Global Well-Posedness for the Lake Equations

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We prove global well-posedness for the lake equations. These equations arise to leading order in a low aspect ratio, low Froude number (i.e. low wave speed) and very small wave amplitude expansion of the three dimensional incompressible Euler equations in a basin with a free upper surface and a spatially varying bottom topography. Well-posedness means that there exists a solution, that it is unique and that it depends continuously on the data, i.e. the initial condition and the bottom topography.
Our approach follows the works of Yudovitch and Bardos in constructing the solutions as the inviscid limit of solutions to a system with artificial viscosity which is the analog of the Navier-Stokes with respect to the Euler equations. One of the main assumptions in the present work is a nondegenerate bottom topography.

Key words: shallow water equations, Euler equations, well-posedness, bottom topography, viscous regularization.

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Article published in Physica D 98 (1996) 492–509

1 Introduction

The objective of shallow water theory is to isolate the large scale motion of a fluid from motions on shorter time and length scales. The basis is a fluid dynamical model that is believed to describe the relevant physics of the problem. The equations are non-dimensionalized and rescaled such that the interesting physical quantities become order one. Provided there are small scaling factors, the rescaled equations can then be asymptotically expanded in these parameters, hopefully giving a simpler set of equations at the lowest orders of the expansion. Since these manipulations are entirely formal, it is *a priori* not clear in what sense solutions of the resulting shallow water equations represent solutions of the original model. In order to answer this question, one must establish a setting in which the simpler equations are globally well-posed. This paper establishes global well-posedness for the lake equations, a particular type of shallow water equation.

In the following we will explore one of the simplest underlying physical situations, namely an inviscid fluid of constant density contained in a shallow basin with vertical lateral boundaries in a uniform gravitational field. The full dynamics is thus described by the Euler equations in three spatial dimensions. We consider the regime where the aspect ratio δ (the ratio of vertical to horizontal length scales) and the Froude number ε (the ratio of typical horizontal speeds to gravity wave speeds) are small, and the wave amplitude is very small. In this limit, the evolution of the vertically averaged velocity vector $\boldsymbol{u}(\boldsymbol{x},t)$ is governed to leading order in δ and ε by the set of nondimensional equations

$$\begin{aligned}
&\partial_t \boldsymbol{u} + (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u} + \boldsymbol{\nabla} h = 0 \quad && \text{in } \Omega\,, && \text{(1a)}\\
&\boldsymbol{\nabla}\cdot(b\boldsymbol{u}) = 0 && \text{in } \Omega\,, && \text{(1b)}\\
&\boldsymbol{u}\cdot\boldsymbol{n} = 0 && \text{on } \partial\Omega\,, && \text{(1c)}\\
&\boldsymbol{u}(0) = \boldsymbol{u}^{\text{in}}\,. && && \text{(1d)}
\end{aligned}$$

Here Ω denotes the horizontal domain, $h(\boldsymbol{x},t)$ the surface height variation, $b(\boldsymbol{x})$ the depth of the basin and $\boldsymbol{n}(\boldsymbol{x})$ a unit vector normal on $\partial\Omega$. This system, which we call the *lake equations*, has essentially been described by Greenspan [12, page 235]. It will be the main focus of our subsequent study. In [6,7] it is also shown how to obtain the system to next order in δ^2, the so-called *great lake equations*, which are

$$\begin{aligned}
&\partial_t \boldsymbol{v} - \boldsymbol{u}^\perp\, \boldsymbol{\nabla}\wedge\boldsymbol{v} + \boldsymbol{\nabla}(h - \tfrac{1}{2}\boldsymbol{u}\cdot\boldsymbol{u} + \boldsymbol{u}\cdot\boldsymbol{v}) = 0 \quad && \text{in } \Omega\,, && \text{(2a)}\\
&\boldsymbol{v} = \mathcal{L}\boldsymbol{u} && \text{in } \Omega\,, && \text{(2b)}\\
&\boldsymbol{\nabla}\cdot(b\boldsymbol{u}) = 0 && \text{in } \Omega\,, && \text{(2c)}\\
&\boldsymbol{u}\cdot\boldsymbol{n} = 0 && \text{on } \partial\Omega\,, && \text{(2d)}\\
&\boldsymbol{u}(0) = \boldsymbol{u}^{\text{in}}\,. && && \text{(2e)}
\end{aligned}$$

We use the notation $\boldsymbol{u}^\perp = (u_2, -u_1)$ and $\boldsymbol{\nabla} \wedge \boldsymbol{v} = \partial_1 v_2 - \partial_2 v_1$. The linear operator $\mathcal{L}$ is given by

$$\begin{aligned}\boldsymbol{v} = \mathcal{L}\boldsymbol{u} = \boldsymbol{u} + \delta^2\big((\boldsymbol{u}\cdot\boldsymbol{\nabla} b)\boldsymbol{\nabla} b + \tfrac{1}{2} b(\boldsymbol{\nabla}\cdot\boldsymbol{u})\,\boldsymbol{\nabla} b \\ - \tfrac{1}{2} b^{-1}\boldsymbol{\nabla}(b^2\,\boldsymbol{u}\cdot\boldsymbol{\nabla} b) - \tfrac{1}{3} b^{-1}\boldsymbol{\nabla}(b^3\boldsymbol{\nabla}\cdot\boldsymbol{u})\big)\,. \end{aligned} \tag{3}$$

It is easily verified that for a flat bottom both the lake and the great lake equations formally reduce to the two-dimensional Euler equations. Moreover, in reference [7] it is shown that these shallow water models have a Hamiltonian structure, convect potential vorticity and enjoy a Kelvin circulation theorem in complete analogy with the pure Euler case. We extend this analogy herein by showing that the techniques employed by Yudovitch [20] and Bardos [4] for proving existence and uniqueness for the two dimensional Euler equations can be generalized to apply to the lake equations.

The construction of the solution employs a two-step regularization procedure. First, artificial viscosity is added to the system, and a solution to the viscous equations is obtained as the limit of a Galerkin approximating sequence [13]. This is essentially a generalization of Leray's theory of weak solutions to the Navier Stokes equations. In the second step one shows that the solutions so obtained converge to solutions of the inviscid lake equations in the limit of vanishing viscosity. The uniqueness of the solutions is proved by estimating the time evolution of the difference of two solutions in a suitable Sobolev norm. This technique can also be used to prove continuous dependence of the solution of the initial data and on the bottom topography. The main complication in the latter estimates stems from the fact that the two solutions whose difference is to be estimated correspond to different bottom topographies, satisfy different weighted incompressibility conditions (1b) and are thereby members of different function classes.

The paper is organized as follows. The derivation of the lake equations will be summarized in Section 2. In Section 3 we will introduce the vorticity formulation of the equations. The main section of this paper, Section 4, is devoted to proving existence and uniqueness of solutions to an abstract version of the viscosity equation. Estimates on how the solutions to the lake equation depend on the initial data and on the bottom topography will be given in the final section. Altogether these results affirm that the lake equations are well-posed.

2 Derivation of the Equations

The geometry under consideration is depicted in Figure 1. The horizontal position coordinates are denoted $\boldsymbol{x} = (x, y)$ and range over the horizontal domain Ω, so that the fixed lateral boundaries are located at $\boldsymbol{x} \in \partial\Omega$. The

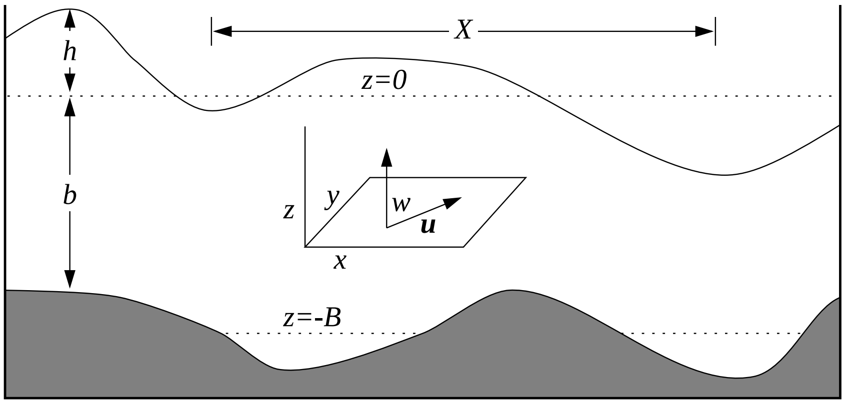

Figure 1. The geometry of the basin. The amplitude of the surface height h is greatly exaggerated. The lateral boundaries are always assumed to be vertical.

vertical position coordinate z ranges from the fixed bottom at $z = -b(\boldsymbol{x})$ to $z = h(\boldsymbol{x})$, the height of the fluid over the mean free surface at $z = 0$. All quantities are assumed to vary horizontally on typical scales X and vertically on scales on the order of the mean depth B.

