COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **00**, Number **0**, Xxxx **XXXX**

Website: http://AIMsciences.org
pp. **000–000**

THE LAGRANGIAN AVERAGED EULER EQUATIONS AS THE SHORT-TIME INVISCID LIMIT OF THE NAVIER–STOKES EQUATIONS WITH BESOV CLASS DATA IN $\mathbb{R}^2$

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ABSTRACT. We compare the vorticity corresponding to a solution of the Lagrangian averaged Euler equations on the plane to a solution of the Navier–Stokes equation with the same initial data, assuming that the averaged Euler potential vorticity is in a certain Besov class of regularity. Then the averaged Euler vorticity stays close to the Navier–Stokes vorticity for a short interval of time as the respective smoothing parameters tend to zero with natural scaling.

1. **Introduction.** There are a large number of results which establish the Euler equations as the inviscid limit of the Navier–Stokes equations on domains without boundary (see, for example, [3] for a list of references). In two dimensions, the natural space for unique weak vorticity solutions of the Euler equations is the "Yudovich space" $L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. However, such solutions are too weak to establish an inviscid limit in the same space—only L^2 convergence of the corresponding velocity fields can be shown.

The best known result is due to Constantin and Wu [3], who prove that for as long as a solution to the Euler equations maintains a certain Besov space regularity (namely $B_s^{2,\infty} \cap B_{s/2}^{4,\infty}$ for some $0 < s < 1$—loosely speaking, a space marginally larger than Sobolev class H^s), the Navier–Stokes vorticity converges in L^p to a solution of the Euler equations which is mollified *a posteriori*. Its main limitation is that the Besov spaces in question are generally not persistence classes for the Euler equations [13]—only the subclass of vortex patches with $C^{1,\alpha}$ boundary is known to persist [1, 4].

In the following I will replace the *a posteriori* mollified Euler equations with the isotropic Lagrangian averaged Euler (LAE) equations (also called the Euler-α equations, see [5, 8, 10] and references cited therein), which can be written in the form

$$\partial_t q^\alpha + u^\alpha \cdot \nabla q^\alpha = 0\,, \tag{1.1a}$$

$$q^\alpha = (1 - \alpha^2 \Delta)\,\mathrm{curl}_{2\mathrm{D}}\, u^\alpha\,. \tag{1.1b}$$

If we now compare the averaged Euler vorticity $\omega^\alpha = \mathrm{curl}_{2\mathrm{D}}\, u^\alpha$ with the Navier–Stokes vorticity, we have, as it turns out, sufficient mollification already built into the model. Moreover, the Besov spaces $B_s^{p,\infty}$ are persistence classes for the averaged Euler equations. We then find that, for a short time, the averaged Euler vorticity

Date: June 18, 2001.

remains close to the corresponding solution of the Navier–Stokes equations

$$\partial_t \omega^{\mathrm{NS}} + u^{\mathrm{NS}} \cdot \nabla \omega^{\mathrm{NS}} = \nu \Delta \omega^{\mathrm{NS}} \,, \tag{1.2}$$

with assumptions only on the initial data, and provided the relationship between dispersivesmoothing and viscosity is close to the natural scaling $\alpha = \sqrt{\nu}$.

For further reference, we denote the Green's kernel of $1 - \alpha^2 \Delta$ by

$$G^\alpha(x) = \frac{1}{\alpha^2} \, G\Bigl(\frac{x}{\alpha}\Bigr) = -\frac{1}{\alpha^2} \, \frac{1}{2\pi} \, K_0\Bigl(\frac{|x|}{\alpha}\Bigr) \,, \tag{1.3}$$

where K_0 is a modified Bessel function of the second kind. The two-dimensional Biot–Savart kernel is denoted

$$K(x) = -\frac{1}{2\pi} \, \frac{x^\perp}{|x|^2} \,. \tag{1.4}$$

The precise statement of the result is then the following.

Theorem 1. *Assume that $q_0 \in L^1 \cap L^\infty \cap B^{2,\infty}_s \cap B^{4,\infty}_s$ for some $s \in (0,1)$. Let q^α be a sequence of solutions to the averaged Euler equations* (1.1) *parameterized by α with initial potential vorticity q_0. Let ω^{NS} denote the corresponding solution to the Navier–Stokes equations* (1.2) *with $\nu = \alpha^2$ and initial vorticity $\omega_0^{\mathrm{NS}} = \omega_0^\alpha = G^\alpha * q_0$. Then for every $\varepsilon > 0$ there exists a time T such that*

$$\sup_{t \in [0,T]} \left\| \omega^\alpha(t) - \omega^{\mathrm{NS}}(t) \right\|_{L^2} \le C \, \alpha^{s-\varepsilon} = C \, \nu^{\frac{s-\varepsilon}{2}} \,. \tag{1.5}$$

The constant C is independent of α, but may depend on s, ε, T, and on the norm of q_0 in the stated spaces.

Remark 1. The result is only local in time, because the growth of the Besov-seminorms is controlled by α^{-Kt} for some constant K. In the proof of the theorem these bounds should ideally be uniform in α—this is made an assumption in the work of Constantin and Wu [3]. However, one can get away with algebraic growth in α provided the exponent is not too large—hence the requirement that Kt be small.

Remark 2. Theorem 3.4 of Constantin and Wu [3], which states the result for the standard Euler equations that corresponds to our Theorem 1, is formulated using the Besov space $B^{4,\infty}_{s/2}$ in place of $B^{4,\infty}_s$. As a result, their rate of convergence differs by a square root from ours. However, it is straightforward to reformulate each proof in terms of the Besov spaces used in the other.

Remark 3. It is not difficult to prove a result in any L^p as in [3]. However, as little additional insight is gained, we restrict ourselves to the notationally simpler L^2 case.

Remark 4. It is possible to reduce the requirement on the initial data to $q_0 \in L^1 \cap L^\infty \cap B^{2,\infty}_s$. However, the estimates become considerably more involved, so that we only sketch the key commutator estimate in the appendix.

Remark 5. It is an interesting open problem if a similar result could be proved on a domain with boundary. On the one hand, our proof makes crucial use of the symmetry of the 2D Biot–Savart kernel. On the other hand, the averaged Euler equations can be endowed with no-slip boundary conditions [7, 11], so that some form of boundary layer formation is to be expected as $\alpha \to 0$.

