REMARK ON THE RATE OF DECAY OF HIGHER ORDER DERIVATIVES FOR SOLUTIONS TO THE NAVIER–STOKES EQUATIONS IN $\mathbb{R}^n$

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Abstract. We present a new derivation of upper bounds for the decay of higher order derivatives of solutions to the unforced Navier–Stokes equations in $\mathbb{R}^n$. The method, based on so-called Gevrey estimates, also yields explicit bounds on the growth of the radius of analyticity of the solution in time. Moreover, under the assumption that the Navier–Stokes solution stays sufficiently close to a solution of the heat equation in the L^2 norm—a result known to be true for a large class of initial data—lower bounds on the decay of higher order derivatives can be obtained.

1. Introduction

We study the decay of solutions to the unforced Navier–Stokes equations in $\mathbb{R}^n$,

$$\partial_t u - \nu \Delta u + u \cdot \nabla u + \nabla p = 0 \,, \tag{1a}$$

$$\nabla \cdot u = 0 \,, \tag{1b}$$

$$u(0) = u_0 \,. \tag{1c}$$

Previous works by Kato [10], Schonbek [12, 13, 14, 15, 16], Kajikiya and Miyakawa [9], Wiegner [19], Schonbek and Wiegner [18], and others have established upper and lower bounds on the rate of decay for various norms, spatial dimensions and classes of initial data. Similar results can be established in exterior domains (see [18] for references); in this article, however, we exclusively consider the case where the domain is the whole of $\mathbb{R}^n$.

On a periodic domain, the generic decay is exponential with a rate given by the lowest eigenvalue of the Stokes operator. For special initial data the solution may decay faster: Foias and Saut [5] have shown that every solution to the two-dimensional Navier–Stokes equations with a potential forcing in a periodic domain decays to zero exponentially fast with a rate which is equal to one of the eigenvalues of the Stokes operator. Furthermore, the set of initial data whose solutions decay with the same rate lie on an analytic manifold.

On $\mathbb{R}^n$, due to the lack of a positive lowest eigenvalue of the Stokes operator—in other words, due to the absence of a Poincaré inequality—the rates of decay are generally algebraic rather than exponential in time. However, there are special classes of initial data that lead to exponential decay [14, 15].

The key for estimating the asymptotic decay rate is the formal smallness of the quadratic nonlinearity $u \cdot \nabla u$ in (1) when the solution u is small. One therefore

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hopes that solutions to the heat equation

$$\partial_t v = \nu \Delta v \,, \tag{2a}$$

$$v(t_0) = u(t_0) \,, \tag{2b}$$

approximate the Navier–Stokes solution u arbitrarily well for t_0 sufficiently large. Note that since $\nabla \cdot v = 0$ initially, v will remain divergence free for all $t \geq t_0$. This is easily proved by taking the divergence of (2a) and integrating against $\nabla \cdot v$.

The purpose of this paper is to present a method which allows to easily translate bounds on the L^2 decay of u and v into bounds on the decay of higher order derivatives. We state the result in terms of the operator $A = \sqrt{-\Delta}$ which, like $-\Delta$ itself, is an unbounded, self-adjoint, and nonnegative operator on $L^2(\mathbb{R}^n)$. Powers of A can be expressed in terms of the Fourier transform by $(A^r w)\hat{}\,(\xi) = |\xi|^r \hat{w}(\xi)$, and the canonical H^r norm (7) is equivalent to the norm $\| \cdot \| + \|A^r \cdot \|$, where $\| \cdot \| \equiv \| \cdot \|_{L^2}$.

Our main assumption is that algebraic decay with rate $\gamma > 0$ has already been proved for some class of initial data. More precisely, we suppose the following.

Assumption 1. *There exist positive real numbers M_1 and γ which may depend on u_0 such that*

$$\|u(t)\|^2 \leq \frac{M_1}{(1+t)^\gamma} \,, \qquad \textit{for all} \quad t \geq 0 \,, \tag{3}$$

where $u(t)$ is a solution to the Navier–Stokes equations (1).

Assumption 2. *There exist constants M_2, M_3 and M_4, which may depend on u_0, so that for every $\varepsilon > 0$ there exist t_0 and t_1, with $t_1 \geq t_0 \geq 0$, so that for all $t \geq t_1$,*

$$\|u(t) - v(t)\|^2 \leq \frac{\varepsilon \, M_2}{(1+t)^\gamma} \,, \tag{4}$$

and, for every $m \in \mathbb{N}$,

$$\frac{M_3(m)}{(1+t)^{\gamma+m}} \leq \|A^m v(t)\|^2 \leq \frac{M_4(m)}{(1+t)^{\gamma+m}} \,, \tag{5}$$

where $u(t)$ is a solution to the Navier-Stokes equations (1), and $v(t)$ solves the heat equation (2).

In addition, we require that

$$\liminf_{t \to \infty} \|u(t)\|_{H^r} < \infty \,, \tag{6}$$

where $r > \frac{n}{2}$ is such that $n \in [r, r+2)$; $H^r(\mathbb{R}^n)$ is the usual Sobolev space of order r endowed with the norm

$$\|w\|_{H^r}^2 = \int_{\mathbb{R}^n} (1 + |\xi|^2)^r \, |\hat{w}|^2 \, \mathrm{d}x \,. \tag{7}$$

Remark 1. Assumption 1 has been proved for initial data $u_0 \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ when $n \geq 2$ and $\gamma = \frac{n}{2}$ [10, 12, 13, 19]. If, in addition, $n = 2, 3$ and u_0 has non-zero average, Schonbek [13] has shown that Assumption 2 holds for $m = 0$. On the other hand, for special initial data one can prove faster decay rate. For instance, if the Fourier transform of the initial data, $\hat{u}_0(\xi)$, has a zero of order one at $\xi = 0$, then Assumption 1 can be improved to $\gamma = \frac{n}{2} + 1$. A lower bound with the same order of decay can be established if the initial data lies outside a set of functions of radially equidistributed energy (Schonbek [14, 15]).

