Accelerating fronts in autocatalysis

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We consider a reaction-diffusion system modelling propagating fronts of an autocatalytic reaction of order m in a one-dimensional, infinitely extended medium. The Lewis number, i.e. the ratio of the molecular diffusivity of the autocatalyst to that of the reactant, is arbitrary. We prove that if the initial profile of the front decays exponentially or algebraically with exponent $\mu > 1/(m-1)$, the speed of the front is bounded for all times. Our method relies on weighted Lebesgue and Sobolev-space estimates, from which we can reconstruct pointwise results for the decay of the front via interpolation. The result gives both, a functional analytic foundation, and an extension to arbitrary Lewis numbers, to the numerical studies of Sherratt & Marchant (IMA J. Appl. Math. **56**, 1996, pp. 289–302) and the asymptotic analysis of Needham & Barnes (Nonlinearity **12**, 1999, pp. 41–58).

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1. Introduction

We study solutions to the reaction-diffusion system

$$\partial_t \psi = \Delta\psi + (1-\psi)\, f(\theta)\,, \tag{1.1a}$$

$$\partial_t \theta = \ell\, \Delta\theta + (1-\psi)\, f(\theta)\,, \tag{1.1b}$$

on the real line with front-type boundary data where

$$\psi \to 1 \quad \text{and} \quad \theta \to 1 \quad \text{as } x \to -\infty, \tag{1.1c}$$

$$\psi \to 0 \quad \text{and} \quad \theta \to 0 \quad \text{as } x \to +\infty, \tag{1.1d}$$

for all $t \geq 0$, and compatible smooth initial front profiles $\psi(x,0) \equiv \psi_0(x)$ and $\theta(x,0) \equiv \theta_0(x)$. This equation describes propagating fronts of autocatalytic reactions. It has also been proposed as a simple model for combustion, population genetics, and epidemiological infections—see, for example, Aronson & Weinberger (1978), Volpert *el al.* (1994), Billingham & Needham (1991), and references cited therein.

In the case of autocatalysis, the dynamic variables represent the concentration of the autocatalyst, $\theta = \theta(x,t)$, and the concentration of another reactant expressed in the form $1-\psi$, where $\psi = \psi(x,t)$. All variables are non-dimensionalized. The positive parameter ℓ is the Lewis number, which is the ratio of the molecular diffusivity of the autocatalyst to that of the reactant. Here we consider chemical reactions of order m, so that $f(\theta) = \theta^m$ for $\theta \geq 0$ and zero otherwise. In our analysis, we suppose that $m \geq 2$, but m need not be integer. The physical interpretation of ψ and θ requires that

$$0 \leq \psi \leq 1 \quad \text{and} \quad 0 \leq \theta \,. \tag{1.2}$$

Indeed, the parabolic maximum principles imply that these properties hold globally in time provided they are true initially. We assume such bounds throughout the paper. Finally, there exist unique global classical solutions (Collet & Xin 1997). With this in mind, we shall only present formal estimates, which can be easily made rigorous.

The boundary condition at $-\infty$ describes the state where all the reactant has burned up and reaction has ceased, while at $+\infty$ no reaction has yet taken place. Therefore the *front*, which, roughly speaking, is the region where the gradients of ψ and θ are largest, must move to the right. In this paper we address the question of whether the speed at which the front is moving remains finite for all times, or increases without bound as $t \to \infty$, in which case we say that the front *accelerates.*

Our investigation is motivated by the work of Sherratt & Marchant (1996), who numerically analysed the scalar Fisher–Kolmogorov equation,

$$\partial_t \theta = \partial_{xx}\theta + (1-\theta)\,\theta^m \,. \tag{1.3}$$

This equation can be seen as an instance of system (1.1) in the special case when $\ell = 1$ and the initial profiles for ψ and θ are identical. They found that when the initial profile decays exponentially or algebraically with exponent $\mu > 1/(m-1)$, the solution evolves into a steadily propagating travelling-wave. However, when $\mu < 1/(m-1)$, the front accelerates. Moreover, the front region appears to stretch, with the overall slope of the front decreasing as portions of the front where the concentration field θ is small propagate at a faster rate. Needham & Barnes (1999) have recently provided a detailed asymptotic analysis of this effect. The case of quadratic nonlinearity has been previously discussed by Billingham & Needham (1992).

Some heuristic understanding can be gleamed from a scaling argument. Rewrite the Fisher–Kolmogorov equation in a translating coordinate frame $\xi = x - ct$,

$$\partial_t \theta = \partial_{\xi\xi}\theta + c\,\partial_\xi \theta + (1-\theta)\,\theta^m \,. \tag{1.4}$$

Suppose that initially, $\theta \sim \xi^{-\mu}$ as $\xi \to \infty$. Then for large values of ξ we have that $\partial_\xi \theta \sim -\xi^{-\mu-1}$ and $(1-\theta)\theta^m \sim \xi^{-\mu m}$, while the diffusion term is formally of lower order. Hence when $\mu > 1/(m-1)$, the negative linear term will dominate and θ will decrease, possibly under the condition that c be large enough to balance the influence of diffusion over long time scales. On the other hand, if $\mu < 1/(m-1)$, the reaction term dominates near infinity no matter how large we choose the translating speed c.

The results for the Fisher–Kolmogorov model can also be understood rigorously as follows. A family of travelling-wave solutions $\theta(x,t) = \theta_c(\xi)$, invariant under translation, exists on a half-axis of wave-speeds. The travelling-wave of minimum speed decays exponentially, $\theta_c(\xi) \sim \exp(-c\xi)$ as $\xi \to \infty$, whereas those of higher wave-speed decay algebraically

$$\theta_c(\xi) \sim \left(\frac{c}{(m-1)\xi} \right)^{\frac{1}{m-1}} \,. \tag{1.5}$$

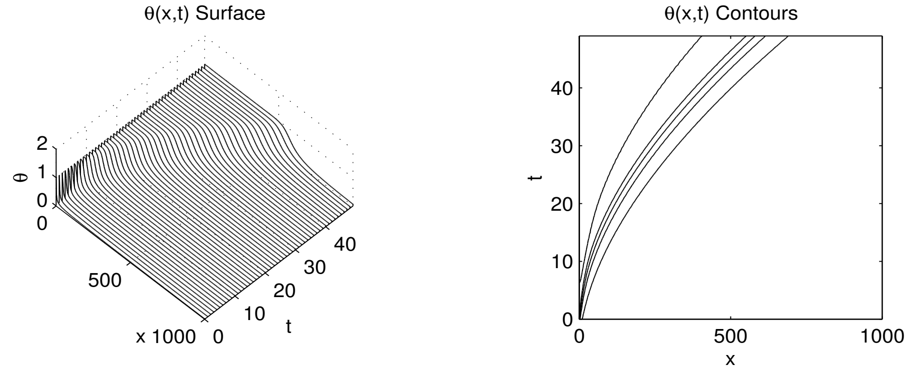

Figure 1. Accelerating front with $\mu = 0.6$, $m = 2$, $\ell = 4$, and $x_{\max} = 2000$. All simulations are initialized with $\psi_0(x) = \theta_0(x) = (1+x)^{-\mu}$ on the interval $[0, x_{\max}]$ and have numerical boundary conditions $\psi = \theta = 1$ at $x = 0$ and $\psi = \theta = 0$ at $x = x_{\max}$. The code is based on EPDCOL by Keast & Muir (1991) with 1000 grid points. We plot the surface $\theta(x,t)$ on the left and the corresponding isoconcentration contours on the right.

