The Complex Ginzburg-Landau Equation as a Model Problem

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Abstract. The generalized complex Ginzburg-Landau (CGL) equation describes the evolution of a complex-valued field $u = u(x,t)$ by

$$\partial_t u = Ru + (1+i\nu)\Delta u - (1+i\mu)|u|^{2\sigma} u \, .$$

It has a long history in physics as a generic amplitude equation near the onset of instabilities that lead to chaotic dynamics in fluid mechanical systems, as well as in the theory of phase transitions and superconductivity. In this role it and equations like it have had remarkable success in describing evolution phenomena in a broad range of physical systems, from fluids to optics. More recently, it has been proposed and studied as a model for "turbulent" dynamics in nonlinear partial differential equations. It is a particularly interesting model in this respect because it is a dissipative version of the nonlinear Schrödinger equation, a Hamiltonian equation which can possess solutions that form localized singularities in finite time.

This article summarizes the status of well-posedness and regularity questions for the CGL equation subject to periodic boundary conditions in any spatial dimension. It also discusses the relationship of these results to the above investigations and, in particular, to analogous issues for Navier-Stokes fluid turbulence. Appropriately defined weak solutions exist globally in time and unique classical solutions are found locally. Conditions are given under which these classical solutions exist globally in time and have bounds on all derivatives that are uniform over large times, thereby providing the setting in which ideas from finite dimensional dynamical systems can be applied to the CGL equation. Moreover, by using a characterization of Gevrey classes in terms of decay of Fourier coefficients, these solutions are shown to be analytic for positive times. Refinements in both some of the theorems and some of the proofs are introduced.

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1. Introduction

The description of chaotic dynamics or, more dramatically, turbulence in continuum systems by nonlinear evolution equations (often partial differential equations) is one of the central challenges facing theoretical physics and applied mathematics. We would like to be able to apply ideas from the studies of dynamical systems or statistical physics to these evolution equations in order to gain a deeper understanding of these phenomena and, as a byproduct, to hopefully achieve simpler or more tractable descriptions. One of the main mathematical stumbling blocks to studying these questions often lies in our lack of understanding of even the most basic issues of existence, uniqueness and regularity for solutions of the underlying evolution equations. This situation has led mathematicians and physicists either to derive, usually through systematic asymptotics, or to propose, usually in an ad hoc manner based on intuition and symmetries, simpler equations for which these mathematical issues can be addressed with greater success.

One such equation is the generalized complex Ginzburg-Landau (CGL) equation, which describes the evolution of a complex-valued field $u = u(x,t)$ over a d-dimensional spatial domain by

$$\partial_t u = Ru + (1+i\nu)\Delta u - (1+i\mu)|u|^{2\sigma}u\,. \tag{1.1}$$

Here the parameters R and σ are positive while ν and μ are real. Without loss of generality, the equation has been normalized so that the coefficients of the linear and nonlinear dissipation (damping) terms are unity. The R is the coefficient of the linear driving term, without which all solutions would decay to zero. Clearly, σ sets the degree of the nonlinearity. The ν and μ are the coefficients of the linear and nonlinear dispersive terms respectively.

The CGL equation with a cubic nonlinearity ($\sigma = 1$) has a long history as a generic amplitude equation derived asymptotically near the onset of instabilities in fluid dynamical systems. The case with real coefficients was first derived by Newell and Whitehead [**65**] and Segel [**74**] to describe Bénard convection. The case with complex coefficients was put forth in a general setting by Newell and Whitehead [**66**] and DiPrima, Eckhaus, and Segel [**20**], and was shown by Stewartson and Stuart [**76**] to apply to plane Poiseuille flow. The reader may also want to consult more comprehensive reviews of the application of amplitude equations, such as the CGL equation to chemical reactions [**51**] and to pattern formation [**17, 64**]. Here we remove it from any particular physical context and investigate it as a mathematical model of a variety of possible phenomena in nonlinear partial differential equations. Considered over the d-dimensional torus $\mathbb{T}^d$, the CGL equation is both theoretically and numerically tractable, and has proved fruitful for illustration of connections between infinite dimensional and finite dimensional dynamical systems [**3, 24, 73**]. We shall therefore focus our study on the spatially periodic CGL equation (1.1) and point out those results that extend to $\mathbb{R}^d$.

