JUSTIFICATION OF THE SHALLOW WATER LIMIT FOR A RIGID LID FLOW WITH BOTTOM TOPOGRAPHY

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ABSTRACT. The so-called lake equations arise as the shallow water limit of the rigid lid equations—three dimensional Euler equations with a rigid lid upper boundary condition—in a horizontally periodic basin with bottom topography. We prove an *a priori* estimate in the Sobolev space H^m for $m \geq 3$ which shows that a solution to the rigid lid equations can be approximated by a solution of the lake equations for an interval of time which can be estimated in terms of the initial deviation from a columnar configuration and the magnitude of the initial data in H^m, the gradient of the bottom topography in H^{m+1}, and the aspect ratio of the basin. In particular, any solution to the lake equations remains close to some solution of the rigid lid equations for an interval of time that can be made arbitrarily large by choosing the aspect ratio of the basin small.

1. INTRODUCTION

Shallow water theory is a technique widely applied to fluid dynamical problems in which it is important to separate motion on large scales from motion on smaller time and length scales. The basic procedure can be summarized as follows. One has a set of equations that are believed to accurately describe the physical processes of a system, and which we henceforth refer to as the *primitive equations*. The primitive equations are asymptotically expanded in powers of one or more small parameters, one of which is typically the aspect ratio of the system. The approximate equations at the leading order(s) of the expansion are thus naturally termed *shallow water equations*.

Derived in this way, the approximation is entirely formal. In other words, it is *a priori* not clear if and in what sense solutions to the shallow water equations represent solutions to the primitive equations. To make a mathematically rigorous statement, one proceeds in two steps. One must first establish a setting in which the simpler equations are globally, and the primitive equations are at least locally well posed. The second step is to derive so-called *a priori estimates*—formal estimates of (Sobolev) norms on the spatial domain—with the goal of obtaining bounds on the growth of the difference between solutions to the primitive and the shallow water equations corresponding to "similar" initial data.

The primitive equations under consideration here are—from a physical standpoint—the simplest possible, namely, the Euler equations describing the motion of an inviscid, incompressible fluid in three spatial dimensions. The fluid basin is a "periodic lake," i.e., it is periodic in the x_1 and x_2 direction, has a solid bottom boundary at $x_3 = -b(x_1, x_2)$ and an upper free surface boundary at $x_3 = h(x_1, x_2, t)$. The bottom function b is assumed to be nondegenerate, i.e., there are

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constants $b_{\min}$ and $b_{\max}$ such that, for every (x_1, x_2) on the two-torus $\mathbb{T}^2$,

$$0 < b_{\min} \le b(x_1, x_2) \le b_{\max} \,. \tag{1}$$

Gravity is acting in the negative x_3 direction, and the free upper surface is assumed to be at constant pressure. We are interested in the physical regime where the aspect ratio δ (the ratio of typical vertical to typical horizontal length scales) is small, the Froude number ε (the ratio of typical horizontal speeds to the speed of gravity waves) is small and the ratio of the surface amplitude to typical vertical length scales is of order ε^2. Lastly, we assume that the initial configuration to leading order in δ is columnar, i.e., is independent of the x_3 coordinate.

Then the evolution of the vertically averaged velocity field $u(x_1, x_2, t)$ is governed to leading order in δ and ε by the set of nondimensional equations

$$\partial_t u + (u \cdot \nabla) u + \nabla h = 0 \,, \tag{2a}$$

$$\nabla \cdot (bu) = 0 \,, \tag{2b}$$

$$u(0) = u^{\mathrm{in}} \,. \tag{2c}$$

This system, which we call the *lake equations*, has essentially been described by Greenspan [6, page 235]. Moreover, the evolution of $u(x_1, x_2, t)$ through order δ^2 is governed by the *great lake equations*, which are

$$\partial_t v - u^{\perp} \, \nabla \wedge v + \nabla (h - \tfrac{1}{2} u \cdot u + u \cdot v) = 0 \,, \tag{3a}$$

$$v = \mathcal{L} u \,, \tag{3b}$$

$$\nabla \cdot (bu) = 0 \,, \tag{3c}$$

$$u(0) = u^{\mathrm{in}} \,. \tag{3d}$$

We use the notation $u^{\perp} = (u_2, -u_1)$ and $\nabla \wedge v = \partial_1 v_2 - \partial_2 v_1$. The linear operator $\mathcal{L}$ is given by

$$\begin{aligned} v = \mathcal{L} u = u + \delta^2 \big(&(u \cdot \nabla b) \nabla b + \tfrac{1}{2} b (\nabla \cdot u) \, \nabla b \\ &- \tfrac{1}{2} b^{-1} \nabla (b^2 \, u \cdot \nabla b) - \tfrac{1}{3} b^{-1} \nabla (b^3 \nabla \cdot u) \big) \,. \end{aligned} \tag{4}$$

The great lake equations have been proposed by Camassa, Holm and Levermore [3, 4] who show that lake and great lake equations have a Hamiltonian structure, convect potential vorticity, and enjoy a Kelvin circulation theorem in complete analogy with the two dimensional Euler equations. These structural similarities have been exploited to prove that lake and great lake equations possess global weak solutions [12, 13, 14], global classical solutions [14, 15], and that analytic data propagates analytically [11, 14].

As we will see later, the existence of global classical solutions is of crucial importance for the current work. We state this result as a theorem.

Theorem 1. *Provided $b \in \mathrm{C}^{m+1}(\mathbb{T}^2)$ for some $m \ge 3$ and $u^{\mathrm{in}} \in \mathrm{H}^m(\mathbb{T}^2, \mathbb{R}^2)$, there exists a unique global classical solution u to the lake or great lake equations such that*

$$u \in \bigcap_{k=0}^{m-2} \mathrm{C}^k([0,\infty); \mathrm{H}^{m-k}(\mathbb{T}^2, \mathbb{R}^2)) \,. \tag{5}$$

Moreover, the H^m norm of u satisfies a bound that grows like $\exp(\exp t)$.

The lake and the great lake equations can be formally derived from the three dimensional Euler equations by first taking the low Froude number limit, and then asymptotically expanding in δ^2. The equations at the intermediate level are called the *rigid lid approximation*. In non-dimensional variables, posed on the thin domain

$$\{(x_1, x_2, x_3) \colon x_1, x_2 \in \mathbb{T}; -\delta b(x_1, x_2) < x_3 < 0\}, \tag{6}$$

they take the form

$$\partial_t u + u \cdot \nabla u + w \partial_3 u + \nabla p = 0\,, \tag{7a}$$

$$\partial_t w + u \cdot \nabla w + w \partial_3 w + \partial_3 p = 0\,, \tag{7b}$$

$$\nabla \cdot u + \partial_3 w = 0\,, \tag{7c}$$

$$p = h \qquad \text{for } x_3 = 0\,, \tag{7d}$$

$$w = 0 \qquad \text{for } x_3 = 0\,, \tag{7e}$$

$$w = -\delta\, u \cdot \nabla b \qquad \text{for } x_3 = -\delta b(x_1, x_2)\,. \tag{7f}$$

As before, u denotes the horizontal components of the velocity field; w is the vertical velocity component. The rigid lid equations differ from the three dimensional Euler equations only in the boundary conditions: The domain is fixed at the equilibrium surface height, while an approximation to the surface height h can be recovered through the pressure boundary condition (7d).

The justification of the rigid lid equations as the low Froude number limit of the three dimensional Euler equations with a free upper surface is still an open problem, which rests on finding a suitable setting in which the free surface problem is at least locally well-posed.

In this paper we show that the formal derivation of the lake equation as the low aspect ratio (shallow water) limit of the rigid lid equations can be made rigorous. We take advantage of the fact that the rigid equations are posed on a fixed domain, and are therefore known to be locally well-posed—for example in the Sobolev space H^m when $m \geq 3$ (see Kato [10]). By Theorem 1 we have the necessary global well-posedness for the lake equations in the same space. This permits us to prove that, in the H^m norm, a solution to the lake equations remains close to solutions of the rigid lid equations—endowed with corresponding initial data—on a time interval that can be made arbitrarily large by choosing the aspect ratio δ small. Vice versa, any solution to the rigid lid equations which is initially close to being columnar—and thus close to an admissible configuration of the lake equations—can be approximated by a solution to the lake equations for an interval of time which can be estimated in terms of the initial deviation from columnarity in H^m, the magnitude of the initial data in H^m, the gradient of the bottom function b in H^{m+1}, and the aspect ratio of the basin δ. This is summarized in the following theorem.