Remark 2. Assumption 2 with $m = 0$ directly implies a lower bound on the L^2 decay of u. In fact, it is the key step in Schonbek's [14] derivation of such bounds.

Remark 3. The dependence of M_1 and M_2 on u_0 is generally not trivial. For details see [14, 15].

A sufficient condition for upper and lower bounds on derivatives of the heat solution to decay at the same algebraic rate is given in the following.

Proposition 3. *Estimates* (5) *holds for any real number $m \geq 0$ provided that the Fourier transform of the initial data $\hat{u}_0 \in H^m(\mathbb{R}^n)$, and satisfies*

$$\rho(r) \equiv \int_{\mathbb{S}^{n-1}} |\hat{u}_0(r\omega)|^2 \,\mathrm{d}\omega = c_1\, r^{2\gamma-n} + o(r^{2\gamma-n}) \tag{8}$$

as $r \to 0$.

Proof. We follow the steps in [15]. To simplify notation, take $t_0 = 0$. Using the Fourier representation of the heat solution and the Plancherel theorem, we find

$$\begin{aligned} \|A^m v(t)\|^2 &= \int_{\mathbb{R}^n} \mathrm{e}^{-2\nu|\xi|^2 t}\, |\xi|^{2m}\, |\hat{u}_0(\xi)|^2 \,\mathrm{d}\xi \\ &= \mathrm{Vol}(n) \int_0^\infty \mathrm{e}^{-2\nu r^2 t}\, r^{2m+n-1}\, \rho(r)\,\mathrm{d}r \qquad (9) \\ &\geq \frac{c_2}{t^{m+\gamma}} \int_0^1 s^{m+\gamma-1}\,\mathrm{e}^{-s}\,\mathrm{d}s + o\left(\frac{1}{t^{m+\gamma}}\right) \qquad (10) \end{aligned}$$

as $t \to \infty$. This implies the lower bound in (5), where the choice of M_3 depends on the initial data. For the upper bound, split (9) into two parts before changing variables:

$$\begin{aligned} \|A^m v(t)\|^2 &\leq \mathrm{Vol}(n) \int_0^{t^{-1/4}} \mathrm{e}^{-2\nu r^2 t}\, r^{2m+n-1}\, \rho(r)\,\mathrm{d}r \\ &\quad + \mathrm{Vol}(n)\,\mathrm{e}^{-2\nu\sqrt{t}} \int_{t^{-1/4}}^\infty r^{2m+n-1}\, \rho(r)\,\mathrm{d}r \\ &\leq \frac{c_3}{t^{m+\gamma}} \int_0^{2\nu\sqrt{t}} s^{m+\gamma-1}\,\mathrm{e}^{-s} \left(1 + \left(\frac{o(\frac{s}{t})}{\frac{s}{t}}\right)^{\gamma-\frac{n}{2}}\right) \mathrm{d}s \\ &\quad + \mathrm{Vol}(n)\,\mathrm{e}^{-2\nu\sqrt{t}}\, \|A^m u_0\|^2 \,. \qquad (11) \end{aligned}$$

For t sufficiently large, the term in parentheses will be bounded independent of t. This implies the upper bound in (5). □

We now state the main result of this paper.

Theorem 4. *Let u be a solution to the Navier–Stokes equations* (1)*, with initial value $u_0 \in L^2(\mathbb{R}^n)$, which satisfies* (6) *and Assumption 1. Then there exists a constant $c_1 = c_1(M_1, \gamma, n)$ such that for every real number $m > 0$*

$$\|A^m u(t)\|^2 \leq c_1 \left(\frac{2m}{\mathrm{e}}\right)^{2m} \frac{1}{(1+t)^{\gamma+m}} \,. \tag{12}$$

If, in addition, $u_0 \in L^1(\mathbb{R}^n)$ and Assumption 2 is satisfied, then there exists a positive constant $c_2 = c_2(M_2, \gamma, n, m)$ such that

$$\|A^m u(t)\|^2 \geq \frac{c_2}{(1+t)^{\gamma+m}} \,. \tag{13}$$

Remark 4. All claims in this paper are formal. I.e., they apply to sufficiently differentiable global strong solutions of the Navier–Stokes equations, or to appropriately constructed sequences of approximate solutions. See [15, 18] for a discussion.

Remark 5. Theorem 4 and its proof hold true in any spatial dimension n. Note, however, that the underlying L^2 bounds exhibit strong dependence on n via γ and c_1. In particular, the validity of Assumption 2 is not known for $n > 3$.

The upper bound (12) has previously been derived for $n = 2$ by Schonbek [16], and more generally by Schonbek and Wiegner [18], using a Fourier splitting approach. We believe the lower bound (13) has not been published before.

The proof of Theorem 4 is based on a characterization of real analytic functions in terms of decay of the Fourier transform, which we detail in Section 2. The key idea is that we introduce a seminorm which contains bounds on all derivatives weighted as terms of an exponential sum. Therefore, a single estimate in this seminorm is sufficient to *a posteriori* reconstruct bounds on all derivatives of u.

In Section 3 we derive a preliminary result, namely that solutions to the Navier–Stokes equations enter a subclass of the real analytic functions in an arbitrarily short time. This section is the $\mathbb{R}^n$ version of work by Foias and Temam [6] (also see [4]) on the Navier–Stokes equations with periodic boundary conditions. We include it for completeness. Our main estimate, which implies upper bounds on the decay of the higher order derivatives, is proved in Section 4. The estimate for the lower bounds is very similar and is given in the final Section 5.