2. **Littlewood–Paley Decomposition and Besov Spaces.** Besov spaces on $\mathbb{R}^n$ can be characterized via a partition of unity in Fourier space—the Littlewood–Paley decomposition. It is constructed as follows. Let $\hat\phi$ be a radial, non-negative, smooth function supported on the annulus $\frac{3}{5} \le |\xi| \le \frac{5}{3}$ and strictly positive in the interior of this set. For $j \in \mathbb{Z}$ let $\hat\phi_j(\xi) = \hat\phi(2^{-j}\xi)$, which corresponds to the delta sequence scaling $\phi_j(x) = 2^{nj}\,\phi(2^j x)$, and set

$$\hat\varphi(\xi) = \frac{\hat\phi(\xi)}{\sum_{j=-\infty}^{\infty} \hat\phi_j(\xi)} \,. \tag{2.1}$$

Then the functions

$$\hat\varphi_k(\xi) = \hat\varphi(2^{-k}\xi) \tag{2.2}$$

for $k = 1, 2, \dots$, and

$$\hat\varphi_0(\xi) = \begin{cases} \displaystyle\sum_{j=-\infty}^{0} \hat\varphi_j(\xi) & \text{for } \xi \neq 0\,, \\ 1 & \text{for } \xi = 0 \end{cases} \tag{2.3}$$

are a partition of unity, i.e. $1 = \sum_0^\infty \hat\varphi_k(\xi)$, with

$$\operatorname{supp}\hat\varphi_0 = \{\xi \in \mathbb{R}^n \colon |\xi| \le \tfrac{5}{3}\} \qquad \text{and} \tag{2.4}$$

$$\operatorname{supp}\hat\varphi_k = \{\xi \in \mathbb{R}^n \colon \tfrac{3}{5}\,2^k \le |\xi| \le \tfrac{5}{3}\,2^k\} \qquad \text{for } k \ge 1\,. \tag{2.5}$$

Moreover, the support of non-neighboring partition functions is non-overlapping, and the partition functions are bounded away from zero on possibly smaller sets that still cover all of $\mathbb{R}^n$.

For $s \ge 0$ and $1 \le p < \infty$, the Besov space $B_s^{p,\infty}$ is defined as the space of all $L^p(\mathbb{R}^n)$ functions for which the semi-norm

$$\|f\|_{B_s^{p,\infty}} = \sup_{k \ge 0} 2^{sk}\,\|\varphi_k * f\|_{L^p} \tag{2.6}$$

is finite. There are a number of alternative characterizations. For example, for every integer $m > s$, an equivalent semi-norm is given by

$$\|f\|_{B_s^{p,\infty}}^{(m)} = \sup_{h \in \mathbb{R}^n} \frac{\|\Delta_h^m f\|_{L^p}}{|h|^s}\,, \tag{2.7}$$

where Δ_h^m is the mth order centered difference operator

$$\Delta_h^m f(x) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k}\, f(x + (k - m/2)h)\,. \tag{2.8}$$

A detailed proof of the equivalence can be found in [2]; for other characterizations and more generality see also [9, 12].

Lemma 2. *Let $f \in B_s^{p,\infty}$ with $0 < s < 1$ and $1 \le p \le \infty$, and let η be a Lipshitz measure-preserving homeomorphism on $\mathbb{R}^n$. Then there exists a constant $c = c(s)$ such that*

$$\|f \circ \eta\|_{B_s^{p,\infty}} \le c\,\|\eta\|_{\mathrm{Lip}}^s\,\|f\|_{B_s^{p,\infty}}\,. \tag{2.9}$$

Proof. The proof is similar to the proof of Theorem 4.2 in Vishik [13]. Writing $f = \sum_0^\infty \varphi_k * f$, we see that it is sufficient to find estimates for each of the terms on the right side of the estimate

$$\|f \circ \eta\|_{B_s^{p,\infty}} = \sup_{j \geq 0} 2^{sj} \, \|\varphi_j * (f \circ \eta)\|_{L^p} \leq \sup_{j \geq 0} 2^{sj} \sum_{k=0}^\infty \|\varphi_j * ((\varphi_k * f) \circ \eta)\|_{L^p} \, . \tag{2.10}$$

This is done as follows. Define a vector valued partition function $\theta \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$ through $\varphi = \nabla \cdot \theta$, so that

$$\hat{\varphi}(\xi) = \mathrm{i}\xi \cdot \hat{\theta}(\xi) \qquad \text{and} \qquad \hat{\theta}(\xi) = \hat{a}(\xi) \, \hat{\varphi}(\xi) \tag{2.11}$$

for some suitably chosen function $\hat{a} \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$. Then, for $k \geq 1$,

$$\varphi_k = (\nabla \cdot \theta)_k = 2^{-k} \, \nabla \cdot \theta_k \, . \tag{2.12}$$

We extend this relationship to $k = 0$ by defining the function θ_0 appropriately. Let us now estimate $\varphi_j * ((\varphi_k * f) \circ \eta)$ in three different ways (in our context we can actually get away with using only the first two). A direct use of the L^p convolution inequality gives

$$\|\varphi_j * ((\varphi_k * f) \circ \eta)\|_{L^p} \leq \|\varphi_j\|_{L^1} \, \|(\varphi_k * f) \circ \eta\|_{L^p} = c \, \|\varphi_k * f\|_{L^p} \, . \tag{2.13}$$

We can also rewrite the expression on the left using (2.12) and integration by parts:

$$\begin{aligned} \varphi_j * ((\varphi_k * f) \circ \eta) &= \int \varphi_j(x-y) \, (\varphi_k * f)(\eta(y)) \, \mathrm{d}y \\ &= -2^{-j} \int \nabla_y \cdot \theta_j(x-y) \, (\varphi_k * f)(\eta(y)) \, \mathrm{d}y \\ &= 2^{-j} \int \theta_j(x-y) \cdot \nabla_y (\varphi_k * f)(\eta(y)) \, \mathrm{d}y \\ &= 2^{-j} \int (\nabla \varphi_k * f)(\eta(y)) \cdot \nabla \eta \cdot \theta_j(x-y) \, \mathrm{d}y \, . \end{aligned} \tag{2.14}$$

The convolution inequality then yields

$$\begin{aligned} \|\varphi_j * ((\varphi_k * f) \circ \eta)\|_{L^p} &\leq 2^{-j} \, \|(\nabla \varphi_k * f) \circ \eta\|_{L^p} \, \|\nabla \eta\|_{L^\infty} \, \|\theta_j\|_{L^1} \\ &\leq c \, 2^{k-j} \, \|\varphi_k * f\|_{L^p} \, \|\eta\|_{\mathrm{Lip}} \, . \end{aligned} \tag{2.15}$$

Alternatively, we can apply (2.12) to the convolution with φ_k:

$$\begin{aligned} \varphi_j * ((\varphi_k * f) \circ \eta) &= \int \varphi_j(x - \eta^{-1}(y)) \, (\varphi_k * f)(y) \, \mathrm{d}y \\ &= 2^{-k} \int \varphi_j(x - \eta^{-1}(y)) \, (\nabla \cdot \theta_k * f)(y) \, \mathrm{d}y \\ &= -2^{j-k} \int (\nabla \varphi)_j(x - \eta^{-1}(y)) \cdot \nabla \eta^{-1} \cdot (\theta_k * f)(y) \, \mathrm{d}y \, . \end{aligned} \tag{2.16}$$

This form of the expression yields the estimate

$$\begin{aligned} \|\varphi_j * ((\varphi_k * f) \circ \eta)\|_{L^p} &\leq 2^{j-k} \, \|(\nabla \varphi)_j\|_{L^1} \, \|\nabla \eta^{-1}\|_{L^\infty} \, \|\theta_k * f\|_{L^p} \\ &\leq c \, 2^{j-k} \, \|\varphi_k * f\|_{L^p} \, \|\eta^{-1}\|_{\mathrm{Lip}} \, . \end{aligned} \tag{2.17}$$

To see that the last inequality is correct, note that by (2.11) θ is related to φ through convolution with an L^1 function.