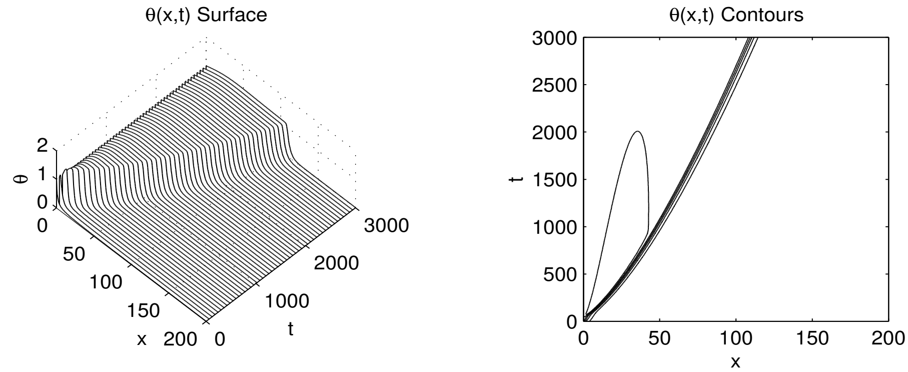

Figure 2. Non-accelerating front with $\mu = 1$, $m = 10$, $\ell = 0.1$, and $x_{\max} = 500$.

Further, a solution with monotone initial data, exponentially close to a travelling-wave for large ξ, will converge to a translate of that wave (see Volpert *et al.* 1994). Hence, when θ_0 is sufficiently well-behaved at $x = -\infty$ and decays exponentially or decays algebraically faster than, or even critically at $x = \infty$, it will develop into a travelling-wave. By using a comparison principle in the spirit of Rothe (1978), we also know that if a solution initially dominates another one on the whole space, this order relationship is preserved under the evolution. We can thus compare any solution against travelling-wave solutions: If the rate of decay is slower than critical, then θ_0 dominates translates of travelling-waves of any wave-speed, so the front accelerates.

The main concern of this paper is the case when $\ell \neq 1$. The basic phenomenology is similar to $\ell = 1$. For initial data of slow algebraic decay, the front accelerates as in Figure 1. When the decay at infinity is sufficiently fast, the evolving front may approach a steady travelling-wave as in Figure 2—the $\ell \neq 1$ travelling-waves

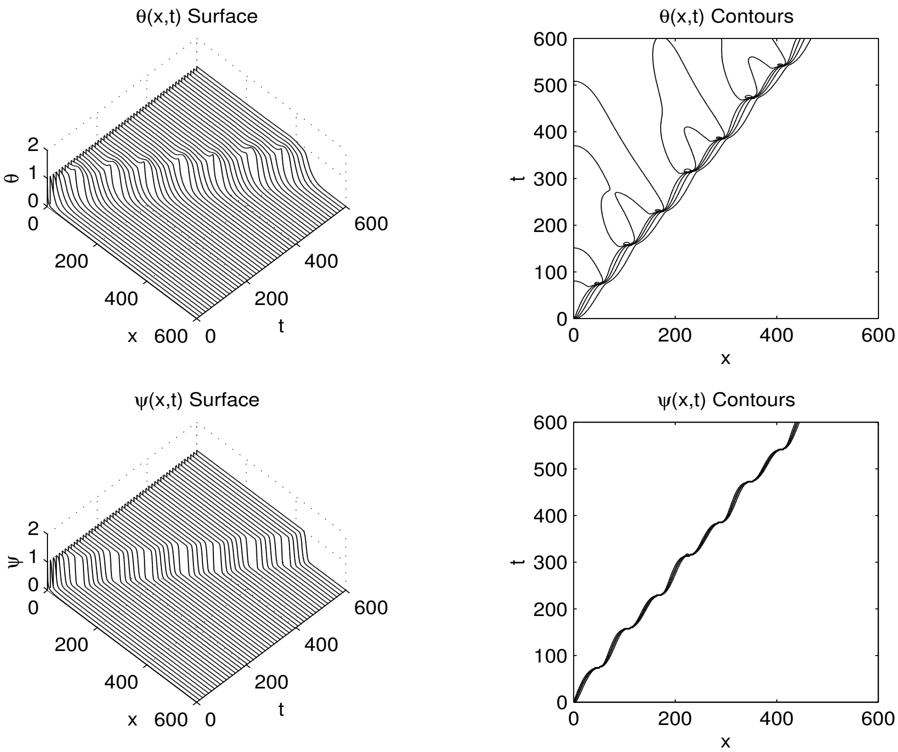

Figure 3. Pulsating, non-accelerating front with $\mu = 1$, $m = 10$, $\ell = 10$, and $x_{\max} = 2000$. The front pulsates, with $\theta(x,t)$ oscillating above and below $\theta = 1$ while the unsteady front velocity appears to remain uniformly bounded. The front is steeper in the ψ variable because $\ell > 1$.

have similar properties to the Fisher-Kolmogorov travelling-waves (Billingham & Needham 1991). However, when the Lewis number and the order of reaction are large, such data may also evolve into a front that starts to pulsate as illustrated in Figure 3. In this parameter regime, the underlying travelling-waves become unstable to small perturbations, a phenomenon known as the pulsating instability (Metcalfe *et al.* 1994; Balmforth *et al.* 1999). As the order of reaction m is further increased, period-doubling bifurcations occur, and the front pulsations eventually become chaotic. On the other hand, for any given set of parameter values for ℓ and m, travelling-waves of sufficiently fast wave-speed $c > c_*(\ell, m)$, where $c_*^2 \geq \sqrt{32}m \cdot \max\{1, \ell\}$, are asymptotically stable with shift (Takase & Sleeman 1999). Of course, when ℓ and m are large, then $c_* > c_{\min}$, where $c_{\min}$ denotes the minimum speed of at which travelling-waves exist, and instabilities occur.

The existence of the pulsating instability precludes using a comparison principle to bound solutions componentwise by translates of travelling-waves. In fact, if this were possible, one would have Lyapunov stability of travelling-waves at once. This difficulty can also be understood as follows. When $\ell \neq 1$, the system is of mixed-monotone, monostable type. Comparison principles readily extend to monotone systems (Volpert *et al.* 1994), and have also been proved for mixed-monotone systems by Lu & Sleeman (1993). However, the character of these results is such that travelling-waves do not qualify as 'sub' or 'super' solutions unless the 'sub' and 'super' solution coincide. Hence, a direct application of the comparison principle only yields trivial information.