We finally remark that our technique is applicable to a wide range of dissipative unforced nonlinear partial differential equations, provided they have an analytic nonlinearity which is of formally higher order as the solution decays towards zero. Once bounds on the L^2 decay are established, the higher derivatives will decay faster with the natural scaling given by the linear part of the equation. Examples are the Boussinesq equations for weakly compressible fluids, the equations of magnetohydrodynamics whose L^2 decay has been studied by Schonbek *et al.* [17], the generalized viscous KdV equation considered by Bona *et al.* [1], and a model for internal waves in a two-layer fluid derived by Choi and Camassa [2], which will be the subject of a joint forthcoming paper with R. Camassa.

2. Characterization of Real Analytic Functions

A function $w \in C^\infty(\mathbb{R}^n)$ is said to be real analytic if for every bounded subdomain $\Omega \subset \mathbb{R}^n$ there are constants $\rho > 0$ and $M > 0$ (which may depend on w and Ω) such that for every $x \in \Omega$ and every $\alpha \in \mathbb{N}^n$ one has

$$|\partial^\alpha w(x)| \le M \frac{\alpha!}{\rho^{|\alpha|}} \,. \tag{14}$$

We employ the usual multi-index notation in which

$$|\alpha| \equiv \sum_{j=1}^n \alpha_j \,, \qquad \alpha! \equiv \prod_{j=1}^n \alpha_j! \,, \qquad \partial^\alpha \equiv \prod_{j=1}^n \partial_j^{\alpha_j} \,. \tag{15}$$

The constant $\rho = \rho(\Omega)$ is a lower bound on the radius of analyticity on a given subdomain $\Omega \subset \mathbb{R}^n$ (see, e.g., John [8]). The class of real analytic functions on $\mathbb{R}^n$ is denoted $C^\omega(\mathbb{R}^n)$.

We introduce the spaces, parametrized by $\tau \geq 0$,

$$\mathcal{D}(\mathrm{e}^{\tau A} \colon H^r) = \{w \in H^r(\mathbb{R}^n) \colon \mathrm{e}^{\tau A} w \in H^r(\mathbb{R}^n)\} \,, \tag{16}$$

which can be endowed with the norm $\|\cdot\| + \|A^r \mathrm{e}^{\tau A} \cdot\|$. The union of these spaces over all positive τ is a subclass of the real analytic functions.

Theorem 5. *For any $r \geq 0$,*

$$C^\omega(\mathbb{R}^n) \supset \bigcup_{\tau > 0} \mathcal{D}(\mathrm{e}^{\tau A} \colon H^r) \,. \tag{17}$$

Remark 6. On the periodic box $\mathbb{T}^n$, (17) actually holds as an equality—see [11] for a review of this case. On $\mathbb{R}^n$, however, the spaces on the right enforce decay at infinity, whereas analytic functions generally do not decay at infinity (consider, for example, the constant function).

Proof. Let $w \in \mathcal{D}(\mathrm{e}^{\tau A} \colon H^r)$ for some $\tau > 0$, and fix $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq r$ and a bounded domain $\Omega \subset \mathbb{R}^n$, sufficiently smooth such that the Sobolev embedding theorem applies. For simplicity, assume that $r > \frac{n}{2}$. We then have

$$\|\partial^\alpha w\|_{L^\infty(\Omega)} \leq c(\Omega) \, \|\partial^\alpha w\|_{H^r(\Omega)} \leq c(\Omega) \, \|\partial^\alpha w\|_{H^r(\mathbb{R}^n)} \,. \tag{18}$$

We set $|\alpha| = m$ and note that, by the polynomial identity,

$$\biggl(\sum_{i=1}^n |\xi_i|^2 \biggr)^m = \sum_{|\beta| = m} \frac{m!}{\beta!} \prod_{i=1}^n |\xi_i|^{2\beta_i} \geq \frac{m!}{\alpha!} \prod_{i=1}^n |\xi_i|^{2\alpha_i} \,. \tag{19}$$

Therefore,

$$\begin{aligned}
\|\partial^\alpha w\|_{H^r(\mathbb{R}^n)}^2 &= \int_{\mathbb{R}^n} \prod_{i=1}^n |\xi_i|^{2\alpha_i} \, (1 + |\xi|^2)^r \, |\hat{w}(\xi)|^2 \, \mathrm{d}\xi \\
&\leq \frac{\alpha!}{m!} \int_{\mathbb{R}^n} |\xi|^{2m} \, (1 + |\xi|^2)^r \, |\hat{w}(\xi)|^2 \, \mathrm{d}\xi \\
&\leq \frac{\alpha!}{m!} \left(\frac{m!}{\tau^m} \right)^2 \int_{\mathbb{R}^n} \mathrm{e}^{2\tau|\xi|} \, (1 + |\xi|^2)^r \, |\hat{w}(\xi)|^2 \, \mathrm{d}\xi \\
&= \left(\frac{\alpha!}{\tau^m} \right)^2 \frac{m!}{\alpha!} \, \|\mathrm{e}^{\tau A} w\|_{H^r(\mathbb{R}^n)}^2 \,.
\end{aligned} \tag{20}$$

We estimate the ratio $m!/\alpha!$ for $m \geq 2$ by using Stirling's formula:

$$\frac{m!}{\alpha!} \leq \frac{m!}{\bigl(\Gamma(m/n + 1)\bigr)^n} \leq \frac{\bigl(\frac{m}{\mathrm{e}}\bigr)^m \sqrt{2\pi m} \, \exp\bigl(\frac{1}{12(m-1)}\bigr)}{\biggl(\bigl(\frac{m}{n\mathrm{e}}\bigr)^{\frac{m}{n}} \sqrt{2\pi \frac{m}{n}} \biggr)^n} \leq c \, n^m \,. \tag{21}$$