We now use estimate (2.15) for the first N terms of the sum in (2.10), and (2.13) for the remainder, with N to be chosen later. We obtain

$$
\begin{aligned}
2^{sj} & \sum_{k=0}^{\infty} \|\varphi_j * ((\varphi_k * f) \circ \eta)\|_{L^p} \\
& \le c\, 2^{sj} \biggl(\|\eta\|_{\mathrm{Lip}} \sum_{k<N} 2^{k-j} \, \|\varphi_k * f\|_{L^p} + \sum_{k \ge N} \|\varphi_k * f\|_{L^p} \biggr) \\
& \le c \, \|f\|_{B^{p,\infty}_s} \biggl(\|\eta\|_{\mathrm{Lip}} \, 2^{-(1-s)j} \sum_{k<N} 2^{(1-s)k} + 2^{sj} \sum_{k \ge N} 2^{-sk} \biggr) \\
& \le c(s) \, \|f\|_{B^{p,\infty}_s} \Bigl(\|\eta\|_{\mathrm{Lip}} \, 2^{-(1-s)j} \, (2^{(1-s)N} - 1) + 2^{s(j-N)} \Bigr) \, . \qquad (2.18)
\end{aligned}
$$

We now choose N such that both terms on the right side of this last expression contribute equally, namely

$$
N = \begin{cases} 0 & \text{when } j \le \log_2(\|\eta\|_{\mathrm{Lip}}) \, , \\ j - \log_2(\|\eta\|_{\mathrm{Lip}}) & \text{otherwise} \, . \end{cases} \qquad (2.19)
$$

(We may round to the nearest integer.) Inserting this choice back into (2.18) yields the statement of the lemma. □

3. **Besov space regularity for averaged Euler solutions.** As is obvious from Lemma 2, an estimate on the Lipshitz continuity of the flow map will immediately imply that Besov space regularity is preserved.

Lemma 3. *For $q_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $0 < \alpha \le \frac{1}{2}$, the flow map of the averaged Euler equation satisfies*

$$
\|\eta(t)\|_{\mathrm{Lip}} \le \Bigl(\frac{1}{\alpha} \Bigr)^{Kt} \qquad \textit{and} \qquad \|\eta^{-1}(t)\|_{\mathrm{Lip}} \le \Bigl(\frac{1}{\alpha} \Bigr)^{Kt} \, , \qquad (3.1)
$$

where K is a constant proportional to $\|q_0\|_{L^1 \cap L^\infty}$.

Since $q(x,t) = q_0(\eta^{-1}(x,t))$, Lemma 2 and Lemma 3 immediately imply the following.

Corollary 4. *Let $q_0 \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap B^{p,\infty}_s$ for $0 \le s < 1$ and $1 \le p \le \infty$. Then $q \in L^\infty_{\mathrm{loc}}([0,\infty); B^{p,\infty}_s)$ and for $0 < \alpha \le \frac{1}{2}$,*

$$
\|q(t)\|_{B^{p,\infty}_s} \le c \Bigl(\frac{1}{\alpha} \Bigr)^{sKt} \|q_0\|_{B^{p,\infty}_s} \, , \qquad (3.2)
$$

where K is as in Lemma 3.

The proof of Lemma 3 is straightforward once we have determined the Lipshitz constant for the averaged Biot–Savart kernel, which is the content of the following lemma.

Lemma 5. *Let $q \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$ and $0 < \alpha < 1$. Then there exists a constant c independent of α such that*

$$
\biggl| \int \bigl(K^\alpha(x,y) - K^\alpha(x',y) \bigr) \, q(y) \, \mathrm{d}y \biggr| \le c \, (1 - \ln \alpha) \, \|q\|_{L^1 \cap L^\infty} \, |x - x'| \, . \qquad (3.3)
$$

Proof. The averaged Biot–Savart kernel in $\mathbb{R}^2$ is known to be

$$K^\alpha(x,y) = \frac{1}{2\pi} \left(\frac{1}{\alpha} K_1\Big(\frac{r}{\alpha}\Big) - \frac{1}{r} \right) \frac{(x-y)^\perp}{r} \tag{3.4}$$

where $r = |x-y|$ and K_1 is a modified Bessel function of the second kind.

Set $R = |x - x'|$. When $r \leq 2R$, we estimate

$$\begin{aligned}
&\Bigg| \int\limits_{|x-y|\leq 2R} \big(K^\alpha(x,y) - K^\alpha(x',y)\big)\, q(y)\,\mathrm{d}y \Bigg| \\
&\quad \leq \frac{1}{2\pi} \int\limits_{|x-y|\leq 2R} \Bigg(\Bigg| \frac{1}{\alpha} K_1\Big(\frac{|x-y|}{\alpha}\Big) - \frac{1}{|x-y|} \Bigg| \\
&\qquad\qquad + \Bigg| \frac{1}{\alpha} K_1\Big(\frac{|x'-y|}{\alpha}\Big) - \frac{1}{|x'-y|} \Bigg| \Bigg) \,\mathrm{d}y\, \|q\|_{L^\infty} \\
&\quad \leq \frac{1}{\pi} \|q\|_{L^\infty} \int\limits_{|x-y|\leq 4R} \Bigg| \frac{1}{\alpha} K_1\Big(\frac{|x-y|}{\alpha}\Big) - \frac{1}{|x-y|} \Bigg| \,\mathrm{d}y \\
&\quad = 2\, \|q\|_{L^\infty} \int_0^{4R} \Big| 1 - \frac{\rho}{\alpha} K_1\Big(\frac{\rho}{\alpha}\Big) \Big| \,\mathrm{d}\rho\,.
\end{aligned} \tag{3.5}$$

Note that $r/\alpha\, K_1(r/\alpha)$ is positive, monotonically decreasing for $r > 0$, and

$$\lim_{r\to 0} \frac{r}{\alpha} K_1\Big(\frac{r}{\alpha}\Big) = 1\,. \tag{3.6}$$

Thus the remaining integral in (3.5) is bounded above by $4R$.