This lecture will present to nonspecialists one overview of how the generalized CGL equation can be viewed as a dynamical system. We have therefore tried not to presume familiarity with material beyond that from standard graduate level analysis and functional analysis courses. The style will be expository. We will give a big picture along with enough detail to clarify the strategies that are used to prove many basic theorems. Without any pretense of being exhaustive, we review several of the avenues of investigation taken in the literature, all of which have application far beyond the context of the CGL equation.

The material is organized as follows. Section 2 begins to paint the CGL phase portrait by identifying some special solutions, examining their stability, and describing how chaotic behavior might arise. Section 3 presents the attributes of the CGL equation that allow it to be a focus for general studies of turbulent behavior. In particular its relation to the nonlinear Schrödinger (NLS) equation,

$$\partial_t u = i\nu\Delta u - i\mu|u|^{2\sigma}u \,, \tag{1.2}$$

is discussed as an analogue to the relation of the driven Navier-Stokes equations to the Euler equations in fluid dynamics. The existence of global (in time) weak solutions to the CGL equation for L^2 initial data is proved in Section 4. Such solutions exist in any spatial dimension and degree of nonlinearity. The result is similar in spirit to that of Leray's existence theorem for global weak solutions to the Navier-Stokes equations. Also similarly, the question of uniqueness for weak solutions remains open. Section 5 discusses the existence and uniqueness of local (in time) classical solutions—even C^∞ solutions provided σ is a positive integer. Initial data in the classes L^p and H^q are considered. In cases when s can be negative this will include initial data that are distributions. Section 6 presents local estimates which show that when σ is an integer the solutions are not only C^∞, but also contained in a subclass of the real analytic functions. The derivation of these estimates employs a characterization of Gevrey classes in terms of decay of Fourier coefficients. Our inability to unconditionally extend the local solutions to global ones is a manifestation of the instability in the inviscid dynamics. One cannot generally exclude the possibility of a finite-time blow-up, even in dissipative systems. Section 7 establishes global classical solutions through *a priori* bounds of L^p norms and does not restrict σ to integer values. When σ is a positive integer, direct global bounds on every H^n norm are derived in Section 8. The role of these results in applying dynamical systems methods to the study of the long-time behavior of CGL solutions is discussed.

2. The CGL Equation as a Model for the Onset of Chaos

This investigation has two main themes. The first theme is to gain as complete an understanding as possible of the CGL "phase portrait" in any infinite dimensional phase space in which the flow is defined globally in time. The second theme is to justify the asymptotic derivation of the CGL equation as an amplitude equation and to determine whether the CGL phase portrait can be related to the onset of chaos in solutions of the original dynamical system. This question will be taken up in the lecture of A. Mielke [**63**] and is also reviewed in [**17, 51**], so it will not be discussed here.

Crucial to the understanding of the phase portrait of any dynamical system is the nature of the long-time asymptotics. Our main focus will be to establish existence and properties of a globally attracting set. Such a set, the *global attractor*, is compact, invariant under the flow, and attracts all orbits uniformly over bounded sets of initial data [**29, 78**]. Naturally, the attractor contains the ω-limit set and the *unstable manifold* of the ω-limit set of every orbit. The nature of orbits on and about the attractor provides valuable information for understanding the mechanism for the onset of chaos.

Our investigation begins by identifying classes of solutions that are known explicitly and which are therefore amenable to local analysis using linear and nonlinear methods. First, the number of linearly unstable directions gives a lower bound on the dimension of the unstable manifold, thereby providing a lower bound on the dimension of the attractor. This analysis was first carried out in detail for the cubic CGL equation for $d = 1$ by Doering, Gibbon, Holm and Nicolaenko [**24**] and for $d = 1$, 2 by Ghidaglia and Héron [**35**]. Second, we will indicate the nonlinear techniques used to paint a picture of the bifurcations from these solutions as R increases.