In this paper we avoid these difficulties by replacing maximum principles, i.e. L^∞ estimates, with energy-type estimates. The basic idea is to prove that the solution to (1.1) remains bounded in a weighted integral norm, so that integrability against an exponentially (or algebraically) growing weight proves exponential (or algebraic) decay of the solution at infinity in the sense that the 'area under the front' becomes exponentially (or algebraically) small far ahead of the front.

In §2 we implement the idea in a simple L^2 setting, and are able to derive a first result for exponential weights. However, to close up this type of estimate in the case of algebraic weights one needs pointwise control on the solution, something that can only be achieved by taking estimates in higher spaces, specifically weighted H^1 and L^{2n} spaces for n sufficiently large. Pointwise bounds can then be reconstructed via an interpolation inequality.

Our main result, valid for all positive Lewis numbers, is the following. If the initial front profile has a pointwise exponential upper bound, the solution will decay exponentially for all times. If the initial front profile decays algebraically with exponent $\mu > 1/(m-1)$, the evolving front will decay faster than any $\nu \in (1/(m-1), \mu)$. In both cases the speed of the front is bounded uniformly in time. For a more formal statement we refer the reader to Theorems 3.3 and 3.4 respectively.

The use of energy estimates has two intrinsic drawbacks. First, the reconstruction of pointwise bounds from integral bounds, which is an essential step in our closure, fails *at* the critical exponent, even for $\ell = 1$. However, we can get results *up to* the critical exponent with bounds that diverge in the limit. As a consequence, we do not expect our methods to give sharp, or even close, upper bounds on the wave speed. Second, we cannot prove that for $\mu < 1/(m-1)$ the front must accelerate. Such a result would require lower bounds on our norms, which are notoriously difficult to achieve via energy-type estimates. We believe these limitations are technical rather than intrinsic properties of the system. An interesting open question, therefore, is whether there is a comparison principle for certain carefully constructed 'sub' and 'super functionals' of the solution. This would allow a more direct proof parallel to the proof available when $\ell = 1$.

2. L^2 estimates for exponentially decaying fronts

To simplify notation, we write $u = (\psi, \theta)^T$ and rescale the equation so that the coefficients of molecular diffusion in equations (1.1a) and (1.1b) are $1/p$ and $1/p^*$, respectively, where $p = 1 + \ell$ and $p^* = 1 + 1/\ell$ are Hölder conjugate exponents. Then the system expressed in the travelling-wave coordinate $\xi = x - ct$ is

$$\partial_t u + Lu = F(u)\,, \tag{2.1}$$

where

$$L = -\Lambda \partial_{\xi\xi} - c\partial_\xi\,, \quad \Lambda = \operatorname{diag}\left(\frac{1}{p}, \frac{1}{p^*}\right), \quad \text{and} \quad F(u) = \begin{pmatrix}1\\1\end{pmatrix}(1-\psi)\,\theta^m\,. \tag{2.2}$$

We introduce weighted L^2 spaces whose weight function ω is continuous, piecewise continuously differentiable, positive, and strictly monotonically increasing. Furthermore, we require that the growth of ω is at most exponential, i.e. there exists a positive constant k such that for almost every $\xi \in \mathbb{R}$

$$\omega'(\xi) \leq k\, \omega(\xi)\,. \tag{2.3}$$

For a scalar function ϕ we set

$$\|\phi\|^2_{L^2(\omega)} = \int_{\mathbb{R}} |\phi(\xi)|^2\, \omega(\xi)\, \mathrm{d}\xi\,. \tag{2.4}$$

In particular, $L^2(\omega)$ is the completion of C_0^∞ in this norm. For vector-valued functions $u, v \colon \mathbb{R} \to \mathbb{R}^2$ we have the canonical weighted inner product,

$$\langle u, v \rangle_{L^2(\omega)} = \int_{\mathbb{R}} (u^T v)(\xi)\, \omega(\xi)\, \mathrm{d}\xi\,, \tag{2.5}$$

as well as an inner product with a matrix weight Ω,

$$\langle u, v \rangle_{\mathbb{L}^2(\Omega)} = \int_{\mathbb{R}} u^T(\xi)\, \Omega(\xi)\, v(\xi)\, \mathrm{d}\xi\,. \tag{2.6}$$

The corresponding norms and spaces are defined as in the scalar case. In the following, we set $\Omega = B\omega$, where ω is as above, and

$$B = \begin{pmatrix} p^3\, p^* & -p \\ -p & 1 \end{pmatrix}\,. \tag{2.7}$$

Here B has been chosen so that it is positive definite and also satisfies two other properties that will become apparent below.

By taking the $\mathbb{L}^2(\Omega)$ inner product of equation (2.1) with u, we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathbb{L}^2(\Omega)} + \langle u, Lu \rangle_{\mathbb{L}^2(\Omega)} = \int_{\mathbb{R}} u^T \Omega F(u)\, \mathrm{d}\xi\,. \tag{2.8}$$

The contribution of the linear operator L is rewritten through integration by parts,

$$\begin{aligned} \langle u, Lu \rangle_{\mathbb{L}^2(\Omega)} &= -\int_{\mathbb{R}} u^T \Omega \Lambda \partial_{\xi\xi} u\, \mathrm{d}\xi - c \int_{\mathbb{R}} u^T \Omega \partial_\xi u\, \mathrm{d}\xi \\ &= \int_{\mathbb{R}} \partial_\xi u^T \Omega \Lambda \partial_\xi u\, \mathrm{d}\xi + \int_{\mathbb{R}} u^T \Omega' \Lambda \partial_\xi u\, \mathrm{d}\xi + \frac{c}{2} \int_{\mathbb{R}} u^T \Omega' u\, \mathrm{d}\xi\,. \end{aligned} \tag{2.9}$$

Noting that $pp^* \geq 4$, one can easily check that the matrix $B\Lambda$ is positive definite, and therefore the first integral on the right is equivalent to the canonical ω-weighted L^2 norm of $\partial_\xi u$.

Further, since ω is monotonically increasing, the last integral on the right of (2.9) is positive definite. Thus, the mixed term can be estimated in terms of the others by using the Cauchy–Schwarz inequality for vectors and then for integrals,

$$\begin{aligned} \int_{\mathbb{R}} u^T (\Omega')^{1/2}\, (\Omega')^{1/2} \Lambda \partial_\xi u\, \mathrm{d}\xi &\leq \int_{\mathbb{R}} \bigl(u^T \Omega' u\bigr)^{1/2}\, \bigl(\partial_\xi u^T \Lambda \Omega' \Lambda \partial_\xi u\bigr)^{1/2}\, \mathrm{d}\xi \\ &\leq \|u\|_{\mathbb{L}^2(\Omega')}\, \|\partial_\xi u\|_{\mathbb{L}^2(\Lambda\Omega'\Lambda)} \\ &\leq k\, \|u\|_{\mathbb{L}^2(\Omega')} \|\partial_\xi u\|_{\mathbb{L}^2(\Lambda\Omega\Lambda)}\,. \end{aligned} \tag{2.10}$$

In the last step we have also used (2.3). The matrix weights, $\Lambda\Omega\Lambda$ and $\Omega\Lambda$, induce equivalent norms. So applying the Young inequality and inserting the result back into (2.9) finally gives

$$2\,\langle u, Lu\rangle_{\mathbb{L}^2(\Omega)} \geq \|\partial_\xi u\|^2_{\mathbb{L}^2(\Omega\Lambda)} + (c - c_1)\,\|u\|^2_{\mathbb{L}^2(\Omega')}\,, \tag{2.11}$$

where c_1 depends on k, B, and Λ.

