Shallow Water Models: Well-Posedness, Regularity, and Justification

Marcel Oliver

Department of Mathematics
University of California
Irvine, CA 92697, USA
E-mail: oliver@math.uci.edu

ABSTRACT

We consider two prototypical models of shallow water, called the *lake* and the *great lake equations*, which describe the long-time motion of an inviscid, incompressible fluid contained in a shallow basin with a slowly spatially varying bottom, a free upper surface and vertical side walls, under the influence of gravity and in the limit of small characteristic velocities and very small surface amplitude.

It is shown that these equations are globally well-posed, i.e. that they possess unique global weak solutions that depend continuously on the initial data and on the bottom topography. Provided the initial data is in a class of sufficiently differentiable functions, it remains a member of that class for all times. In other words, the lake and great lake equations have global classical solutions. Moreover, if the equations are posed on a space-periodic domain and the initial data is real analytic, the solution remains real analytic for all times. The proof is based on a characterization of Gevrey classes in terms of decay of Fourier coefficients.

Finally, a partial mathematical justification of the formal derivation of the lake equations is given. It is shown that solutions of the lake equation stay close to solutions of the rigid lid equations—the three dimensional Euler equations in the limit of small surface wave amplitude—in the following sense: For every error bound E there exists a time $T = T(\mathrm{E})$ such that for all times $t \in [0, T]$ the difference between a solution to the lake equations and the solution to the rigid lid equation corresponding to the same initial data is less than E in a suitably chosen norm. Moreover, T can be chosen arbitrarily large as the aspect ratio of the basin tends to zero.

1 INTRODUCTION

Mathematical models of physical systems are often derived through formal limits or asymptotic expansions. A set of equations that is believed to accurately represent the relevant physics, and which we henceforth refer to as the *primitive equations*, is reduced to a "simpler" set of equations, which we will call the *model*. The reasons for doing so are the following. The primitive equations may be too expensive computationally, or one would like to separate certain effects or scales in the system from others of lesser interest. Naturally, the question of validity of the model arises. From the perspective of applied analysis, two points need to be addressed.

A first necessary step in the justification of any model is to find a setting in which the model is globally, and the primitive equations are at least locally well posed. In a second step one can derive so-called *a priori estimates*—formal estimates of (Sobolev) norms on the spatial domain—with the goal of obtaining bounds on the growth of the difference between solutions of the primitive equations and of the model corresponding to "similar" initial data.

In this paper we discuss two simple models describing weakly non-hydrostatic effects of bottom topography on the motion of shallow water, and summarize the techniques and results that lead to a partial justification of their derivation.

2 SHALLOW WATER MODELS FOR LAKES

The primitive equations considered here are—from a physical standpoint—the simplest possible, namely the Euler equations describing the motion of an inviscid, incompressible fluid in three spatial dimensions. The fluid basin is a "pool" rather that a "lake", in the sense that while we do allow for nontrivial bottom topography, we require the lateral boundaries to be vertical. In other words, the model excludes beaches. The upper surface of the fluid is free and assumed to be at a constant pressure.

We are interested in the physical regime where the aspect ratio δ (the ratio of typical vertical to typical horizontal length scales) is small, the Froude number ε (the ratio of typical horizontal speeds to the speed of gravity waves) is small and the ratio of the surface amplitude to typical vertical length scales is of order ε^2. Lastly, we assume that the initial configuration to leading order in δ is columnar, i.e. is independent of the x_3 coordinate.

The evolution of the vertically averaged velocity field $\overline{u}(x,t)$ of the primitive equations is governed, to leading order in δ and ε, by the shallow water type equations

$$\partial_t u + (u \cdot \nabla) u + \nabla h = 0 \quad \text{in } \Omega\,, \tag{1a}$$
$$\nabla \cdot (bu) = 0 \quad \text{in } \Omega\,, \tag{1b}$$
$$u \cdot n = 0 \quad \text{on } \partial\Omega\,, \tag{1c}$$
$$u(0) = u^{\text{in}}\,. \tag{1d}$$