To illustrate the procedure, we first linearize the CGL equation (1.1) about the zero solution and obtain

$$\partial_t \tilde{u} = R\tilde{u} + (1 + i\nu)\Delta\tilde{u} \,. \tag{2.1}$$

Then a small perturbation with a pure Fourier mode $\tilde{u}(x,t) = \tilde{a}(t)\exp(i\xi{\cdot}x)$ where $\xi \in 2\pi\mathbb{Z}^d$ $(= (2\pi\mathbb{Z})^d)$ evolves according to

$$\partial_t \tilde{a} = R\tilde{a} - (1 + i\nu)|\xi|^2\tilde{a} \,. \tag{2.2}$$

This shows that with respect to this perturbation the zero solution is linearly unstable when $|\xi|^2 < R$, linearly asymptotically stable when $|\xi|^2 > R$, and linearly neutrally stable when $|\xi|^2 = R$. Hence, the dimension of the subspace of linear instability, which gives a lower bound on the dimension of the globally attracting set, grows like $R^{d/2}$ as R becomes large.

The simplest class of nontrivial spatially periodic solutions is comprised of rotating wave solutions, which have the form

$$u(x,t) = a\exp\big(i(\xi{\cdot}x - \omega t)\big) \,, \tag{2.3}$$

where $a \in \mathbb{C}$, $\omega \in \mathbb{R}$, and $\xi \in 2\pi\mathbb{Z}^d$ satisfy

$$\omega = \mu R + (\nu - \mu)|\xi|^2 \,, \qquad |a|^{2\sigma} = R - |\xi|^2 \,. \tag{2.4}$$

The largest of these, corresponding to $\xi = 0$, is the spatially homogeneous solution, sometimes referred to as the Stokes solution. For each wave vector ξ such that $|\xi|^2 < R$, formulas (2.3) and (2.4) define a circle of rotating wave solutions that is parameterized by the phase of a. These solutions that play an important role in many physical applications and their existence is one of the reasons that the CGL initial-value problem posed in $L^p(\mathbb{T}^d)$ is more interesting than that posed in $L^p(\mathbb{R}^d)$. Notice that, as R increases, the circle of rotating waves corresponding to wave vector ξ bifurcates off the zero solution at $R = |\xi|^2$, thereby corresponding to the addition of two real dimensions to the subspace of linear instability of the zero solution for each such ξ as R increases beyond $|\xi|^2$. There are on the order of $R^{d/2}$ such circles as R becomes large.

Due to the simple form of the rotating waves (2.3), their linear stability can be analyzed exactly. This is not something one can do for any nontrivial solution of any partial differential equation because the associated linearized equation will generally have coefficients that depend on both space and time. Indeed, a naive linearization of the CGL equation (1.1) about a rotating wave (2.3) leads to a variable coefficient equation. However, if one introduces a perturbation of the form

$$u(x,t) = a\exp\big(i(\xi{\cdot}x - \omega t)\big)\big(1 + \epsilon\,\tilde{u}(x,t)\big) \,, \tag{2.5}$$

then the resulting linearization is

$$\partial_t \tilde{u} = 2(1+i\nu)i\xi \cdot \nabla\tilde{u} + (1+i\nu)\Delta\tilde{u} - (1+i\mu)\sigma|a|^{2\sigma}(\tilde{u}+\tilde{u}^*)\,. \tag{2.6}$$

This equation can be viewed as a real linear system for the real and imaginary parts of $\tilde{u}$ (alternatively, for $\tilde{u}$ and $\tilde{u}^*$) with constant coefficients, which can therefore be analyzed using Fourier modes. Due to the multiplicative form of the perturbation (2.5), the Fourier mode of $\tilde{u}$ with wave vector $\tilde{\xi}$ corresponds to the Fourier mode of the perturbation with wave vector $\xi + \tilde{\xi}$. Hence, this procedure is often called a side-band analysis and $\tilde{\xi}$ a side-band wave vector.