For $r \geq 2R$, note that

$$\begin{aligned}
\big|K^\alpha(x,y) - K^\alpha(x',y)\big| &\leq |x-x'| \sup_{x''\in B(x,R)} |\nabla K^\alpha(x'',y)| \\
&\leq \frac{R}{2\pi} \sup_{x''\in B(x,R)} \Bigg| \frac{1}{\alpha^2} K_0\Big(\frac{\rho}{\alpha}\Big) - \frac{1}{\rho^2} \Big(1 - \frac{\rho}{\alpha} K_1\Big(\frac{\rho}{\alpha}\Big)\Big) \Bigg|_{\rho=|x''-y|} \\
&\leq \frac{R}{2\pi} \Bigg[\frac{1}{\alpha^2} K_0\Big(\frac{r}{2\alpha}\Big) + \frac{4}{r^2} \Big(1 - \frac{2r}{\alpha} K_1\Big(\frac{2r}{\alpha}\Big)\Big) \Bigg]\,.
\end{aligned} \tag{3.7}$$

In the last step we used that $|x''-y| \geq \frac{1}{2}|x-y|$, and all the properties of $\rho/\alpha\, K_1(\rho/\alpha)$ stated above. Since

$$\frac{1}{2\pi} \int\limits_{|x-y|\geq 2R} \frac{1}{\alpha^2} K_0\Big(\frac{|x-y|}{2\alpha}\Big)\,\mathrm{d}y \leq \int_0^\infty \frac{\rho}{\alpha^2} K_0\Big(\frac{\rho}{2\alpha}\Big)\,\mathrm{d}\rho = 4 \int_0^\infty \rho\, K_0(\rho)\,\mathrm{d}\rho < \infty\,, \tag{3.8}$$

the problem reduces to finding a bound for

$$\begin{aligned}
&\int\limits_{|x-y|\geq 2R} \frac{1}{|x-y|^2} \Big(1 - \frac{2|x-y|}{\alpha} K_1\Big(\frac{2|x-y|}{\alpha}\Big)\Big) |q(y)|\,\mathrm{d}y \\
&\qquad \leq \int_{2R}^1 \Big(\frac{1}{\rho} - \frac{2}{\alpha} K_1\Big(\frac{2\rho}{\alpha}\Big)\Big) \,\mathrm{d}\rho\, \|q\|_{L^\infty} + \|q\|_{L^1}\,.
\end{aligned} \tag{3.9}$$

But

$$\int_{2R}^{1} \Big(\frac{1}{\rho} - \frac{2}{\alpha}\, K_1\Big(\frac{2\rho}{\alpha}\Big) \Big)\, \mathrm{d}\rho = \Big(\ln 2 + K_0\Big(\frac{2}{\alpha}\Big) \Big) + \Big(- \ln 4R - K_0\Big(\frac{4R}{\alpha}\Big) \Big) \,. \tag{3.10}$$

The first term is bounded independent of α. The second term is monotonically decreasing in r, thus

$$\sup_{R>0} \Big(- \ln 4R - K_0\Big(\frac{4R}{\alpha}\Big) \Big) = - \lim_{\rho \to 0} \Big(\ln \rho + K_0\Big(\frac{\rho}{\alpha}\Big) \Big) = C + \ln \frac{1}{2\alpha} \,, \tag{3.11}$$

where C is the Euler–Gamma constant. By combining the estimates we obtain the statement of the lemma. □

Proof of Lemma 3. We find by direct calculation that

$$\begin{aligned} \big|\eta(x,t) - \eta(x',t)\big| &= \Big| \int_0^t \int_{\mathbb{R}^2} \big(K^\alpha(\eta(x,s),y) - K^\alpha(\eta(x',s),y) \big)\, q(y,s)\, \mathrm{d}y\, \mathrm{d}s \Big| \\ &\le c\,(1-\ln\alpha) \int_0^t \big|\eta(x,s) - \eta(x',s)\big|\, \mathrm{d}s\, \|q_0\|_{L^1\cap L^\infty} \,. \end{aligned} \tag{3.12}$$

Since $|\eta(x,0) - \eta(x',0)| = |x - x'|$, the Gronwall inequality implies

$$\big|\eta(x,t) - \eta(x',t)\big| \le |x-x'| \, \exp\!\big(c\,(1-\ln\alpha)\, \|q_0\|_{L^1\cap L^\infty}\, t \big) \le |x-x'| \, \Big(\frac{1}{\alpha}\Big)^{Kt} \,. \tag{3.13}$$

Since K depends only on conserved quantities and the flow is time reversible, the same estimate must hold for η^{-1}. □

4. **Further kinematic estimates.**

Lemma 6. *If $q \in B_s^{p,\infty}$ for $1 < p < \infty$, and $\omega = G^\alpha * q$, then*

$$\|q - \omega\|_{L^p} \le c(p)\, \alpha^s\, \|q\|_{B_s^{p,\infty}} \,, \tag{4.1}$$

and

$$\|\nabla \omega\|_{L^p} \le c(p)\, \alpha^{s-1}\, \|q\|_{B_s^{p,\infty}} \,. \tag{4.2}$$

Proof. Change variables and apply the Minkowski inequality to find that

$$\begin{aligned} \|q - \omega\|_{L^p} &= \Big(\int \Big| \int G^\alpha(x-y)\, (q(x) - q(y))\, \mathrm{d}y \Big|^p \mathrm{d}x \Big)^{\frac{1}{p}} \\ &= \Big(\int \Big| \int G(z)\, (q(x) - q(x-\alpha z))\, \mathrm{d}z \Big|^p \mathrm{d}x \Big)^{\frac{1}{p}} \\ &\le \int G(z)\, \|\Delta^1_{\alpha z} q\|_{L^p}\, \mathrm{d}z \,. \end{aligned} \tag{4.3}$$

Now (4.1) follows directly from (2.7). The proof of (4.2) is similar. Notice that $\nabla_y q(x) = 0$, so that

$$\|\nabla \omega\|_{L^p} = \Big(\int \Big| \int G^\alpha(x-y)\, \nabla_y (q(y) - q(x))\, \mathrm{d}y \Big|^p \mathrm{d}x \Big)^{\frac{1}{p}}$$

$$= \left(\int \left| \int \nabla_x G^\alpha(x-y)\,(q(y)-q(x))\,\mathrm{d}y \right|^p \mathrm{d}x \right)^{\frac{1}{p}} . \tag{4.4}$$

From differentiating the kernel we obtain a factor of α^{-1}; all further steps are identical to the ones used for proving (4.1). □

Lemma 7. *Let $u = K * \omega$ with $\omega \in L^p(\mathbb{R}^2)$. Then there exists a constant c independent of p such that for $2 \le p < \infty$,*

$$\|\Delta_h^1 u\|_{L^p} \le c\,p\,|h|\,\|\omega\|_{L^p} , \tag{4.5}$$

and

$$\|\Delta_h^1 u\|_{L^\infty} \le c\,\varphi(h)\,\|\omega\|_{L^1\cap L^\infty} \tag{4.6}$$

where

$$\varphi(x) = \begin{cases} |x|\,(1-\ln|x|) & \text{for } |x|<1\,, \\ 1 & \text{for } |x| \ge 1\,. \end{cases} \tag{4.7}$$