Here $u = (u_1, u_2)$, $x = (x_1, x_2)$; $\Omega \subset \mathbb{R}^2$ denotes the horizontal domain, $h(x,t)$ the surface height variation, $b(x)$ the depth of the basin and $n(x)$ a unit outward normal vector on $\partial\Omega$. This system, which is called the *lake equations*, has essentially been described by Greenspan [6, page 235]. Moreover, by including terms through order δ^2 one obtains the *great lake equations*, which are

$$\partial_t v - u^{\perp}\, \nabla \wedge v + \nabla(h - \tfrac{1}{2} u \cdot u + u \cdot v) = 0 \quad \text{in } \Omega\,, \tag{2a}$$
$$v = \mathcal{L} u \quad \text{in } \Omega\,, \tag{2b}$$
$$\nabla \cdot (bu) = 0 \quad \text{in } \Omega\,, \tag{2c}$$
$$u \cdot n = 0 \quad \text{on } \partial\Omega\,, \tag{2d}$$
$$u(0) = u^{\text{in}}\,. \tag{2e}$$

We use the notation $u^{\perp} = (u_2, -u_1)$ and $\nabla \wedge v = \partial_1 v_2 - \partial_2 v_1$. An auxiliary velocity field v is related to u through the linear, self-adjoint operator $\mathcal{L}$ given by

$$\begin{aligned} v = \mathcal{L} u = u + \delta^2 \big(& (u \cdot \nabla b) \nabla b + \tfrac{1}{2} b (\nabla \cdot u)\, \nabla b \\ & - \tfrac{1}{2} b^{-1} \nabla (b^2 u \cdot \nabla b) - \tfrac{1}{3} b^{-1} \nabla (b^3 \nabla \cdot u) \big)\,. \end{aligned} \tag{3}$$

When $\delta = 0$, and thus $u = v$, equation (2) reduces to (1). For comments on the physical role of v see Camassa and Levermore [4].

The great lake equations were derived by Camassa, Holm and Levermore [2, 3], who also show that lake and great lake equations have a Hamiltonian structure, convect potential vorticity, and enjoy a Kelvin circulation theorem in complete analogy with the two dimensional Euler equations.

The structural similarities to the two dimensional Euler equations are most easily expressed and exploited by noting that either system can be written in the abstract vorticity form

$$\partial_t \omega + u \cdot \nabla \omega = 0\,, \tag{4a}$$
$$u = K \omega\,, \tag{4b}$$
$$\omega(0) = \omega^{\text{in}}\,. \tag{4c}$$

The *potential vorticity* ω is defined by $\omega = \nabla \wedge u$ for the lake, and $\omega = \nabla \wedge v$ for the great lake equations. When the bottom is flat, both models reduce to the two dimensional Euler equations, and K is simply the appropriately defined inverse of the curl operator. In general, K will satisfy a weighted divergence condition and certain regularity properties, as long as the bottom topography is non-degenerate, i.e., provided that there are constants $b_{\min}$ and $b_{\max}$ such that for every $x \in \Omega$

$$0 < b_{\min} \le b(x) \le b_{\max} \,. \tag{5}$$

Under these assumptions we will study existence and regularity of solutions to (4).

3 WEAK SOLUTIONS

The construction of weak solutions employs a two-step regularization procedure. First, artificial viscosity is added to the system, and a solution to the viscous equations is obtained as the limit of a Galerkin approximating sequence. In the second step one shows that the solutions so obtained converge to solutions of the inviscid lake equations in the limit of vanishing viscosity. The uniqueness of the solutions is proved by estimating the time evolution of the difference of two solutions in a suitable Sobolev norm. This technique can also be used to prove continuous dependence of the solution on the initial data and on the bottom topography. Technically, the weighted divergence condition (1b) makes it natural to conduct the proof in b-weighted Sobolev spaces, thereby extending techniques developed by Yudovitch [17] and Bardos [1] for the Euler equations.