The contribution to (2.8) from the nonlinear term, written out explicitly, is

$$\int_{\mathbb{R}} u^T \Omega F(u)\,\mathrm{d}\xi = \int_{\mathbb{R}} \bigl[p\,(p^2p^* - 1)\,\psi - (p-1)\,\theta\bigr]\,(1-\psi)\,\theta^m\,\omega\,\mathrm{d}\xi\,. \tag{2.12}$$

Since $0 \leq \psi \leq 1$, the term in square brackets can only be positive for $\theta < \kappa \equiv p(p^2p^* - 1)/(p-1)$. For those values of θ we neglect the term $(p-1)\theta$ so that

$$\begin{aligned}\int_{\mathbb{R}} u^T \Omega F(u)\,\mathrm{d}\xi &\leq p\,(p^2p^*-1) \int_{\mathbb{R}} (1-\psi)\,\theta^m\,\omega\,\mathrm{d}\xi \\ &\leq p\,(p^2p^*-1)\,\kappa^{m-2}\,\|u\|^2_{\mathbb{L}^2(\Omega')}\,\sup_{\xi\in\mathbb{R}} \frac{\omega(\xi)}{\omega'(\xi)}\,. \end{aligned} \tag{2.13}$$

For example, if the choose $\omega(\xi) = \exp(k\xi)$, we can close the differential inequality for the $\mathbb{L}^2(\Omega)$-norm;

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_\xi u\|^2_{\mathbb{L}^2(\Omega\Lambda)} + (c - c_2)\,\|u\|^2_{\mathbb{L}^2(\Omega')} \leq 0\,. \tag{2.14}$$

Thus, for translating velocities $c > c_2$, the $\mathbb{L}^2(\Omega)$-norm is exponentially decreasing. In other words, the frame of reference of our exponential weight moves faster than the front. Thus we have proved the following.

Theorem 2.1. *Assume that u_0 is square integrable against an exponential weight function. Then there exists a maximal front speed $c_{\max}$ so that all features moving faster than $c_{\max}$ are decaying in the exponentially weighted L^2 sense.*

This statement is unsatisfactory for two reasons. First, it does not give pointwise bounds on the shape of the front, and second, the result is not easily extended to the case of algebraic decay. If we simply replaced ω by an algebraically increasing function, we cannot close up the estimate because the supremum in (2.13) becomes infinite. More severely, however, for decay near the critical exponent found by Sherratt & Marchant, the estimate would require a decreasing weight, thereby reversing the sign of every term that contains ω'. Thus a weighted L^2-norm is too restrictive, but does nonetheless, introduces the basic idea without the technicalities that appear for higher order norms.

In particular, the L^2 estimate illustrates how the matrix B was chosen: In addition to B being positive definite, we also used that $B\Lambda$ is positive definite so that the contribution from the diffusive part of L is of positive sign. Finally, the coefficients of ψ and θ in (2.12) must be positive so that large values of θ dampen rather than drive the estimate. We will re-encounter generalized versions of these three conditions when we construct higher order norms in Lemma 3.1 below.

3. H^1 and L^{2n} estimates

Consider the Ω-weighted inner product of $\partial_\xi u$ with the ξ-derivative of the vector form of the reaction-diffusion equation (2.1). One gets

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_\xi u\|^2_{\mathbb{L}^2(\Omega)} + \langle \partial_\xi u, L\partial_\xi u\rangle_{\mathbb{L}^2(\Omega)} = \int_{\mathbb{R}} \partial_\xi u^T \Omega \partial_\xi F(u)\,\mathrm{d}\xi\,. \tag{3.1}$$

The contribution from the linear part is estimated literally as in the previous section,

$$2\,\langle \partial_\xi u, L\partial_\xi u\rangle_{\mathbb{L}^2(\Omega)} \geq \|\partial_{\xi\xi} u\|^2_{\mathbb{L}^2(\Omega\Lambda)} + (c - c_1)\,\|\partial_\xi u\|^2_{\mathbb{L}^2(\Omega')}\,. \tag{3.2}$$

In the integral on the right of (3.1), we differentiate through and take absolute values. Since ψ is bounded, we need only keep the contributions from θ, i.e.,

$$\begin{aligned}
\int_{\mathbb{R}} \partial_\xi u^T \Omega \partial_\xi F(u)\,\mathrm{d}\xi &= \int_{\mathbb{R}} \partial_\xi u^T \Omega\,(DF)(u)\,\partial_\xi u\,\mathrm{d}\xi \\
&\leq K_1 \int_{\mathbb{R}} |\partial_\xi u|^2 \left(\theta^{m-1} + \theta^m\right)\omega\,\mathrm{d}\xi \\
&\leq K_2(\kappa) \int_{\theta\leq\kappa} |\partial_\xi u|^2\,\theta^{m-1}\,\omega\,\mathrm{d}\xi + K_3(\kappa)\int_{\theta\geq\kappa} |\partial_\xi u|^2\,\theta^m\,\omega\,\mathrm{d}\xi\,.
\end{aligned} \tag{3.3}$$

For the moment, we suppose κ to be any positive constant—we will fix it later on when we close the estimate. The goal is to absorb the contribution from $\theta \leq \kappa$ into the convecting term of (3.1). To do this, consider the following estimate

$$\int_{\theta\leq\kappa} |\partial_\xi u|^2\,\theta^{m-1}\,\omega\,\mathrm{d}\xi \leq \int_{\theta\leq\kappa} |\partial_\xi u|^2\,\omega'\,\mathrm{d}\xi \cdot \sup_{\xi\,:\,\theta\leq\kappa} \frac{\theta^{m-1}\,\omega}{\omega'}\,. \tag{3.4}$$

Hence, we can achieve this closure provided we can find a bound on the supremum on the right of (3.4). Since we cannot yet handle the last term in (3.4), we summarize the differential inequality for the H^1-seminorm:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_\xi u\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_{\xi\xi} u\|^2_{\mathbb{L}^2(\Omega\Lambda)} + (c - c_2)\,\|\partial_\xi u\|^2_{\mathbb{L}^2(\Omega')} \leq K_3 \int_{\theta\geq\kappa} |\partial_\xi u|^2\,\theta^m\,\omega\,\mathrm{d}\xi\,, \tag{3.5}$$

where

$$c_2 = c_1 + \sup_{\xi\,:\,\theta\leq\kappa} \frac{\theta^{m-1}\,\omega}{\omega'}\,. \tag{3.6}$$

The most difficult part of the programme is to obtain an L^q-estimate for u, where q must be large enough to gain control over the remaining term on the right in estimate (3.5). A general expression for a weighted L^{2n}-norm, for n integer, is

$$\|u\|^{2n}_{\mathbb{L}^{2n}(\varpi)} = \int_{\mathbb{R}} P_{2n}(u(\xi))\,\varpi(\xi)\,\mathrm{d}\xi\,, \tag{3.7}$$

where $P \equiv P_{2n}$ is a homogeneous polynomial in ψ and θ of degree $2n$. We impose several conditions on P, one is that P be positive definite, so that the norm defined above is equivalent to the canonical weighted norm on L^{2n}. However, the structure of our problem makes it convenient to impose further conditions on P, which make the choice of its coefficients non-trivial. This technique was introduced by Malham & Xin (1998) and is summarized in the following lemma.