Substituting (21) back into (20), we obtain an estimate of the form (14) with $\rho = \tau/\sqrt{n}$. The case when $r \leq \frac{n}{2}$ can be treated in much the same way by translating the index m in (20) by a fixed integer. □

Remark 7. In particular, the proof shows that if $w \in \mathcal{D}(\mathrm{e}^{\tau A} \colon H^r)$, its radius of analyticity is $\tau/\sqrt{n}$ on the whole of $\mathbb{R}^n$. We could therefore call the right side of (17) the class of *uniformly analytic functions.*

When $r > \frac{n}{2}$, $\mathcal{D}(\mathrm{e}^{\tau A} \colon H^r(\mathbb{T}^n))$ is known to be a multiplicative topological algebra [4], and is therefore well suited for the study of analytic nonlinear partial differential equations with periodic boundary conditions. This property can easily be shown to carry over to functions defined on $\mathbb{R}^n$. However, our main result relies—through Lemma 9 below—on obtaining estimates in the seminorm $\|A^r \mathrm{e}^{\tau A} \cdot \|$, so that the inequality for products is more subtle.

Lemma 6. *Let $\tau \geq 0$, $r > \frac{n}{2}$, and $s < \frac{n}{2}$. Then there exists a constant $C = C(n,r,s)$ such that any two functions v and w in $\mathcal{D}(\mathrm{e}^{\tau A} \colon H^r)$ satisfy the inequality*

$$\|A^r \mathrm{e}^{\tau A}(vw)\|_{L^2} \leq C(n,r,s)\Big(\|A^r \mathrm{e}^{\tau A} v\|_{L^2} \, \|A^s \mathrm{e}^{\tau A} w\|_{H^{r-s}} + \|A^s \mathrm{e}^{\tau A} v\|_{H^{r-s}} \, \|A^r \mathrm{e}^{\tau A} w\|_{L^2} \Big) . \tag{22}$$

Proof. Use the Plancherel theorem, the triangle inequality, the inequality $(x+y)^r \leq 2^{r-1}(x^r + y^r)$, and the convolution estimate $\|f * g\|_{L^2} \leq \|f\|_{L^1} \|g\|_{L^2}$ to obtain

$$\begin{aligned}
\|A^r \mathrm{e}^{\tau A}(vw)\|^2 &= \int_{\mathbb{R}^n} |\xi|^{2r} \, \mathrm{e}^{2\tau|\xi|} \left| \int_{\mathbb{R}^n} \hat{v}(\eta) \, \hat{w}(\xi - \eta) \, \mathrm{d}\eta \right|^2 \mathrm{d}\xi \\
&\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\hat{v}(\eta)| \, |\hat{w}(\xi-\eta)| \, |\eta + (\xi - \eta)|^r \, \mathrm{e}^{\tau|\eta + (\xi-\eta)|} \, \mathrm{d}\eta \right)^2 \mathrm{d}\xi \\
&\leq c_1 \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} |\eta|^r \mathrm{e}^{\tau|\eta|} \, |\hat{v}(\eta)| \, \mathrm{e}^{\tau|\xi-\eta|} \, |\hat{w}(\xi-\eta)| \, \mathrm{d}\eta \right)^2 \mathrm{d}\xi \\
&\quad + c_1 \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^n} \mathrm{e}^{\tau|\eta|} \, |\hat{v}(\eta)| \, |\xi - \eta|^r \mathrm{e}^{\tau|\xi-\eta|} \, |\hat{w}(\xi-\eta)| \, \mathrm{d}\eta \right)^2 \mathrm{d}\xi \\
&\leq c_1 \left(\int_{\mathbb{R}^n} \mathrm{e}^{\tau|\xi|} \, |\hat{w}(\xi)| \, \mathrm{d}\xi \right)^2 \|A^r \mathrm{e}^{\tau A} v\|^2 \\
&\quad + c_1 \left(\int_{\mathbb{R}^n} \mathrm{e}^{\tau|\xi|} \, |\hat{v}(\xi)| \, \mathrm{d}\xi \right)^2 \|A^r \mathrm{e}^{\tau A} w\|^2 .
\end{aligned} \tag{23}$$

The L^1 norm of a Fourier transform $\hat{u}$ is easily estimated,

$$\begin{aligned}
\int_{\mathbb{R}^n} |\hat{u}(\xi)| \, \mathrm{d}\xi &= \int_{\mathbb{R}^n} |\xi|^{-s} \, (1 + |\xi|^2)^{-\frac{r-s}{2}} \, |\xi|^s \, (1 + |\xi|^2)^{\frac{r-s}{2}} \, |\hat{u}(\xi)| \, \mathrm{d}\xi \\
&\leq \left(\int_{\mathbb{R}^n} |\xi|^{-2s} \, (1 + |\xi|^2)^{s-r} \, \mathrm{d}\xi \right)^{\frac{1}{2}} \|A^s u\|_{H^{r-s}} .
\end{aligned} \tag{24}$$

The remaining integral is finite for $r > \frac{n}{2}$ and $s < \frac{n}{2}$. By applying this estimate to (23), we obtain (22). □

In the remainder of this section, we prove a series of technical estimates which relate Gevrey norms to Sobolev norms of finite order.