Proof. The result is a direct consequence of the classical L^p theory for elliptic partial differential equations. Namely, since

$$u(x+h)-u(x) = \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} u(x+th)\,\mathrm{d}t = h \cdot \int_0^1 \nabla u(x+th)\,\mathrm{d}t\,, \tag{4.8}$$

we see that, by the Minkowski inequality,

$$\|\Delta_h^1 u\|_{L^p} \le |h| \left(\int_{\mathbb{R}^2} \left(\int_0^1 |\nabla u(x+th)|\,\mathrm{d}t \right)^p \mathrm{d}x \right)^{\frac{1}{p}} \le |h|\,\|\nabla u\|_{L^p} . \tag{4.9}$$

So (4.5) follows from a standard $W^{2,p}$ estimate for the stream function, and (4.6) is literally the well-known quasi-Lipshitz estimate for the Euler velocity field. □

5. Estimates on the Commutator of Convolution and Advection.

The goal of this section is to derive estimates for the commutator

$$\nabla \cdot W \equiv G^\alpha * \nabla \cdot (uq) - \nabla \cdot (u\omega)\,, \tag{5.1}$$

where u, q, and ω correspond to a solution of the averaged Euler equation, i.e. $q = (1-\alpha^2\Delta)\omega$ and $\omega = \nabla^\perp \cdot u$, or $\omega = G^\alpha * q$ and $u = K * \omega$; G^α denotes the Green's kernel of $1-\alpha^2\Delta$ and K denotes the Biot–Savart kernel in two dimensions.

By changing variables $x-y \mapsto \alpha z$ first, and then $z \mapsto -z$, we can write

$$\begin{aligned} W &= \int G^\alpha(x-y)\,(u(y)-u(y))\,q(y)\,\mathrm{d}y \\ &= \int G(z)\,(u(x+\alpha z)-u(x))\,q(x+\alpha z)\,\mathrm{d}z \\ &= \int G(z)\,(u(x-\alpha z)-u(x))\,q(x-\alpha z)\,\mathrm{d}z\,. \end{aligned} \tag{5.2}$$

By averaging the last two expressions and a careful re-grouping of terms, we obtain

$$\begin{aligned} W = \frac{1}{4} &\int G(z)\,(u(x+\alpha z)-u(x-\alpha z))\,(q(x+\alpha z)-q(x-\alpha z))\,\mathrm{d}z \\ -\frac{1}{4} &\int G(z)\,(-u(x-\alpha z)+2u(x)-u(x+\alpha z))\,(q(x+\alpha z)+q(x-\alpha z))\,\mathrm{d}z \end{aligned}$$

$$= \frac{1}{4} \int G(z) \left(\Delta^1_{2\alpha z} u(x) \, \Delta^1_{2\alpha z} q(x) - \Delta^2_{\alpha z} u(x) \, M^1_{2\alpha z} q(x) \right) \mathrm{d}z \tag{5.3}$$

where Δ^m_h is the difference operator (2.8) and M^m_h is the corresponding averaging operator

$$M^m_h f(x) = \sum_{k=0}^{m} \binom{m}{k} f(x + (k - m/2)h) \,. \tag{5.4}$$

Let p and p' be Hölder conjugates with $1 < p, p' < \infty$. By using the Minkowski and Hölder inequalities, Lemma 7, and the characterization of Besov spaces (2.7), we find that

$$\begin{aligned} \|W\|_{L^2} &\leq \frac{1}{4} \biggl(\int \biggl(\int G(z) \left(|\Delta^1_{2\alpha z} u(x) \, \Delta^1_{2\alpha z} q(x)| + |\Delta^2_{\alpha z} u(x) \, M^1_{2\alpha z} q(x)| \right) \mathrm{d}z \biggr)^2 \mathrm{d}x \biggr)^{\frac{1}{2}} \\ &\leq \frac{1}{4} \int G(z) \left(\|\Delta^1_{2\alpha z} u(x) \, \Delta^1_{2\alpha z} q(x)\|_{L^2} + \|\Delta^2_{\alpha z} u(x) \, M^1_{2\alpha z} q(x)\|_{L^2} \right) \mathrm{d}z \\ &\leq c(p) \, \alpha^{1+s} \int G(z) \, |z|^{1+s} \, \mathrm{d}z \left(\|\omega\|_{L^{2p}} \|q\|_{B^{2p',\infty}_s} + \|\omega\|_{B^{2p,\infty}_s} \|q\|_{L^{2p'}} \right) . \end{aligned} \tag{5.5}$$

In particular, for $p = p' = 2$,

$$\|W\|_{L^2} \leq c \, \alpha^{1+s} \, \|q\|_{L^4} \, \|q\|_{B^{4,\infty}_s} \,. \tag{5.6}$$

6. **The inviscid limit for the velocity.** To estimate the L^2 difference between the Navier–Stokes velocity and the averaged Euler velocity field, we employ an H^{-1} estimate on the vorticities. It turns out that for non-standard two-dimensional fluids, the vorticity estimate is much easier to handle than a direct estimate on the velocity (see [6], for example).

Recall the Navier–Stokes vorticity equation

$$\partial_t \omega^{\text{NS}} + u^{\text{NS}} \cdot \nabla \omega^{\text{NS}} = \nu \Delta \omega^{\text{NS}} \,, \tag{6.1}$$

and note that the averaged Euler vorticity equation can be written

$$\partial_t \omega^\alpha + u^\alpha \cdot \nabla \omega^\alpha = \nabla \cdot \int G^\alpha(x-y) \, (u^\alpha(x) - u^\alpha(y)) \, q^\alpha(y) \, \mathrm{d}y \equiv \nabla \cdot W \,. \tag{6.2}$$

Setting $\theta = \omega^{\text{NS}} - \omega^\alpha$ and $w = u^{\text{NS}} - u^\alpha$, we find that θ satisfies the equation

$$\partial_t \theta + w \cdot \nabla \omega^{\text{NS}} + u^\alpha \cdot \nabla \theta = \nu \Delta \omega^{\text{NS}} - \nabla \cdot W \,. \tag{6.3}$$

We now multiply the equation with $\psi \equiv (1 - \Delta)^{-1} \theta$, integrate in x, and integrate by parts to obtain the equation for the evolution of the H^{-1} norm of θ,

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\theta\|^2_{H^{-1}} &= \int \omega^{\text{NS}} \, w \cdot \nabla \psi \, \mathrm{d}x \\ &\quad + \int \theta \, u^\alpha \cdot \nabla \psi \, \mathrm{d}x - \nu \int \nabla \psi \cdot \nabla \omega^{\text{NS}} \, \mathrm{d}x + \int \nabla \psi \cdot W \, \mathrm{d}x \,. \end{aligned} \tag{6.4}$$

