CLASSICAL SOLUTIONS FOR A GENERALIZED EULER EQUATION IN TWO DIMENSIONS

MARCEL OLIVER

ABSTRACT. It is well known that the Euler equations in two spatial dimensions have global classical solutions. We provide a new proof which is analytic rather than geometric. It is set in an abstract framework that applies to the so-called lake and the great lake equations describing weakly non-hydrostatic effects of bottom topography on the motion of shallow water. The key ingredient is a new L^p estimate on the nonlinear term. The estimate is used to develop a global H^m theory for bounded domains in $\mathbb{R}^2$ which is similar in spirit to a 1975 paper by R. Temam. It also provides explicit bounds on the H^m norm which grow like $\exp(\exp t)$.

1. INTRODUCTION

The Euler equations on a two-dimensional domain Ω have global classical solutions. For example, it is well known that if the vorticity ω is contained in the Sobolev space H^m for $m \geq 2$ initially, it will remain therein for all times, and its H^m norm satisfies a bound that grows like $\exp(\exp t)$. However, the usual proofs do not easily extend to more general systems as arise, for exmple, in the study of shallow water when bottom topography effects are included.

The purpose of this paper is to give a short new proof which applies to the two-dimensional Euler equations as well as to the shallow water models in question. It is based on the observation that these equations have a common structure, namely the abstract vorticity formulation

$$\partial_t \omega + \boldsymbol{u} \cdot \boldsymbol{\nabla} \omega = 0\,, \tag{1a}$$

$$\boldsymbol{u} = K\omega\,, \tag{1b}$$

$$\omega(0) = \omega^{\text{in}}\,. \tag{1c}$$

We use $\boldsymbol{u}$ and ω to denote the velocity and vorticity fields, respectively. For the two-dimensional Euler equations, K is the appropriately defined inverse of the curl operator. However, for our purposes it is sufficient to presume K is linear, satisfies a divergence condition and has certain regularity properties which will be detailed in Section 2. In this setting it is natural to think in terms of elliptic L^p theory, so that classical L^p regularity results can be used to considerably simplify the argument.

The model equations which motivated the present work are called the *lake equations* and the *great lake equations*. They describe the motion of an inviscid, incompressible fluid in a basin with a free upper surface in a regime where the aspect ratio δ (the ratio of typical horizontal to typical vertical length scales) is small,

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the Froude number ε (the ratio of typical horizontal speeds to the speed of gravity waves) is low, and the ratio of the surface amplitude to typical vertical length scales is of order ε^2. Moreover, the initial configuration of the fluid is assumed to be columnar (meaning independent of the vertical coordinate) to leading order in δ. The lake and great lake equations have been derived by Camassa, Holm, and Levermore [8, 9] and are shown to be globally well-posed in [19, 20].

The theory of classical solutions presented in this paper provides the necessary foundation for two recent results: Solutions to (1) posed on the two-dimensional torus $\mathbb{T}^2$ remain real analytic if they are so initially [18]. In the case of the lake equations, they also remain H^m-close to solutions of the three-dimensional Euler equations with a rigid lid upper boundary for an interval of time that can be estimated in terms of their initial closeness [23, 24]. The latter result provides a rigorous justification of the small aspect ratio limit in the derivation of the lake equations.

The history of results concerning existence of global classical solutions to the Euler equations in $\mathbb{R}^2$ goes back to Wolibner [30], Hölder [14], Schaeffer [25] and McGrath [21], while the case of bounded two-dimensional domains has been treated by Kato [15], Bardos and Frisch [4], and—more abstractly—by Ebin and Marsden [10]. These results are obtained in spaces of continuous functions or Hölder spaces, use the Lagrangian flow formulation and rely on a so-called quasi-Lipschitz condition on K, i.e. an estimate of the form

$$\left|K\omega(\boldsymbol{x}) - K\omega(\boldsymbol{y})\right| \leq c \, \|\omega\|_{\mathrm{L}^\infty} \, |\boldsymbol{x}-\boldsymbol{y}| \left(1 + \ln_+ \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\right) , \tag{2}$$

where $\ln_+ a = \max\{0, \ln a\}$. For a good exposition, see the book by Marchioro and Pulvirenti [22].