Lemma 7. *For every $r \geq 0$ and $\tau \geq 0$ one has the inequality*

$$\|A^r \mathrm{e}^{\tau A} u\|^2 \leq 2 \, \|A^r u\|^2 + 2 \, \tau^2 \, \|A^{r+1} \mathrm{e}^{\tau A} u\|^2 . \tag{25}$$

Proof. For every $x \geq 0$ one has $\mathrm{e}^x \leq 1 + x\mathrm{e}^x$. Therefore, by the Plancherel theorem,

$$\begin{aligned} \|A^r \mathrm{e}^{\tau A} u\|^2 &= \int_{\mathbb{R}^n} |\xi|^{2r} \mathrm{e}^{2\tau|\xi|} \, |\hat{u}(\xi)|^2 \, \mathrm{d}\xi \\ &\leq \int_{\mathbb{R}^n} |\xi|^{2r} \bigl(1 + \tau|\xi| \, \mathrm{e}^{\tau|\xi|}\bigr)^2 \, |\hat{u}(\xi)|^2 \, \mathrm{d}\xi \\ &\leq 2 \int_{\mathbb{R}^n} |\xi|^{2r} \bigl(1 + \tau^2 |\xi|^2 \, \mathrm{e}^{2\tau|\xi|}\bigr) |\hat{u}(\xi)|^2 \, \mathrm{d}\xi \, . \end{aligned} \tag{26}$$

This last expression is equal to the right side of (25). □

Lemma 8. *For all nonnegative p, q, and τ,*

$$\|A^p \mathrm{e}^{\tau A} u\|^2 \leq \mathrm{e} \, \|A^p u\|^2 + (2\tau)^{2q} \, \|A^{p+q} \mathrm{e}^{\tau A} u\|^2 \, . \tag{27}$$

Proof. For every $x \geq 0$ and $m > 0$ one has $\mathrm{e}^x \leq \mathrm{e} + x^m \mathrm{e}^x$, since $e^x \leq \mathrm{e}$ on $[0, 1]$ and $\mathrm{e}^x \leq x^m \mathrm{e}^x$ for $x \geq 1$. Now proceed as in the proof of Lemma 7. □

Lemma 9. *Provided that $2q \geq p \geq 0$ and $\tau > 0$,*

$$\|A^q u\|^2 \leq c(p, q) \, \tau^{p-2q} \, \|u\| \, \|A^p \mathrm{e}^{\tau A} u\| \, . \tag{28}$$

Proof. Again with the Plancherel theorem, we obtain

$$\begin{aligned} \|A^q u\|^2 &= \int_{\mathbb{R}^n} |\xi|^{2q} \, |\hat{u}|^2 \, \mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} |\xi|^{2q-p} \mathrm{e}^{-\tau|\xi|} \, |\hat{u}| \, |\xi|^p \mathrm{e}^{\tau|\xi|} \, |\hat{u}| \, \mathrm{d}\xi \\ &\leq \max_{|\xi| \geq 0} |\xi|^{2q-p} \mathrm{e}^{-\tau|\xi|} \, \|u\| \, \|A^p \mathrm{e}^{\tau A} u\| \, . \end{aligned} \tag{29}$$

An elementary calculation shows that the maximum is attained at $|\xi| = (2q - p)/\tau$, and thus

$$c(q, p) = \begin{cases} (2q - p)^{2q-p} \mathrm{e}^{-(2q-p)} & \text{for } 2q > p \, , \\ 1 & \text{for } 2q = p \, . \end{cases} \tag{30}$$

□

3. Short-time Uniform Analyticity

In this section we demonstrate that a solution to the Navier–Stokes equations enters some class $\mathcal{D}(\mathrm{e}^{\tau A} \colon H^r)$ in an arbitrarily short time. This has been proved by Foias and Temam [6] in the case of periodic boundary conditions, and in [3] for the Navier–Stokes equations on the two-dimensional rotating sphere; corresponding results for the Navier–Stokes equations in L^p spaces have recently been obtained by Kukavica and Grujic [7].

On $\mathbb{R}^n$, the algebra inequality (22) contributes two extra terms, so our task is to show that these do not affect the validity of the basic result. We assume that the existence of a solutions $u \in L^\infty([0, T]; H^r(\mathbb{R}^n))$, $r > \frac{n}{2}$, is known for some $T > 0$.

To simplify notation, we set

$$J_r = \|A^r u\|_{L^2}^2 \, , \tag{31a}$$

$$G_r = \|A^r \mathrm{e}^{\tau A} u\|_{L^2}^2 \, , \tag{31b}$$

where $\tau = \tau(t)$ is to be specified later.

By direct calculation one finds

$$\tfrac{1}{2}\,\dot{G}_r = \dot{\tau}\,G_{r+1/2} - \nu\,G_{r+1} - \int_{\mathbb{R}^n} A^r \mathrm{e}^{\tau A}(u\cdot\nabla u)\,A^r\mathrm{e}^{\tau A}u\,\mathrm{d}x\,. \tag{32}$$

Note that the contribution of the pressure term is zero because A commutes with the Leray projection onto divergence free vector fields. Let us first estimate the contribution from the nonlinearity. Noting that

$$\|A^s\mathrm{e}^{\tau A}u\|_{H^{r-s}} \le c\,\bigl(G_r^{\frac12} + G_s^{\frac12}\bigr)\,, \tag{33}$$

we employ the Cauchy–Schwarz inequality and Lemma 6, and write

$$\begin{aligned}\left|\int_{\mathbb{R}^n} A^r\mathrm{e}^{\tau A}(u\cdot\nabla u)\,A^r\mathrm{e}^{\tau A}u\,\mathrm{d}x\right| &\le \|A^r\mathrm{e}^{\tau A}(u\cdot\nabla u)\|\,\|A^r\mathrm{e}^{\tau A}u\|\\ &\le c_1\bigl(G_s^{\frac12}+G_r^{\frac12}\bigr)G_{r+1}^{\frac12}\,G_r^{\frac12} + c_1\bigl(G_{s+1}^{\frac12}+G_{r+1}^{\frac12}\bigr)G_r\,.\end{aligned} \tag{34}$$

Now interpolate G_s by using Lemma 8 with $p=s$ and $q=r-s$; similarly, interpolate G_{s+1} by using using Lemma 8 with $p=s+1$ and $q=r-s$. After applying the Young inequality, one finds

$$\begin{aligned}\left|\int_{\mathbb{R}^n} A^r\mathrm{e}^{\tau A}(u\cdot\nabla u)\,A^r\mathrm{e}^{\tau A}u\,\mathrm{d}x\right| &\le \|A^r\mathrm{e}^{\tau A}(u\cdot\nabla u)\|\,\|A^r\mathrm{e}^{\tau A}u\|\\ &\le c_2\,J_s^{\frac12}\,G_r^{\frac12}\,G_{r+1}^{\frac12} + c_3\,J_{s+1}^{\frac12}\,G_r + c_4\,(1+\tau^{r-s})\,G_r\,G_{r+1}^{\frac12}\,,\end{aligned} \tag{35}$$

where $\frac{n}{4} < \frac{r}{2} \le s < \frac{n}{2}$.