All but the second term on the right can be estimated directly by Cauchy–Schwarz or Hölder inequalities. But

$$\int \theta \, u^\alpha \cdot \nabla \psi \, \mathrm{d}x = - \int \nabla \psi \cdot \nabla (u^\alpha \cdot \nabla \psi) \, \mathrm{d}x = - \int \nabla \psi \cdot (\nabla u^\alpha) \cdot \nabla \psi \, \mathrm{d}x \tag{6.5}$$

due to incompressibility. Altogether,

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{H^{-1}}^2 \leq \|\omega^{\rm NS}\|_{L^\infty}\|w\|_{L^2}\|\nabla\psi\|_{L^2} \\ + \|\nabla u^\alpha\|_{L^\infty}\|\nabla\psi\|_{L^2}^2 + \nu\|\nabla\omega^{\rm NS}\|_{L^2}\|\nabla\psi\|_{L^2} + \|W\|_{L^2}\|\nabla\psi\|_{L^2} \tag{6.6}$$

or, after two applications of the Young inequality,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{H^{-1}}^2 \leq \kappa(\alpha)\|\nabla\psi\|_{L^2}^2 + \nu^2\|\nabla\omega^{\rm NS}\|_{L^2}^2 + \|W\|_{L^2}^2\,, \tag{6.7}$$

where, by Lemma 5, for $0 < \alpha \leq \frac{1}{2}$,

$$\kappa(\alpha) = 1 + \|\omega^{\rm NS}\|_{L^\infty} + \|\nabla u^\alpha\|_{L^\infty} \leq 1 + \|\omega_0^{\rm NS}\|_{L^\infty} - c\ln\alpha\,\|q_0^\alpha\|_{L^1\cap L^\infty}\,. \tag{6.8}$$

Integration in time gives

$$\|\theta(t)\|_{H^{-1}}^2 \leq \|\theta_0\|_{H^{-1}}^2\,\mathrm{e}^{\kappa t} + \mathrm{e}^{\kappa t}\int_0^t \left(\nu^2\|\nabla\omega^{\rm NS}\|_{L^2}^2 + \|W\|_{L^2}^2\right)\mathrm{d}\tau\,, \tag{6.9}$$

where we dropped the factor $\exp(-\kappa t)$ from the integrand. Note that for Navier–Stokes solutions

$$\nu\int_0^\infty \|\nabla\omega^{\rm NS}\|_{L^2}^2\,\mathrm{d}t \leq \|\omega_0\|_{L^2}^2\,. \tag{6.10}$$

Further, by choosing $\omega_0^{\rm NS} = G^\alpha * q_0^\alpha$, corresponding to $\theta_0 = 0$, and using (5.6) as well as Corollary 4, we obtain

$$\begin{aligned}\|\theta(t)\|_{H^{-1}}^2 &\leq \left(\frac{1}{\alpha}\right)^{\kappa_1 t}\mathrm{e}^{\kappa_2 t}\left[\nu\|\omega_0\|_{L^2}^2 + c\,\alpha^{2(1+s)}\int_0^t\left(\frac{1}{\alpha}\right)^{sK\tau}\mathrm{d}\tau\,\|q_0^\alpha\|_{L^4}^2\|q_0^\alpha\|_{B_s^{4,\infty}}^2\right] \\ &\leq \left(\frac{1}{\alpha}\right)^{\kappa_1 t}\mathrm{e}^{\kappa_2 t}\left[\nu\|\omega_0\|_{L^2}^2 + c\,\alpha^{2(1+s-sKt)}\|q_0^\alpha\|_{L^4}^2\|q_0^\alpha\|_{B_s^{4,\infty}}^2\right],\end{aligned} \tag{6.11}$$

or, for $s = 0$,

$$\|\theta(t)\|_{H^{-1}}^2 \leq \left(\frac{1}{\alpha}\right)^{\kappa_1 t}\mathrm{e}^{\kappa_2 t}\left[\nu\|\omega_0\|_{L^2}^2 + c\,\alpha t\,\|q_0^\alpha\|_{L^4}^2\right] \tag{6.12}$$

where $\kappa_1 = c\,\|q_0^\alpha\|_{L^1\cap L^\infty}$ and $\kappa_2 = 2 + \|q_0^\alpha\|_{L^\infty}$.

Remark 6. If we know that $\|\nabla u^\alpha\|_{L^\infty}$ is bounded independently of α, which is the case for vortex patch initial data for example, the $1/\alpha$ factor disappears and we obtain a global-in-time result.

7. **The inviscid limit for the vorticity.** We now estimate the L^2 difference between the Navier–Stokes vorticity and the averaged Euler vorticity (not the averaged Euler *potential* vorticity q). The calculation closely follows that of Constantin and Wu [3].

Note that (6.3) can be re-written

$$\partial_t\theta + u^{\rm NS}\cdot\nabla\theta - \nu\Delta\theta = \nu\Delta\omega^\alpha - w\cdot\nabla\omega^\alpha - \nabla\cdot W\,. \tag{7.1}$$

To obtain an L^2 estimate on θ, we need to control the terms on the right side of

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta\|_{L^2}^2 + \nu\|\nabla\theta\|_{L^2}^2 = -\nu\int\nabla\theta\cdot\nabla\omega^\alpha\,\mathrm{d}x - \int\theta\,w\cdot\nabla\omega^\alpha\,\mathrm{d}x + \int\nabla\theta\cdot W\,\mathrm{d}x\,. \tag{7.2}$$

The first term on the right is bounded by using the Cauchy–Schwarz inequality, Lemma 6, and the Young inequality,

$$
\begin{aligned}
-\nu \int \nabla\theta \cdot \nabla\omega^\alpha \, \mathrm{d}x &\le \nu \, \|\nabla\theta\|_{L^2} \, c \, \alpha^{s-1} \, \|q^\alpha\|_{B^{2,\infty}_s} \\
&\le \frac{\nu}{2} \, \|\nabla\theta\|_{L^2}^2 + c \, \nu \, \alpha^{2(s-1)} \, \|q^\alpha\|_{B^{2,\infty}_s}^2 \, . \qquad (7.3)
\end{aligned}
$$

Similarly, the second term on the right of (7.2) is estimated

$$
- \int \theta \, w \cdot \nabla\omega^\alpha \, \mathrm{d}x \le c \, \alpha^{s-1} \, \|\theta\|_{L^\infty} \, \|w\|_{L^2} \, \|q^\alpha\|_{B^{2,\infty}_s} \qquad (7.4)
$$

The third term on the right of (7.2) is estimated again by using the Cauchy–Schwarz and Young inequalities and estimate (5.6) for W,

$$
\int \nabla\theta \cdot W \, \mathrm{d}x \le \|\nabla\theta\|_{L^2} \, \|W\|_{L^2} \le \frac{\nu}{2} \, \|\nabla\theta\|_{L^2}^2 + c \, \frac{\alpha^{2(1+s)}}{\nu} \, \|q^\alpha\|_{L^4}^2 \, \|q^\alpha\|_{B^{4,\infty}_s}^2 \, . \qquad (7.5)
$$

Altogether, using Corollary 4 and (6.11), we obtain

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \|\theta\|_{L^2}^2 &\le C_1 \, \nu \, \alpha^{2(s-1-sKt)} \\
&\quad + C_2 \, \alpha^{s-1-sKt-\kappa_1 t} \left(\nu + \alpha^{2(1+s-sKt)} \right) + C_3 \, \frac{\alpha^{2(1+s-sKt)}}{\nu} \qquad (7.6)
\end{aligned}
$$

where the C_i depend on various norms of the initial data, and C_2 is also an increasing function of t. With the natural scaling $\nu = \alpha^2$ and

$$
T = \frac{\varepsilon}{sK + \kappa_1} \, , \qquad (7.7)
$$

integration of (7.6) completes the proof of Theorem 1.