Complementary to these—implicitly or explicitly—geometrical techniques are Sobolev space methods that have first been used by Golovkin [13]. Swann [26] and Kato [16] developed an H^m theory for local classical solutions of the Euler equations in $\mathbb{R}^3$, and Temam [27, 28] obtained the corresponding result for bounded domains in three spatial dimensions (see [29] for supplements and corrections in conjunction with these two references). The three dimensional estimates are relatively simple, because the spaces $\mathrm{H}^s(\mathbb{R}^3)$ for $s > \frac{3}{2}$ are Banach algebras with respect to pointwise multiplication, but they are too crude to obtain global control even in two dimensions. The necessary refinement is commonly written in the form

$$\|\boldsymbol{\nabla u}\|_{\mathrm{L}^\infty} \leq c \, \|\omega\|_{\mathrm{L}^\infty} \left(1 + \ln_+ \|\boldsymbol{u}\|_{\mathrm{H}^3}\right) . \tag{3}$$

This estimate holds in two and three dimensions (where it has been used, for example, in [7]). However, in two dimensions the vorticity is known to be bounded *a priori*, which is sufficient for obtaining global H^3 bounds on $\boldsymbol{u}$. Inequality (3) is intimately related to the quasi-Lipschitz condition (2); see, for example, the review by Kato [17]. As such, a direct verification for a general operator K is difficult as its Green's function is typically not known. To overcome this difficulty, we abstractly characterize K by using general results from the classical L^p theory for elliptic partial differential operators. This requires an estimate on the nonlinear term in (1a) in terms of L^p rather than L^∞ norms.

Our proof splits into two main steps. First, local classical solutions are constructed as the limit of a Galerkin approximating sequence (Section 3). The method is mostly standard. Second, we prove a global *a priori* estimate in H^m which implies that the classical solutions are global (Section 4). The crucial L^p estimate on the nonlinear term is given in Section 5.

2. Preliminaries

In the following we think of Ω as a bounded and simply connected domain in $\mathbb{R}^2$. However, the specifics of the domain enter primarily through the operator K, and the main sections of this paper apply equally to non-simply connected domains as well as to $\Omega = \mathbb{R}^2$ and $\Omega = \mathbb{T}^2$.

Let b be a nondegenerate weight function. This means that there are constants $b_{\min}$ and $b_{\max}$ such that for every $\boldsymbol{x} \in \Omega$,

$$0 < b_{\min} \leq b(\boldsymbol{x}) \leq b_{\max} \,. \tag{4}$$

In the context of shallow water theory, b describes the depth of the basin. It enters the lake and the great lake equations in particular through the continuity equation

$$\boldsymbol{\nabla} \cdot (b\boldsymbol{u}) = 0 \,. \tag{5}$$

Mathematically, this constraint makes $b(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$ the invariant measure of the Lagrangian flow $\dot{\boldsymbol{x}} = \boldsymbol{u}$, where $\mathrm{d}\boldsymbol{x}$ denotes the two-dimensional Lebesgue measure. Hence it is natural to pose (1) in Sobolev spaces with weight b. We abbreviate the weighted integral over the domain by $\langle\cdot\rangle$, i.e.,

$$\langle\varphi\rangle \equiv \int_\Omega \varphi(\boldsymbol{x})\, b(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x} \,. \tag{6}$$

The H^m inner product can then be written

$$\langle\varphi,\psi\rangle_m = \sum_{0\leq|\alpha|\leq m} \langle \partial^\alpha\varphi\,\partial^\alpha\psi\rangle \,, \tag{7}$$

where $\alpha = (\alpha_1, \alpha_2)$ is a multi-index and

$$|\alpha| \equiv \sum_{j=1}^{2} \alpha_j \,, \qquad \partial^\alpha \equiv \prod_{j=1}^{2} \partial_j^{\alpha_j} \,. \tag{8}$$

Similarly, the Lebesgue spaces $\mathrm{L}^p(\Omega)$ and the Sobolev spaces $\mathrm{W}^{m,p}(\Omega)$—the spaces of functions whose derivatives up to order m are in L^p—are defined with respect to the b-weighted integral. The nondegeneracy of b assures that the weighted norms are equivalent to the norms in the unweighted case, so that all well-known embeddings as can be found in [1] still hold.

We fix $m \geq 2$ such that $b \in \mathrm{C}^{m+1}(\overline{\Omega})$ and $\partial\Omega \in \mathrm{C}^{m+1}$. Then our requirements on K can be formulated as follows:

(i) K is linear;
(ii) $\boldsymbol{\nabla}\cdot(b\,K\omega) = 0$ for $\omega \in \mathrm{H}^{m-1}(\Omega)$;
(iii) K is continuous from $\mathrm{W}^{m-1,p}(\Omega)$ into $\mathrm{W}^{m,p}(\Omega)$ and satisfies the estimate

$$\|K\omega\|_{\mathrm{W}^{m,p}} \leq c\,p\,\|\omega\|_{\mathrm{W}^{m-1,p}} \,, \tag{9}$$

for all $p \geq p_0$ where the constant c depends only on p_0, m, and Ω.