Theorem 2. *Let $\bar{u}$ denote a fixed solution to the lake equations* (2) *corresponding to initial data $\bar{u}^{\mathrm{in}}$ and bottom topography b, and let $\bar{w}$ be its reconstructed vertical velocity component. Let $(\hat{u}, \hat{w})$ denote some solution to the rigid lid equations* (7) *posed on the thin domain* (6) *with initial data $(\hat{u}^{\mathrm{in}}, \hat{w}^{\mathrm{in}})$. Let $m \geq 3$ be fixed.*

Then for every $T > 0$ and every error bound $\mathrm{E} > 0$ there exist Δ_1 and Δ_2 such that if

$$\|(\bar{u}^{\mathrm{in}}, \bar{w}^{\mathrm{in}}) - (\hat{u}^{\mathrm{in}}, \hat{w}^{\mathrm{in}})\|_{\mathrm{H}^m} < \Delta_1\,, \tag{8a}$$

and

$$\delta \left\| \nabla b \right\|_{\mathrm{H}^{m+1}} < \Delta_2 \,, \tag{8b}$$

then

$$\left\| (\bar{u}(t), \bar{w}(t)) - (\hat{u}(t), \hat{w}(t)) \right\|_{\mathrm{H}^m} < \mathrm{E} \tag{9}$$

for every $t \in [0, T]$.

In Section 3.4 we give an explicit bound on the H^m difference between lake and rigid lid solutions. Its rapid growth reflects the unstable nature of solutions to the lake equations as well as the finite time blow-up of all known H^m bounds for solutions of the rigid lid equations.

The paper is organized as follows. In Section 2 we derive the rigid lid equations and the lake equations in "equilibrium coordinates", which are a natural setting for the comparison of their respective solutions. *A priori* estimates on the difference of lake and rigid lid solutions are obtained in Section 3. The main difficulty is obtaining estimates on the difference of the lake and the rigid lid pressure, which solve different elliptic boundary value problems with data given in terms of the velocity fields. Section 4 concludes with a discussion of the result, and mentions several open problems.

2. Derivation of Rigid Lid and Lake Equations

The main point of this section is to set up notation that facilitates the comparison of rigid lid and lake equations. This is achieved through a choice of coordinates in which both systems assume the same structure. This approach leads to an alternative and very natural derivation of the lake equations. Since our result concerns only the aspect ratio scaling, we refrain from introducing the full set of physical scaling relations, and refer the reader instead to references [3, 4, 12].

2.1. Notation.

It is convenient to adopt the following notation conventions. Three dimensional column vectors are denoted by bold faced letters, while their horizontal components are set in normal type. Thus

$$\boldsymbol{x} = \begin{pmatrix} x \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \,, \qquad \boldsymbol{\nabla} = \begin{pmatrix} \nabla \\ \partial_3 \end{pmatrix} = \begin{pmatrix} \partial_1 \\ \partial_2 \\ \partial_3 \end{pmatrix} \,. \tag{10}$$

We systematically employ linear algebra notation where $\boldsymbol{x}^{\mathrm{t}}$ denotes the transpose of $\boldsymbol{x}$.

The possibly time dependent physical domain is given by

$$\Omega_t = \{ \boldsymbol{x} \in \mathbb{R}^3 : x \in \Omega_{\text{lake}}, -b(x) < x_3 < h(x,t) \} \tag{11}$$

and has total depth $d(x,t) = b(x) + h(x,t)$. Its top and bottom surface are denoted Γ_t^{top} and Γ_t^{bot}, respectively. Our goal is to convert the Euler problem to one on the flat domain

$$\Omega_{\text{eq}} = \Omega_{\text{lake}} \times [-1, 0] \tag{12}$$

with top and bottom surfaces again denoted by $\Gamma_{\text{eq}}^{\text{top}}$ and $\Gamma_{\text{eq}}^{\text{bot}}$. The horizontal domain of the basin Ω_{lake} is assumed to be the two-dimensional torus $\mathbb{T}^2$ to avoid technical problems arising from corner singularities. The formal calculations however can also be carried out on more general domains $\Omega_{\text{lake}} \subset \mathbb{R}^2$ with zero normal flow boundary conditions.

In this notation, the non-dimensionalized Euler equations in three spatial dimensions read

$$\partial_t \boldsymbol{u} + \boldsymbol{u}^{\mathrm{t}} \boldsymbol{\nabla} \boldsymbol{u} + \boldsymbol{\nabla} p = 0\,, \tag{13a}$$

$$\boldsymbol{\nabla}^{\mathrm{t}} \boldsymbol{u} = 0\,; \tag{13b}$$

and the top and bottom boundary conditions are

$$p = h \qquad \text{on } \Gamma_t^{\text{top}}\,, \tag{13c}$$

$$u_3 = \partial_t h + u^{\mathrm{t}} \nabla h \qquad \text{on } \Gamma_t^{\text{top}}\,, \tag{13d}$$

$$u_3 = -u^{\mathrm{t}} \nabla b \qquad \text{on } \Gamma_t^{\text{bot}}\,. \tag{13e}$$

2.2. **Transformation of the Continuity Equation.** The spatial coordinates are transformed simply by compression or stretching in the vertical direction, i.e., through the map $\boldsymbol{\theta}\colon \Omega_{\text{eq}} \to \Omega_t$

$$\boldsymbol{\theta}(\boldsymbol{x}) = \begin{pmatrix} x_1 \\ x_2 \\ h + x_3 d \end{pmatrix}\,. \tag{14}$$

To avoid losing the structure of the zero divergence condition (13b), we do not transform the velocity field directly by composition with $\boldsymbol{\theta}$, but rather define a vector field $\boldsymbol{v}$ on Ω_{eq} through

$$\boldsymbol{u}(\boldsymbol{\theta}) = M \boldsymbol{v}\,, \qquad \text{where } M = (\boldsymbol{\nabla} \boldsymbol{\theta}^{\mathrm{t}})^{\mathrm{t}}\,. \tag{15}$$

With this choice of M, the vector field $\boldsymbol{u}$ is divergence-free on Ω_t if and only if $\boldsymbol{v}$ satisfies a weighted divergence condition on Ω_{eq}, namely

$$\boldsymbol{\nabla}^{\mathrm{t}} (\det M\, \boldsymbol{v}) = \boldsymbol{\nabla}^{\mathrm{t}} (d \boldsymbol{v}) = 0\,. \tag{16}$$

A proof is easily achieved by noting that divergence free functions are L^2-orthogonal to gradients, so that for arbitrary ϕ,

$$\begin{aligned} 0 &= \int_{\Omega_t} \boldsymbol{u}^{\mathrm{t}}\, \boldsymbol{\nabla} \phi \, \mathrm{d}\boldsymbol{x} \\ &= \int_{\Omega_{\text{eq}}} \boldsymbol{u}^{\mathrm{t}}(\boldsymbol{\theta})\, (\boldsymbol{\nabla} \phi)(\boldsymbol{\theta})\, \det(\boldsymbol{\nabla} \boldsymbol{\theta}^{\mathrm{t}})\, \mathrm{d}\boldsymbol{x} \\ &= \int_{\Omega_{\text{eq}}} \boldsymbol{v}^{\mathrm{t}}\, \boldsymbol{\nabla}\big(\phi(\boldsymbol{\theta})\big)\, \det M \, \mathrm{d}\boldsymbol{x}\,. \end{aligned} \tag{17}$$

As ϕ was arbitrary, this shows that $\boldsymbol{v}$ is orthogonal to gradients with respect to the $(\det M)$-weighted L^2 product. This statement is equivalent to (16).