Appendix A. L^2-L^∞ **splitting.** It is possible to replace the L^4-L^4 splitting in (5.6) by an L^2-L^∞ splitting, which is somewhat more natural given that the potential vorticity q is an advected quantity. However, the estimates of Section 5 are not valid with $p = \infty$, and we have to work much harder to prove the corresponding result. The benefit is that this allows us to drop the requirement that the initial vorticity is in $B^{4,\infty}_s$ (we still need that the initial data is in $B^{2,\infty}_s$) on the expense of some other logarithmic correction in α in our main result. Here I will present only the commutator estimate. The necessary modifications to the other parts of the argument are rather straightforward and shall be omitted.

Lemma 8. *Let u, ω, and q denote the velocity, vorticity, and potential vorticity fields of a solution to the averaged Euler equations, i.e. $q = (1 - \alpha^2\Delta)\omega$ and $\omega = \nabla^\perp \cdot u$, and suppose that $q \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \cap B^{2,\infty}_s$. Further, let $\theta \in H^1(\mathbb{R}^2)$ and $0 < \alpha \le 1$. Then*

$$
\begin{aligned}
&\iint G^\alpha(x-y) \, \nabla\theta(x) \cdot (u(x) - u(y)) \, q(y) \, \mathrm{d}x \, \mathrm{d}y \\
&\qquad \le c \, \alpha^{1+s} \, (1 - \ln\alpha) \, \|\nabla\theta\|_{L^2} \, \|q\|_{L^1 \cap L^\infty} \, \|q\|_{B^{2,\infty}_s} \, , \qquad (A.1)
\end{aligned}
$$

where G^α is the Green's kernel of $1 - \alpha^2\Delta$, or some other radial kernel with sufficiently weak singularity at $x = y$.

Proof. Througout the proof we assume that all functions are smooth, the statement as claimed will immediately follow by density. Notice first that

$$u(x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \ln|x-z| \, \nabla^\perp \omega(z) \, \mathrm{d}z \,, \tag{A.2}$$

so that the left side of (A.1) can be written

$$\begin{aligned} I &\equiv \frac{1}{2\pi} \iiint G^\alpha(x-y) \, \nabla\theta(x) \cdot \nabla_z^\perp \omega(z) \, \ln \frac{|x-z|}{|y-z|} \, q(y) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\ &= \frac{1}{2\pi} \iiint G(z) \, \nabla\theta(x) \cdot \left(\frac{(y+\alpha z)^\perp}{|y+\alpha z|^2} - \frac{y^\perp}{|y|^2} \right) \omega(x+y) \, q(x-\alpha z) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \,. \end{aligned} \tag{A.3}$$

The second line has been obtained from the first by the change of variables $(x-y)/\alpha \mapsto z$ and $z-x \mapsto y$ and integration by parts. We now change variables $y \mapsto -y$ and $z \mapsto -z$, and average of both versions of the integral I,

$$\begin{aligned} I = \frac{1}{2} \iiint G(z) \, \nabla\theta(x) \cdot \left(\frac{(y+\alpha z)^\perp}{|y+\alpha z|^2} - \frac{y^\perp}{|y|^2} \right) \Big[\omega(x+y) \, (q(x-\alpha z) - q(x+\alpha z)) \\ + (\omega(x+y) - \omega(x-y)) \, q(x+\alpha z) \Big] \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \,. \end{aligned} \tag{A.4}$$

Let us now split the integral into a part near the singularities, and a part away from the singularities. For the former, assume that $|y| < 2\alpha|z|$, and estimate

$$\begin{aligned} I_{\text{near}} &\leq \frac{1}{2} \iiint_{|y|<2\alpha|z|} |G(z)| \, |\nabla\theta(x)| \left(\frac{1}{|y+\alpha z|} + \frac{1}{|y|} \right) \times \\ &\quad \Big[|\omega(x+y)| \, |q(x-\alpha z) - q(x+\alpha z)| + |\omega(x+y) - \omega(x-y)| \, |q(x+\alpha z)| \Big] \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \\ &\leq c \iint_{|y|<2\alpha|z|} |G(z)| \left(\frac{1}{|y+\alpha z|} + \frac{1}{|y|} \right) (\alpha^s \, |z|^s + |y|^s) \, \mathrm{d}y \, \mathrm{d}z \, \|\nabla\theta\|_{L^2} \, \|q\|_{L^\infty} \, \|q\|_{B_s^{2,\infty}} \end{aligned} \tag{A.5}$$

where, in the second step, we have used characterization (2.7) of $B_q^{2,\infty}$ as well as the continuity of $q \mapsto \omega$ in L^p and Besov spaces. The remaining integral is bounded from above by

$$2 \int |G(z)| \int_{|y|<4\alpha|z|} \frac{\alpha^s \, |z|^s + |y|^s}{|y|} \, \mathrm{d}y \, \mathrm{d}z = c \, \alpha^{1+s} \int_{\mathbb{R}^2} |G(z)| \, |z|^{1+s} \, \mathrm{d}z \,, \tag{A.6}$$

so that

$$I_{\text{near}} \leq c \, \alpha^{1+s} \, \|\nabla\theta\|_{L^2} \, \|q\|_{L^\infty} \, \|q\|_{B_s^{2,\infty}} \,. \tag{A.7}$$

Away from the singularities, we expand the difference of the two quotient terms,

$$\frac{(y+\alpha z)^\perp}{|y+\alpha z|^2} - \frac{y^\perp}{|y|^2} = \frac{\alpha|y|^2 z^\perp - 2\alpha \, y \cdot z \, y^\perp - \alpha^2 |z|^2 y^\perp}{|y|^2 \, |y+\alpha z|^2} \,. \tag{A.8}$$