In the case of the two-dimensional Euler equations, $b \equiv 1$ and K is the inverse of the curl operator. It can be defined via the *stream function* ψ which is the solution of the elliptic problem

$$\omega = -\Delta\psi \qquad \text{in } \Omega\,, \tag{10a}$$

$$\psi = 0 \qquad \text{on } \partial\Omega\,, \tag{10b}$$

by setting $\boldsymbol{u} = K\omega \equiv \boldsymbol{\nabla}^\perp\psi$ where $\boldsymbol{\nabla}^\perp = (\partial_2, -\partial_1)$. Thus (i) and (ii) can be verified by inspection, and (iii) is a direct consequence of the classical L^p theory for elliptic equations [2, 12].

The case of the lake and great lake equations is discussed in detail in references [20, 23].

Provided K is as above, the vorticity equation (1) is known to be globally well posed. In particular, the supremum of the vorticity ω is bounded by its initial value for all times. This is summarized in the following theorem, which has been proved for the Euler equations by Yudovitch [31] and Bardos [3], and in our abstract setting in references [20, 23].

Theorem 1. *Given $\omega^{\mathrm{in}} \in \mathrm{L}^\infty(\Omega)$, there exists a unique function*

$$\omega \in \mathrm{C}([0,\infty); \mathrm{w}^*\text{-}\mathrm{L}^\infty(\Omega)) \cap \mathrm{L}^\infty([0,\infty)\times\Omega) \tag{11}$$

that satisfies the weak form of system (1)*, given by*

$$\langle\psi\omega(t_2)\rangle - \langle\psi\omega(t_1)\rangle - \int_{t_1}^{t_2} \big\langle(\boldsymbol{u}\cdot\boldsymbol{\nabla}\psi)\,\omega\big\rangle\,\mathrm{d}t = 0\,, \tag{12a}$$

$$\boldsymbol{u} = K\omega\,, \tag{12b}$$

$$\omega(0) = \omega^{\mathrm{in}}\,, \tag{12c}$$

for every $[t_1,t_2] \subset [0,\infty)$ and every test function $\psi \in \mathrm{H}^1_0(\Omega)$, and the vorticity bound

$$\|\omega(t)\|_{\mathrm{L}^\infty} \le \|\omega^{\mathrm{in}}\|_{\mathrm{L}^\infty}\,, \tag{13}$$

for every $t \in [0,\infty)$. Moreover, $\omega(t)$ depends continuously on ω^{in} and b.

3. Local Classical Solutions

We begin the construction of classical solutions with a local existence theorem.

Theorem 2. *Assume that $b \in \mathrm{C}^{m+2}(\overline{\Omega})$ and $\partial\Omega \in \mathrm{C}^{m+2}$ with $m \ge 2$. Then for every $\rho > 0$ there exists a time $T(\rho) > 0$ such that for every initial datum $\omega^{\mathrm{in}} \in \mathrm{H}^m(\Omega)$ with $\|\omega^{\mathrm{in}}\|_{\mathrm{H}^m} \le \rho$ there exists a unique local classical solution to the abstract vorticity equation* (1) *which satisfies*

$$\omega \in \bigcap_{k=0}^{m-2} \mathrm{C}^k([0,T); \mathrm{H}^{m-k}(\Omega))\,. \tag{14}$$

Proof. Uniqueness comes for free as local classical solutions are unique—even in the larger class of global weak solutions—by Theorem 1.

To prove existence, we construct a solution as the limit of a Galerkin approximating sequence as in Temam [27]. Some of the technical details are inspired by Kato [16]. The proof follows five distinct steps.

Step 1. *Construct a family of approximate solutions $\{\omega_n\}$.*

One can avoid technical difficulties by constructing the Galerkin approximating sequence with respect to a special basis. By the Lax-Milgram theorem, the mapping

$$\begin{aligned} \mathrm{H}^m(\Omega) &\to \mathrm{H}^m(\Omega)' \,, \\ \varphi &\mapsto \langle \varphi, \cdot \rangle_m \end{aligned} \tag{15}$$

is invertible. Moreover, the first embedding in

$$\mathrm{H}^m(\Omega) \hookrightarrow \mathrm{L}^2(\Omega) \hookrightarrow \mathrm{H}^m(\Omega)'$$

is dense and compact, so that the inverse of (15) is in fact a compact, positive and self-adjoint operator on $\mathrm{L}^2(\Omega)$. As such, it possesses a complete orthonormal family of eigenvectors $\{\varphi_k\} \subset \mathrm{H}^m(\Omega)$ which satisfy

$$\langle \varphi_k, \psi \rangle_m = \lambda_k \, \langle \varphi_k \, \psi \rangle \qquad \text{for every } \psi \in \mathrm{H}^m(\Omega). \tag{16}$$