Our basic equations are the three dimensional Euler equations which we write as separate equations for the horizontal velocity vector $\boldsymbol{u}(\boldsymbol{x}, z, t)$ and the vertical velocity $w(\boldsymbol{x}, z, t)$,

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + w \partial_z \boldsymbol{u} + \frac{1}{\rho} \boldsymbol{\nabla} p = 0\,, \tag{4a}$$

$$\partial_t w + \boldsymbol{u} \cdot \boldsymbol{\nabla} w + w \partial_z w + \frac{1}{\rho} \partial_z p = 0\,, \tag{4b}$$

where ρ denotes the constant fluid density and $p = p_{\text{tot}} + \rho g z$ is the modified pressure, which is the total pressure minus the hydrostatic pressure associated with the stationary solution, as the latter does not accelerate the fluid [5]. The constant of gravity g is taken to be positive. For a fluid of constant density, the continuity equation reads

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} + \partial_z w = 0\,. \tag{4c}$$

We assume that the total pressure p_{tot} at the free surface is zero. Furthermore, no fluid crosses the top, bottom and lateral boundaries, so that the boundary conditions we must impose are

$$p = \rho g h \qquad \text{for } \boldsymbol{x} \in \Omega \quad \text{and } z = h(\boldsymbol{x}, t)\,, \tag{4d}$$

$$w = \partial_t h + \boldsymbol{u} \cdot \boldsymbol{\nabla} h \qquad \text{for } \boldsymbol{x} \in \Omega \quad \text{and } z = h(\boldsymbol{x}, t)\,, \tag{4e}$$

$$w = -\boldsymbol{u} \cdot \boldsymbol{\nabla} b \qquad \text{for } \boldsymbol{x} \in \Omega \quad \text{and } z = -b(\boldsymbol{x})\,, \tag{4f}$$

$$\boldsymbol{n} \cdot \boldsymbol{u} = 0 \qquad \text{for } \boldsymbol{x} \in \partial\Omega \text{ and } -b(\boldsymbol{x}) < z < h(\boldsymbol{x}, t)\,. \tag{4g}$$

Using capital letters to denote the typical magnitude of all quantities, and setting $t = Tt^*$, $w = Ww^*$ etc., we introduce a new set of dimensionless variables whose magnitude is order one. In these new variables, after dropping the stars, the equations now read

$$\frac{U}{T}\partial_t \boldsymbol{u} + \frac{U^2}{X}\boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \frac{UW}{B}w\partial_z\boldsymbol{u} + \frac{1}{\rho}\frac{P}{X}\boldsymbol{\nabla}p = 0\,, \tag{5a}$$

$$\frac{W}{T}\partial_t w + \frac{UW}{X}\boldsymbol{u}\cdot\boldsymbol{\nabla}w + \frac{W^2}{B}w\partial_z w + \frac{1}{\rho}\frac{P}{B}\partial_z p = 0\,, \tag{5b}$$

$$\frac{U}{X}\boldsymbol{\nabla}\cdot\boldsymbol{u} + \frac{W}{B}\partial_z w = 0\,, \tag{5c}$$

$$\begin{aligned}
&Pp = \rho g H h && \text{for } \boldsymbol{x}\in\Omega \quad \text{and } Bz = Hh(\boldsymbol{x},t)\,, && \text{(5d)}\\
&Ww = \frac{H}{T}\partial_t h + \frac{UH}{X}\boldsymbol{u}\cdot\boldsymbol{\nabla}h && \text{for } \boldsymbol{x}\in\Omega \quad \text{and } Bz = Hh(\boldsymbol{x},t)\,, && \text{(5e)}\\
&Ww = -\frac{UB}{X}\boldsymbol{u}\cdot\boldsymbol{\nabla}b && \text{for } \boldsymbol{x}\in\Omega \quad \text{and } Bz = -Bb(\boldsymbol{x})\,, && \text{(5f)}\\
&\boldsymbol{n}\cdot\boldsymbol{u} = 0 && \text{for } \boldsymbol{x}\in\partial\Omega \text{ and } -Bb(\boldsymbol{x}) < Bz < Hh(\boldsymbol{x},t)\,. && \text{(5g)}
\end{aligned}$$

All the scaling factors are still completely undetermined, and we will use physical considerations to fix these free parameters. First, both terms in the incompressibility condition (5c) should be of the same order because we want the displacement of the fluid caused by the total height variation of a column of water to enter the equations at leading order, hence

$$\frac{U}{X} = \frac{W}{B}\,. \tag{6}$$

Second, we want to look at the problem on a convective time scale, i.e. on the time scale in which a fluid parcel travels a typical length scale. This means that the time derivative and the nonlinear term in the momentum equations (5a) and (5b) enter at the same order, or

$$\frac{1}{T} = \frac{U}{X}\,. \tag{7}$$

The relative order in which the pressure terms enter the momentum equations depends only on the aspect ratio of the basin. In the shallow water situation which we are considering here, the aspect ratio is assumed to be the small quantity

$$\delta = \frac{B}{X}\,. \tag{8}$$

This also means that horizontal motion is dominant. The pressure term as the horizontal accelerator should therefore enter the momentum equation (5a) at leading order, which implies the scaling

$$P = \rho U^2 \,. \tag{9}$$

The assumptions made so far fix the scalings in all of (5) but the two free top boundary conditions. Indeed, the remaining free scaling factor in (5e) is the ratio of wave amplitude to depth H/B, and the free scaling factor in (5d) is then recognized to be $\varepsilon^2 B/H$. The quantity

$$\varepsilon = \frac{U}{\sqrt{gB}} \tag{10}$$

is the so-called *Froude number*, defined to be the ratio of typical horizontal wave speed to the speed of gravity waves. We will consider the low Froude number, very low wave amplitude regime. More precisely, we demand that

$$\frac{H}{B} = \varepsilon^2 \,. \tag{11}$$

This is consistent with the characteristic scales of global ocean flows ($U =$ 2m/s, $B =$ 4000m, $g =$ 10m/s^2 and $H =$ 0.4m). Of course we do not claim to model a real ocean because quantities that crucially affect the dynamics of the ocean (such as Coriolis force, temperature, salinity and stratification) are excluded from the model. On the other hand, the scaling of the quantities that are present is physical and can be expected to carry over to more comprehensive models. On the level of the rescaled Euler equations (5), the scaling (11) implies that the hydrostatic imbalance enters the horizontal momentum equation to leading order, and thus assures nontrivial leading order dynamics. We remark further that this scaling allows for order one depth variations over horizontal lengths X so long as the depth remains order one, i.e. the scaling (11) is not violated. This is realistically required in many physical situations. Other commonly used scalings need the assumption of small variations in depth (see e.g. [16]).

Altogether the rescaled system becomes

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + w \partial_z \boldsymbol{u} + \boldsymbol{\nabla} p = 0 \,, \tag{12a}$$

$$\delta^2 \big(\partial_t w + \boldsymbol{u} \cdot \boldsymbol{\nabla} w + w \partial_z w \big) + \partial_z p = 0 \,, \tag{12b}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} + \partial_z w = 0 \,, \tag{12c}$$

$$p = h \qquad \text{for } \boldsymbol{x} \in \Omega \quad \text{and } z = \varepsilon^2 h(\boldsymbol{x}, t) \,, \tag{12d}$$

$$w = \varepsilon^2 (\partial_t h + \boldsymbol{u} \cdot \boldsymbol{\nabla} h) \qquad \text{for } \boldsymbol{x} \in \Omega \quad \text{and } z = \varepsilon^2 h(\boldsymbol{x}, t) \,, \tag{12e}$$

$$w = -\boldsymbol{u} \cdot \boldsymbol{\nabla} b \qquad \text{for } \boldsymbol{x} \in \Omega \quad \text{and } z = -b(\boldsymbol{x}) \,, \tag{12f}$$

$$\boldsymbol{n} \cdot \boldsymbol{u} = 0 \qquad \text{for } \boldsymbol{x} \in \partial\Omega \text{ and } -b(\boldsymbol{x}) < z < \varepsilon^2 h(\boldsymbol{x}, t) \,. \tag{12g}$$

In order to obtain the leading order approximation, we first hold δ fixed and let $\varepsilon \to 0$. Since condition (12e) becomes $w = 0$ at $z = 0$, we can integrate the incompressibility condition in the z variable to obtain

$$w(\boldsymbol{x}, z, t) = \int_z^0 \boldsymbol{\nabla} \cdot \boldsymbol{u}(\boldsymbol{x}, z', t) \, \mathrm{d}z' \,. \tag{13}$$

Furthermore, by using this identity, one obtains

$$\boldsymbol{\nabla} \cdot \left(\int_{-b}^0 \boldsymbol{u}(\boldsymbol{x}, z', t) \, \mathrm{d}z' \right) = w(\boldsymbol{x}, -b, t) + \boldsymbol{u} \cdot \boldsymbol{\nabla} b \,, \tag{14}$$

so that boundary condition (12f) is equivalent to

$$\boldsymbol{\nabla} \cdot \left(\int_{-b}^0 \boldsymbol{u}(\boldsymbol{x}, z', t) \, \mathrm{d}z' \right) = 0 \,. \tag{15}$$

Second, we let $\delta \to 0$. By integrating the resulting vertical momentum equation (12b) and using boundary condition (12d) we then find $p = p(\boldsymbol{x}, t) = h(\boldsymbol{x}, t)$.

The main assumption we make to reduce the dynamics to a two dimensional problem is that the initial velocity field, and thus the velocity field at any time t is independent of z, i.e. that we have columnar motion. Condition (15) immediately reduces to $\boldsymbol{\nabla} \cdot (b\boldsymbol{u}) = 0$ and the equations to leading order in δ become exactly the lake equations (1). In references [6,7] it is also shown how to obtain the great lake equations (2) which are closed equations of the approximation to order δ^2. At this order, the shallow water equations allow for noncolumnar corrections to the velocity field.

3 The Vorticity Formulation

In the remainder of this paper we will restrict ourselves to discussing the lake equations, and remark that the great lake equations can be treated without difficulty within the framework presented here [13].

The goal of this section is to recast the lake equations into their so-called vorticity formulation. This serves several purposes. Most importantly, the proof of the existence and uniqueness theorem makes substantial use of the fact that the *potential vorticity* $\omega \equiv b^{-1} \boldsymbol{\nabla} \wedge \boldsymbol{u}$ is bounded for all times, which can be established directly from the vorticity equation. Moreover, the vorticity formulation allows one to write the argument in a more compact way and, lastly, to treat the great lake equations in complete analogy.