Lemma 3.1. *Assume that $0 \le \psi \le 1$ and $0 \le \theta$ and set $u = (\psi, \theta)^T$. Then for every $n \ge 1$ there exists a homogeneous polynomial $P(u)$ of degree $2n$ in ψ and θ such that the following are true.*

(P1) $P(u)$ is positive definite.

(P2) There exists a constant $\kappa > 0$ such that $\partial_\theta P + \partial_\psi P \le 0$ whenever $\theta \ge \kappa$.

(P3) The polynomial $v^T D^2 P \Lambda v$ is positive definite as a $2n$-form, i.e. there exists a constant $\delta > 0$ such that for every $v \in \mathbb{R}^2$,

$$v^T D^2 P \Lambda v \ge \delta\, |v|^2\, |u|^{2n-2}\,. \tag{3.8}$$

The proof is not difficult, but technical, and will be given in §4. We use that $\partial_t(P(u)) = (DP)(u)\partial_t u$, where $DP = (\partial_\psi P, \partial_\theta P)$, and insert the expression for $\partial_t u$ from our reaction-diffusion model (2.1). Then integration against ϖ gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^{2n}_{\mathbb{L}^{2n}(\varpi)} + \int_{\mathbb{R}} (DP)(u)\, Lu\, \varpi\, \mathrm{d}\xi = \int_{\mathbb{R}} (DP)(u)\, F(u)\, \varpi\, \mathrm{d}\xi\,. \tag{3.9}$$

We integrate by parts to rewrite the contribution from the linear term,

$$\begin{aligned}\int_{\mathbb{R}} (DP)(u)\, Lu\, \varpi\, \mathrm{d}\xi &= -\int_{\mathbb{R}} (DP)(u)\, \Lambda\, \partial_{\xi\xi} u\, \varpi\, \mathrm{d}\xi - c\int_{\mathbb{R}} \partial_\xi (P(u))\, \varpi\, \mathrm{d}\xi \\ &= \int_{\mathbb{R}} \partial_\xi u^T\, (D^2 P \Lambda)(u)\, \partial_\xi u\, \varpi\, \mathrm{d}\xi + \int_{\mathbb{R}} (DP)(u)\, \Lambda\, \partial_\xi u\, \varpi'\, \mathrm{d}\xi + c\, \|u\|^{2n}_{\mathbb{L}^{2n}(\varpi')}\,.\end{aligned} \tag{3.10}$$

The first integral is nonnegative due to (P3), while the second has to be estimated:

$$\begin{aligned}\int_{\mathbb{R}} (DP)(u)\, \Lambda\, \partial_\xi u\, \varpi'\, \mathrm{d}\xi &\le K \int_{\mathbb{R}} |u|^{2n-1}\, |\partial_\xi u|\, \varpi'\, \mathrm{d}\xi \\ &\le K \left(\int_{\mathbb{R}} |u|^{2n-2}\, |\partial_\xi u|^2\, \varpi'\, \mathrm{d}\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |u|^{2n}\, \varpi'\, \mathrm{d}\xi \right)^{\frac{1}{2}}.\end{aligned} \tag{3.11}$$

We apply property (P3) and the bound on ϖ', equation (2.3), to the first integral. Due to (P1), the second integral defines a norm that is equivalent to the $\mathbb{L}^{2n}(\varpi')$-norm. Hence, by applying the Young inequality, we find that

$$\int_{\mathbb{R}} (DP)(u)\, Lu\, \varpi\, \mathrm{d}\xi \ge \frac{\delta}{2} \int_{\mathbb{R}} |\partial_\xi u|^2\, |u|^{2n-2}\, \varpi\, \mathrm{d}\xi + (c - c_3)\, \|u\|^{2n}_{\mathbb{L}^{2n}(\varpi')}\,. \tag{3.12}$$

Let us now turn to the contribution from the nonlinearity. Due to (P2),

$$\begin{aligned}\int_{\mathbb{R}} (DP)(u)\, F(u)\, \varpi \,\mathrm{d}\xi &\le \int_{\theta\le\kappa} (\partial_\theta P + \partial_\psi P)\,(1-\psi)\,\theta^m\,\varpi\,\mathrm{d}\xi \\ &\le K\, \|u\|^{2n}_{\mathbb{L}^{2n}(\varpi')} \cdot \sup_{\xi\colon\,\theta\le\kappa} \frac{\theta^{m-1}\,\varpi}{\varpi'}\,. \end{aligned} \tag{3.13}$$

Altogether, the $\mathbb{L}^{2n}$ estimate now reads

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u\|^{2n}_{\mathbb{L}^{2n}(\varpi)} + \frac{\delta}{2}\int_{\mathbb{R}} |\partial_\xi u|^2\, |u|^{2n-2}\,\varpi\,\mathrm{d}\xi + (c-c_4)\,\|u\|^{2n}_{\mathbb{L}^{2n}(\varpi')} \le 0\,, \tag{3.14}$$

where

$$c_4 = c_3 + K\cdot \sup_{\xi\colon\,\theta\le\kappa} \frac{\theta^{m-1}\,\varpi}{\varpi'}\,. \tag{3.15}$$

In the following we will combine this $\mathbb{L}^{2n}$ estimate with the H^1-seminorm estimate given by inequality (3.5). Since we are only concerned with the behaviour of the solution at $+\infty$, it is convenient to introduce truncated weights

$$\omega^{[r]}(\xi) = \begin{cases} \omega^r(\xi) & \text{for } \xi \ge 1, \\ \omega^r(1)\,\mathrm{e}^{\xi-1} & \text{for } \xi < 1. \end{cases} \tag{3.16}$$

Thus, if ω is an admissible weight (see the beginning of §2) on $[1,\infty)$, then $\omega^{[r]}$ is an admissible weight on $\mathbb{R}$ for every $r>0$.