For the remainder of this section, set $\tau = t$. Interpolate the first term on the right of (32), then use estimate (35) and, once more, the Young inequality, to obtain a differential inequality of the form

$$\dot{G}_r \le c(\|u\|_{H^1})\,G_r + c(r,s,T)\,G_r^2\,. \tag{36}$$

Since $G_r(0) = \|u_0\|_{H^r}^2$ is assumed to be finite, there exists a $\sigma\in(0,T]$ such that $G_r(t)$ is finite for $t\in[0,\sigma)$.

4. Upper Bounds

In this section we derive a more subtle differential inequality for G_r which, in contrast to (36), is valid for long times. The key observation is that the radius of uniform analyticity, $\rho = \tau/\sqrt{n}$, increases like $\sqrt{t}$ as $t\to\infty$ as for solutions of the heat equation. Once we have established the optimal decay rate for the Gevrey norm, we can use Lemma 9 to immediately reconstruct the decay rates for norms of finite order derivatives of u.

The basic structure of the argument is determined by a rather straightforward estimate involving only contributions from the linear terms of the Navier–Stokes equations. Consider the first two terms on the right of (32), and assume that $\dot{\tau} > 0$. Then we can use an interpolation inequality and the Young inequality for the first term, while breaking up the second term into several fractions. Since Lemma 7 implies

$$\frac{G_r - 2\,J_r}{2\,\tau^2} \le G_{r+1}\,, \tag{37}$$

we altogether obtain

$$\begin{aligned}\dot\tau\, G_{r+1/2} - \nu\, G_{r+1} &\le \frac{1}{2}\frac{\dot\tau}{\tau}\, G_r + \frac{1}{2}\,\dot\tau\tau\, G_{r+1} - \frac{\nu}{2}\, G_{r+1} - \frac{\nu}{2}\frac{G_r - 2\,J_r}{2\,\tau^2} \\ &= \left(\frac{1}{2}\frac{\dot\tau}{\tau} - \frac{\nu}{8}\frac{1}{\tau^2}\right) G_r + \left(\frac{1}{2}\,\dot\tau\tau - \frac{\nu}{8}\right) G_{r+1} \\ &\quad - \frac{\nu}{8}\frac{1}{\tau^2}\, G_r + \frac{\nu}{2}\frac{1}{\tau^2}\, J_r - \frac{3\nu}{8}\, G_{r+1}\,. \end{aligned} \tag{38}$$

Combining Lemma 9 with $q = r$ and the Young inequality gives

$$J_r \le c_3\, \frac{1}{\tau^{2r}}\, J_0 + \frac{1}{8}\, G_r\,. \tag{39}$$

Moreover, we set

$$\tau = \sqrt{\tau_0^2 + \alpha\, t}\,, \tag{40}$$

where $\tau_0 > 0$ and $0 < \alpha \le \nu/2$ are to be determined later. In any case, we immediately find that

$$\frac{1}{2}\,\dot\tau\tau = \frac{\alpha}{4} \le \frac{\nu}{8}\,, \tag{41}$$

so that the first two terms on the right of (38) are nonpositive and can be neglected.

The main technical complication now arises from the contributions of the nonlinearity. Our goal is to show that these do not affect the decay properties of the solution to leading order. We thus revisit the estimate on the nonlinear term, equation (35), and interpolate J_s by using Lemma 9 with $p = r$ and $q = s$; J_{s+1} is interpolated in an analogous manner. After applying the Young inequality, we find

$$\begin{aligned}&\left| \int_{\mathbb{R}^n} A^r \mathrm{e}^{\tau A} (u\cdot\nabla u)\, A^r \mathrm{e}^{\tau A} u \,\mathrm{d}x \right| \\ &\quad\le c_5\, \tau^{\frac{r}{2}-s}\, J_0^{\frac14}\, G_r^{\frac34}\, G_{r+1}^{\frac12} + c_6\, \tau^{\frac{r}{2}-s-1}\, J_0^{\frac14}\, G_r^{\frac54} + c_4\,(1+\tau^{r-s})\, G_r\, G_{r+1}^{\frac12} \\ &\quad\le c_7\, \tau^{r-2s}\, J_0^{\frac12}\, G_r^{\frac32} + c_6\, \tau^{\frac{r}{2}-s-1}\, J_0^{\frac14}\, G_r^{\frac54} + c_8\,(1+\tau^{2(r-s)})\, G_r^2 + \frac{3\nu}{8}\, G_{r+1}\,. \end{aligned} \tag{42}$$

Altogether, we obtain the differential inequality

$$\begin{aligned}\dot G_r &\le -\frac{\nu}{8}\frac{1}{\tau^2}\, G_r + c_3\, \frac{\nu}{\tau^{2(r+1)}}\, J_0 \\ &\quad + 2c_7\, \tau^{r-2s}\, J_0^{\frac12}\, G_r^{\frac32} + 2c_6\, \tau^{\frac{r}{2}-s-1}\, J_0^{\frac14}\, G_r^{\frac54} + 2c_8\,\bigl(1+\tau^{2(r-s)}\bigr)\, G_r^2\,. \end{aligned} \tag{43}$$