An evolution equation for the potential vorticity is obtained by taking the curl of the momentum equation (1a). The nonlinear term simplifies through use of the identity

$$\begin{aligned}(\boldsymbol{u}\cdot\boldsymbol{\nabla})(b^{-1}\boldsymbol{\nabla}\wedge\boldsymbol{u}) &= -b^{-2}(\boldsymbol{u}\cdot\boldsymbol{\nabla}b)\boldsymbol{\nabla}\wedge\boldsymbol{u}+b^{-1}(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{\nabla}\wedge\boldsymbol{u}\\ &= b^{-1}(\boldsymbol{\nabla}\cdot\boldsymbol{u}+\boldsymbol{u}\cdot\boldsymbol{\nabla})(\boldsymbol{\nabla}\wedge\boldsymbol{u})\\ &= b^{-1}\boldsymbol{\nabla}\wedge\big((\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\big)\,, \end{aligned}\tag{16}$$

which is true due to the weighted incompressibility condition $\boldsymbol{\nabla}\cdot(b\boldsymbol{u}) = 0$. The resulting potential vorticity equation is therefore

$$\partial_t\omega+\boldsymbol{u}\cdot\boldsymbol{\nabla}\omega = 0\,. \tag{17}$$

In order to close this equation, we need to be able to compute $\boldsymbol{u}$ given ω. To this end we introduce the stream function ψ by solving the—as we will show below—elliptic problem

$$\omega = b^{-1}\,\boldsymbol{\nabla}\wedge(b^{-1}\,\boldsymbol{\nabla}^{\perp}\psi) \quad \text{in } \Omega\,, \tag{18a}$$

$$\psi = 0 \quad \text{on } \partial\Omega\,. \tag{18b}$$

The notation used here is $\boldsymbol{\nabla}^{\perp} = (\partial_2, -\partial_1)$. We then let

$$\boldsymbol{u} = b^{-1}\,\boldsymbol{\nabla}^{\perp}\psi\,. \tag{19}$$

This relationship between $\boldsymbol{u}$ and ω shall be denoted by the abstract operator K, so that the full set of vorticity equations becomes

$$\partial_t\omega+\boldsymbol{u}\cdot\boldsymbol{\nabla}\omega = 0\,, \tag{20a}$$

$$\boldsymbol{u} = K\omega\,, \tag{20b}$$

$$\omega(0) = \omega^{\mathrm{in}}\,. \tag{20c}$$

The fact that $\boldsymbol{u}$ satisfies the weighted incompressibility condition (1b) is not explicitly contained in (20), but rather has to be verified given the explicit form of K above. This condition makes it natural to pose (20) in Sobolev spaces [1] with weight b, i.e. to endow $\mathrm{H}^n(\Omega)$ for $n\in\mathbb{N}$ with the scalar product

$$(\varphi,\vartheta)_{\mathrm{H}^n} = \sum_{0\le|\alpha|\le n}\int_{\Omega} D^{\alpha}\varphi(\boldsymbol{x})\,D^{\alpha}\vartheta(\boldsymbol{x})\,b(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,. \tag{21}$$

The domain Ω is bounded and for simplicity assumed to be simply connected (i.e. no islands in the lake) with a C^2 boundary $\partial\Omega$. Its closure is denoted $\overline{\Omega}$. The multiply connected case is discussed in [13]. Our main assumption is that the bottom function b is nondegenerate, i.e. that there are constants $b_{\min}$ and $b_{\max}$ such that for every $\boldsymbol{x}\in\Omega$,

$$0 < b_{\min} \le b(\boldsymbol{x}) \le b_{\max}\,. \tag{22}$$

The nondegeneracy of b means that the norm associated with (21) is equivalent to the canonical norm on $\mathrm{H}^n(\Omega)$, i.e. the weight b affects the geometry, but not the topology. The degenerate case, namely $b(\boldsymbol{x}) = 0$ on $\partial\Omega$ or part of $\partial\Omega$ causes additional technical complications and will be studied in subsequent work.

A frequently used subspace of $\mathrm{H}^1(\Omega)$ is $\mathrm{H}^1_0(\Omega)$, defined as the completion of the infinitely differentiable and compactly supported functions $\mathrm{C}^\infty_0(\Omega)$ in the H^1 norm. Its dual $\mathrm{H}^1_0(\Omega)'$ is denoted $\mathrm{H}^{-1}(\Omega)$, and we will in fact show below that the operator defined by (18) is an isomorphism from $\mathrm{H}^1_0(\Omega)$ onto $\mathrm{H}^{-1}(\Omega)$.

The weighted integral over the domain is abbreviated by $\langle\cdot\rangle$, i.e.

$$\langle\varphi\rangle = \int_\Omega \varphi(\boldsymbol{x})\, b(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \,, \tag{23}$$

so that the $\mathrm{L}^2(\Omega)$ scalar product can be written $(\varphi,\vartheta)_{\mathrm{L}^2} \equiv \langle\varphi\vartheta\rangle$. Similarly, the Lebesgue spaces $\mathrm{L}^p(\Omega)$ and the Sobolev spaces $\mathrm{W}^{n,p}(\Omega)$—the spaces of functions whose derivatives up to order n are in L^p—are defined with respect to the b-weighted integral. The non-degeneracy of b again assures us that the weighted norms are equivalent to the norms in the unweighted case, so that all well-known embeddings as can be found in [1] still hold. Finally, we introduce the space of infinitely differentiable and compactly supported functions which satisfy our weighted incompressibility condition,

$$\mathbb{D} = \{\boldsymbol{u} \in \mathrm{C}^\infty_0(\Omega,\mathbb{R}^2) : \boldsymbol{\nabla}\cdot(b\boldsymbol{u}) = 0 \text{ in } \Omega\} \,, \tag{24}$$

and denote by $\mathbb{H}$ and $\mathbb{V}$ its completions in the L^2 and H^1 norm respectively. The L^2-orthogonal (with respect to the b-weighted scalar product) projector onto $\mathbb{H}$ shall be denoted by P. Its complement, $Q\mathrm{L}^2(\Omega,\mathbb{R}^2)$ where $Q \equiv 1 - P$, are the gradients of scalar potential functions. For the unweighted case, this is well known and details can be found in [8,10,17]. The decomposition carries over naturally to our weighted spaces.

We conclude this section by showing that K has exactly the smoothing properties that we need in order to proceed with proving the abstract existence theorem in the next section, and that it imposes the correct boundary and incompressibility conditions on $\boldsymbol{u}$. We formulate this as a lemma.

Lemma 1 *For $b \in \mathrm{C}^2(\overline{\Omega})$ and every $\omega \in \mathrm{H}^{-1}(\Omega)$ there exists a unique function $\boldsymbol{u} = K\omega \in \mathbb{H}$. Moreover, K is continuous as a mapping between the spaces $\mathrm{H}^{-1}(\Omega) \to \mathbb{H}$, $\mathrm{L}^2(\Omega) \to \mathbb{V}$ and $\mathrm{H}^1(\Omega) \to \mathbb{V} \cap \mathrm{H}^2(\Omega,\mathbb{R}^2)$, and for some $p_0 > 1$ satisfies the estimate*

$$\|K\omega\|_{\mathrm{W}^{1,p}} \le c\, p\, \|\omega\|_{\mathrm{L}^p} \,, \tag{25}$$

for all $p \ge p_0$ where the constant c depends only on p_0 and Ω.

Proof. Let φ, $\psi \in C_0^\infty(\Omega)$ and define $\boldsymbol{u} \equiv b^{-1}\boldsymbol{\nabla}^\perp\varphi$ and $\boldsymbol{v} \equiv b^{-1}\boldsymbol{\nabla}^\perp\psi$. The fact that the classical functions $\boldsymbol{u}$ and $\boldsymbol{v}$ are also members of $\mathbb{H}$ can be verified with a direct calculation. Now integrate by parts to obtain

$$(\varphi, b^{-1}\,\boldsymbol{\nabla}\wedge(b^{-1}\,\boldsymbol{\nabla}^\perp\psi))_{\mathrm{L}^2} = \int_\Omega (b^{-1}\,\boldsymbol{\nabla}^\perp\varphi)\cdot(b^{-1}\,\boldsymbol{\nabla}^\perp\psi)\, b\; \mathrm{d}\boldsymbol{x}\,. \tag{26}$$

Due to the Poincaré inequality and the nondegeneracy of b, $\|b^{-1}\boldsymbol{\nabla}^\perp\cdot\|_{\mathrm{L}^2}$ is an equivalent norm on $\mathrm{H}_0^1(\Omega)$, so that we can bound the left side of (26) from above and below by

$$(\varphi, b^{-1}\,\boldsymbol{\nabla}\wedge(b^{-1}\,\boldsymbol{\nabla}^\perp\psi))_{\mathrm{L}^2} \le c_1\,\|\varphi\|_{\mathrm{H}^1}\,\|\psi\|_{\mathrm{H}^1}\,, \tag{27a}$$

and

$$\|\varphi\|_{\mathrm{H}^1}^2 \le c_2\,(\varphi, b^{-1}\,\boldsymbol{\nabla}\wedge(b^{-1}\,\boldsymbol{\nabla}^\perp\varphi))_{\mathrm{L}^2}\,. \tag{27b}$$

This proves continuity and coercivity of the bilinear form. We can now extend (27) to $\mathrm{H}_0^1(\Omega)$ functions by a density argument, so that the Lax-Milgram theorem implies unique solvability of problem (18) for $\omega \in \mathrm{H}^{-1}(\Omega) \equiv \mathrm{H}_0^1(\Omega)'$. By continuously extending the operator $b^{-1}\boldsymbol{\nabla}^\perp$ to $\mathrm{H}_0^1(\Omega)$ functions, we can conclude that $\boldsymbol{u} \in \mathbb{H}$, and that $K\colon \mathrm{H}^{-1}(\Omega) \to \mathbb{H}$ is continuous.

Provided that $\omega \in \mathrm{L}^p(\Omega)$, well-known regularity theorems for elliptic equations allow us to conclude that $\psi \in \mathrm{W}^{2,p}(\Omega)$ and satisfies an estimate

$$\|\psi\|_{\mathrm{W}^{2,p}} \le c\,p\,\|\omega\|_{\mathrm{L}^p}\,, \tag{28}$$

where it is possible to choose the constant c independent of p as long as $p \ge p_0$ for some $p_0 > 1$, see references [2,11,20]. Estimate (25) follows directly.

In particular if $\omega \in \mathrm{L}^2(\Omega)$, then one can conclude that $\psi \in \mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega)$ which implies $\boldsymbol{u} \in \mathbb{V}$. Furthermore, if $\omega \in \mathrm{H}^1(\Omega)$, then $\psi \in \mathrm{H}^3(\Omega) \cap \mathrm{H}_0^1(\Omega)$ and $\boldsymbol{u} \in \mathrm{H}^2(\Omega, \mathbb{R}^2) \cap \mathbb{V}$. It is for this step that the regularity assumptions on $\partial\Omega$ and b need to be as strong as stated, see [11,19].