The transformation (15) has, for example, been used by Heywood [9] and Beale [2] with $M = (\boldsymbol{\nabla} \boldsymbol{\theta}^{\mathrm{t}})^{\mathrm{t}} / \det(\boldsymbol{\nabla} \boldsymbol{\theta}^{\mathrm{t}})$ as to preserve the zero divergence condition under transformation. However, because our goal is to justify the lake equations which have a weighted divergence condition, (15) is more natural here.

2.3. **Transformation of Momentum Equation and Boundary Conditions.** Starting from equation (15), we find by direct calculation

$$(\boldsymbol{\nabla}\boldsymbol{u}^{\mathrm{t}})(\boldsymbol{\theta}) = (\boldsymbol{\nabla}\boldsymbol{\theta}^{\mathrm{t}})^{-1}\,\boldsymbol{\nabla}(M\boldsymbol{v})^{\mathrm{t}}\,, \tag{18}$$

and

$$(\partial_t \boldsymbol{u})(\boldsymbol{\theta}) = \partial_t(M\boldsymbol{v}) - \frac{(1+x_3)\,\partial_t d}{d}\partial_3(M\boldsymbol{v})\,. \tag{19}$$

We define the pressure in the new coordinates by

$$p(\boldsymbol{\theta}) = q\,, \tag{20}$$

so that

$$(\boldsymbol{\nabla}p)(\boldsymbol{\theta}) = (\boldsymbol{\nabla}\boldsymbol{\theta}^{\mathrm{t}})^{-1}\boldsymbol{\nabla}q\,. \tag{21}$$

All terms in

$$\Big(\partial_t\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u}^{\mathrm{t}})^{\mathrm{t}}\boldsymbol{u} + \boldsymbol{\nabla}p\Big)\,(\boldsymbol{\theta}) = 0$$

can now be replaced, using (19), (18), (15) and (21) respectively, so that the momentum equation in the equilibrium coordinates reads

$$\partial_t(M\boldsymbol{v}) - \frac{(1+x_3)\,\partial_t d}{d}\partial_3(M\boldsymbol{v}) + \boldsymbol{v}^{\mathrm{t}}\,\boldsymbol{\nabla}(M\boldsymbol{v}) + (M^{-1})^{\mathrm{t}}\,\boldsymbol{\nabla}q = 0\,. \tag{22}$$

Finally, the boundary conditions become

$$\begin{aligned} q &= h && \text{on } \Gamma_{\mathrm{eq}}^{\mathrm{top}}\,, && (23\mathrm{a})\\ \partial_t h &= d\,v_3 && \text{on } \Gamma_{\mathrm{eq}}^{\mathrm{top}}\,, && (23\mathrm{b})\\ v_3 &= 0 && \text{on } \Gamma_{\mathrm{eq}}^{\mathrm{bot}}\,. && (23\mathrm{c}) \end{aligned}$$

2.4. **The Rigid Lid Equations on a Flat Domain.** The rigid lid equations are obtained from the Euler equations by letting the Froude number tend to zero. The precise scaling is explained in references [3, 4]. For our purposes it suffices to note that the rigid lid equations can be obtained from (13) by setting $h = 0$ everywhere but in the dynamical top boundary condition (23a). In the new variables, this gives the system

$$\begin{aligned} &M\partial_t\boldsymbol{v} + \boldsymbol{v}^{\mathrm{t}}\,\boldsymbol{\nabla}(M\boldsymbol{v}) + (M^{-1})^{\mathrm{t}}\,\boldsymbol{\nabla}q = 0 && \text{in } \Omega_{\mathrm{eq}}\,, && (24\mathrm{a})\\ &\boldsymbol{\nabla}^{\mathrm{t}}(b\boldsymbol{v}) = 0 && \text{in } \Omega_{\mathrm{eq}}\,, && (24\mathrm{b})\\ &q = h && \text{on } \Gamma_{\mathrm{eq}}^{\mathrm{top}}\,, && (24\mathrm{c})\\ &v_3 = 0 && \text{on } \Gamma_{\mathrm{eq}} \equiv \Gamma_{\mathrm{eq}}^{\mathrm{top}} \cup \Gamma_{\mathrm{eq}}^{\mathrm{bot}}\,, && (24\mathrm{d}) \end{aligned}$$

where M now takes the form

$$M = \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ x_3\partial_1 b & x_3\partial_2 b & b \end{pmatrix}\,. \tag{25}$$

2.5. **The Rigid Lid Equations on a Thin Domain.** We introduce a thin domain scaling into the rigid lid equations (24) by making the substitution $b \to \delta b$, where b is of order one and δ is small. As the horizontal scales are of order one, δ directly corresponds to the physical aspect ratio. The rescaled equations still only depend implicitly on δ through the transformation matrix M. However, it is useful to make this dependence explicit by defining new quantities

$$\boldsymbol{v}_\delta = \begin{pmatrix} v \\ \delta v_3 \end{pmatrix} , \qquad \boldsymbol{\nabla}_\delta = \begin{pmatrix} \nabla \\ \delta^{-1} \partial_3 \end{pmatrix} ;$$

and by introducing a corresponding transformation M_δ such that $M\boldsymbol{v} = M_\delta \, \boldsymbol{v}_\delta$. Thus

$$M_\delta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \delta x_3 \partial_1 b & \delta x_3 \partial_2 b & b \end{pmatrix} , \tag{26}$$

i.e., M_δ is a small perturbation of an order one diagonal matrix. In terms of the delta-indexed quantities, the rigid lid equations on a thin domain are

$$M_\delta \partial_t \boldsymbol{v}_\delta + \boldsymbol{v}_\delta^{\mathrm{t}} \boldsymbol{\nabla}_\delta (M_\delta \boldsymbol{v}_\delta) + (M_\delta^{-1})^{\mathrm{t}} \boldsymbol{\nabla}_\delta q = 0 \, , \tag{27a}$$

$$\boldsymbol{\nabla}_\delta^{\mathrm{t}} (b \boldsymbol{v}_\delta) = 0 \, . \tag{27b}$$

The boundary conditions (24c,d) remain unchanged.

2.6. **Derivation of the Lake Equations.** Upon taking the formal limit $\delta \to 0$ in (27), we see that

$$M_\delta \to \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b \end{pmatrix} \equiv M_0 \, . \tag{28}$$

Moreover, the vertical momentum equation formally requires $\partial_3 q = 0$ for all t, so that the horizontal momentum equation becomes

$$\partial_t v + \boldsymbol{v}^{\mathrm{t}} \boldsymbol{\nabla} v + \nabla h = 0 \, . \tag{29}$$

Columnar initial data, i.e. $\partial_3 \boldsymbol{v}^{\mathrm{in}} = 0$, remains columnar. This can be seen by taking ∂_3 of (29) and regarding the resulting equation as a linear evolution equation for $\partial_3 v$ which has $\partial_3 v = 0$ as its trivial solution.

In addition, columnar motion implies $v_3 = 0$, as can be seen by taking ∂_3 of the continuity equation (27b),

$$\partial_3^2 v_3 = -\partial_3 \nabla^{\mathrm{t}} (bv) = 0 \, ,$$

and applying boundary condition (24d).

In summary, in the limit $\delta \to 0$ and under the assumption of columnar motion, the rigid lid equations (27) reduce to the lake equations

$$\partial_t v + v^{\mathrm{t}} \nabla v + \nabla h = 0 \, , \tag{30a}$$

$$\nabla^{\mathrm{t}} (bv) = 0 \, . \tag{30b}$$

As we are to compare the lake with the rigid lid equations, it is practical to write the lake equations in terms of three-vectors,

$$M_0 \partial_t \boldsymbol{v} + \boldsymbol{v}^{\mathrm{t}} \boldsymbol{\nabla}_\delta (M_0 \boldsymbol{v}) + (M_0^{-1})^{\mathrm{t}} \boldsymbol{\nabla}_\delta q = 0 \, , \tag{31a}$$

$$\boldsymbol{\nabla}_\delta^{\mathrm{t}} (b \boldsymbol{v}) = 0 \, , \tag{31b}$$

where $q = q(x,t) = h(x,t)$ and $v_3 = 0$.