Theorem 1 ([9, 10]). *Given* $\omega^{\mathrm{in}} \in \mathrm{L}^\infty(\Omega)$*, there exists a unique function*

$$\omega \in \mathrm{C}([0,\infty);\mathrm{w}^*\text{-}\mathrm{L}^\infty(\Omega)) \cap \mathrm{L}^\infty([0,\infty) \times \Omega) \tag{6}$$

that satisfies the weak form of system (4)*, given by*

$$\int_\Omega \psi\omega(t_2)\,\mathrm{d}x - \int_\Omega \psi\omega(t_1)\,\mathrm{d}x - \int_{t_1}^{t_2}\int_\Omega (u\cdot\nabla\psi)\,\omega\,\mathrm{d}x\,\mathrm{d}t = 0\,, \tag{7a}$$

$$u = K\omega\,, \tag{7b}$$

$$\omega(0) = \omega^{\mathrm{in}}\,, \tag{7c}$$

for every $[t_1,t_2] \subset [0,\infty)$ *and every test function* $\psi \in \mathrm{H}^1_0(\Omega)$*, and the vorticity bound*

$$\|\omega(t)\|_{\mathrm{L}^\infty} \le \|\omega^{\mathrm{in}}\|_{\mathrm{L}^\infty}\,, \tag{8}$$

for every $t \in [0,\infty)$*. Moreover, for every fixed time* T *and every* $q > 1$*, the mapping* $(b,\omega^{\mathrm{in}}) \mapsto \omega(T)$ *is continuous in the spaces* $\mathrm{W}^{1,2q}(\Omega) \times \mathrm{H}^{-1}(\Omega) \to \mathrm{H}^{-1}(\Omega)$.

4 GLOBAL CLASSICAL SOLUTIONS AND THEIR REGULARITY

The solutions given by Theorem 1 are too weak to be used for establishing rigorous estimates on their closeness to solutions of the three dimensional Euler equations. However, as the two dimensional Euler equations are known to have global classical solutions, we expect the same to be true for the lake and the great lake equations. This is the contents of the following theorem.

Theorem 2 ([11]). *Provided* $b \in \mathrm{C}^{m+2}(\Omega)$ *for some* $m \ge 3$ *and* $\omega^{\mathrm{in}} \in \mathrm{H}^m(\Omega)$*, there exists a unique global classical solution* ω *to the vorticity equation* (4) *such that*

$$\omega \in \bigcap_{k=0}^{m-2} \mathrm{C}^k([0,\infty);\mathrm{H}^{m-k}(\Omega))\,. \tag{9}$$

Moreover, the H^m *norm of* ω *satisfies a bound that grows like* $\exp(\exp t)$.

A proof which is inspired by the method of Temam [15, 16] can be found in [11].

When the problem is posed on a periodic domain, we can use so-called Gevrey class techniques to prove existence of real analytic solutions (see Foias and Temam [5]). The method is based on a characterization of the real analytic functions as the infinite union of normed spaces

$$G_\tau \equiv \mathcal{D}(\exp(\tau\sqrt{-\Delta})) \tag{10}$$

with $\tau > 0$. This representation makes the problem accessible for functional analytic tools such as Sobolev inequalities and embedding theorems, and also yields explicit bounds on the radius of the domain of analyticity τ.

Theorem 3 ([8]). *For real analytic ω^{in} and b there exists a unique global analytic solution to the abstract vorticity equations* (4)*, i.e.*

$$\omega \in \mathrm{C}^\omega([0,\infty) \times \mathbb{T}^2)\,. \tag{11}$$

5 JUSTIFICATION OF THE LOW ASPECT RATIO LIMIT

The lake and the great lake equations can be formally derived from the three dimensional Euler equations by first taking the low Froude number limit, and then asymptotically expanding in δ^2. The equations at the intermediate level are called the *rigid lid approximation*. In non-dimensional variables, posed on the horizontally periodic thin domain

$$\{(x_1,x_2,x_3)\colon x_1,x_2 \in \mathbb{T}; -\delta b(x_1,x_2) < x_3 < 0\}, \tag{12}$$

they take the form

$$\partial_t u + u\cdot\nabla u + w\partial_3 u + \nabla p = 0\,, \tag{13a}$$
$$\partial_t w + u\cdot\nabla w + w\partial_3 w + \partial_3 p = 0\,, \tag{13b}$$

$$\nabla\cdot u + \partial_3 w = 0\,, \tag{13c}$$

$$p = h \qquad \text{for } x_3 = 0\,, \tag{13d}$$
$$w = 0 \qquad \text{for } x_3 = 0\,, \tag{13e}$$
$$w = -\delta u\cdot\nabla b \qquad \text{for } x_3 = -\delta b(x_1,x_2)\,. \tag{13f}$$

As before, u denotes the horizontal components of the velocity field; w is the vertical velocity component. The rigid lid equations differ from the three dimensional Euler equations only in the boundary conditions: The domain is fixed at the equilibrium surface height, while an approximation to the surface height h can be recovered through the pressure boundary condition (13d).