For exponentially decaying initial data, we set $\zeta = \exp\xi$, and consider (3.5) with $\omega(\xi) = \zeta^{[2\nu]}$, and also (3.14) with $\varpi(x) = \zeta^{[2n\nu]}$. With this choice, the suprema occurring in each of the inequalities are immediately bounded in terms of a constant that depends only on κ, n, and ν. Further, if $n \ge m/2+1$ and inequality (3.5) is multiplied with a sufficiently small positive constant ε, then the term on the right of (3.5) is controlled by the dissipation term in the $\mathbb{L}^{2n}$ inequality. Altogether,

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u\|^{2n}_{\mathbb{L}^{2n}(\varpi)} + \varepsilon\,\|\partial_\xi u\|^2_{\mathbb{L}^2(\Omega)}\Big) + (c-c_5)\,\Big(\|u\|^{2n}_{\mathbb{L}^{2n}(\varpi')} + \varepsilon\,\|\partial_\xi u\|^2_{\mathbb{L}^2(\Omega')}\Big) \le 0\,. \tag{3.17}$$

Since $\varpi' \le k\varpi$ with $k = \min\{1,\nu\}$, this last inequality implies the exponential bound

$$\Big(\|u\|^{2n}_{\mathbb{L}^{2n}(\varpi)} + \varepsilon\,\|\partial_\xi u\|^2_{\mathbb{L}^2(\Omega)}\Big) \le \Big(\|u_0\|^{2n}_{\mathbb{L}^{2n}(\varpi)} + \varepsilon\,\|\partial_\xi u_0\|^2_{\mathbb{L}^2(\Omega)}\Big)\,\mathrm{e}^{(c_5-c)kt}\,. \tag{3.18}$$

By choosing a translating velocity $c > c_5$, we 'outrun' the propagating front. So if the solution is initially exponentially decaying in the averaged sense of the norms in (3.18), this property is maintained for all times.

Moreover, we can obtain a pointwise result as follows. Suppose that $|u_0(\xi)| \le A_0\exp(-\mu\xi)$ and satisfies the boundary condition of the equation. Then $u_0 \in L^{2n}(\zeta^{[2n\nu]})$ and $\partial_\xi u_0 \in L^2(\zeta^{[2\nu]})$ for every $\nu < \mu$. For c sufficiently large, independent of A_0, the norm of u in these spaces decays exponentially in time. We can then reconstruct a pointwise exponentially decaying bound with any spatial decay exponent less than ν by applying the following interpolation lemma with $\omega = \zeta$, $\lambda = n\nu$, and $\sigma < \nu$.

Lemma 3.2. *Let ω be an admissible weight function, defined on $[1,\infty)$, and let $\phi \in C^1(\mathbb{R})$ with $\phi(\xi) \to$ const as $\xi \to -\infty$. Let $n > 1$ and $\sigma, \lambda \geq 0$ be real numbers. Then*

$$\sup_{\xi\in\mathbb{R}} \bigl|\phi^{n+1}(\xi)\,\omega^{[\lambda+\sigma]}(\xi)\bigr| \leq C_1\,\|\partial_\xi \phi\|^2_{L^2(\omega^{[2\sigma]})} + C_2\,\|\phi\|^{2n}_{L^{2n}(\omega^{[2\lambda]})} + C_3\,M(\omega)\,, \tag{3.19}$$

provided that

$$M(\omega) = \omega^{\frac{2n\sigma-2\lambda}{n-1}}(1) + \int_1^\infty |\partial_\xi \omega|^{\frac{2n}{n-1}}\,\omega^{-2\frac{\lambda+n-\sigma n}{n-1}}(\xi)\,\mathrm{d}\xi \tag{3.20}$$

is finite. A possible choice of constants is $C_1 = (n+1)/2$, $C_3 = \max\{1, \lambda+\sigma\}$, and $C_2 = C_1 + C_3$.

The proof of this lemma is given in §4. We now summarize our result for exponentially decaying fronts.

Theorem 3.3. *Let $\zeta = \exp\xi$. Suppose that u solves equation (1.1) with initial data $u_0 \in L^{2n}(\zeta^{[2n\nu]})$ for some $n \geq m/2+1$ and that $\partial_\xi u_0 \in L^2(\zeta^{[2\nu]})$. Then the velocity of the front is bounded by a constant $c_{\max}$ which may depend on n and ν, but is otherwise independent of u_0, in the sense that*

$$\|u\|_{L^{2n}(\zeta^{[2n\nu]})} \to 0 \quad \textit{and} \quad \|\partial_\xi u\|_{L^2(\zeta^{[2\nu]})} \to 0 \tag{3.21}$$

exponentially as $t \to \infty$ whenever $c > c_{\max}$.

Moreover, if $u_0(\xi) \leq A_0 \exp(-\mu\xi)$ for $\xi > 0$, then for every $\nu < \mu$ there exists a function $A(\nu,t)$ and a constant $c_{\max}$ such that

$$|u(\xi,t)| \leq A(\nu,t)\,\mathrm{e}^{-\nu\xi} \tag{3.22}$$

and $A(\nu,t) \to 0$ for $t \to \infty$ whenever $c > c_{\max}$; in this case $c_{\max}$ may depend on μ but is independent of A_0 and ν.

For algebraically decaying fronts we combine the H^1-seminorm estimate with $\omega = \xi^{[2\nu+1]}$ and the $\mathbb{L}^{2n}$ estimate with $\varpi = \xi^{[2n\nu-1]}$. That this choice of weights is the 'right' one can be seen immediately from a simple scaling argument: Assume that $|u| \sim \xi^{-\mu}$, then the exponents on ξ in the integrands of the two norms become identical exactly in the critical case when $\nu = \mu$, and integrability fails.

To absorb the term on the right of (3.5) into the diffusion term of the $\mathbb{L}^{2n}$-estimate, we must now require, in addition to $n \geq m/2+1$, that $2\nu+1 \leq 2n\nu-1$, which can easily be satisfied. Specifically, if $\nu > 1/(m-1)$, a sufficient condition is $n \geq m$.

The suprema in (3.6) and (3.15) both have a bound of the form

$$\sup_{\xi\colon \theta\leq\kappa} \frac{\theta^{m-1}\,\xi^{[r]}}{\partial_\xi \xi^{[r]}} \leq K(r)\,\max\Bigl\{\kappa^{m-1}, \sup_{\xi\geq 1}\bigl|\theta^{m-1}\,\xi\bigr|\Bigr\}\,. \tag{3.23}$$