3. Estimates on the Deviation of the Lake from the Rigid Lid Flow

3.1. Derivation of the Pressure Equations and Elliptic Estimates. At any fixed time t, the pressure q of the rigid lid equations is given as the solution of an elliptic boundary value problem in terms of the velocity field $\boldsymbol{v}$. Written out explicitly, this system exhibits a rather complicated dependence on the bottom topography. Thus the main task of this section is to show how to bring the pressure equations into a simple abstract form.

Note that from now on all δ subscripts are dropped for easier reading, even though we will continue to carry all δ dependences through the calculation. In particular, all norms and gradients are defined with respect to the partial derivative $\delta^{-1}\partial_3$ in x_3 direction, so that

$$\langle \varphi, \psi \rangle_m \equiv \sum_{|\alpha| \leq m} \int_{\Omega_{\text{eq}}} D^\alpha \varphi \, D^\alpha \psi \, b \, \mathrm{d}\boldsymbol{x} \,, \tag{32}$$

where $D^\alpha \equiv \partial_1^{\alpha_1} \partial_2^{\alpha_2} (\delta^{-1}\partial_3)^{\alpha_3}$. We also set $\Omega = \Omega_{\text{eq}}$ and $\Gamma = \Gamma_{\text{eq}}$.

To solve for the pressure in the rigid lid equation, we left-multiply (27) with $\boldsymbol{\nabla}^{\mathrm{t}}(bM^{-1}\cdot)$ and $\boldsymbol{n}^{\mathrm{t}}(bM^{-1}\cdot)$ respectively, $\boldsymbol{n}$ being the outward normal on Γ, to obtain

$$\partial_t\big(\boldsymbol{\nabla}^{\mathrm{t}}(b\boldsymbol{v})\big) + \boldsymbol{\nabla}^{\mathrm{t}}\Big(bM^{-1}\big(\boldsymbol{v}^{\mathrm{t}}\boldsymbol{\nabla}(M\boldsymbol{v})\big)\Big) + \boldsymbol{\nabla}^{\mathrm{t}}\big(b(MM^{\mathrm{t}})^{-1}\boldsymbol{\nabla}q\big) = 0 \qquad \text{in } \Omega\,, \tag{33a}$$

$$\partial_t\big(\boldsymbol{n}^{\mathrm{t}}(b\boldsymbol{v})\big) + \boldsymbol{n}^{\mathrm{t}}\Big(bM^{-1}\big(\boldsymbol{v}^{\mathrm{t}}\boldsymbol{\nabla}(M\boldsymbol{v})\big)\Big) + \boldsymbol{n}^{\mathrm{t}}\big(b(MM^{\mathrm{t}})^{-1}\boldsymbol{\nabla}q\big) = 0 \qquad \text{on } \Gamma\,. \tag{33b}$$

The time derivative terms vanish due to the weighted divergence condition (27b) and the kinematic boundary condition (24d), respectively. An elementary calculation, again including use of the weighted divergence condition, shows that the second term in (33a) can be brought into the form

$$\begin{aligned}
\boldsymbol{\nabla}^{\mathrm{t}}\big(bM^{-1}(\boldsymbol{v}^{\mathrm{t}}\boldsymbol{\nabla})(M\boldsymbol{v})\big) &= \delta(\delta^{-1}\partial_3)\Big(x_3\, v^{\mathrm{t}}(\nabla\nabla^{\mathrm{t}} b)v + v_3\, v^{\mathrm{t}}\nabla b\Big) \\
&\quad + \Big(-\nabla^{\mathrm{t}}\big((v^{\mathrm{t}}\nabla b)v\big) + (\boldsymbol{\nabla}\boldsymbol{v}^{\mathrm{t}}) : \big(\boldsymbol{\nabla}(b\boldsymbol{v})\big)^{\mathrm{t}}\Big) \\
&\equiv \delta F'(\boldsymbol{v},\boldsymbol{v}) + F(\boldsymbol{v},\boldsymbol{v})\,.
\end{aligned} \tag{34}$$

Next, we decompose the negative of last term in (33a) into an elliptic second order operator L and a small perturbation $\delta L'$, given by

$$Lq = -\nabla^{\mathrm{t}}(b\nabla q) + \delta^{-1}\partial_3(b^{-1}\delta^{-1}\partial_3 q) \tag{35}$$

$$L'q = \nabla^{\mathrm{t}}(x_3\nabla b\delta^{-1}\partial_3 q) + \delta^{-1}\partial_3(x_3\nabla^{\mathrm{t}} b\nabla q) - \partial_3\Big(\frac{x_3^2}{b}|\nabla b|^2\delta^{-1}\partial_3 q\Big)\,. \tag{36}$$

The second and third terms in the boundary equation (33b) can be treated in a similar way. By using that $v_3 = 0$ on Γ we get

$$\begin{aligned}
\boldsymbol{n}^{\mathrm{t}}\big(bM^{-1}(\boldsymbol{v}^{\mathrm{t}}\boldsymbol{\nabla})(M\boldsymbol{v})\big) &= \begin{cases} 0 & \text{at } x_3 = 0 \\ \delta\, v^{\mathrm{t}}(\nabla\nabla^{\mathrm{t}} b)v & \text{at } x_3 = -1 \end{cases} \\
&\equiv \delta G'(\boldsymbol{v},\boldsymbol{v})\,.
\end{aligned} \tag{37}$$

Finally, the boundary operator $-\boldsymbol{n}^{\mathrm{t}}\big(b(MM^{\mathrm{t}})^{-1}\boldsymbol{\nabla}\cdot\big)$ on Γ decomposes into a normal boundary operator B and a small perturbation $\delta B'$, given by

$$Bq = \pm\delta^{-1}b^{-1}(\partial_3 q) \tag{38}$$

$$B'q = \begin{cases} 0 & \text{at } x_3 = 0 \\ (\nabla^{\mathrm{t}} b \nabla q) + \delta b^{-1}|\nabla b|^2(\delta^{-1}\partial_3 q) & \text{at } x_3 = -1\,. \end{cases} \tag{39}$$

In summary, we have found that the rigid lid velocity $\boldsymbol{v}$ and pressure q satisfy a relation of the form

$$(L + \delta L')q = F(\boldsymbol{v},\boldsymbol{v}) + \delta F'(\boldsymbol{v},\boldsymbol{v}) \qquad \text{in } \Omega\,, \tag{40a}$$

$$(B + \delta B')q = \delta G'(\boldsymbol{v},\boldsymbol{v}) \qquad \text{on } \Gamma\,, \tag{40b}$$

where L, L', B and B' are linear operators; F, F' and G' are bilinear forms. By applying standard Sobolev and trace inequalities to equations (34) to (39), one obtains the following set of estimates on the "data functionals" of (40).

Lemma 3. *For $m \geq 3$ and sufficiently smooth $\boldsymbol{u}$, $\boldsymbol{v}$ and q one has*

$$\|F(\boldsymbol{u},\boldsymbol{v})\|_{\mathrm{H}^{m-1}} \leq c_3\,\|b\|_{\mathrm{H}^{m+1}}\,\|\boldsymbol{u}\|_{\mathrm{H}^m}\,\|\boldsymbol{v}\|_{\mathrm{H}^m}\,, \tag{41a}$$

$$\|F'(\boldsymbol{v},\boldsymbol{v})\|_{\mathrm{H}^{m-1}} \leq c_4\,\|\nabla b\|_{\mathrm{H}^{m+1}}\,\|\boldsymbol{v}\|_{\mathrm{H}^m}^2\,, \tag{41b}$$

$$\|L'q\|_{\mathrm{H}^{m-1}} \leq c_5\big(\|\nabla b\|_{\mathrm{H}^m} + \delta c_6\,\|\nabla b\|_{\mathrm{H}^m}^2\big)\|q\|_{\mathrm{H}^{m+1}}\,, \tag{41c}$$

$$\|G'(\boldsymbol{v},\boldsymbol{v})\|_{\mathrm{H}^{m-1/2}(\Gamma)} \leq c_7\,\|\nabla\nabla^{\mathrm{t}} b\|_{\mathrm{H}^{m-1/2}}\,\|\boldsymbol{v}\|_{\mathrm{H}^m}^2\,, \tag{41d}$$

$$\|B'q\|_{\mathrm{H}^{m-1/2}(\Gamma)} \leq c_8\big(\|\nabla b\|_{\mathrm{H}^{m-1/2}} + \delta c_9\,\|\nabla b\|_{\mathrm{H}^{m-1/2}}^2\big)\|q\|_{\mathrm{H}^{m+1}}\,. \tag{41e}$$

On the other hand, we can investigate the left sides of (40) as an inhomogeneous elliptic boundary value problem.