Ghidaglia [11] has shown that a maximal regularity result holds for the operator defined through (15), which implies that $\{\varphi_k\} \subset \mathrm{H}^{2m}(\Omega)$. In the following, however, we only use that $\{\varphi_k\} \subset \mathrm{H}^{m+1}(\Omega)$.

Let P_n denote the H^m-orthogonal projector onto the span of the first n eigenvectors $\{\varphi_1, \ldots, \varphi_n\}$. We consider the vorticity equation (1) restricted to $P_n \mathrm{H}^m(\Omega)$,

$$\partial_t \omega_n + P_n(\boldsymbol{u}_n \cdot \boldsymbol{\nabla} \omega_n) = 0 \,, \tag{17a}$$

$$\boldsymbol{u}_n = K \omega_n \,, \tag{17b}$$

$$\omega_n(0) = \omega_n^{\mathrm{in}} \equiv P_n \omega^{\mathrm{in}} \,. \tag{17c}$$

The Picard-Lindelöf existence theorem for ordinary differential equations guarantees the existence of a solution ω_n for some interval of time. As a consequence of the *a priori* estimates in Step 2 below, this time interval can be chosen independent of n.

Step 2. *Show that $\{\omega_n\}$ is a relatively compact set in $\mathrm{C}([0,T]; \text{w-}\mathrm{H}^m(\Omega))$ for some $T > 0$.*

Proof of Step 2. We first show that $\{\omega_n\}$ is a bounded set in $\mathrm{L}^\infty([0,T); \mathrm{H}^m(\Omega))$, and characterize the minimum time T up to which we have such control.

To obtain an *a priori* estimate in H^m, take the H^m inner product of (17a) with ω_n. The Galerkin projector P_n is H^m-orthogonal, so that Lemma 4 (which is proved at the end of this article) is directly applicable:

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\omega_n\|_{\mathrm{H}^m}^2 = -2 \langle \boldsymbol{u}_n \cdot \boldsymbol{\nabla} \omega_n, \omega_n \rangle_m \leq c \, \|\omega_n\|_{\mathrm{H}^m}^3 \,. \tag{18}$$

By integrating this differential inequality and noting that $\|\omega_n^{\mathrm{in}}\|_{\mathrm{H}^m} \leq \|\omega^{\mathrm{in}}\|_{\mathrm{H}^m}$, one sees that there exists a continuous function $\theta(t)$ such that $\theta(0) = \|\omega^{\mathrm{in}}\|_{\mathrm{H}^m}$ and

$$\|\omega_n\|_{\mathrm{H}^m} \leq \theta(t) \qquad \text{for every } t \leq T = \frac{1}{c \, \|\omega^{\mathrm{in}}\|_{\mathrm{H}^m}} \,, \tag{19}$$

independent of n.

It remains to be shown that $\{\omega_n\}$ is a relatively compact set in the space $\mathrm{C}([0,T); \text{w-}\mathrm{H}^m(\Omega))$. According to the Arzela-Ascoli theorem this is equivalent to the following.

(i) $\{\omega_n(t)\}$ is a relatively compact set in $\text{w-}\mathrm{H}^m(\Omega)$ for every $t \in [0,T)$;

(ii) $\{\omega_n\}$ is equicontinuous in $\mathrm{C}([0,T);\text{w-}\mathrm{H}^m(\Omega))$, i.e. for every $\psi \in \mathrm{H}^m(\Omega)$ the sequence $\{\langle \psi, \omega_n \rangle_m\}$ is equicontinuous in $\mathrm{C}([0,T))$.

Condition (i) is equivalent to $\{\omega_n(t)\}$ being bounded in $\mathrm{H}^m(\Omega)$ for every $t \in [0,T)$, and hence is a consequence of (19).