4 Existence and Uniqueness

We begin with establishing the functional setting and providing some identities that will be used frequently. We will assume throughout that $b \in \mathrm{C}^1(\overline{\Omega})$.

First define an operator A by

$$\omega = -b^{-1}\boldsymbol{\nabla}\cdot(b\,\boldsymbol{\nabla}\psi) \equiv A\psi \quad \text{on } \Omega\,, \tag{29a}$$
$$\psi = 0 \quad \text{on } \partial\Omega\,, \tag{29b}$$

i.e., $-A$ can be thought of as a Laplace operator on Sobolev spaces with weight b. For $\omega \in \mathrm{C}_0^\infty(\Omega)$,

$$\langle A\omega\, \varphi \rangle = \langle \boldsymbol{\nabla}\omega \cdot \boldsymbol{\nabla}\varphi \rangle \leq \|\omega\|_{\mathrm{H}^1} \, \|\varphi\|_{\mathrm{H}^1} \, , \tag{30}$$

which shows that $\langle \boldsymbol{\nabla}\omega \cdot \boldsymbol{\nabla}\varphi \rangle$ can be extended to a symmetric, continuous and—again due to the Poincaré inequality—coercive bilinear form on $\mathrm{H}_0^1(\Omega)$, a closed subspace of $\mathrm{H}^1(\Omega)$. By the Lax-Milgram theorem, $A\colon \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)' \equiv \mathrm{H}^{-1}(\Omega)$ is an isomorphism. Its positivity allows us to define arbitrary powers of A, so that we can identify the H^1 norm on $\mathrm{H}_0^1(\Omega)$ with $\|A^{1/2}\cdot\|_{\mathrm{L}^2}$ and the H^2 norm on $\mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega)$ with $\|A\cdot\|_{\mathrm{L}^2}$. Similarly, $\mathrm{H}^{-1}(\Omega) = \mathrm{H}_0^1(\Omega)'$ can be endowed with norm $\|A^{-1/2}\cdot\|_{\mathrm{L}^2}$ and scalar product $(A^{-1/2}\cdot, A^{-1/2}\cdot)_{\mathrm{L}^2}$.

For $\boldsymbol{u} \in \mathbb{D}$ and $\omega \in \mathrm{C}^\infty(\overline{\Omega})$ one establishes through integration by parts (i.e. the Gauss integral formula) that

$$\big\langle (\boldsymbol{u} \cdot \boldsymbol{\nabla}\omega)\, \omega \big\rangle = \int_{\partial\Omega} \boldsymbol{n} \cdot \boldsymbol{u}\, \omega^2 \, b \, \mathrm{d}\sigma - \int_{\Omega} \boldsymbol{\nabla} \cdot (b\boldsymbol{u})\, \omega^2 \, \mathrm{d}\boldsymbol{x} - \big\langle (\boldsymbol{u} \cdot \boldsymbol{\nabla}\omega)\, \omega \big\rangle \, . \tag{31}$$

The first two terms on the right side clearly vanish so that the remaining terms must vanish, too. This implies

$$\big\langle (\boldsymbol{u} \cdot \boldsymbol{\nabla}\omega)\, \varphi \big\rangle = -\big\langle (\boldsymbol{u} \cdot \boldsymbol{\nabla}\varphi)\, \omega \big\rangle \, , \tag{32}$$

for every $\boldsymbol{u} \in \mathbb{D}$ and ω, $\varphi \in \mathrm{C}^\infty(\overline{\Omega})$. This relation can be extended by continuity to all $\boldsymbol{u} \in \mathbb{H}$ and ω, $\varphi \in \mathrm{H}^1(\Omega)$.

We first consider the following system which consists of the potential vorticity equation (20) with an extra viscosity term,

$$\partial_t\omega + \nu\, A\omega + \boldsymbol{u} \cdot \boldsymbol{\nabla}\omega = 0 \quad \text{in } \Omega \, , \tag{33a}$$
$$\boldsymbol{u} = K\omega \quad \text{in } \Omega \, , \tag{33b}$$
$$\omega = 0 \quad \text{on } \partial\Omega \, , \tag{33c}$$
$$\omega(0) = \omega^{\mathrm{in}} \, . \tag{33d}$$

We assume that K is a linear operator which satisfies all the properties stated in Lemma 1. The following theorem then establishes global existence of weak solutions to problem (33).

Theorem 2 *Given $\omega^{\mathrm{in}} \in \mathrm{H}_0^1(\Omega)$, there exists a unique function*

$$\omega \in \mathrm{C}([0,\infty); \mathrm{H}_0^1(\Omega)) \cap \mathrm{L}^2_{\mathrm{loc}}([0,\infty); \mathrm{H}^2(\Omega) \cap \mathrm{H}_0^1(\Omega)) \tag{34}$$

that satisfies the weak form of the system (33) *given by*

$$\langle \psi\omega(t_2)\rangle - \langle \psi\omega(t_1)\rangle + \nu \int_{t_1}^{t_2} \big\langle \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\omega \big\rangle \,\mathrm{d}t - \int_{t_1}^{t_2} \big\langle (\boldsymbol{u}\cdot\boldsymbol{\nabla}\psi)\,\omega \big\rangle \,\mathrm{d}t = 0\,, \tag{35a}$$

$$\boldsymbol{u} = K\omega\,, \tag{35b}$$

$$\omega(0) = \omega^{\mathrm{in}}\,, \tag{35c}$$

for every $[t_1, t_2] \subset [0,\infty)$ *and every test function* $\psi \in \mathrm{H}^1_0(\Omega)$. *Moreover, it satisfies the "energy relation"*

$$\|\omega(t)\|_{\mathrm{L}^p} \le \|\omega^{\mathrm{in}}\|_{\mathrm{L}^p}\,, \tag{36}$$

for every $t \in [0,\infty)$ *and* $2 \le p < \infty$.

Remark 3 *Physically, for* $p = 2$ *equation* (36) *is an inequality for the* potential enstrophy. *Formally, however, it plays the role of the energy in Leray's theory of weak solutions to the Navier-Stokes equations.*

Remark 4 *As the* L^p *bound* (36) *holds uniformly for all* $p \in [2,\infty)$, *it also holds for* $p = \infty$ *provided the initial data is* L^∞.

The proof follows standard techniques for equations of Navier-Stokes type [8,17,18] and can be found in some detail in [13].

Let us now proceed with showing that it is possible to pass to the limit $\nu \to 0$. The result is the following theorem on the inviscid equations.

Theorem 5 *Given* $\omega^{\mathrm{in}} \in \mathrm{L}^2(\Omega)$, *there exists a function*

$$\omega \in \mathrm{C}([0,\infty); \mathrm{w}\text{-}\mathrm{L}^2(\Omega)) \cap \mathrm{L}^2_{\mathrm{loc}}([0,\infty); \mathrm{L}^2(\Omega))\,, \tag{37}$$

not necessarily unique, that satisfies the weak form of the system (20), *given by*

$$\langle \psi\omega(t_2)\rangle - \langle \psi\omega(t_1)\rangle - \int_{t_1}^{t_2} \big\langle (\boldsymbol{u}\cdot\boldsymbol{\nabla}\psi)\,\omega \big\rangle \,\mathrm{d}t = 0\,, \tag{38a}$$

$$\boldsymbol{u} = K\omega\,, \tag{38b}$$

$$\omega(0) = \omega^{\mathrm{in}}\,, \tag{38c}$$

for every $[t_1, t_2] \subset [0,\infty)$ *and every test function* $\psi \in \mathrm{H}^1_0(\Omega)$. *It satisfies the "energy relation"*

$$\|\omega(t)\|_{\mathrm{L}^2} \le \|\omega^{\mathrm{in}}\|_{\mathrm{L}^2}\,, \tag{39}$$

for every $t \in [0,\infty)$. *Moreover, if* $\omega^{\mathrm{in}} \in \mathrm{L}^\infty(\Omega)$ *then the weak form has a* *unique solution*

$$\omega \in \mathrm{C}([0,\infty);\mathrm{w}^*\text{-}\mathrm{L}^\infty(\Omega)) \cap \mathrm{L}^\infty([0,\infty)\times\Omega) \tag{40}$$

that satisfies

$$\|\omega(t)\|_{\mathrm{L}^\infty} \le \|\omega^{\mathrm{in}}\|_{\mathrm{L}^\infty}\,, \tag{41}$$

for every $t \in [0,\infty)$.

Remark 6 *The sense of convergence in the spaces we are using can be summarized as follows. Given a topological vector space* X *with dual* X'*, a sequence of functions* $\varphi_\nu\colon [0,\infty) \to \mathrm{X}$ *converges to* φ *in* $\mathrm{C}([0,\infty);\mathrm{w}\text{-}\mathrm{X})$*, where* $\mathrm{w}\text{-}\mathrm{X}$ *denotes* X *endowed with its weak topology, when for every* $\psi \in \mathrm{X}'$

$$\langle \psi, \varphi_\nu(t)\rangle_{\mathrm{X}',\mathrm{X}} \to \langle \psi, \varphi(t)\rangle_{\mathrm{X}',\mathrm{X}} \tag{42}$$

uniformly on compact subsets of $[0,\infty)$*. Similarly, a sequence of functions* $\varphi_\nu\colon [0,\infty) \to \mathrm{X}'$ *converges to* φ *in* $\mathrm{C}([0,\infty);\mathrm{w}^*\text{-}\mathrm{X}')$*, where* $\mathrm{w}^*\text{-}\mathrm{X}'$ *denotes* X' *endowed with its weak-$*$ topology, when for every* $\psi \in \mathrm{X}$

$$\langle \varphi_\nu(t), \psi\rangle_{\mathrm{X}',\mathrm{X}} \to \langle \varphi(t), \psi\rangle_{\mathrm{X}',\mathrm{X}} \tag{43}$$

uniformly on compact subsets of $[0,\infty)$.