Lemma 4. *There exists a δ_0 so that for every $\delta \leq \delta_0$, $m \geq 1$, $f \in \mathrm{H}^{m-1}(\Omega)$ and $g \in \mathrm{H}^{m-1/2}(\Gamma)$ the problem*

$$(L + \delta L')p = f \qquad \text{in } \Omega\,, \tag{42a}$$

$$(B + \delta B')p = g \qquad \text{on } \Gamma\,, \tag{42b}$$

has a solution p, unique up to a constant, which satisfies the estimate

$$\|p\|_{\mathrm{H}^{m+1}} \leq c_1\,\|f\|_{\mathrm{H}^{m-1}} + c_2\,\|g\|_{\mathrm{H}^{m-1/2}(\Gamma)}\,, \tag{43}$$

where c_1 and c_2 can be chosen independent of δ.

Proof. The statement follows essentially from the theory of elliptic boundary value problems as considered by Agmon, Douglis and Nirenberg [1] and Schechter [17].

Strong ellipticity of L is a direct consequence of the nondegeneracy of b. Moreover, B imposes a standard Neumann boundary condition on Γ. Thus we have an estimate of the form (43) for the problem

$$Lp = f \qquad \text{in } \Omega\,, \tag{44a}$$

$$Bp = g \qquad \text{on } \Gamma\,. \tag{44b}$$

The independence of the constants on δ can be seen by transforming the problem back onto the domain $\mathbb{T}^2 \times [-\delta, 0]$, effectively replacing $\delta^{-1}\partial_3$ by ∂_3 and thus completely removing δ from the equations and the norms. Moreover, the constants in the elliptic regularity estimate may depend on properties the boundary, but do not depend on the size of the domain [5].

This result is still true for the perturbed problem (42), because estimates (41c) and (41e) show that L' and B' are regular perturbations which can be made arbitrarily small by choosing δ_0 small. □

Lemmas 3 and 4 can be combined into the following.

Corollary 5. *There exists a δ_0 so that for every $\delta \le \delta_0$, $m \ge 3$ and $\boldsymbol{v} \in \mathrm{H}^m(\Omega)$, problem (40) has a solution q, unique up to a constant, which satisfies*

$$\|q\|_{\mathrm{H}^{m+1}} \le c\,\|\boldsymbol{v}\|_{\mathrm{H}^m}^2\,, \tag{45}$$

where c can be chosen independent of δ.

Corollary 6. *For the lake equations (31), q and $\boldsymbol{v}$ satisfy the problem*

$$Lq = F(\boldsymbol{v},\boldsymbol{v}) \qquad \text{in } \Omega\,, \tag{46a}$$

$$Bq = 0 \qquad \text{on } \Gamma\,. \tag{46b}$$

Proof. It is sufficient to observe that the explicit δ dependence of (40) is exclusively due to the δ dependence of M, and that we have $M = M_0$ for the lake equations. Moreover, all partial derivatives $\delta^{-1}\partial_3$ vanish, so that we can simply set $\delta = 0$ in (40) without any terms becoming singular. □

3.2. **Estimate on the Difference Between Rigid Lid and Lake Pressure.** Let $\hat q$ and $\hat{\boldsymbol{v}}$ denote the pressure and velocity fields of a solution to the rigid lid equations (27), i.e.

$$L\hat q = F(\hat{\boldsymbol{v}},\hat{\boldsymbol{v}}) + \delta F'(\hat{\boldsymbol{v}},\hat{\boldsymbol{v}}) - \delta L'\hat q \qquad \text{in } \Omega\,, \tag{47a}$$

$$B\hat q = \delta G'(\hat{\boldsymbol{v}},\hat{\boldsymbol{v}}) - \delta B'\hat q \qquad \text{on } \Gamma\,. \tag{47b}$$

The corresponding quantities $\bar q$ and $\bar{\boldsymbol{v}}$ for the lake equations (31) satisfy the equations

$$L\bar q = F(\bar{\boldsymbol{v}},\bar{\boldsymbol{v}}) \qquad \text{in } \Omega\,, \tag{48a}$$

$$B\bar q = 0 \qquad \text{on } \Gamma\,. \tag{48b}$$

By subtracting (48) from (47), we obtain an elliptic system for the pressure difference:

$$L(\hat q - \bar q) = F(\hat{\boldsymbol{v}},\hat{\boldsymbol{v}}) - F(\bar{\boldsymbol{v}},\bar{\boldsymbol{v}}) + \delta F'(\hat{\boldsymbol{v}},\hat{\boldsymbol{v}}) - \delta L'\hat q \qquad \text{in } \Omega\,, \tag{49a}$$

$$B(\hat q - \bar q) = \delta G'(\hat{\boldsymbol{v}},\hat{\boldsymbol{v}}) - \delta B'\hat q \qquad \text{on } \Gamma\,. \tag{49b}$$

Lemma 4 provides an estimate for $\hat q - \bar q$ in terms of the right sides of (49), which can be written by using the bilinearity of F and all the estimates provided by Lemma 3 and Corollary 5 as

$$\begin{aligned} \|\hat q - \bar q\|_{\mathrm{H}^{m+1}} &\le c_1 \|F(\hat{\boldsymbol{v}} - \bar{\boldsymbol{v}},\hat{\boldsymbol{v}}) + F(\bar{\boldsymbol{v}},\hat{\boldsymbol{v}} - \bar{\boldsymbol{v}})\|_{\mathrm{H}^{m-1}} \\ &\quad + \delta c_1 \|F'(\hat{\boldsymbol{v}},\hat{\boldsymbol{v}}) - L'\hat q\|_{\mathrm{H}^{m-1}} + \delta c_2 \|G'(\hat{\boldsymbol{v}},\hat{\boldsymbol{v}}) - B'\hat q\|_{\mathrm{H}^{m-1/2}(\Gamma)} \\ &\le c_3 \|\hat{\boldsymbol{v}} - \bar{\boldsymbol{v}}\|_{\mathrm{H}^m} \left(\|\bar{\boldsymbol{v}}\|_{\mathrm{H}^m} + \|\hat{\boldsymbol{v}}\|_{\mathrm{H}^m}\right) + \delta c_4 \|\nabla b\|_{\mathrm{H}^{m+1}} \|\hat{\boldsymbol{v}}\|_{\mathrm{H}^m}^2\,. \end{aligned} \tag{50}$$

3.3. **Estimate on the Nonlinear Term.** This estimate has been proved by Kato [10]. We only need make sure the constant does not depend on δ.

Lemma 7. *For $m \geq 3$, $\psi \in \mathrm{H}^m(\Omega)$ and*

$$\boldsymbol{v} \in \left\{\boldsymbol{u} \in \mathrm{H}^m(\Omega, \mathbb{R}^3) \colon \boldsymbol{\nabla}^{\mathrm{t}}(b\boldsymbol{u}) = 0 \text{ in } \Omega, \boldsymbol{n}^{\mathrm{t}}\boldsymbol{u} = 0 \text{ on } \partial\Omega\right\},$$

the following estimates holds:

$$\langle \boldsymbol{v}^{\mathrm{t}} \boldsymbol{\nabla}\psi, \psi\rangle_m \leq c\, \|\boldsymbol{v}\|_{\mathrm{H}^m} \|\psi\|_{\mathrm{H}^m}^2 , \tag{51}$$

where c is independent of $\delta < \delta_0$ for some fixed δ_0.