In order to show (ii), we begin by considering the case where $\psi \in \mathrm{C}^\infty(\overline{\Omega})$. One obtains, again with the aid of Lemma 4,

$$\begin{aligned} \left| \langle \psi, \omega_n(t_2) \rangle_m - \langle \psi, \omega_n(t_1) \rangle_m \right| &= \left| \int_{t_1}^{t_2} \left\langle P_n \psi, \boldsymbol{u}_n \cdot \boldsymbol{\nabla} \omega_n \right\rangle_m \mathrm{d}t \right| \\ &\leq \int_{t_1}^{t_2} \| P_n \psi \|_{\mathrm{H}^{m+1}} \, \| \omega_n \|_{\mathrm{H}^m}^2 \, \mathrm{d}t \, . \end{aligned} \tag{20}$$

The projectors P_n are uniformly bounded in H^{m+1}, and we have already shown that $\{\omega_n\}$ is bounded in $\mathrm{L}^\infty([0,T);\mathrm{H}^m(\Omega))$. Thus the right side of (20) has an upper bound which is proportional to $|t_2 - t_1|$ independent of n, which implies that the set $\{\langle \psi, \omega_n \rangle_m\}$ is equicontinuous in $\mathrm{C}([0,T))$.

Finally, the class of test functions can be extended to $\psi \in \mathrm{H}^m(\Omega)$ by a straightforward density argument. □

Step 3. *Pass to the limit.*

Step 2 asserts that there exists a subsequence of $\{\omega_n\}$ that converges to a function

$$\omega \in \mathrm{C}([0,T);\text{w-}\mathrm{H}^m(\Omega)) \, .$$

We first show that ω satisfies the weak vorticity equation (12a). Without loss of generality, let $\psi \in \mathrm{C}_0^\infty(\Omega)$. As the embedding $\text{w-}\mathrm{H}^m(\Omega) \hookrightarrow \mathrm{L}^2(\Omega)$ is continuous, $\omega_n \to \omega$ in $\mathrm{C}([0,T);\mathrm{L}^2(\Omega))$, so that

$$\langle \psi \omega_n(t) \rangle \to \langle \psi \omega(t) \rangle \, , \tag{21}$$

for every $t \geq 0$ as $n \to \infty$. Similarly,

$$\begin{aligned} &\int_{t_1}^{t_2} \Big(\big\langle (\boldsymbol{u}_n \cdot \boldsymbol{\nabla} \psi) \, \omega_n \big\rangle - \big\langle (\boldsymbol{u} \cdot \boldsymbol{\nabla} \psi) \, \omega \big\rangle \Big) \mathrm{d}t \\ &\quad = \int_{t_1}^{t_2} \Big(\big\langle ((\boldsymbol{u}_n - \boldsymbol{u}) \cdot \boldsymbol{\nabla} \psi) \, \omega_n \big\rangle + \big\langle (\boldsymbol{u} \cdot \boldsymbol{\nabla} \psi) \, (\omega_n - \omega) \big\rangle \Big) \mathrm{d}t \\ &\quad \leq \| \boldsymbol{\nabla} \psi \|_{\mathrm{L}^\infty} \int_{t_1}^{t_2} \Big(\| \boldsymbol{u}_n - \boldsymbol{u} \|_{\mathrm{L}^2} \, \| \omega_n \|_{\mathrm{L}^2} + \| \boldsymbol{u} \|_{\mathrm{L}^2} \, \| \omega_n - \omega \|_{\mathrm{L}^2} \Big) \mathrm{d}t \\ &\quad \to 0 \end{aligned} \tag{22}$$

as $n \to \infty$. Therefore, once we show that ω is indeed a sufficiently smooth classical solution, namely

$$\omega \in \mathrm{C}([0,T);\mathrm{H}^m(\Omega)) \cap \mathrm{C}^1([0,T);\mathrm{H}^{m-1}(\Omega)) \, ,$$

we know that it must satisfy the vorticity equations (1) in a classical sense.

Step 4. *Show that $\omega \in \mathrm{C}([0,T);\mathrm{H}^m(\Omega))$.*

Proof of Step 4. From (19) we have that $\| \omega_n(t) \|_{\mathrm{H}^m}$ is bounded by $\theta(t)$, so that the limiting function must also satisfy $\| \omega(t) \|_{\mathrm{H}^m} \leq \theta(t)$. This implies

$$\limsup_{t \searrow 0} \| \omega(t) \|_{\mathrm{H}^m} \leq \theta(0) = \| \omega^{\mathrm{in}} \|_{\mathrm{H}^m} \, . \tag{23}$$

On the other hand, the weak continuity into H^m implies

$$\liminf_{t \searrow 0} \|\omega(t)\|_{\mathrm{H}^m} \geq \|\omega^{\mathrm{in}}\|_{\mathrm{H}^m} \tag{24}$$

and thus

$$\lim_{t \searrow 0} \|\omega(t)\|_{\mathrm{H}^m} = \|\omega^{\mathrm{in}}\|_{\mathrm{H}^m} . \tag{25}$$

In a Hilbert space, continuity of the norm and weak continuity imply strong continuity, i.e., ω is strongly continuous at $t = 0$.