Proof. We will use an equicontinuity argument as has been applied to the complex Ginzburg-Landau equation in [9]. The proof proceeds in five distinct steps and focuses on the case of L^∞ initial data. The first statement of the theorem holds even for $\omega^{\mathrm{in}} \in \mathrm{L}^2(\Omega)$, and its proof can be abstracted from Steps 1–4 below by recasting appropriate L^p or L^∞ statements in L^2.

Step 1 *Use Theorem 2 to construct a family of approximate solutions*

$$\{\omega_\nu\} \subset \mathrm{C}([0,\infty);\mathrm{w}\text{-}\mathrm{H}^1_0(\Omega)) \cap \mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{H}^2(\Omega)\cap\mathrm{H}^1_0(\Omega)) \tag{44}$$

that satisfy the weak form of the viscous equations (33)*, given by*

$$\langle \psi\omega_\nu(t_2)\rangle - \langle \psi\omega_\nu(t_1)\rangle + \nu\int_{t_1}^{t_2} \big\langle \boldsymbol{\nabla}\psi\cdot\boldsymbol{\nabla}\omega_\nu\big\rangle\,\mathrm{d}t - \int_{t_1}^{t_2}\big\langle (\boldsymbol{u}_\nu\cdot\boldsymbol{\nabla}\psi)\,\omega_\nu\big\rangle\,\mathrm{d}t = 0\,, \tag{45a}$$

$$\boldsymbol{u}_\nu = K\omega_\nu\,, \tag{45b}$$

$$\omega_\nu(0) = \omega_\nu^{\mathrm{in}}\,, \tag{45c}$$

for every $[t_1,t_2] \subset [0,\infty)$ *and every test function* $\psi \in \mathrm{H}^1_0(\Omega)$*, as well as the "energy relation"*

$$\|\omega_\nu(t)\|_{\mathrm{L}^p} \le \|\omega_\nu^{\mathrm{in}}\|_{\mathrm{L}^p}\,, \tag{46}$$

for every $t \in [0, \infty)$ *and* $2 \le p < \infty$.

The regularized initial data necessary for the application of Theorem 2 can for example be achieved by extending ω^{in} as zero outside

$$\Omega_{2\nu} = \{\boldsymbol{x} \in \Omega : \operatorname{dist}(\boldsymbol{x}, \partial\Omega) > 2\nu\}, \tag{47}$$

and setting

$$\omega_\nu^{\text{in}} \equiv j_\nu * (\omega^{\text{in}}\, \chi(\Omega_{2\nu})) \in \mathrm{C}_0^\infty(\Omega), \tag{48}$$

where j_ν is a Friedrichs mollifier. The regularized function ω_ν^{in} converges to ω^{in} in $\mathrm{L}^p(\Omega)$ and satisfies $\|\omega_\nu^{\text{in}}\|_{\mathrm{L}^p} \le \|\omega^{\text{in}}\|_{\mathrm{L}^p}$.

Step 2 *Show that* $\{\omega_\nu\}$ *is a relatively compact set in* $\mathrm{C}([0,\infty); \text{w-}\mathrm{L}^2(\Omega))$*, and if* $\omega^{\text{in}} \in \mathrm{L}^\infty(\Omega)$*, it moreover is relatively compact in* $\mathrm{C}([0,\infty); \text{w}^*\text{-}\mathrm{L}^\infty(\Omega))$.

Proof of Step 2 We prove only the second statement, which by the Arzela-Ascoli theorem is equivalent to the following.

(i) $\{\omega_\nu(t)\}$ is a relatively compact set in $\text{w}^*\text{-}\mathrm{L}^\infty(\Omega)$ for every $t \ge 0$;
(ii) $\{\omega_\nu\}$ is equicontinuous in $\mathrm{C}([0,\infty); \text{w}^*\text{-}\mathrm{L}^\infty(\Omega))$, i.e. for every $\psi \in \mathrm{L}^1(\Omega)$ the sequence $\{\langle \psi\omega_\nu\rangle\}$ is equicontinuous in $\mathrm{C}([0,\infty))$.

Since $\omega^{\text{in}} \in \mathrm{L}^\infty(\Omega)$, (46) holds uniformly for all $2 \le p < \infty$, and hence

$$\|\omega_\nu(t)\|_{\mathrm{L}^\infty} \le \|\omega_\nu^{\text{in}}\|_{\mathrm{L}^\infty}. \tag{49}$$

This directly implies (i).

In order to show (ii), we begin by considering the case where $\psi \in \mathrm{C}_0^\infty(\Omega)$. One obtains

$$\begin{aligned} &|\langle\psi\omega_\nu(t_2)\rangle - \langle\psi\omega_\nu(t_1)\rangle| \\ &\qquad = \left|\nu \int_{t_1}^{t_2} \big\langle \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\omega_\nu \big\rangle \,\mathrm{d}t - \int_{t_1}^{t_2} \big\langle (\boldsymbol{u}_\nu \cdot \boldsymbol{\nabla}\psi)\, \omega_\nu \big\rangle \,\mathrm{d}t \right| \\ &\qquad \le \nu\, \|A\psi\|_{\mathrm{L}^\infty} \left(\int_{t_1}^{t_2} \mathrm{d}t\right)^{\frac{1}{2}} \left(\int_{t_1}^{t_2} \|\omega_\nu\|_{\mathrm{L}^2}^2 \,\mathrm{d}t\right)^{\frac{1}{2}} \\ &\qquad\quad + \|\boldsymbol{\nabla}\psi\|_{\mathrm{L}^\infty} \left(\int_{t_1}^{t_2} \mathrm{d}t\right)^{\frac{1}{4}} \left(\int_{t_1}^{t_2} \|\boldsymbol{u}_\nu\|_{\mathrm{L}^4}^4 \,\mathrm{d}t\right)^{\frac{1}{4}} \left(\int_{t_1}^{t_2} \|\omega_\nu\|_{\mathrm{L}^2}^2 \,\mathrm{d}t\right)^{\frac{1}{2}}, \end{aligned} \tag{50}$$

The sequence $\{\omega_\nu\}$ is bounded in $\mathrm{L}^\infty_{\text{loc}}([0,\infty); \mathrm{L}^2(\Omega))$ so that $\{\boldsymbol{u}_\nu\}$ is bounded in $\mathrm{L}^4_{\mathrm{loc}}([0,\infty);\mathrm{L}^4(\Omega,\mathbb{R}^2))$. Hence for every $\nu > 0$, $T < \infty$ and $[t_1, t_2] \subset [0, T]$,

$$|\langle \psi\omega_\nu(t_2)\rangle - \langle \psi\omega_\nu(t_1)\rangle| \le c(\psi, T, \omega^{\mathrm{in}})\big(|t_2 - t_1|^{\frac{1}{2}} + |t_2 - t_1|^{\frac{1}{4}}\big)\,. \tag{51}$$

Because the right side of this equation is independent of ν, the set $\{\langle \psi\omega_\nu\rangle\}$ is equicontinuous in $\mathrm{C}([0,\infty))$.

We extend the class of test functions to $\mathrm{L}^1(\Omega)$ with a density argument. Let $\varepsilon > 0$. Choose $\psi_\varepsilon \in \mathrm{C}_0^\infty(\Omega)$ such that

$$\|\psi - \psi_\varepsilon\|_{\mathrm{L}^1} < \frac{\varepsilon}{3\,\|\omega^{\mathrm{in}}\|_{\mathrm{L}^\infty} + 1}\,. \tag{52}$$

By the triangle inequality

$$\begin{aligned} |\langle \psi\omega_\nu(t_2)\rangle - \langle \psi\omega_\nu(t_1)\rangle| &= \big|\big\langle(\psi - \psi_\varepsilon)(\omega_\nu(t_2) - \omega_\nu(t_1))\big\rangle + \big\langle \psi_\varepsilon(\omega_\nu(t_2) - \omega_\nu(t_1))\big\rangle\big| \\ &\le \|\psi - \psi_\varepsilon\|_{\mathrm{L}^1}\Big(\|\omega_\nu(t_2)\|_{\mathrm{L}^\infty} + \|\omega_\nu(t_1)\|_{\mathrm{L}^\infty}\Big) + \big|\big\langle \psi_\varepsilon(\omega_\nu(t_2) - \omega_\nu(t_1))\big\rangle\big|\,. \end{aligned} \tag{53}$$

From (52) and the uniform L^∞ bound (49) we infer that the first term on the right is smaller that $\frac{2}{3}\varepsilon$. Applying (51) and choosing $|t_2 - t_1|$ small enough, the second term on the right can be made smaller than $\frac{1}{3}\varepsilon$, so that the whole expression is less than ε. This establishes (ii) and completes the proof of Step 2.

Step 3 *Show that* $\{\omega_\nu\}$ *is a relatively compact set in* $\mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{H}^{-1}(\Omega))$.

Proof of Step 3 The statement follows from Step 2 and the continuity of the embedding

$$\mathrm{C}([0,\infty);\text{w-}\mathrm{L}^2(\Omega)) \hookrightarrow \mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{H}^{-1}(\Omega))\,, \tag{54}$$

which are proved as in [9].

Step 4 *Show that there is a subsequence of* $\{\omega_\nu\}$ *converging to a function* ω *which satisfies the weak form* (38) *and the energy relation* (41).

Proof of Step 4 Steps 2 and 3 assert that there exists a subsequence of $\{\omega_\nu\}$ which we will still denote by $\{\omega_\nu\}$ that converges to a function

$$\omega \in \mathrm{C}([0,\infty);\text{w}^*\text{-}\mathrm{L}^\infty(\Omega)) \cap \mathrm{L}^\infty([0,\infty)\times\Omega) \tag{55}$$

in the following sense:

(i) $\omega_\nu \to \omega$ in $\mathrm{C}([0,\infty);\text{w}^*\text{-}\mathrm{L}^\infty(\Omega))$;
(ii) $\omega_\nu \to \omega$ in $\text{w}^*\text{-}\mathrm{L}^\infty([0,\infty)\times\Omega)$;

(iii) $\omega_\nu \to \omega$ in $\mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{H}^{-1}(\Omega))$.