Proof. As usual, we first assume $\boldsymbol{v}$ and ψ to be smooth, and extend the estimate by density later. A direct calculation shows that

$$\begin{aligned} \langle \boldsymbol{v}^{\mathrm{t}} \boldsymbol{\nabla}\psi, \psi\rangle_m &\equiv \sum_{|\alpha|\leq m} \left\langle \left(D^\alpha(\boldsymbol{v}^{\mathrm{t}}\boldsymbol{\nabla}\psi)\right) D^\alpha\psi\right\rangle \\ &= \sum_{|\alpha|\leq m} \left(\left\langle (\boldsymbol{v}^{\mathrm{t}}\boldsymbol{\nabla} D^\alpha\psi)\, D^\alpha\psi\right\rangle + \sum_{0<\beta\leq\alpha} c_{\alpha,\beta} \left\langle (D^\beta\boldsymbol{v}^{\mathrm{t}}\, \boldsymbol{\nabla} D^{\alpha-\beta}\psi)\, D^\alpha\psi\right\rangle \right). \end{aligned} \tag{52}$$

Here we employ the usual multi-index notation in which

$$|\alpha| \equiv \sum_{j=1}^{3} \alpha_j , \qquad D^\alpha \equiv \partial_1^{\alpha_1}\partial_2^{\alpha_2}(\delta^{-1}\partial_3)^{\alpha_3} .$$

The first term in the parenthesis on the right side of (52) vanishes through integration by parts. Each of the terms is broken up with a Cauchy-Schwarz inequality and can be estimated in terms of a single common expression as follows: For $|\beta| = 1$,

$$\|D^\beta\boldsymbol{v}^{\mathrm{t}}\, \boldsymbol{\nabla} D^{\alpha-\beta}\psi\|_{\mathrm{L}^2} \leq \|\boldsymbol{v}\|_{\mathrm{W}^{1,\infty}} \|\psi\|_{\mathrm{H}^{|\alpha|}} \leq c_1 \|\boldsymbol{v}\|_{\mathrm{H}^3} \|\psi\|_{\mathrm{H}^{|\alpha|}} ; \tag{53}$$

for $|\beta| = 2$,

$$\|D^\beta\boldsymbol{v}^{\mathrm{t}}\, \boldsymbol{\nabla} D^{\alpha-\beta}\psi\|_{\mathrm{L}^2} \leq \|\boldsymbol{v}\|_{\mathrm{W}^{2,4}} \|\psi\|_{\mathrm{W}^{|\alpha|-1,4}} \leq c_2 \|\boldsymbol{v}\|_{\mathrm{H}^3} \|\psi\|_{\mathrm{H}^{|\alpha|}} ; \tag{54}$$

for $|\beta| > 2$,

$$\|D^\beta\boldsymbol{v}^{\mathrm{t}}\, \boldsymbol{\nabla} D^{\alpha-\beta}\psi\|_{\mathrm{L}^2} \leq \|\boldsymbol{v}\|_{\mathrm{H}^{|\beta|}} \|\psi\|_{\mathrm{W}^{|\alpha|-|\beta|+1,\infty}} \leq c_1 \|\boldsymbol{v}\|_{\mathrm{H}^{|\beta|}} \|\psi\|_{\mathrm{H}^{|\alpha|-|\beta|+3}} . \tag{55}$$

This directly implies the statement of the lemma, since the constants c_1 and c_2 can be chosen independent of δ for small δ. □

3.4. **The Time-Dependent Estimate.** We now estimate the time evolution of the difference between a solution to the rigid lid equations (27) and a solution to the lake equations (31) in the H^m norm. Again, we denote the quantities associated with the rigid lid equation $\hat{\boldsymbol{v}} \equiv \boldsymbol{v}_\delta$, $\hat{q}$, and $\hat{M} \equiv M_\delta$. Correspondingly, the quantities associated with the lake equation are denoted $\bar{\boldsymbol{v}}$, $\bar{q} \equiv h$, and $\bar{M} \equiv M_0$. We define a difference between the velocity fields by $\boldsymbol{w} \equiv \hat{M}\hat{\boldsymbol{v}} - \bar{M}\bar{\boldsymbol{v}}$.

A direct calculation shows

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{w}\|_{\mathrm{H}^m}^2 &= \big\langle \boldsymbol{w}, \hat{M}\partial_t\hat{\boldsymbol{v}} - \bar{M}\partial_t\bar{\boldsymbol{v}}\big\rangle_m \\
&= -\big\langle \boldsymbol{w}, \hat{\boldsymbol{v}}^{\mathrm{t}}\,\boldsymbol{\nabla}(\hat{M}\hat{\boldsymbol{v}}) + (\hat{M}^{-1})^{\mathrm{t}}\,\boldsymbol{\nabla}\hat{q}\big\rangle_m + \big\langle \boldsymbol{w}, \bar{\boldsymbol{v}}^{\mathrm{t}}\,\boldsymbol{\nabla}(\bar{M}\bar{\boldsymbol{v}}) + (\bar{M}^{-1})^{\mathrm{t}}\,\boldsymbol{\nabla}\bar{q}\big\rangle_m \\
&= -\big\langle \boldsymbol{w}, (\hat{\boldsymbol{v}}-\bar{\boldsymbol{v}})^{\mathrm{t}}\,\boldsymbol{\nabla}(\hat{M}\hat{\boldsymbol{v}})\big\rangle_m - \big\langle \boldsymbol{w}, \bar{\boldsymbol{v}}^{\mathrm{t}}\,\boldsymbol{\nabla}(\hat{M}\hat{\boldsymbol{v}} - \bar{M}\bar{\boldsymbol{v}})\big\rangle_m \\
&\quad - \big\langle \boldsymbol{w}, (\hat{M}^{-1})^{\mathrm{t}}\,\boldsymbol{\nabla}(\hat{q}-\bar{q})\big\rangle_m - \big\langle \boldsymbol{w}, (\hat{M}^{-1} - \bar{M}^{-1})^{\mathrm{t}}\,\boldsymbol{\nabla}\bar{q}\big\rangle_m \\
&\equiv I_1 + I_2 + I_3 + I_4\,.
\end{aligned} \tag{56}
$$

We now derive bounds on the integrals I_1–I_4. I_4 simply vanishes, as can be checked by direct calculation, because $\partial_3\bar{q} = 0$. I_3 is estimated with a Cauchy-Schwarz inequality where—due to the nondegeneracy of b—the contribution of $\hat{M}^{-1}$ can be absorbed into the prefactor c_1:

$$
I_3 \le c_1\,\|\boldsymbol{w}\|_{\mathrm{H}^m}\,\|\hat{q}-\bar{q}\|_{\mathrm{H}^{m+1}}\,. \tag{57}
$$

We then apply the main result of Section 3.1, equation (50), and obtain

$$
I_3 \le c_2\,\|\boldsymbol{w}\|_{\mathrm{H}^m}\,\|\hat{\boldsymbol{v}}-\bar{\boldsymbol{v}}\|_{\mathrm{H}^m}\big(\|\bar{\boldsymbol{v}}\|_{\mathrm{H}^m} + \|\hat{\boldsymbol{v}}\|_{\mathrm{H}^m}\big) + \delta c_3\,\|\boldsymbol{w}\|_{\mathrm{H}^m}\,\|\nabla b\|_{\mathrm{H}^{m+1}}\,\|\hat{\boldsymbol{v}}\|_{\mathrm{H}^m}^2\,. \tag{58}
$$