By considering the initial value problem with $\omega^{\mathrm{in}} \equiv \omega(t_0)$, the above argument readily shows that ω is continuous from the right at any $t_0 \in [0, T)$. But since all our estimates are invariant under time reversal, it must also be continuous from the left. This completes the proof of Step 4. □

Step 5. *Show that $\omega \in \mathrm{C}^k([0,T); \mathrm{H}^{m-k}(\Omega))$ for $k \leq m-2$.*

Proof of Step 5. First consider the case $k = 1$. By assumption H^{m-1} is a Banach algebra, so that

$$\|\boldsymbol{u} \cdot \boldsymbol{\nabla} \omega\|_{\mathrm{H}^{m-1}} \leq c_1 \|\boldsymbol{u}\|_{\mathrm{H}^{m-1}} \|\omega\|_{\mathrm{H}^m} \leq c_2 \|\omega\|_{\mathrm{H}^{m-2}} \|\omega\|_{\mathrm{H}^m} . \tag{26}$$

This shows that $\omega \mapsto \boldsymbol{u} \cdot \boldsymbol{\nabla} \omega$ is continuous as a map from H^m into H^{m-1}. Since

$$\partial_t \omega = -\boldsymbol{u} \cdot \boldsymbol{\nabla} \omega , \tag{27}$$

and $\omega \in \mathrm{C}([0,T); \mathrm{H}^m(\Omega))$, one also has $\partial_t \omega \in \mathrm{C}([0,T); \mathrm{H}^{m-1}(\Omega))$.

The cases when $k > 1$ can be obtained by successive time differentiation of (27) and repetition of the argument, so long as H^{m-k} is an algebra. □

Remark 1. It is not possible to construct classical solutions via a viscosity regularization. The reason is that the viscous equations need an additional (artificial) boundary condition $\omega = 0$ on $\partial\Omega$, which persists in spaces of classical functions as the viscosity tends to zero. In contrast, functions in L^∞ have no traces, so that the viscosity method can well be used to prove Theorem 1.

4. Global Classical Solutions

The result of the last section gives us enough local control on the solution to carry out a more subtle estimate on the nonlinear term which establishes that classical solutions actually persist globally in time.

Theorem 3. *Assume that $b \in \mathrm{C}^{m+2}(\overline{\Omega})$ and $\partial\Omega \in \mathrm{C}^{m+2}$ with $m \geq 2$. Then for $\omega^{\mathrm{in}} \in \mathrm{H}^m(\Omega)$, there exists a unique global classical solution to the abstract vorticity equation* (1) *which satisfies*

$$\omega \in \bigcap_{k=0}^{m-2} \mathrm{C}^k([0,\infty); \mathrm{H}^{m-k}(\Omega)) . \tag{28}$$

Moreover,

$$\|\omega(t)\|_{\mathrm{H}^m} \leq \|\omega^{\mathrm{in}}\|_{\mathrm{H}^m}^{\exp(ct)} , \tag{29}$$

where c depends on m, Ω, K, and $\|\omega^{\mathrm{in}}\|_{\mathrm{L}^\infty}$.

Proof. Given that we already proved Theorem 2, it is now sufficient to derive the global H^m bound (29). In contrast to (18), we use estimate (33b) of Lemma 4 to control the growth of the H^m norm of ω:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{\mathrm{H}^m}^2 &= -2\langle \boldsymbol{u}\cdot\boldsymbol{\nabla}\omega, \omega\rangle_m \\ &\leq c_1\, p\, \|\omega\|_{\mathrm{L}^p}^{1-\mu}\, \|\omega\|_{\mathrm{H}^m}^{2+\mu} \\ &\leq \frac{c_2}{(m+1)\mu}\big(1+\|\omega^{\mathrm{in}}\|_{\mathrm{L}^\infty}\big)\, \|\omega\|_{\mathrm{H}^m}^{2+\mu} \end{aligned} \tag{30}$$

where $\mu = 2/(2+(m-1)p)$ and therefore $p \leq 2/((m+1)\mu)$. We can assume that $\|\omega\|_{\mathrm{H}^m} > 1$. Then the right side of (30) is minimized by setting

$$\frac{1}{\mu} = \ln\|\omega\|_{\mathrm{H}^m}\,, \tag{31}$$

whence (30) simplifies to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{\mathrm{H}^m} \leq c_3\, \|\omega\|_{\mathrm{H}^m} \ln\|\omega\|_{\mathrm{H}^m}\,. \tag{32}$$

This differential inequality is easily seen to yield a bound of the form (29), and therefore implies that the classical solution is in fact global. □

Remark 2. A differential inequality of the form (32) has essentially been derived by Golovkin [13], albeit for the stream function and with a different proof. The ideas of optimizing the exponent from elliptic L^p estimates is due to Yudovitch [31].