It remains to be shown that ω satisfies the weak form (38a). Towards this end, we will show that for every test function ψ from a dense subset of H^1_0, each of the terms in the regularized weak form (45a) converges to the corresponding term in (38a). This will prove that ω satisfies the weak form for any such ψ and, by density, for every $\psi \in \mathrm{H}^1_0(\Omega)$.

This said, it is sufficient to assume a test function $\psi \in \mathrm{C}^\infty_0(\Omega)$. Since (i) implies convergence in $\mathrm{C}([0,\infty);\text{w-}\mathrm{L}^2(\Omega))$, we infer that

$$\langle \psi\omega_\nu(t)\rangle \to \langle \psi\omega(t)\rangle\,, \tag{56}$$

for every $t \geq 0$ as $\nu \to 0$. The viscosity term can be bounded as

$$\nu \int_{t_1}^{t_2} \big\langle \boldsymbol{\nabla}\psi \cdot \boldsymbol{\nabla}\omega_\nu \big\rangle \,\mathrm{d}t \leq \nu\, \|\psi\|_{\mathrm{H}^2} \int_{t_1}^{t_2} \|\omega_\nu\|_{\mathrm{L}^2}\,\mathrm{d}t\,. \tag{57}$$

The last two factors on the right are bounded independent of ν, so that the left side must converge to zero as $\nu \to 0$. To show the convergence of the nonlinear term we use the estimate

$$\begin{aligned}
&\int_{t_1}^{t_2} \Big(\big\langle (\boldsymbol{u}_\nu \cdot \boldsymbol{\nabla}\psi)\,\omega_\nu\big\rangle - \big\langle (\boldsymbol{u}\cdot\boldsymbol{\nabla}\psi)\,\omega\big\rangle\Big)\,\mathrm{d}t \\
&\quad= \int_{t_1}^{t_2} \Big(\big\langle ((\boldsymbol{u}_\nu - \boldsymbol{u})\cdot\boldsymbol{\nabla}\psi)\,\omega_\nu\big\rangle + \big\langle (\boldsymbol{u}\cdot\boldsymbol{\nabla}\psi)\,(\omega_\nu-\omega)\big\rangle\Big)\,\mathrm{d}t \\
&\quad\leq \int_{t_1}^{t_2} \|\boldsymbol{u}_\nu - \boldsymbol{u}\|_{\mathrm{L}^2}\, \|\boldsymbol{\nabla}\psi\|_{\mathrm{L}^\infty}\, \|\omega_\nu\|_{\mathrm{L}^2}\,\mathrm{d}t + \left| \int_{t_1}^{t_2} \big\langle (\boldsymbol{u}\cdot\boldsymbol{\nabla}\psi)\,(\omega_\nu-\omega)\big\rangle\,\mathrm{d}t \right|\,.
\end{aligned} \tag{58}$$

Thus, in order to pass to the limit it is sufficient to have strong convergence of $\{\boldsymbol{u}_\nu\}$ in $\mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega,\mathbb{R}^2))$ and weak convergence of $\{\omega_\nu\}$ in $\mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$, properties that follow directly from (ii) and (iii).

Finally, for $p<\infty$, the convergence of the energy relation (46) follows from the fact that $\omega^{\mathrm{in}}_\nu \to \omega^{\mathrm{in}}$ in L^p and that the convergence (i) implies

$$\|\omega(t)\|_{\mathrm{L}^p} \leq \liminf_{\nu\to 0} \|\omega_\nu(t)\|_{\mathrm{L}^p}\,, \tag{59}$$

for every $t \geq 0$. As $\|\omega(t)\|_{\mathrm{L}^p} \leq \|\omega^{\mathrm{in}}\|_{\mathrm{L}^p}$ uniformly for all $p<\infty$, it also holds for $p=\infty$, thus $\omega \in \mathrm{L}^\infty([0,\infty)\times\Omega)$.

Step 5 *Show that the solution ω is unique.*

Proof of Step 5 In order to prove uniqueness, we first need to recover the time derivative of ω in a weak sense. One has the estimate

$$\int_0^t \left|\left\langle(\boldsymbol{u}(t')\cdot\boldsymbol{\nabla}\psi)\,\omega(t')\right\rangle\right|^2\,\mathrm{d}t' \le \|\psi\|_{\mathrm{H}^1}^2 \int_0^t \|\boldsymbol{u}(t')\|_{\mathrm{L}^4}^2\,\|\omega(t')\|_{\mathrm{L}^4}^2\,\mathrm{d}t'\,, \tag{60}$$

and $\omega \in \mathrm{L}^\infty([0,\infty)\times\Omega)$, so that the integral on the right is finite. We conclude that the mapping $(t,\psi)\mapsto\langle(\boldsymbol{u}(t)\cdot\boldsymbol{\nabla}\psi)\,\omega(t)\rangle$ defines a function in $\mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{H}^{-1}(\Omega))$.

Let therefore $\psi\in\mathrm{H}^1_0(\Omega)$. The weak form (38a) includes in particular

$$\langle\psi\omega(t)\rangle = \langle\psi\omega^{\mathrm{in}}\rangle + \int_0^t \left\langle(\boldsymbol{u}\cdot\boldsymbol{\nabla}\psi)\,\omega\right\rangle\,\mathrm{d}t'\,. \tag{61}$$

Take a test function $\theta\in C_0^\infty([0,\infty))$, multiply the previous line with $\mathrm{d}\theta/\,\mathrm{d}t$, integrate from 0 to ∞ and then integrate by parts to obtain

$$\int_0^\infty \langle\psi\omega(t)\rangle\,\frac{\mathrm{d}\theta}{\mathrm{d}t}(t)\,\mathrm{d}t = \int_0^\infty \left\langle(\boldsymbol{u}(t)\cdot\boldsymbol{\nabla}\psi)\,\omega(t)\right\rangle\theta(t)\,\mathrm{d}t\,. \tag{62}$$

Integration by parts is legitimate here because from our earlier remark it follows that the integrand on the right side of (61) is a $\mathrm{L}^1_{\mathrm{loc}}([0,\infty))$ function. Relation (62) is equivalent to saying that

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi\omega(t)\rangle = \left\langle(\boldsymbol{u}(t)\cdot\boldsymbol{\nabla}\psi)\,\omega(t)\right\rangle \tag{63}$$

in the sense of distributions on $[0,\infty)$. Even more, we can write

$$\partial_t\omega = -b^{-1}\boldsymbol{\nabla}\cdot(b\boldsymbol{u}\omega)\in\mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{H}^{-1}(\Omega))\,. \tag{64}$$

We also recall that $\omega\in\mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{L}^2(\Omega))$. It follows that the following identity holds in the sense of distributions on $\mathbb{R}$:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega(t)\|_{\mathrm{H}^{-1}}^2 = (\partial_t\omega,\omega)_{\mathrm{H}^{-1}}\,. \tag{65}$$

This identity is obvious for functions $\omega\in\mathrm{C}_0^\infty(\mathbb{R};\mathrm{H}^{-1}(\Omega))$ and can be extended by continuity to all $\omega\in\mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{H}^{-1}(\Omega))$ with $\partial_t\omega\in\mathrm{L}^2_{\mathrm{loc}}([0,\infty);\mathrm{H}^{-1}(\Omega))$ by a density argument. It then follows that $\omega\in\mathrm{C}([0,\infty);\mathrm{H}^{-1}(\Omega))$, see Wloka [19, Theorem 25.5].

Now assume that $\hat\omega$ and $\bar\omega$ are two solutions to equation (64) that correspond to identical initial data and set $\hat{\boldsymbol{u}}=K\hat\omega$, $\bar{\boldsymbol{u}}=K\bar\omega$ and $\varphi=\hat\omega-\bar\omega$. Our goal is to compute the time evolution of φ in the H^{-1} norm. If we can show that this norm remains zero for all times, the two solutions must be identical. Note that this does not mean that we allow general H^{-1} initial data for $\hat\omega$ or $\bar\omega$.

First observe that for any two functions $\boldsymbol{v} \in \mathrm{L}^2(\Omega, \mathbb{R}^2)$ and $\psi \in \mathrm{L}^2(\Omega)$ one has

$$\big\langle A^{-1}(b^{-1}\boldsymbol{\nabla} \cdot (b\boldsymbol{v}))\, \psi \big\rangle = -\big\langle \boldsymbol{v} \cdot \boldsymbol{\nabla} A^{-1}\psi \big\rangle\,. \tag{66}$$

This is easily verified for smooth functions and extends by continuity. Using (65) and then (66),

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi(t)\|^2_{\mathrm{H}^{-1}} &= -\big\langle A^{-1}\big(b^{-1}\boldsymbol{\nabla} \cdot (b\hat{\boldsymbol{u}}\hat{\omega})\big)\, \varphi\big\rangle + \big\langle A^{-1}\big(b^{-1}\boldsymbol{\nabla} \cdot (b\bar{\boldsymbol{u}}\bar{\omega})\big)\, \varphi\big\rangle \\
&= \big\langle \hat{\omega}\, \hat{\boldsymbol{u}} \cdot \boldsymbol{\nabla} A^{-1}\varphi \big\rangle - \big\langle \bar{\omega}\, \bar{\boldsymbol{u}} \cdot \boldsymbol{\nabla} A^{-1}\varphi \big\rangle \\
&= \big\langle \varphi\, \hat{\boldsymbol{u}} \cdot \boldsymbol{\nabla} A^{-1}\varphi \big\rangle + \big\langle \bar{\omega}\, (\hat{\boldsymbol{u}} - \bar{\boldsymbol{u}}) \cdot \boldsymbol{\nabla} A^{-1}\varphi \big\rangle\,.
\end{aligned} \tag{67}$$