If we want to use this estimate to turn equation (56) into a closed differential inequality, we need to express $\hat{\boldsymbol{v}} - \bar{\boldsymbol{v}}$ in terms of $\boldsymbol{w}$ and $\bar{\boldsymbol{v}}$, the latter of which is known to be bounded *a priori*. Clearly,

$$
\boldsymbol{w} = \hat{M}(\hat{\boldsymbol{v}}-\bar{\boldsymbol{v}}) + (\hat{M}-\bar{M})\bar{\boldsymbol{v}}\,,
$$

so that

$$
\begin{aligned}
\hat{\boldsymbol{v}}-\bar{\boldsymbol{v}} &= \hat{M}^{-1}\boldsymbol{w} - \hat{M}^{-1}(\hat{M}-\bar{M})\bar{\boldsymbol{v}} \\
&= \hat{M}^{-1}\boldsymbol{w} - \delta x_3 b^{-1}\bar{\boldsymbol{v}}^{\mathrm{t}}\,\boldsymbol{\nabla} b\,,
\end{aligned}
$$

and finally—once more due to the nondegeneracy of b—one obtains

$$
\|\hat{\boldsymbol{v}}-\bar{\boldsymbol{v}}\|_{\mathrm{H}^m} \le c_4\,\|\boldsymbol{w}\|_{\mathrm{H}^m} + \delta c_5\,\|\nabla b\|_{\mathrm{H}^m}\,\|\bar{\boldsymbol{v}}\|_{\mathrm{H}^m}\,. \tag{59}
$$

Moreover, one has

$$
\|\hat{\boldsymbol{v}}\|_{\mathrm{H}^m} \le \|\hat{\boldsymbol{v}}-\bar{\boldsymbol{v}}\|_{\mathrm{H}^m} + \|\bar{\boldsymbol{v}}\|_{\mathrm{H}^m} \le c_4\,\|\boldsymbol{w}\|_{\mathrm{H}^m} + c_6\,\|\bar{\boldsymbol{v}}\|_{\mathrm{H}^m}\,, \tag{60}
$$

where c_4 and c_6 can be chosen independent of δ. We combine (58), (59) and (60) into the estimate

$$
\begin{aligned}
I_3 &\le c_7\,\|\boldsymbol{w}\|_{\mathrm{H}^m}^2\big(\|\boldsymbol{w}\|_{\mathrm{H}^m} + \|\bar{\boldsymbol{v}}\|_{\mathrm{H}^m}\big) \\
&\quad + \delta c_8\,\|\nabla b\|_{\mathrm{H}^{m+1}}\,\|\bar{\boldsymbol{v}}\|_{\mathrm{H}^m}\,\|\boldsymbol{w}\|_{\mathrm{H}^m}\,\big(\|\boldsymbol{w}\|_{\mathrm{H}^m} + \|\bar{\boldsymbol{v}}\|_{\mathrm{H}^m}\big)\,.
\end{aligned} \tag{61}
$$

The key to estimating I_2 is the result of Lemma 7, which implies

$$
I_2 \le c_9\,\|\bar{\boldsymbol{v}}\|_{\mathrm{H}^m}\,\|\boldsymbol{w}\|_{\mathrm{H}^m}^2\,. \tag{62}
$$

Finally, rewrite I_1 in the following way:

$$\begin{aligned} I_1 &= -\langle \boldsymbol{w}, (\hat{\boldsymbol{v}} - \bar{\boldsymbol{v}})^{\mathrm{t}} \boldsymbol{\nabla} \boldsymbol{w} \rangle_m - \langle \boldsymbol{w}, (\hat{\boldsymbol{v}} - \bar{\boldsymbol{v}})^{\mathrm{t}} \boldsymbol{\nabla} (\bar{M} \bar{\boldsymbol{v}}) \rangle_m \\ &\leq c_9 \, \|\boldsymbol{w}\|_{\mathrm{H}^m}^2 \, \|\hat{\boldsymbol{v}} - \bar{\boldsymbol{v}}\|_{\mathrm{H}^m} + c_{10} \, \|\boldsymbol{w}\|_{\mathrm{H}^m} \, \|\hat{\boldsymbol{v}} - \bar{\boldsymbol{v}}\|_{\mathrm{H}^m} \|\bar{\boldsymbol{v}}\|_{\mathrm{H}^{m+1}} \\ &\leq c_{11} \, \|\boldsymbol{w}\|_{\mathrm{H}^m}^2 \left(\|\boldsymbol{w}\|_{\mathrm{H}^m} + \|\bar{\boldsymbol{v}}\|_{\mathrm{H}^m} \right) \\ &\quad + c_{12} \, \|\boldsymbol{w}\|_{\mathrm{H}^m} \, \|\bar{\boldsymbol{v}}\|_{\mathrm{H}^{m+1}} \left(\|\boldsymbol{w}\|_{\mathrm{H}^m} + \delta \|\nabla b\|_{\mathrm{H}^m} \, \|\bar{\boldsymbol{v}}\|_{\mathrm{H}^m} \right) . \end{aligned} \tag{63}$$

Altogether, we obtain the following closed differential inequality which controls the time evolution of $\boldsymbol{w}$:

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{w}\|_{\mathrm{H}^m}^2 \leq c_1 \, \|\boldsymbol{w}\|_{\mathrm{H}^m}^3 + c_2 \, \|\bar{\boldsymbol{v}}\|_{\mathrm{H}^{m+1}} \, \|\boldsymbol{w}\|_{\mathrm{H}^m}^2 + \delta c_3 \, \|\nabla b\|_{\mathrm{H}^{m+1}} \, \|\bar{\boldsymbol{v}}\|_{\mathrm{H}^{m+1}}^2 \, \|\boldsymbol{w}\|_{\mathrm{H}^m} \,, \tag{64}$$

or, after dividing through by $\|\boldsymbol{w}\|_{\mathrm{H}^m}$,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{w}\|_{\mathrm{H}^m} \leq c_1 \, \|\boldsymbol{w}\|_{\mathrm{H}^m}^2 + c_2 \, \|\bar{\boldsymbol{v}}\|_{\mathrm{H}^{m+1}} \, \|\boldsymbol{w}\|_{\mathrm{H}^m} + \delta c_3 \, \|\nabla b\|_{\mathrm{H}^{m+1}} \, \|\bar{\boldsymbol{v}}\|_{\mathrm{H}^{m+1}}^2 \, . \tag{65}$$

The first term on the right is inherited from the three dimensional Euler equations, and causes the bound to become singular in finite time. The blow-up time will depend on the initial H^m norm of $\boldsymbol{w}$, i.e., on the "closeness" of the initial velocity field to some solution of the lake equations. We will develop a precise statement of this relationship below. The second term is a bounded growth term. As the lake equations have global classical solutions, there exists a function $\theta(t)$ which depends on $\boldsymbol{v}^{\mathrm{in}}$ such that $\|\bar{\boldsymbol{v}}\|_{\mathrm{H}^{m+1}} \leq \theta(t)$ for every $t \in [0, \infty)$. By Theorem 1, $\theta(t)$ can grow super-exponentially as $t \to \infty$. The third term is an additional driving term which adds a component of growth that is independent of the initial velocity field, and depends only on the geometry of the basin. This term is small for flat or thin domains, and vanishes in the case of a constant bottom function.

A differential inequality of the type (65) does not have a closed form solution. We can, however, make one more approximation and remove the third term from (65) by combining it with the initial value: For a given $\|\boldsymbol{w}(0)\|_{\mathrm{H}^m}$, the solution F to

$$\frac{\mathrm{d}F}{\mathrm{d}t} \leq c_1 \, F^2 + c_2 \, \theta(t) \, F \,, \tag{66a}$$

$$F(0) = \|\boldsymbol{w}(0)\|_{\mathrm{H}^m} + \delta \tfrac{c_3}{c_2} \, \theta(0) \, \|\nabla b\|_{\mathrm{H}^{m+1}} \tag{66b}$$

provides a bound on $\|\boldsymbol{w}\|_{\mathrm{H}^m}$ for as long is F is finite. Equation (66) is of Bernoulli type, and can be solved through the change of variables $G \equiv F^{-1}$. One obtains

$$F(t) \leq \frac{\exp\left(c_2 \displaystyle\int_0^t \theta(t') \mathrm{d}t' \right)}{F^{-1}(0) - c_1 \displaystyle\int_0^t \exp\left(c_2 \int_0^{t'} \theta(t'') \mathrm{d}t'' \right) \mathrm{d}t'} \,. \tag{67}$$

This bound—after scaling the problem back onto the physical domain—implies the main convergence result, Theorem 2.