Remark 3. The separate proof of existence of local classical solutions is necessary to make the formal manipulations in estimate (30) rigorous. Conversely, the trick used here cannot be directly applied to the corresponding estimate (18) in the proof of Theorem 2, because we do not have uniform bounds on the L^∞ norm of the Galerkin approximating sequence.

In contrast, L^∞ bounds are easy to establish for Galerkin truncated viscous flows, which is the main reason why a viscosity regularization is commonly used to construct weak solutions.

5. Estimates on the Nonlinear Term

In this section we prove a set of closely related estimates on the nonlinearity $\boldsymbol{u}\cdot\boldsymbol{\nabla}\omega$ which are the key to the results of this article.

Lemma 4. *For $m \geq 2$, $\omega, \psi \in \mathrm{H}^{m+1}(\Omega)$, and $\boldsymbol{u} \equiv K\omega$, the following estimates hold:*

$$\langle \boldsymbol{u}\cdot\boldsymbol{\nabla}\omega, \psi\rangle_m \leq c_1\, p\, \|\omega\|_{\mathrm{L}^p}^{1-\mu}\, \|\omega\|_{\mathrm{H}^m}^{1+\mu}\, \|\psi\|_{\mathrm{H}^{m+1}}\,, \tag{33a}$$

$$\langle \boldsymbol{u}\cdot\boldsymbol{\nabla}\omega, \omega\rangle_m \leq c_2\, p\, \|\omega\|_{\mathrm{L}^p}^{1-\mu}\, \|\omega\|_{\mathrm{H}^m}^{2+\mu}\,, \tag{33b}$$

where $\mu = 2/(2+(m-1)p)$ and the constants can be chosen independent of p. Moreover,

$$\langle \boldsymbol{u}\cdot\boldsymbol{\nabla}\omega, \psi\rangle_m \leq c_3\, \|\omega\|_{\mathrm{H}^m}^2\, \|\psi\|_{\mathrm{H}^{m+1}}\,, \tag{34a}$$

$$\langle \boldsymbol{u}\cdot\boldsymbol{\nabla}\omega, \omega\rangle_m \leq c_4\, \|\omega\|_{\mathrm{H}^m}^3\,. \tag{34b}$$

Proof. Due to the continuity of the embedding $\mathrm{H}^m \hookrightarrow \mathrm{L}^p$, estimates (34) are a direct consequence of (33).

We first prove (33) for functions $\omega, \psi \in \mathrm{C}^\infty(\overline{\Omega})$, and extend the estimate with a density argument. A direct calculation shows that

$$\begin{aligned} \langle \boldsymbol{u} \cdot \boldsymbol{\nabla}\omega, \psi\rangle_m &\equiv \sum_{|\alpha|\le m} \big\langle \big(\partial^\alpha(\boldsymbol{u}\cdot\boldsymbol{\nabla}\omega)\big)\partial^\alpha\psi\big\rangle \\ &= \sum_{|\alpha|\le m}\bigg(\big\langle (\boldsymbol{u}\cdot\boldsymbol{\nabla}\partial^\alpha\omega)\,\partial^\alpha\psi\big\rangle + \sum_{0<\beta\le\alpha} c_{\alpha,\beta}\,\big\langle (\partial^\beta\boldsymbol{u}\cdot\boldsymbol{\nabla}\partial^{\alpha-\beta}\omega)\,\partial^\alpha\psi\big\rangle\bigg)\,, \end{aligned} \tag{35}$$

where $0<\beta$ if and only if $0<\beta_i$ for $i=1,2$. The first term in the outermost parentheses on the right side of (35) can be rewritten through integration by parts as

$$\big\langle (\boldsymbol{u}\cdot\boldsymbol{\nabla}\partial^\alpha\omega)\,\partial^\alpha\psi\big\rangle = -\big\langle (\boldsymbol{u}\cdot\boldsymbol{\nabla}\partial^\alpha\psi)\,\partial^\alpha\omega\big\rangle\,. \tag{36}$$