The second term on the right is estimated

$$\big|\big\langle \bar{\omega}\, (\hat{\boldsymbol{u}} - \bar{\boldsymbol{u}}) \cdot \boldsymbol{\nabla} A^{-1}\varphi \big\rangle\big| \le \|\bar{\omega}\|_{\mathrm{L}^\infty}\, \|\hat{\boldsymbol{u}} - \bar{\boldsymbol{u}}\|_{\mathrm{L}^2}\, \|\varphi\|_{\mathrm{H}^{-1}} \le c_1\, \|\varphi\|^2_{\mathrm{H}^{-1}}\,, \tag{68}$$

using the global L^∞ bound on $\bar{\omega}$ and the continuity of K. The first term on the right of (67) is estimated by noting that for every $\boldsymbol{u} \in \mathbb{D}$ and $\psi \in \mathrm{C}_0^\infty(\Omega)$

$$\begin{aligned}
\big\langle A(\boldsymbol{u} \cdot \boldsymbol{\nabla}\psi)\, \psi \big\rangle &= \big\langle \boldsymbol{\nabla}(\boldsymbol{u} \cdot \boldsymbol{\nabla}\psi) \cdot \boldsymbol{\nabla}\psi \big\rangle \\
&= \big\langle \boldsymbol{\nabla}\psi \cdot (\boldsymbol{\nabla}\boldsymbol{u}) \cdot \boldsymbol{\nabla}\psi \big\rangle + \big\langle \boldsymbol{u} \cdot (\boldsymbol{\nabla}\boldsymbol{\nabla}\psi) \cdot \boldsymbol{\nabla}\psi \big\rangle\,.
\end{aligned} \tag{69}$$

The second term on the right varnishes due to relation (32). Then, by taking $\psi = A^{-1}\varphi$, still assuming classical functions,

$$\begin{aligned}
\big|\big\langle \varphi\, \boldsymbol{u} \cdot \boldsymbol{\nabla} A^{-1}\varphi \big\rangle\big| &= \big|\big\langle \psi\, A(\boldsymbol{u} \cdot \boldsymbol{\nabla}\psi) \big\rangle\big| = \big|\big\langle \boldsymbol{\nabla}\psi \cdot (\boldsymbol{\nabla}\boldsymbol{u}) \cdot \boldsymbol{\nabla}\psi \big\rangle\big| \\
&\le \|\boldsymbol{u}\|_{\mathrm{W}^{1,p}}\, \|\psi\|^2_{\mathrm{W}^{1,2q}}\,, \qquad \text{where } \tfrac{1}{p} + \tfrac{1}{q} = 1\,.
\end{aligned} \tag{70}$$

This estimate extends by continuity to all $\boldsymbol{u} \in \mathrm{W}^{1,p}(\Omega, \mathbb{R}^2)$ with $\boldsymbol{\nabla} \cdot (b\boldsymbol{u}) = 0$ on Ω and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$, and all $\varphi \in \mathrm{L}^2(\Omega)$. Then

$$\big|\big\langle \varphi\, \hat{\boldsymbol{u}} \cdot \boldsymbol{\nabla} A^{-1}\varphi \big\rangle\big| \le \|\hat{\boldsymbol{u}}\|_{\mathrm{W}^{1,p}}\, \|A^{-1}\varphi\|^2_{\mathrm{W}^{1,2q}} \le p\, c_2\, \|A^{-1/2}\varphi\|^2_{\mathrm{L}^{2q}} \le p\, c_3\, \|\varphi\|^{2-2/p}_{\mathrm{H}^{-1}}\,. \tag{71}$$

The last step is valid because $\varphi \in \mathrm{L}^\infty([0,\infty) \times \Omega)$ implies that $A^{-1/2}\varphi \in \mathrm{L}^\infty([0,\infty) \times \Omega)$. Altogether one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi(t)\|^2_{\mathrm{H}^{-1}} \le p\, c_4\, \|\varphi(t)\|^{2-2/p}_{\mathrm{H}^{-1}}\,. \tag{72}$$

Since $\|\varphi(0)\|^2_{\mathrm{H}^{-1}} = 0$, we have in particular that $\|\varphi(0)\|^2_{\mathrm{H}^{-1}} \le 2^{-p}$ for every p. This allows us to integrate this differential inequality away from the singularity when $\|\varphi(t)\|^2_{\mathrm{H}^{-1}} = 0$ and we obtain the bound

$$\|\varphi(t)\|^2_{\mathrm{H}^{-1}} \le (\tfrac{1}{2} + c_4\, t)^p\,. \tag{73}$$

Taking the limit $p \to \infty$ shows that $\varphi(t) = 0$ on the interval $[0, t_0]$ whenever $\frac{1}{2} + c_4\, t_0 < 1$. Tiling the half line $[0, \infty)$ with intervals of length t_0 and repeating the argument then shows that $\varphi(t) = 0$ for every $t \geq 0$. This completes the proof of Theorem 5.

5 Continuous Dependence on the Data

Now we estimate how the solutions depend on bottom topography and initial data. The strategy is very similar to that of the uniqueness argument in the previous section. Again we consider two solutions $\hat{\omega}$ and $\bar{\omega}$ to the lake equations, but with bottom functions $\hat{b}$ and $\bar{b}$ respectively. Their difference $\varphi = \hat{\omega} - \bar{\omega}$ is then estimated as before, with some additional terms arising due to the difference in the bottom topographies. All quantities associated with bottom topography $\hat{b}$ will also carry the hat. For example, we will denote by $\hat{\mathrm{H}}^n$ the H^n spaces with weight $\hat{b}$. Analogously, quantities associated with $\bar{b}$ will carry the bar.

An estimate on the time evolution of the $\bar{\mathrm{H}}^{-1}$ norm can be obtained in a way similar to the derivation of (67). Note that (66) holds whether or not $\boldsymbol{\nabla} \cdot (b\boldsymbol{v}) = 0$. A direct calculation using (66) gives

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi(t)\|^2_{\bar{\mathrm{H}}^{-1}} &= -\big\langle \bar{A}^{-1}\big(\hat{b}^{-1}\,\boldsymbol{\nabla}\cdot(\hat{b}\hat{\boldsymbol{u}}\hat{\omega})\big)\varphi\big\rangle^{\bar{}} + \big\langle \bar{A}^{-1}\big(\bar{b}^{-1}\,\boldsymbol{\nabla}\cdot(\bar{b}\bar{\boldsymbol{u}}\bar{\omega})\big)\varphi\big\rangle^{\bar{}} \\
&= \big\langle \hat{\omega}\,\hat{\boldsymbol{u}}\cdot\boldsymbol{\nabla}(\bar{b}\hat{b}^{-1}\bar{A}^{-1}\varphi)\big\rangle^{\hat{}} - \big\langle \bar{\omega}\,\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{A}^{-1}\varphi\big\rangle^{\bar{}} \\
&= \big\langle \hat{\omega}\,\hat{\boldsymbol{u}}\cdot\boldsymbol{\nabla}(\bar{b}(\hat{b}^{-1}-\bar{b}^{-1})\bar{A}^{-1}\varphi)\big\rangle^{\hat{}} + \int_\Omega \hat{\omega}\,(\hat{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{A}^{-1}\varphi)(\hat{b}-\bar{b})\,\mathrm{d}\boldsymbol{x} \\
&\quad + \big\langle \hat{\omega}\,(\hat{\boldsymbol{u}}-\bar{\boldsymbol{u}})\cdot\boldsymbol{\nabla}\bar{A}^{-1}\varphi\big\rangle^{\bar{}} + \big\langle \varphi\,\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{A}^{-1}\varphi\big\rangle^{\bar{}} \\
&\equiv I_1 + I_2 + I_3 + I_4\,.
\end{aligned} \tag{74}$$

We now derive bounds on the integrals I_1–I_4. The first two can be estimated directly by Hölder inequalities. We employ the same strategy as in the derivation of (71), so that for $\frac{1}{p} + \frac{1}{q} = 1$ one has

$$\begin{aligned}
|I_1| &\leq c\,\|\hat{\omega}\|_{\mathrm{L}^\infty}\,\|\hat{\boldsymbol{u}}\|_{\mathrm{L}^p}\,\|\hat{b}^{-1}-\bar{b}^{-1}\|_{\mathrm{W}^{1,2q}}\,\|\bar{A}^{-1}\varphi\|_{\mathrm{W}^{1,2q}} \\
&\leq c\,p\,\|\hat{b}-\bar{b}\|_{\mathrm{W}^{1,2q}}\,\|\varphi\|^{1-1/p}_{\bar{\mathrm{H}}^{-1}}\,,
\end{aligned} \tag{75}$$

and similarly,

$$|I_2| \leq c\,p\,\|\hat{b}-\bar{b}\|_{\mathrm{L}^{2q}}\,\|\varphi\|^{1-1/p}_{\bar{\mathrm{H}}^{-1}}\,. \tag{76}$$

In order to bound I_3 we first estimate

$$|I_3| \leq c\,\|\hat{\omega}\|_{\mathrm{L}^\infty}\,\|\hat{\boldsymbol{u}}-\bar{\boldsymbol{u}}\|_{\bar{\mathrm{L}}^2}\,\|\bar{A}^{-1}\varphi\|_{\bar{\mathrm{H}}^1}\,. \tag{77}$$

Here the $\bar{\mathrm{L}}^2$ difference of $\hat{\boldsymbol{u}}$ and $\bar{\boldsymbol{u}}$ requires a more subtle analysis than in the corresponding estimate of the uniqueness argument, equation (68). Recall that $\bar{P}$ denotes the $\bar{\mathrm{L}}^2$-orthogonal projector onto $\bar{\mathbb{H}}$, so that we can write

$$\begin{aligned}\hat{\boldsymbol{u}} - \bar{\boldsymbol{u}} &= (1 - \bar{K}\hat{K}^{-1})\hat{\boldsymbol{u}} + \bar{K}(\hat{\omega} - \bar{\omega}) \\ &= \bar{K}(\bar{K}^{-1}\bar{P} - \hat{K}^{-1}\hat{P})\hat{\boldsymbol{u}} + \bar{Q}\hat{\boldsymbol{u}} + \bar{K}\varphi\,. \end{aligned} \tag{78}$$