Set

$$\lambda = \frac{2n\nu-1}{2\nu(m-1)}\,, \qquad \sigma = \frac{2\nu+1}{2\nu(m-1)}\,, \tag{3.24}$$

to use Lemma 3.2 for the supremum on the right of (3.23):

$$\sup_{\xi\geq 1}\left|\theta^{m-1}\,\xi\right| \leq \Big(\sup_{\xi\in\mathbb{R}}\left|\theta^{n+1}\,\xi^{[\lambda+\sigma]}\right|\Big)^{\frac{m-1}{n+1}}$$
$$\leq \Big(C_1\,\|\partial_\xi\theta\|^2_{L^2(\xi^{[2\sigma]})} + C_2\,\|\theta\|^{2n}_{L^{2n}(\xi^{[2\lambda]})} + C_3\,M\Big)^{\frac{m-1}{n+1}} \tag{3.25}$$

where

$$M = 1 + \int_1^\infty \xi^{-1+\frac{(n+1)(1-\nu(m-1))}{(n-1)\nu(m-1)}}\,\mathrm{d}\xi \tag{3.26}$$

is finite whenever the Sherratt & Marchant bound $\nu > 1/(m-1)$ is satisfied. Under this condition, we also have that $2\lambda < 2n\nu-1$ and $2\sigma < 2\nu+1$, so that we altogether obtain a bound of the form

$$\sup_{\xi\colon\,\theta\leq\kappa}\frac{\theta^{m-1}\,\xi^{[r]}}{\partial_\xi\xi^{[r]}} \leq G\Big(\|\theta\|_{L^{2n}(\xi^{[2n\nu-1]})} + \|\partial_\xi\theta\|_{L^2(\xi^{[2\nu+1]})}\Big)\,, \tag{3.27}$$

where G is a bounded function, and has coefficients which also depend on κ, $\nu/(m-1)$, and n.

This allows us to close up the estimate. For ε sufficiently small, we get (possibly absorbing further constants into the expression for G)

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big(\|u\|^{2n}_{\mathbb{L}^{2n}(\varpi)} + \varepsilon\,\|\partial_\xi u\|^2_{\mathbb{L}^2(\Omega)}\Big) + (c-G)\,\Big(\|u\|^{2n}_{\mathbb{L}^{2n}(\varpi')} + \varepsilon\,\|\partial_\xi u\|^2_{\mathbb{L}^2(\Omega')}\Big) \leq 0\,. \tag{3.28}$$

where

$$G = G\Big(\|u\|_{\mathbb{L}^{2n}(\varpi)} + \varepsilon\,\|\partial_\xi u\|_{\mathbb{L}^2(\Omega)}\Big) \tag{3.29}$$

Hence, provided that c is chosen sufficiently large, depending on the initial magnitude of the arguments of G, these same quantities are non-increasing in time and the estimate extends to arbitrary later times in a self-consistent way. Let us summarize.

Theorem 3.4. *Let $\nu > 1/(m-1)$ and $n \geq m$. Suppose that u solves equation* (1.1) *with initial data $u_0 \in L^{2n}(\xi^{[2n\nu-1]})$ and that $\partial_\xi u_0 \in L^2(\xi^{[2\nu+1]})$. Then the velocity of the front is bounded by a constant $c_{\max}$ which may depend on ν, m, and n, as well as on the norms of u_0 in the given spaces.*

Moreover, if $|u_0(\xi)| \leq A_0\,\xi^{-\mu}$ for $\xi \geq 1$ and $\mu > 1/(m-1)$, then for every $\nu \in (1/(m-1),\mu)$, there exists an $A(\nu)$ and a constant $c_{\max}$ such that $|u(\xi,t)| \leq A(\nu)\,\xi^{-\nu}$ for all $t \geq 0$ and translating velocities $c > c_{\max}$; in this case $c_{\max}$ may depend on m and μ as well as on A_0 and ν.

Remark 3.5. In particular, there is the possibility that $c_{\max} \to \infty$ as $\nu \to \mu$. This is different from the exponential case where, although the constant in the pointwise estimate may grow without bounds as $\nu \to \mu$, $c_{\max}$ is not affected.

Remark 3.6. Similarly, $c_{\max}$ may diverge as ν approaches the Sherratt & Marchant critical exponent $1/(m-1)$. We speculate that this is an artifact of our method because results based on comparison theorems that are available for $\ell = 1$ do not exhibit this behaviour.

4. Proofs of Lemma 3.1 and Lemma 3.2

Proof of Lemma 3.1. We claim that a possible choice is

$$P(\theta,\psi) = \sum_{k=0}^{2n} (-1)^k\, \alpha_k\, \theta^k\, \psi^{2n-k}\,, \tag{4.1}$$

when the positive coefficients $\alpha_0, \dots, \alpha_{2n}$ are given by

$$\alpha_k \equiv \begin{cases} (36\, n^2\, pp^*)^{n-k/2} & \text{for } k \text{ even,} \\ 4n\, \alpha_{k+1} & \text{for } k \text{ odd.} \end{cases} \tag{4.2}$$

Condition (P1) can be checked directly. To verify (P2), we first consider

$$\begin{aligned} \partial_\theta P + \partial_\psi P &= \sum_{k=0}^{2n-1} (-1)^k \big[\alpha_k\, (2n-k) - \alpha_{k+1}\, (k+1)\big]\theta^k\, \psi^{2n-k-1} \\ &= \sum_{\substack{k \text{ even} \\ k \le 2n-2}} \Big[\big(\alpha_k\, (2n-k) - \alpha_{k+1}\, (k+1)\big)\psi \\ &\qquad - \big(\alpha_{k+1}\, (2n-k-1) - \alpha_{k+2}\, (k+2)\big)\theta\Big]\theta^k\, \psi^{2n-k-2}\,. \end{aligned} \tag{4.3}$$

As $\psi \le 1$, we only need make sure that

$$\alpha_{k+1}\, (2n-k-1) - \alpha_{k+2}\, (k+2) > 0 \tag{4.4}$$

for $k \le 2n-2$ even, or, equivalently, that

$$\alpha_k > \frac{k+1}{2n-k}\, \alpha_{k+1} \tag{4.5}$$

for k odd. By taking

$$\alpha_k = 4n\, \alpha_{k+1} \tag{4.6}$$

when k is odd, we always satisfy this condition.

To verify (P3), set $v = (x,y)^T$. A direct calculation gives

$$\begin{aligned} v^T D^2 P \Lambda v &= \sum_{k=0}^{2n} (-1)^k\, \alpha_k\, \theta^{k-2}\, \psi^{2n-k-2} \\ &\quad \cdot \left(\frac{k(k-1)}{p}\, x^2\, \psi^2 + (2n-k)k\, xy\, \psi\theta + \frac{(2n-k)(2n-k-1)}{p^*}\, y^2\, \theta^2 \right)\,. \end{aligned} \tag{4.7}$$

The most difficult task is to bound the cross terms proportional to xy. When xy is positive, the ‘bad’ terms are those for which k is odd. In this case we neglect all cross terms for k even, and control the others with a third fraction of those neighbouring quadratic terms that have the same homogeneity in θ and ψ. We must therefore seek that the sum of their coefficients be non-negative, i.e.,

$$\frac{\alpha_{k-1}}{3}\, \frac{(2n-k+1)(2n-k)}{p^*}\, y^2 - \alpha_k\, (2n-k)k\, xy + \frac{\alpha_{k+1}}{3}\, \frac{(k+1)k}{p}\, x^2 \ge 0\,. \tag{4.8}$$

A sufficient condition for the discriminant of this quadratic form in x and y to be non-negative is that

$$4\,\alpha_{k-1}\,\alpha_{k+1} \geq 9\,pp^*\,\alpha_k^2\,, \tag{4.9}$$

or, taking relation (4.6) into account, that

$$\alpha_{k-1} \geq 36\,n^2\,pp^*\,\alpha_{k+1}\,. \tag{4.10}$$

The case when this last relation is satisfied as an equality, and $\alpha_{2n} \equiv 1$, corresponds to the choice made above in (4.2).