4. Discussion

Our bound on the H^m norm of the difference $\boldsymbol{w}$ between a solution to the lake and a solution to the rigid lid equations, given by equation (67), grows very rapidly. To get a better understanding of its behavior for small initial data, let us assume that θ is constant over time scales in the order of the blow-up time of (67), so that

$$F(t) \leq \frac{1}{F^{-1}(0)\mathrm{e}^{-c_2\theta t} - \frac{c_1}{c_2\theta}} \,. \tag{68}$$

The blow-up time T^* of this bound is given by

$$T^* = \frac{1}{c_2\,\theta} \ln \frac{c_2\,\theta}{c_1\,F(0)} \,. \tag{69}$$

Thus, to increase T^* by an amount of order one, we have to decrease $F(0)$ by an order of magnitude—and this holds only on time scales that are short with respect to the super-exponential growth of θ.

Some final comments are now in order.

1. The basic structure of our estimates—including their rapid growth and finite time blow-up—is inherited from the three dimensional Euler equations. As such, any fundamental improvements would have to come from advances in understanding the dynamics of the Euler equations. This is one of the most difficult and challenging problems in theoretical fluid mechanics.

2. We have achieved small data–thin domain well-posedness for the rigid lid equations: *For any time T, every non-dimensional bottom function b, and integer $m \geq 3$, there exists a scale constant δ so that, for every initial data close enough to permissible initial data for the lake equations with respect to the* H^m *norm on the domain*

$$\{\boldsymbol{x} \in \mathbb{R}^3 : x \in \mathbb{T}^2, -\delta b(x) < x_3 < 0\} \,,$$

the Cauchy problem for the rigid lid equations has a classical solution on the time interval $[0,T]$.

3. As can be seen from (66b), smallness of δ can be traded in for smallness of $\|\nabla b\|_{\mathrm{H}^{m+1}}$. In the limit of a flat bottom, the third term in (65) vanishes. But the lake equations with a flat bottom are nothing but the two-dimensional Euler equations, so that we recover the well known fact that a two-dimensional velocity configuration in the three-dimensional Euler equations remains two-dimensional for all times.

4. The norms in which our results are valid are still the scaled norms (32). This implies that $\|\boldsymbol{v}_\delta\|_{\mathrm{H}^m_\delta}$ generally diverges as $\delta \to 0$ due to the δ^{-1} weight for vertical gradients. In other words, on thin domains the measure of closeness is stricter than on thick domains. Note, however, that horizontal gradients have a scale invariant weight.

5. It is an open question whether the constant c_3 in (65) is proportional to some positive power of δ, i.e. whether one can assert a higher degree of smallness of the last term in this estimate.

Such results would require techniques developed by Hale and Raugel [7, 8] and Raugel and Sell [16] for dissipative partial differential equations on thin domains. Their methods, however, do not naturally generalize to Sobolev spaces of higher order due to boundary effects.

6. The justification of the lake equations is incomplete unless it can be shown that solutions of the rigid lid equations are close in some appropriate sense and for an estimable time interval to solutions of the three dimensional Euler equations on a domain bounded by a free surface under gravity. A necessary first step for obtaining such a result is to find a setting in which the free surface problem is locally well posed, and in which an estimate of the minimum time interval of existence is obtained in terms of some norm of the initial data. This is a formidable problem by itself and has not been done in spaces with a finite degree of smoothness. There is, however, work by Sedenko [18, 19] who uses a *scale of Banach spaces* that characterizes a strict subclass of the analytic functions. As such, the construction is very technical, but may provide a setting in which rigorous estimates can be done.

7. Another open question is whether the rate of convergence can be improved by going to order δ^2 in the asymptotic expansion. In other words, are the great lake equations a better approximation to the vertically averaged velocity field of the rigid lid equations?

8. A very different approach to the justification of shallow water models would be to consider viscous primitive equations and use them to formally derive viscous shallow water equations. Since the well-posedness theory for viscous free surface problems is well developed (see [2] and references therein), we expect that it will be easier to justify the viscous shallow water asymptotics than its inviscid counterpart. As a second step, one might consider taking the inviscid limit on the level of the shallow water approximation, hence using techniques similar to those employed in [12, 13]. The downside is that the modeling aspects for the viscosity, in particular its scaling in aspect ratio and Froude number, are not yet fully understood.

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References

[1] S. Agmon, A. Douglis and L. Nirenberg, *Estimates near the Boundary for Solutions of Elliptic Partial Differential Equations I and II*, Comm. Pure Appl. Math. **12** (1959), 623–727; **17** (1964), 35–92.

[2] J.T. Beale, *Large Time Regularity of Viscous Surface Waves*, Arch. Rational Mech. Anal. **84** (1984), 307–352.

[3] R. Camassa, D.D. Holm and C.D. Levermore, *Long-Time Effects of Bottom Topography in Shallow Water*, Phys. D **98** (1996), 343–378.

[4] R. Camassa, D.D. Holm and C.D. Levermore, *Long-Time Shallow Water Equations with a Varying Bottom*, submitted to J. Fluid Mech. (1995).

[5] D. Gilbarg and N.S. Trudinger, *Elliptic Partial Differential Equations of Second Order*, Springer-Verlag, New York, 1983.

[6] H.P. Greenspan, *The Theory of Rotating Fluids*, Cambridge University Press, London, 1968.

[7] J.K. Hale and G. Raugel, *Reaction-Diffusion Equations on Thin Domains*, J. Math. pures et appl. **71** (1992), 33–95.

[8] J.K. Hale and G. Raugel, *A Damped Hyperbolic Equation on Thin Domains*, Trans. Amer. Math. Soc. **329** (1992), 185–219.

[9] J.G. Heywood, *The Navier-Stokes Equations: On Existence, Regularity and Decay of Solutions*, Indiana Univ. Math. J. **29** (1980), 639–681.

[10] T. Kato, *Nonstationary Flows of Viscous and Ideal Fluids in* $\mathbb{R}^3$, J. Funct. Anal. **9** (1972), 296–305.

[11] C.D. Levermore and M. Oliver. *Analyticity of Solutions for a Generalized Euler Equation*, J. Differential Equations **133** (1997), 321–339.

[12] C.D. Levermore, M. Oliver and E.S. Titi, *Global Well-Posedness for the Lake Equations*, Phys. D **98** (1996), 492–509.

[13] C.D. Levermore, M. Oliver and E.S. Titi, *Global Well-Posedness for Models of Shallow Water in a Basin with a Varying Bottom*, Indiana Univ. Math. J. **45** (1996), 479–510.

[14] M. Oliver, *A Mathematical Investigation of Models of Shallow Water with a Varying Bottom*, Ph.D. dissertation, The University of Arizona, 1996.

[15] M. Oliver, *Classical Solutions for a Generalized Euler Equation in Two Dimensions*, submitted for publication.

[16] G. Raugel and G.R. Sell, *Navier-Stokes Equations on Thin 3D Domains. I: Global Attractors and Global Regularity of Solutions*, J. Amer. Math. Soc. **6** (1993), 503–568.

[17] M. Schechter, *General Boundary Value Problems for Elliptic Partial Differential Equations*, Comm. Pure Appl. Math. **12** (1959), 457–486.

[18] V.I. Sedenko, *Solvability of Initial-Boundary Value Problems for Flows of an Ideal Incompressible Inhomogeneous Fluid Bounded by a Free Surface*, Dokl. Akad. Nauk SSSR **263** (1982), no. 3; English translation in Soviet Math. Dokl. **25** (1982), 395–398.

[19] V.I. Sedenko, *Solvability of initial-boundary value problems for the Euler equations of flows on an ideal incompressible nonhomogeneous fluid and an ideal barotropic fluid that are bounded by free surfaces*, Mat. Sb. **185** (1994), no. 11, 57–78; English translation in Russian Acad. Sci. Sb. Math. **83** (1995), no. 2, 347–368.

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