We will now argue that at least locally and—as we shall see later—for suitably small initial data globally, the "nonlinear" terms in (43) are of lower order compared to to G_r. Specifically, we demand that

$$\frac{\nu}{32}\frac{1}{\tau^2} > c_7\, \tau^{r-2s}\, J_0^{\frac12}\, G_r^{\frac12} + c_6\, \tau^{\frac{r}{2}-s-1}\, J_0^{\frac14}\, G_r^{\frac14} + c_8\bigl(1+\tau^{2(r-s)}\bigr) G_r\,, \tag{44}$$

where $s \in [\frac{r}{2}, \frac{r}{2}+1)$ is fixed. Due to the result of Section 3, we can assume without loss of generality that $u_0 \in \mathcal{D}(\mathrm{e}^{\sigma A}\colon H^r)$ for some $\sigma > 0$. Note that G_r is an increasing function of τ, so that, at the initial time $t = 0$, G_r is bounded between $\|A^r u_0\|^2$ when $\tau = \tau_0 = 0$ and $\|A^r \mathrm{e}^{\sigma A} u\|^2$ when $\tau = \tau_0 = \sigma$. Thus, the left side of (44) diverges faster than the right side as $\tau \to 0$, so that we can satisfy condition (44) *at* $t = 0$ by choosing $\tau_0 \in (0, \sigma]$ small enough. Moreover, the differential inequality (43) admits a local smooth solution, so that (44) is already satisfied *near* $t = 0$. We will consider τ_0 fixed in this way from now on.

So as long as (44) is satisfied, we only need to solve the linear differential inequality

$$\dot{G}_r \leq -\frac{\delta}{\tau^2}\, G_r + \frac{c_4}{\tau^{2(r+1)}}\, J_0\,, \tag{45}$$

where $\delta = \nu/16$. According to Assumption 1, $J_0 \leq M_1 (\tau_0/\tau)^{2\gamma}$ provided α, in (40), is chosen sufficiently small. The final form of the differential inequality is then

$$\dot{G}_r \leq -\frac{\delta}{\tau^2}\, G_r + \frac{k}{\tau^{2(\gamma+r+1)}}\,. \tag{46}$$

The integrating factor for this linear differential inequality is

$$\exp\biggl(\delta \int_0^t \frac{1}{\tau_0^2 + \alpha t'}\, \mathrm{d}t'\biggr) = \biggl(\frac{\tau_0^2 + \alpha t}{\tau_0^2}\biggr)^{\frac{\delta}{\alpha}}\,, \tag{47}$$

so that

$$\frac{\mathrm{d}}{\mathrm{d}t}\bigl(\tau^{2\delta/\alpha}\, G_r\bigr) \leq k\, \tau^{2(\delta/\alpha - \gamma - r - 1)}\,. \tag{48}$$

If we fix α, in (40), small enough so that $\delta > \alpha(\gamma + r)$, we can conclude that

$$G_r(t) \leq \biggl(G_r(0) - \frac{k}{\delta - \alpha(\gamma+r)}\frac{1}{\tau_0^{2(\gamma+r)}}\biggr)\biggl(\frac{\tau_0^2}{\tau^2}\biggr)^{\frac{\delta}{\alpha}} + \frac{k}{(\delta - \alpha(\gamma+r))}\frac{1}{\tau^{2(\gamma+r)}}\,. \tag{49}$$

Provided we can show that condition (44) remains satisfied for all t, estimate (49) will be global in time. Here it is sufficient to show that

$$\frac{32}{\nu}\tau^2\Bigl(c_7\, \tau^{r-2s}\, J_0^{\frac{1}{2}}\, G_r^{\frac{1}{2}} + c_6\, \tau^{\frac{1}{2}-s-1}\, J_0^{\frac{1}{4}}\, G_r^{\frac{1}{4}} + c_8\bigl(1 + \tau^{2(r-s)}\bigr) G_r\Bigr) \leq g(\tau) \tag{50}$$

for some non-increasing function $g(\tau)$. Estimate (49) shows that this is the case whenever $\gamma > 0$ and

$$G_r(0) > \frac{k}{\delta - \alpha(\gamma+r)}\frac{1}{\tau_0^{2(\gamma+r)}}\,; \tag{51}$$

in other words, if the constant M_1 in Assumption 1, which depends on norms of the initial data, is small enough. This can always be achieved by waiting long enough before initializing (49). Note that assumption (6) guarantees that $G_r(0)$, and thus the choice of τ_0, do not need change as we wait to let M_1 decrease.

Altogether, we obtain that

$$G_r(t) \leq \frac{c_6}{\tau^{2(\gamma+r)}} + O\bigl(\tau^{-2\delta/\alpha}\bigr)\,. \tag{52}$$

The upper bound on the decay of $\|A^m u\|$, equation (12), is now a direct consequence of Lemma 9 with $p = r$ and $q = m$, i.e.,

$$\begin{aligned} \|A^m u\|^2 &\leq c(m,r)\, \tau^{r-2m}\, J_0^{\frac{1}{2}}\, G_r^{\frac{1}{2}} \\ &\leq c(m,r)\, \tau^{r-2m}\, M_1^{\frac{1}{2}} \biggl(\frac{\tau_0}{\tau}\biggr)^{\gamma} \biggl(\frac{c_6}{\tau^{2(\gamma+r)}} + O\bigl(\tau^{-2\delta/\alpha}\bigr)\biggr)^{\frac{1}{2}} \\ &\leq c_9\, c(m,r)\, \frac{1}{\tau^{2(\gamma+m)}}\bigl(1 + O(\tau^{\gamma+r-\delta/\alpha})\bigr)\,, \end{aligned} \tag{53}$$

where $c(m,r)$ is given by (30), c_9 is independent of m, and $\gamma + r < \delta/\alpha$.

