm{e}^{\tau A}) \to \mathcal{D}(A^{r+2}\mathrm{e}^{\tau A})'$, defined through

$$L\psi \equiv A^{2r+2}\mathrm{e}^{2\tau A}\big(b\boldsymbol{\nabla}\wedge(b^{-1}\boldsymbol{\nabla}^{\perp}\psi)\big) + \lambda\,\psi\,, \tag{57}$$

for some $\lambda > 0$. Suppose this is true, then the elliptic problem for the lake equations,

$$\omega = b^{-1}\boldsymbol{\nabla}\wedge(b^{-1}\boldsymbol{\nabla}^{\perp}\psi) \tag{58}$$

is equivalent to

$$L\psi = A^{2r+2}\mathrm{e}^{2\tau A}(b^2\omega) + \lambda\,\psi\,. \tag{59}$$

If we represent the duality product of $\mathcal{D}(A^{r+2}\mathrm{e}^{\tau A})$ through the L^2 inner product, it is clear that

$$\mathcal{D}(A^{r+2}\mathrm{e}^{\tau A})' = \big\{\text{the completion of } \mathrm{L}^2 \text{ in the norm } \|A^{-r-2}\mathrm{e}^{\tau A^{-1}}\cdot\|\big\}\,,$$

so that the right side of (57) is bounded in $\mathcal{D}(A^{r+2}\mathrm{e}^{\tau A})'$. More specifically, we can estimate the first term on the right of (57) by using the algebra properties of the norm, and the assumption $b,\omega \in \mathcal{D}(A^{r}\mathrm{e}^{\tau A})$. Moreover, $\psi \in \mathrm{L}^2 \hookrightarrow \mathcal{D}(A^{r+2}\mathrm{e}^{\tau A})'$ is bounded *a priori* in terms of the L^2 norm of ω, for example. Thus the invertibility of L implies continuity of K as stated in the lemma.

To prove invertibility of L, it is sufficient to show that the form $\langle L\cdot,\cdot\rangle$ is continuous and coercive in the norm of $\mathcal{D}(A^{r+2}\mathrm{e}^{\tau A})$. We prove coercivity only, as this is the less trivial property. Assume first that ψ is an entire function and rewrite the form, using the self-adjointness of A and the product rule as follows:

$$\begin{aligned}\langle L\psi,\psi\rangle &= \big\langle A^{r+1}\mathrm{e}^{\tau A}(-\Delta\psi - b\boldsymbol{\nabla}b^{-1}\cdot\boldsymbol{\nabla}\psi), A^{r+1}\mathrm{e}^{\tau A}\psi\big\rangle + \lambda\,\|\psi\|^2 \\ &= \|A^{r+2}\mathrm{e}^{\tau A}\psi\|^2 + \lambda\,\|\psi\|^2 - \big\langle A^{r+\frac12}\mathrm{e}^{\tau A}(b\boldsymbol{\nabla}b^{-1}\cdot\boldsymbol{\nabla}\psi), A^{r+\frac32}\mathrm{e}^{\tau A}\psi\big\rangle\,. \end{aligned}\tag{60}$$

If we can show that the last term on the right is bounded by a fraction of the first two terms, coercivity (and continuity) follows. We use the Cauchy-Schwarz inequality, the fact that $\mathcal{D}(A^{r+1/2}\mathrm{e}^{\tau A})$ is an algebra, and Lemma 13 to find that

$$\begin{aligned}\big\langle A^{r+\frac12}\mathrm{e}^{\tau A}(b\boldsymbol{\nabla}b^{-1}\cdot\boldsymbol{\nabla}\psi), A^{r+\frac32}\mathrm{e}^{\tau A}\psi\big\rangle &\le \|A^{r+\frac12}\mathrm{e}^{\tau A}(b\boldsymbol{\nabla}b^{-1})\|\,\|A^{r+\frac32}\mathrm{e}^{\tau A}\psi\|^2 \\ &\le \tfrac12\,\|A^{r+2}\mathrm{e}^{\tau A}\psi\|^2 + c(b)\,\|\psi\|^2\,.\end{aligned}\tag{61}$$

This estimate can be extended by density to all $\psi \in \mathcal{D}(A^{r+2}\mathrm{e}^{\tau A})$. Hence, for a fixed $\lambda > c(b)$, the form $\langle L\cdot,\cdot\rangle$ is continuous and coercive, i.e. $L\colon \mathcal{D}(A^{r+2}\mathrm{e}^{\tau A}) \to \mathcal{D}(A^{r+2}\mathrm{e}^{\tau A})'$ is an isomorphism by the Lax-Milgram theorem.

The great lake equations can be treated in exactly the same way by using the form of $\mathcal{L}$ given in Lemma 10. This only introduces further terms that depend on b into estimate (61), but leaves the leading order contribution unchanged. □

Acknowledgments

We thank Darryl Holm, Simon Malham, and Edriss Titi for interesting discussions. C.D.L. acknowledges partial support through NSF Grant DMS-9404570; M.O. acknowledges partial support through Department of Energy Grant DE-FG03-94ER25174 and the kind hospitality of the CNLS at Los Alamos National Laboratory during the finalization of the manuscript.

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