Separating the two terms in the square brackets of (A.4) and the α^2-term of (A.8), we see that we have to estimate the integrals

$$
\begin{aligned}
I_{\rm far}^{(1)} \equiv \alpha \iiint\limits_{|y|>2\alpha|z|} G(z)\,\nabla\theta(x)\cdot \frac{|y|^2 z^\perp - 2\,y\cdot z\,y^\perp}{|y|^2\,|y+\alpha z|^2} \\
\times\,\omega(x+y)\,(q(x-\alpha z)-q(x+\alpha z))\,{\rm d}x\,{\rm d}y\,{\rm d}z\,,
\end{aligned} \tag{A.9}
$$

$$
\begin{aligned}
I_{\rm far}^{(2)} \equiv -\alpha^2 \iiint\limits_{|y|>2\alpha|z|} G(z)\,\nabla\theta(x)\cdot \frac{|z|^2 y^\perp}{|y|^2\,|y+\alpha z|^2} \\
\times\,\omega(x+y)\,(q(x-\alpha z)-q(x+\alpha z))\,{\rm d}x\,{\rm d}y\,{\rm d}z\,,
\end{aligned} \tag{A.10}
$$

$$
\begin{aligned}
I_{\rm far}^{(3)} \equiv \alpha \iiint\limits_{|y|>2\alpha|z|} G(z)\,\nabla\theta(x)\cdot \frac{|y|^2 z^\perp - 2\,y\cdot z\,y^\perp}{|y|^2\,|y+\alpha z|^2} \\
\times\,(\omega(x+y)-\omega(x-y))\,q(x+\alpha z)\,{\rm d}x\,{\rm d}y\,{\rm d}z
\end{aligned} \tag{A.11}
$$

and

$$
\begin{aligned}
I_{\rm far}^{(4)} \equiv -\alpha^2 \iiint\limits_{|y|>2\alpha|z|} G(z)\,\nabla\theta(x)\cdot \frac{|z|^2 y^\perp}{|y|^2\,|y+\alpha z|^2} \\
\times\,(\omega(x+y)-\omega(x-y))\,q(x+\alpha z)\,{\rm d}x\,{\rm d}y\,{\rm d}z
\end{aligned} \tag{A.12}
$$

For $I_{\rm far}^{(1)}$ we can simply take the absolute value of all terms, break up the domain of y-integration into a bounded and an unbounded piece, and use the Cauchy–Schwarz inequality for the x-integral:

$$
\begin{aligned}
I_{\rm far}^{(1)} &\le 6\,\alpha \iiint\limits_{1>|y|>2\alpha|z|} |G(z)|\,|\nabla\theta(x)|\,\frac{|z|}{|y|^2}\,|\omega(x+y)|\,|q(x-\alpha z)-q(x+\alpha z)|\,{\rm d}x\,{\rm d}y\,{\rm d}z \\
&\quad + 6\alpha \iiint\limits_{\substack{|y|>1\\|y|>2\alpha|z|}} |G(z)|\,|\nabla\theta(x)|\,|z|\,|\omega(x+y)|\,|q(x-\alpha z)-q(x+\alpha z)|\,{\rm d}x\,{\rm d}y\,{\rm d}z \\
&\le c\,\alpha^{1+s} \int\limits_{2\alpha|z|<1} |G(z)|\,|z|^{1+s} \int\limits_{2\alpha|z|}^{1} \frac{{\rm d}r}{r}\,{\rm d}z\,\|\nabla\theta\|_{L^2}\,\|q\|_{L^\infty}\,\|q\|_{B^{2,\infty}_s} \\
&\quad + c\,\alpha^{1+s} \int |G(z)|\,|z|^{1+s}\,{\rm d}z\,\|\nabla\theta\|_{L^2}\,\|q\|_{L^1}\,\|q\|_{B^{2,\infty}_s} \\
&\le c\,\alpha^{1+s}\,(1-\ln\alpha)\,\|\nabla\theta\|_{L^2}\,\|q\|_{L^1\cap L^\infty}\,\|q\|_{B^{2,\infty}_s}\,.
\end{aligned} \tag{A.13}
$$

The estimation of $I_{\rm far}^{(2)}$ is similar,

$$
\begin{aligned}
I_{\rm far}^{(2)} &\le 2\,\alpha^2 \iiint\limits_{|y|>2\alpha|z|} |G(z)|\,|\nabla\theta(x)|\,\frac{|z|^2}{|y|^3}\,|\omega(x+y)|\,|q(x-\alpha z)-q(x+\alpha z)|\,{\rm d}x\,{\rm d}y\,{\rm d}z \\
&\le c\,\alpha^{2+s} \int |G(z)|\,|z|^{2+s} \int\limits_{2\alpha|z|}^{\infty} \frac{{\rm d}r}{r^2}\,{\rm d}z\,\|\nabla\theta\|_{L^2}\,\|q\|_{L^\infty}\,\|q\|_{B^{2,\infty}_s} \\
&\le c\,\alpha^{1+s}\,\|\nabla\theta\|_{L^2}\,\|q\|_{L^\infty}\,\|q\|_{B^{2,\infty}_s}\,.
\end{aligned} \tag{A.14}
$$

To estimate $I_{\text{far}}^{(3)}$, we make the change of variables $z \mapsto -z$ and take the average of both versions of the integral:

$$I_{\text{far}}^{(3)} \equiv \frac{\alpha}{2} \iiint\limits_{|y|>2\alpha|z|} G(z)\, \nabla\theta(x) \cdot \frac{|y|^2 z^\perp - 2\, y \cdot z\, y^\perp}{|y|^2}\, (\omega(x+y) - \omega(x-y)) \times$$
$$\left[\left(\frac{1}{|y+\alpha z|^2} - \frac{1}{|y-\alpha z|^2} \right) q(x+\alpha z) + \frac{1}{|y-\alpha z|^2} \Big(q(x+\alpha z) - q(x-\alpha z) \Big) \right] \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\,. \tag{A.15}$$

The integral arising from the second term in the square brackets can be estimated in exactly the same way as $I_{\text{far}}^{(1)}$. To estimate the first term in the square brackets, we note that

$$\frac{1}{|y+\alpha z|^2} - \frac{1}{|y-\alpha z|^2} = \frac{4\alpha\, y \cdot z}{|y+\alpha z|^2\, |y-\alpha z|^2}\,, \tag{A.16}$$

and, by taking absolute values inside the integral, we find that the contribution from this term is essentially bounded by

$$\alpha^2 \iiint\limits_{|y|>2\alpha|z|} |G(z)|\, |\nabla\theta(x)|\, \frac{|z|^2}{|y|^3}\, |\omega(x+y) - \omega(x-y)|\, |q(x+\alpha z)|\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z\,. \tag{A.17}$$

Manipulations similar to the above will lead to an expression that is identical to the second step in the estimation of $I_{\text{far}}^{(1)}$, equation (A.13). Finally, taking absolute values inside of $I_{\text{far}}^{(4)}$, we also obtain another integral of the form (A.17). This concludes the estimation of I. □

Acknowledgments. I thank Igor Kukavica, Jerry Marsden, and Steve Shkoller for interesting and stimulating discussions, and the Department of Mathematics at the University of Southern California for their hospitality when this work was begun.

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