This form is particularly useful because we have $K^{-1}P\boldsymbol{u} = b^{-1}\boldsymbol{\nabla}\wedge\boldsymbol{u}$ for every $\boldsymbol{u} \in \mathrm{L}^2(\Omega, \mathbb{R}^2)$. Using this identity with bottom functions $\hat{b}$ and $\bar{b}$ respectively, it is easy to see that

$$(\bar{K}^{-1}\bar{P} - \hat{K}^{-1}\hat{P})\hat{\boldsymbol{u}} = (\bar{b}^{-1} - \hat{b}^{-1})\boldsymbol{\nabla}\wedge\hat{\boldsymbol{u}} \tag{79}$$

so that

$$\begin{aligned}\|\bar{K}(\bar{K}^{-1}\bar{P} - \hat{K}^{-1}\hat{P})\hat{\boldsymbol{u}}\|_{\mathrm{L}^2} &\le \|(\bar{b}^{-1} - \hat{b}^{-1})\boldsymbol{\nabla}\wedge\hat{\boldsymbol{u}}\|_{\mathrm{L}^2} \\ &\le c\,\|\hat{b} - \bar{b}\|_{\mathrm{L}^{2q}}\,\|\hat{\boldsymbol{u}}\|_{\mathrm{W}^{1,2p}} \le c\,p\,\|\hat{b} - \bar{b}\|_{\mathrm{L}^{2q}}\,. \end{aligned} \tag{80}$$

An estimate on $\bar{Q}\hat{\boldsymbol{u}}$ is obtained as follows. The projection is computed by solving the Neumann problem

$$\bar{b}^{-1}\boldsymbol{\nabla}\cdot(\bar{b}\hat{\boldsymbol{u}}) = -\bar{A}\psi \quad \text{in } \Omega\,, \tag{81a}$$

$$\boldsymbol{n}\cdot\boldsymbol{\nabla}\psi = \boldsymbol{n}\cdot\bar{\boldsymbol{u}} \qquad \text{on } \partial\Omega\,, \tag{81b}$$

and setting $\bar{Q}\hat{\boldsymbol{u}} \equiv \boldsymbol{\nabla}\psi$. Then, invoking elliptic regularity, one estimates

$$\begin{aligned}\|\bar{Q}\hat{\boldsymbol{u}}\|_{\bar{\mathrm{L}}^2} \le \|\psi\|_{\bar{\mathrm{H}}^1} &\le c\,\|\boldsymbol{\nabla}\cdot\big((\hat{b} - \bar{b})\hat{\boldsymbol{u}}\big)\|_{\bar{\mathrm{H}}^{-1}} \\ &\le c\,\|\hat{b} - \bar{b}\|_{\mathrm{L}^{2q}}\,\|\hat{\boldsymbol{u}}\|_{\mathrm{L}^{2p}} \le c\,p\,\|\hat{b} - \bar{b}\|_{\mathrm{L}^{2q}}\,. \end{aligned} \tag{82}$$

Altogether, we obtain the following bound:

$$\begin{aligned}|I_3| &\le c\,\|\hat{\boldsymbol{u}} - \bar{\boldsymbol{u}}\|_{\bar{\mathrm{L}}^2}\,\|\varphi\|_{\bar{\mathrm{H}}^{-1}} \\ &\le c\,\big(\|\bar{K}(\bar{K}^{-1}\bar{P} - \hat{K}^{-1}\hat{P})\hat{\boldsymbol{u}}\|_{\bar{\mathrm{L}}^2} + \|\bar{Q}\hat{\boldsymbol{u}}\|_{\bar{\mathrm{L}}^2} + \|\bar{K}\varphi\|_{\bar{\mathrm{L}}^2}\big)\|\varphi\|_{\bar{\mathrm{H}}^{-1}} \\ &\le c\,p\,\|\hat{b} - \bar{b}\|_{\mathrm{L}^{2q}}\,\|\varphi\|_{\bar{\mathrm{H}}^{-1}} + c\,\|\varphi\|^2_{\bar{\mathrm{H}}^{-1}}\,. \end{aligned} \tag{83}$$

Finally, I_4 is estimated by proceeding as in (70), namely

$$\begin{aligned}|I_4| &= \big|\big\langle \varphi\,\bar{\boldsymbol{u}}\cdot\boldsymbol{\nabla}\bar{A}^{-1}\varphi\big\rangle\bar{}\big| = \big|\big\langle(\boldsymbol{\nabla}\bar{A}^{-1}\varphi)\cdot(\boldsymbol{\nabla}\bar{\boldsymbol{u}})\cdot(\boldsymbol{\nabla}\bar{A}^{-1}\varphi)\big\rangle\bar{}\big| \\ &\le \|\bar{\boldsymbol{u}}\|_{\mathrm{W}^{1,p}}\,\|\bar{A}^{-1}\varphi\|^2_{\mathrm{W}^{1,2q}} \\ &\le c\,p\,\|\bar{A}^{-1/2}\varphi\|^2_{\mathrm{L}^{2q}} \le c\,p\,\|\varphi\|^{2-2/p}_{\bar{\mathrm{H}}^{-1}}\,. \end{aligned} \tag{84}$$

One can combine all these estimates for example, by using the L^∞ bound on

φ again, into the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\varphi(t)\|^2_{\bar{\mathrm{H}}^{-1}} \leq c\,p\left(\|\hat{b}-\bar{b}\|_{\mathrm{W}^{1,2q}} + \|\varphi\|_{\bar{\mathrm{H}}^{-1}}\right)\|\varphi\|^{1-2/p}_{\bar{\mathrm{H}}^{-1}}\,. \tag{85}$$

Comparing this differential inequality with the corresponding inequality (72) in the proof of uniqueness, we see that the inequalities have the same structure, albeit here we have an extra driving term depending on the difference of the the bottom topographies. For φ smaller in magnitude than the L^∞ bound on ω guaranteed by the existence result, Theorem 5—which is the range of φ for which the new estimate might be of any interest—the two solutions $\hat{\omega}$ and $\bar{\omega}$ drift apart marginally worse than exponentially in H^{-1}. The fact that this behavior is intrinsic in both (72) and (85) suggests that it is the optimum obtainable by Sobolev estimation techniques. Lastly we remark that H^{-1} estimates on differences on the potential vorticities imply L^2 bounds on the velocity fields.

The foremost consequence of (85) is the continuous dependence of the solution at a fixed time T on the bottom topography and the initial data. This can be seen as follows: Given (85) with initial condition $\|\varphi(0)\|_{\bar{\mathrm{H}}^{-1}}$, the solution of

$$\frac{\mathrm{d}F}{\mathrm{d}t} = c\,p\,F^{1-1/p}\,, \tag{86a}$$

$$F(0) = \left(\|\hat{b}-\bar{b}\|_{\mathrm{W}^{1,2q}} + \|\varphi(0)\|_{\bar{\mathrm{H}}^{-1}}\right)^{\frac{p}{p-1}}\|\varphi(0)\|^{\frac{p-2}{p-1}}_{\bar{\mathrm{H}}^{-1}}\,, \tag{86b}$$

provides an upper bound on $\|\varphi(t)\|_{\bar{\mathrm{H}}^{-1}}$ for all times $t \geq 0$. This equation is easily solved, namely

$$F(t) = \left(F^{1/p}(0) + c\,t\right)^p\,. \tag{87}$$

Now let $\varepsilon > 0$. Then there exists a p such that for every $F(0) \leq \delta \equiv 3^{-p}$ and $t \leq 1/(3c)$ one has $F(t) < \varepsilon$. Together with (86b), this proves continuous dependence of ω in $\mathrm{H}^{-1}(\Omega)$ on b in $\mathrm{W}^{1,2q}(\Omega)$ and ω^{in} in $\mathrm{H}^{-1}(\Omega)$ on the interval $[0, 1/(3c)]$. For arbitrary times T one can tile the time axis with intervals of length $1/(3c)$ and iterate the argument. Thus we have the following.

Theorem 7 *For every fixed time T and every $q > 1$, the mapping $(b, \omega^{\mathrm{in}}) \mapsto \omega(T)$ is continuous on $\mathrm{W}^{1,2q}(\Omega) \times \mathrm{H}^{-1}(\Omega) \to \mathrm{H}^{-1}(\Omega)$.*

6 Conclusion

The lake equations are globally well-posed. This is a necessary first step in rendering the derivation of the lake equations rigorous, which is still an open problem. In addition, there are several other avenues for further investigation:

How regular are the solutions of the lake equations? For example, are there classical solutions? And given real analytic initial and boundary data, does a solution remain real analytic for all times? Such results hold for the two dimensional Euler equations, and are expected to carry over to the lake equations.

From a physical perspective, lakes may have inlets and outlets, the modeling aspects for which are still unresolved. In addition, lakes and oceans have beaches, which requires the treatment of basins with nonvertical lateral boundaries. In that setting, the scaling used in Section 2 needs to be modified near the lateral boundary. In any case, we expect that the dynamics in the interior will still be described by the lake equations, possibly with different boundary conditions, and therefore fall within the mathematical framework presented here. A degeneracy of the bottom function would render the elliptic operator (18) degenerate, thereby introducing additional technical difficulties.

Finally, even in the simpler setting presented in this paper, the lake equations may be sufficient for obtaining acceptable predictions of large scale flow for lakes with beaches. This is because the only assumption on the minimal depth $b_{\min}$ is that it be greater in magnitude than the (very small) typical surface amplitude H. Hence, we allow for situations where $H \ll b_{\min} \ll B$, thereby permitting very shallow shores. It is natural to investigate whether one can study lakes with beaches as the limit of lakes with vanishing shore depth.

Acknowledgement

We thank R. Camassa, C. Doering, D. Holm and L. Margolin for interesting and stimulating discussions. Part of this work was done when the authors were visitors of the CNLS and the IGPP at Los Alamos National Laboratory. The first two authors would also like to thank the Ecole Normale Supérieure de Cachan for their hospitality during the finalization of this manuscript. This work was supported in part by the NSF grants No. DMS-9308774 and DMS-9404570 and by the Joint University of California and Los Alamos National Laboratory INCOR program. E.S. Titi also acknowledges the kind hospitality of the Newton Institute, the partial support of the EPSRC Grant No. GR J82041 and the University of California-Irvine Faculty Research Fellowship.

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