This general analysis will not be carried out here (see **[24]** for details), but some of its features will be illustrated by analyzing the linear stability of the homogeneous solution. In that case (2.6) simplifies to

$$\partial_t \tilde{u} = (1+i\nu)\Delta\tilde{u} - (1+i\mu)\sigma R(\tilde{u}+\tilde{u}^*)\,. \tag{2.7}$$

When it is written as a real linear system for $\tilde{u}_R$ and $\tilde{u}_I$, the real and imaginary parts of $\tilde{u}$, this equation becomes

$$\partial_t \begin{pmatrix} \tilde{u}_R \\ \tilde{u}_I \end{pmatrix} = \begin{pmatrix} \Delta - 2\sigma R & -\nu\Delta \\ \nu\Delta - 2\mu\sigma R & \Delta \end{pmatrix} \begin{pmatrix} \tilde{u}_R \\ \tilde{u}_I \end{pmatrix}. \tag{2.8}$$

It is easily checked that the Fourier mode with wave vector $\tilde{\xi}$ will decay exponentially provided

$$(1+\nu^2)|\tilde{\xi}\,|^4 + 2\sigma R(1+\mu\nu)|\tilde{\xi}\,|^2 > 0\,. \tag{2.9}$$

This will certainly be satisfied for every nonzero $\tilde{\xi}$, no matter what values the degree σ and forcing R may have, whenever μ and ν lie in the region of the $\mu\nu$-plane where $(1+\mu\nu) \geq 0$, commonly referred to as the modulationally stable region (illustrated in Figure 7.1 on page 179).

A slightly more general class than the rotating wave solutions are the harmonic (or monochromatic) solutions, which have the form

$$u(x,t) = a(t)\exp(i\xi{\cdot}x)\,, \tag{2.10}$$

where the complex-valued amplitude $a = a(t)$ satisfies

$$\frac{da}{dt} = \big(R - (1+i\nu)|\xi|^2\big)a - (1+i\mu)|a|^{2\sigma}a\,. \tag{2.11}$$

Multiplying this equation by a^* and separating real and imaginary parts gives

$$\frac{1}{2}\frac{d}{dt}|a|^2 = \big(R - |\xi|^2\big)|a|^2 - |a|^{2\sigma+2}\,, \tag{2.12}$$

$$\frac{1}{2i}\left(a^*\frac{da}{dt} - a\frac{da^*}{dt}\right) = -\nu|\xi|^2\,|a|^2 - \mu|a|^{2\sigma+2}\,. \tag{2.13}$$

It is easily seen that when $R - |\xi|^2 < 0$ these solutions decay exponentially to zero. In this case, these solutions are part of the stable manifold of the trivial solution. When $R - |\xi|^2 = 0$ these solutions decay algebraically to zero, reflecting the marginal stability of the zero solution at the given value of R to perturbations with wave vector ξ. When $R - |\xi|^2 > 0$ these solutions are attracted exponentially to a rotating wave solution corresponding to the wave vector ξ. In this case, these solutions lie on the intersection of the unstable manifold of the zero solution with the stable manifold of the rotating wave solution of wave vector ξ.

The rotating waves are also part of the more general class of traveling wave solutions, each of which has the form

$$u(x,t) = \exp(-i\alpha t)\, v(x-ct)\,, \tag{2.14}$$

where $\alpha \in \mathbb{R}$ is the phase frequency, $c \in \mathbb{R}^d$ is the wave velocity and $v = v(z)$ is the complex-valued profile function over $z = x - ct \in \mathbb{T}^d$ that satisfies

$$(R + i\alpha)v + (1 + i\nu)\Delta_z v - c \cdot \nabla_z v - (1 + i\mu)|v|^{2\sigma} v = 0\,. \tag{2.15}$$

Within the class of traveling waves are the so-called standing waves, for which $c = 0$, and which possess a nontrivial spatial structure in both amplitude and phase. A rotating wave (2.3) will solve (2.15) provided

$$\omega = \alpha + \xi \cdot c\,. \tag{2.16}$$

More general traveling wave solutions will bifurcate off these rotating wave solutions as R increases [**77**]. In particular, standing waves bifurcate off the spatially homogeneous solution [**67**]. Traveling wave solutions have also been shown to exist when $d = 1$ near the NLS limit [**18**]. However, a complete understanding of all solutions of (2.15) is still lacking.