To derive the lake equations from (13), one must first rescale the vertical velocity field and space coordinate in accordance with the scaling assumptions given in Section 2, effectively replacing $w \to \delta w$, $x_3 \to \delta x_3$ and $\partial_3 \to \delta^{-1}\partial_3$. Then one obtains the lake equations in the formal limit $\delta \to 0$ under the assumption of columnar initial data.

Our point here is that this formal procedure can be made rigorous. We take advantage of the fact that the rigid equations are posed on a fixed domain, and are therefore known to be locally well-posed—for example in the Sobolev space H^m when $m \geq 3$ (see Kato [7]). By Theorem 2 we have the necessary global well-posedness for the lake equations in the same space. This permits us to prove that, in the H^m norm, a solution to the lake equations remains close to solutions of the rigid lid equations—endowed with corresponding initial data—on a time interval that can be made arbitrarily large by choosing the aspect ratio δ small. Vice versa, any solution to the rigid lid equations which is initially close to being columnar—and thus close to an admissible configuration of the lake equations—can be approximated by a solution to the lake equations for an interval of time which can be estimated in terms of the initial deviation from columnarity in H^m, the magnitude of the initial data in H^m, the gradient of the bottom function b in H^{m+1}, and the aspect ratio of the basin δ. This is summarized in the following theorem.

Theorem 4 ([12]). *Let $\bar{u}$ denote a fixed solution to the lake equations* (1) *corresponding to initial data $\bar{u}^{\mathrm{in}}$ and bottom topography b, and let $\bar{w}$ be its reconstructed vertical velocity component. Let $(\hat{u},\hat{w})$ denote some solution to the rigid lid equations* (13) *posed on the horizontally periodic thin domain* (12) *with initial data $(\hat{u}^{\mathrm{in}},\hat{w}^{\mathrm{in}})$. Let $m \geq 3$ be fixed.*

Then for every $T > 0$ and every error bound $\mathrm{E} > 0$ there exist Δ_1 and Δ_2 such that if

$$\|(\bar{u}^{\mathrm{in}},\bar{w}^{\mathrm{in}}) - (\hat{u}^{\mathrm{in}},\hat{w}^{\mathrm{in}})\|_{\mathrm{H}^m} < \Delta_1\,, \quad \textit{and} \quad \delta\|\nabla b\|_{\mathrm{H}^{m+1}} < \Delta_2\,, \tag{14a}$$

then

$$\|(\bar{u}(t),\bar{w}(t)) - (\hat{u}(t),\hat{w}(t))\|_{\mathrm{H}^m} < \mathrm{E} \tag{15}$$

for every $t \in [0,T]$.

Due to the nature of the three dimensional Euler equations, all known bounds on gradient norms of $(\hat{u},\hat{w})$ blow up in finite time. Therefore we cannot expect the left side of (15) to remain finite as $t \to \infty$. Explicit bounds on its growth as well as the relationship between E, Δ_1 and Δ_2 are given in [12].

6 OPEN PROBLEMS

The justification of the lake equations is incomplete unless one can show that solutions of the rigid lid equations are close in some appropriate sense and for an estimable time interval to solutions of the three dimensional Euler equations on a domain bounded by a free surface under gravity. One difficulty in obtaining such a result is to find a suitable setting in which the free surface problem is locally well posed (see, e.g., Sedenko [13, 14]).

Another open question is whether one can improve the rate of convergence by going to order δ^2 in the asymptotic expansion. In other words, do the great lake equations give a better approximation to the vertically averaged rigid lid velocity field than the lake equations?

From a physical perspective, the problems of including beaches (i.e. degenerate bottom topography) and viscous effects (with the accompanying bottom boundary layer) into the model remain to be solved.

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