Remark 8. Estimate (52) shows that the radius of uniform analyticity of a decaying solution to the Navier–Stokes equations increases like $\sqrt{t}$.

Remark 9. The restriction $\frac{n}{2} > s \geq \frac{r}{2}$, which we imposed in the estimation of the nonlinear term, equation (42), and is subsequently used in the context of (44) to determine the initial value for τ, is technical and can be removed, for example, by interpolating the nonlinear term down to $\|A^m u\|$ for some real number $m > 0$, and using (53) to obtain the result for general r. In particular, it will be important in the next section that (52) holds with r replaced by $r+1$.

5. Lower Bounds

Let v denote the solution to the heat equation (2). Suppose Assumption 2 is satisfied. Without loss of generality we can take $t_0 = 0$, and assume that $u_0 \in \mathcal{D}(\mathrm{e}^{\sigma A}\colon H^r) \cap L^1(\mathbb{R}^n)$ for some $\sigma > 0$. Our goal is to prove that the difference $w \equiv u - v$ between Navier–Stokes and heat solution in $\|A^m \cdot\|$ can be made sufficiently small so that u must decay at the same rate.

We first derive an estimate on the difference w in $\mathcal{D}(\mathrm{e}^{\tau A}\colon H^r)$. Clearly, w satisfies

$$\partial_t w = \nu \Delta w - u \cdot \nabla u - \nabla p \tag{54a}$$

$$\nabla \cdot w = 0\,. \tag{54b}$$

As the heat equation preserves the divergence condition, we also have $\nabla \cdot w = 0$ for all $t \geq 0$. We now proceed exactly as in the previous section. Setting

$$\mathcal{J}_r = \|A^r w\|_{L^2}^2\,, \tag{55}$$

$$\mathcal{G}_r = \|A^r \mathrm{e}^{\tau A} w\|_{L^2}^2\,, \tag{56}$$

and repeating the steps leading to equation (38), we find

$$\begin{aligned}
\tfrac{1}{2}\,\dot{\mathcal{G}}_r &\leq \dot\tau\, \mathcal{G}_r^{\frac{1}{2}}\, \mathcal{G}_{r+1}^{\frac{1}{2}} - \nu\, \mathcal{G}_{r+1} + c_1 \big(G_s^{\frac{1}{2}} + G_r^{\frac{1}{2}}\big) G_{r+1}^{\frac{1}{2}}\, G_r^{\frac{1}{2}} + c_1 \big(G_{s+1}^{\frac{1}{2}} + G_{r+1}^{\frac{1}{2}}\big) G_r \\
&= \left(\frac{1}{2}\,\frac{\dot\tau}{\tau} - \frac{\nu}{8}\,\frac{1}{\tau^2}\right)\mathcal{G}_r + \left(\frac{1}{2}\,\dot\tau\tau - \frac{\nu}{8}\right)\mathcal{G}_{r+1} \\
&\quad - \frac{\nu}{16}\,\frac{1}{\tau^2}\,\mathcal{G}_r + \frac{c_3}{2}\,\frac{\nu}{\tau^{2(r+1)}}\,\mathcal{J}_0 + O\left(\frac{1}{\tau^{3\gamma+5r/2+1}}\right).
\end{aligned} \tag{57}$$

The order of decay of the "nonlinear" terms arises from (52) by choosing the smallest possible $s = \frac{r}{2}$. Recall that, through appropriate choice of α in Section 4, the second term on the right of (52) is of higher order. Thus, for $r \leq 2$, the second of the nonlinear terms can be estimated by

$$\begin{aligned}
\big(G_{s+1}^{\frac{1}{2}} + G_{r+1}^{\frac{1}{2}}\big) G_r &= \big(G_{r/2+1}^{\frac{1}{2}} + G_{r+1}^{\frac{1}{2}}\big) G_r \\
&= G_{r/2+1}^{\frac{1}{2}}\, G_r + \text{higher order terms} \\
&\leq O\left(\frac{1}{\tau^{\gamma+r/2+1}}\right) O\left(\frac{1}{\tau^{2(\gamma+r)}}\right) + \text{higher order terms}\,.
\end{aligned} \tag{58}$$

When $r > 2$, $G_{r/2+1}$ has to be interpolated first, for example, through use of Lemmas 8 and 9. The first of the nonlinear terms is estimated in a similar way.

With the choice of τ as in Section 4, the first two terms on the right of (57) can be neglected. Moreover, by using (4) we obtain the differential inequality

$$\dot{\mathcal{G}}_r \leq -\frac{\delta}{\tau^2}\,\mathcal{G}_r + \frac{\varepsilon\, c_8}{\tau^{2(\gamma+r+1)}} + O\biggl(\frac{1}{\tau^{3\gamma+5r/2+1}}\biggr)\,, \tag{59}$$

which can be integrated in exactly the same way as in Section 4. We find that

$$\mathcal{G}_r(t) \leq \frac{\varepsilon\, c_9}{\tau^{2(\gamma+r)}} + O\biggl(\frac{1}{\tau^{3\gamma+5r/2+3}}\biggr) + O\biggl(\frac{1}{\tau^{2\alpha/\delta}}\biggr)\,. \tag{60}$$

Finally, we employ Lemma 9 exactly as in the derivation of estimate (53) at the end of Section 4, thereby obtaining

$$\|A^m w\|^2 \leq \frac{\varepsilon\, c_{11}(m,r)}{\tau^{2(\gamma+m)}} + \text{higher order terms}\,. \tag{61}$$

For a given m we only need choose ε small enough so that $M_3(m) > \varepsilon c_{11}$, whence the triangle inequality implies a lower bound on the decay of $\|A^m u\|$ as stated in Theorem 4.

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