Hence, it vanishes if $\omega=\psi$. Furthermore, in the general case

$$\begin{aligned} \big\langle (\boldsymbol{u}\cdot\boldsymbol{\nabla}\partial^\alpha\omega)\,\partial^\alpha\psi\big\rangle &\le c_1\,\|\boldsymbol{u}\|_{\mathrm{L}^\infty}\,\|\omega\|_{\mathrm{H}^m}\,\|\psi\|_{\mathrm{H}^{m+1}} \\ &\le c_2\,\|\boldsymbol{u}\|_{\mathrm{L}^p}^{1-\mu}\,\|\boldsymbol{u}\|_{\mathrm{H}^m}^{\mu}\,\|\omega\|_{\mathrm{H}^m}\,\|\psi\|_{\mathrm{H}^{m+1}} \\ &\le c_3\,p^{1-\mu}\,\|\omega\|_{\mathrm{L}^p}^{1-\mu}\,\|\omega\|_{\mathrm{H}^m}^{1+\mu}\,\|\psi\|_{\mathrm{H}^{m+1}}\,. \end{aligned} \tag{37}$$

Note that this estimate is not sharp; however, it is sufficiently sharp to gain control of the same order as is possible with the remaining terms in (35).

Each of the terms in (35) is broken up with a Cauchy-Schwarz inequality and can be estimated in terms of a single common expression as follows:

$$\begin{aligned} \|\partial^\beta\boldsymbol{u}\cdot\boldsymbol{\nabla}\partial^{\alpha-\beta}\omega\|_{\mathrm{L}^2} &\le \|\partial^\beta\boldsymbol{u}\|_{\mathrm{L}^q}\,\|\partial^{\alpha-\beta}\omega\|_{\mathrm{W}^{1,q^*}} \qquad \text{where } \frac{1}{q}+\frac{1}{q^*}=\frac{1}{2} \\ &\le \|\boldsymbol{u}\|_{\mathrm{W}^{|\beta|,q}}\,\|\omega\|_{\mathrm{W}^{1+m-|\beta|,q^*}} \\ &\le c_3\,\|\boldsymbol{u}\|_{\mathrm{W}^{1,p}}^{(1-\lambda)(1-\mu)}\,\|\boldsymbol{u}\|_{\mathrm{H}^{m+1}}^{\lambda+\mu(1-\lambda)}\,\|\omega\|_{\mathrm{L}^p}^{\lambda(1-\mu)}\,\|\omega\|_{\mathrm{H}^m}^{1-\lambda(1-\mu)} \\ &\le c_4\,p^{(1-\lambda)(1-\mu)}\,\|\omega\|_{\mathrm{L}^p}^{1-\mu}\,\|\omega\|_{\mathrm{H}^m}^{1+\mu}\,. \end{aligned} \tag{38}$$

The Gagliardo-Nirenberg inequalities used in the third step of (38) are best written in terms of two parameters $\lambda,\mu\in[0,1]$ which are defined by

$$\lambda = \frac{|\beta|-1}{m}\,, \qquad \mu = \frac{2}{2+(m-1)p}\,. \tag{39}$$

Moreover, we make the special choice of $q=2/\lambda$. In this notation it is straightforward to check that the exponents of the Gagliardo-Nirenberg inequalities are indeed admissible. Finally, note that the last inequality in (38) uses the continuity properties of K as stated in (9).

The prefactor on the right side of (38) grows with a power of p which is less or equal to 1. Thus for every $\beta>0$,

$$\big\langle (\partial^\beta\boldsymbol{u}\cdot\boldsymbol{\nabla}\partial^{\alpha-\beta}\omega)\,\partial^\alpha\psi\big\rangle \le c_6\,p\,\|\omega\|_{\mathrm{L}^p}^{1-\mu}\,\|\omega\|_{\mathrm{H}^m}^{1+\mu}\,\|\psi\|_{\mathrm{H}^m}\,. \tag{40}$$

Once we insert this expression back into (35), the proof is complete. □

Remark 4. This lemma improves on the estimates used in [16, 27] through the application of "sharp" Gagliardo-Nirenberg inequalities to each term of a Leibniz expansion of a nonlinearity. We have used similar ideas in [6] to improve on earlier estimates [5] for the nonlinear term in the complex Ginzburg-Landau equation.

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PROGRAM IN APPLIED MATHEMATICS, UNIVERSITY OF ARIZONA, TUCSON, AZ 85721, USA
Current address: Department of Mathematics, University of California, Irvine, CA 92697
E-mail address: `oliver@member.ams.org`