On the other hand, when xy is negative, the 'bad' cross terms correspond to k even, with $2 \leq k \leq 2n-2$. Each of these terms can be controlled by half of the neighbouring 'good' cross terms. Therefore, for the corresponding sum of coefficients we must require that

$$\frac{\alpha_{k-1}}{2}\,(2n-k+1)(k-1)\,\psi^2 - \alpha_k\,(2n-k)k\,\psi\theta + \frac{\alpha_{k+1}}{2}\,(2n-k-1)(k+1)\,\theta^2 \geq 0\,. \tag{4.11}$$

The discriminant of this quadratic form in ψ and θ is non-negative whenever

$$\alpha_{k-1}\,\alpha_{k+1} \geq \frac{(2n-k)^2}{(2n-k-1)(2n-k+1)}\,\frac{k^2}{(k-1)(k+1)}\,\alpha_k^2\,, \tag{4.12}$$

which is weaker than (4.9).

It remains to be shown that the negative quadratic terms can be bounded as well. We first consider each negative term proportional to x^2 and bound it using a third of those neighbouring positive terms that are proportional to x^2 as well. Here k is odd, $k \geq 3$, and we seek that

$$\frac{\alpha_{k-1}}{3}\,\frac{(k-1)(k-2)}{p}\,\psi^2 - \alpha_k\,\frac{k(k-1)}{p}\,\psi\theta + \frac{\alpha_{k+1}}{3}\,\frac{(k+1)k}{p}\,\theta^2 \geq 0\,. \tag{4.13}$$

The discriminant of this quadratic form in ψ and θ is non-negative whenever

$$4\,\alpha_{k-1}\,\alpha_{k+1} \geq 9\,\frac{k}{k+1}\,\frac{k-1}{k-2}\,\alpha_k^2\,. \tag{4.14}$$

Since $pp^* \geq 4$, this condition is also weaker than (4.9). Finally, the negative terms proportional to y^2 can be estimated in exactly the same way due to the symmetry of the summation in (4.7).

By carefully keeping track of all the terms in (4.7), we notice that in the process of cancelling out the negative terms, we have used up all the positive quadratic terms but a third of each of the corner terms in the scheme, namely

$$\frac{2n(2n-1)}{p^*}\,y^2\,\psi^{2n-2}, \quad \frac{2}{p}\,x^2\,\psi^{2n-2}, \quad \frac{2n(2n-1)}{p}\,x^2\,\theta^{2n-2}, \quad \text{and}\ \frac{2}{p^*}\,y^2\,\theta^{2n-2}. \tag{4.15}$$

These remaining terms imply the bound on the right side of (3.8). $\square$

Proof of Lemma 3.2. By the fundamental theorem of calculus,

$$
\begin{aligned}
\big|\phi^{n+1}(\xi)\,\omega^{[\lambda+\sigma]}(\xi)\big| &\le \int_{\mathbb{R}} \left|\frac{\mathrm{d}}{\mathrm{d}\eta}\big(\phi^{n+1}(\eta)\,\omega^{[\lambda+\sigma]}(\eta)\big)\right| \mathrm{d}\eta \\
&\le (n+1)\int_{-\infty}^{1} |\partial_\xi\phi|\,|\phi|^n\,\omega^{\lambda+\sigma}(1)\,\mathrm{e}^{\xi-1}\,\mathrm{d}\xi + (n+1)\int_1^\infty |\partial_\xi\phi|\,|\phi|^n\,\omega^{\lambda+\sigma}(\xi)\,\mathrm{d}\xi \\
&\quad + \int_{-\infty}^{1} |\phi|^{n+1}\,\omega^{\lambda+\sigma}(1)\,\mathrm{e}^{\xi-1}\,\mathrm{d}\xi + (\lambda+\sigma)\int_1^\infty |\phi|^{n+1}\,|\partial_\xi\omega|\,\omega^{\lambda+\sigma-1}(\xi)\,\mathrm{d}\xi\,. \qquad (4.16)
\end{aligned}
$$

We apply first the Cauchy–Schwarz inequality, and then the Young inequality with conjugate exponents $1/2$ to each of the first two integrals. This yields the following four integrals,

$$
\begin{aligned}
&\frac{1}{2}\int_{-\infty}^{1} |\partial_\xi\phi|^2\,\omega^{2\sigma}(1)\,\mathrm{e}^{\xi-1}\,\mathrm{d}\xi + \frac{1}{2}\int_{-\infty}^{1} |\phi|^{2n}\,\omega^{2\lambda}(1)\,\mathrm{e}^{\xi-1}\,\mathrm{d}\xi \\
&\qquad + \frac{1}{2}\int_1^\infty |\partial_\xi\phi|^2\,\omega^{2\sigma}(\xi)\,\mathrm{d}\xi + \frac{1}{2}\int_1^\infty |\phi|^{2n}\,\omega^{2\lambda}(\xi)\,\mathrm{d}\xi\,, \qquad (4.17)
\end{aligned}
$$

which can be recombined into the norms proportional to C_1 on the right side of the interpolation inequality.

To estimate the third integral in (4.16), insert the product $\omega^q(1)\,\omega^{-q}(1)$ with $q=\lambda/n-\sigma$, and apply the Hölder and Young inequalities with conjugate exponents $2n/(n+1)$ and $2n/(n-1)$. One of the integrals can be solved exactly, and we obtain the terms

$$
\frac{n+1}{2n}\int_{-\infty}^{1} |\phi|^{2n}\,\omega^{2\lambda}(1)\,\mathrm{e}^{\xi-1}\,\mathrm{d}\xi + \frac{n-1}{2n}\,\omega^{\frac{2n\sigma-2\lambda}{n-1}}(1)\,. \qquad (4.18)
$$

For the last integral in (4.16), insert the product $\omega^q(\xi)\,\omega^{-q}(\xi)$ with $q=1-\sigma+\lambda/n$ into the second integral, and apply the Hölder and Young inequalities, again with conjugate exponents $2n/(n+1)$ and $2n/(n-1)$. This yields the terms

$$
\frac{n+1}{2n}\int_1^\infty |\phi|^{2n}\,\omega^{2\lambda}(\xi)\,\mathrm{d}\xi + \frac{n-1}{2n}\int_1^\infty |\partial_\xi\omega|^{\frac{2n}{n-1}}\,\omega^{-2\frac{\lambda+n-\sigma n}{n-1}}\,\mathrm{d}\xi \qquad (4.19)
$$

By combining (4.18) and (4.19), we obtain the remaining terms on the right of (3.19). □

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