Stability is not as easy to assess for these traveling waves as it was for the rotating waves. For example, to study their linear stability, introduce a perturbation of the form

$$u(x,t) = \exp(-i\alpha t)\big(v(x-ct) + \epsilon\, \tilde{v}(x-ct,t)\big)\,, \tag{2.17}$$

into the CGL equation and retain only terms of order ϵ to obtain

$$\begin{aligned} \partial_t \tilde{v} = {} & (R + i\alpha)\tilde{v} + (1 + i\nu)\Delta_z \tilde{v} - c \cdot \nabla_z \tilde{v} \\ & - (1 + i\mu)|v|^{2\sigma}\tilde{v} - \sigma(1 + i\mu)|v|^{2\sigma-2} v(v^* \tilde{v} + v \tilde{v}^*)\,. \end{aligned} \tag{2.18}$$

The operator on the right side is independent of t, so the linear stability analysis is thereby reduced to studying the semigroup generated by that operator, which is not easy because the operator generally has coefficients with a complicated dependence on z.

Harder still is analytically determining the bifurcation sequence of solutions in the attractor. Some insight into these questions has been gained through numerical studies [**57**]. For the parameter choices studied in [**57**], as R is increased, the standing wave solutions undergo supercritical Hopf bifurcations, creating two stable periodic orbits. Further increase in R gradually deforms these orbits into a pair of homoclinic orbits which connect the homogeneous solution to itself (the homogeneous solution is really a fixed point of the CGL system). When this homoclinic orbit is broken, a single stable periodic orbit appears which looks like a combination of the two periodic orbits that formed the homoclinic orbit. This bifurcation sequence is known as a "gluing bifurcation" (see [**33**] and the references therein); it is not a period doubling bifurcation, but rather a length doubling that can be understood in a three dimensional context. At this homoclinic connection, a chaotic set is not created because of the stable nature of the periodic orbits involved: the orbits bring their attracting neighborhoods to the homoclinic orbit, which is consequently a *stable* object in the phase space, and therefore cannot produce a hyperbolic chaotic set of trajectories when the homoclinic connection is broken by a change in R [**57**]. As R is increased beyond this initial bifurcation, a series of unstable homoclinic orbits occur that connect the homogeneous solution to itself.

These homoclinic orbits lead to chaotic attractors through so-called "homoclinic explosions", similar to those found in the Lorenz equations.

A special case of the CGL equation for which the attractor can, in principle, be completely characterized is when $\mu = \nu$. In that case it is natural to introduce the new complex-valued field $v = v(x,t)$ by

$$u(x,t) = \exp(-iR\mu t)\, v(x,t)\,. \tag{2.19}$$

The evolution of v is then governed by

$$\partial_t v = -(1+i\mu)\frac{\delta G}{\delta v^*}\,, \tag{2.20}$$

where

$$\frac{\delta G}{\delta v^*} = -\Delta v + |v|^{2\sigma} v - Rv\,, \tag{2.21}$$

which is the classical variational partial derivative with respect to v^* of the so-called Ginzburg-Landau functional G defined by

$$G = \int_{\mathbb{T}^d} |\nabla v|^2 + \frac{1}{\sigma+1}|v|^{2\sigma+2} - R|v|^2\, dx\,. \tag{2.22}$$

Computing the time derivative of this functional under the flow (2.20) gives

$$\frac{dG}{dt} = \int_{\mathbb{T}^d} \frac{\delta G}{\delta v}\,\partial_t v + \frac{\delta G}{\delta v^*}\,\partial_t v^*\, dx = -2\int_{\mathbb{T}^d}\left|\frac{\delta G}{\delta v}\right|^2 dx\,, \tag{2.23}$$

which shows that G is a monotonically decreasing function of time except at points where $\delta G/\delta v^*$ vanishes. Such points are stationary points of the flow (2.20) and, hence, are standing waves of the CGL flow. They include as a special case the circle of stationary harmonic waves given by

$$v(x) = b\exp(i\xi\cdot x)\,, \qquad |b|^{2\sigma} = R - |\xi|^2 > 0\,, \tag{2.24}$$

for some $b \in \mathbb{C}$ and $\xi \in 2\pi\mathbb{Z}^d$. When $d = 1$, the stationary points generally satisfy the ordinary differential equation

$$-\partial_{xx} v + |v|^{2\sigma} v - Rv = 0\,, \tag{2.25}$$

so their existence may be determined through a phase plane analysis. Furthermore, these solutions may be expressed explicitly in terms of elliptic functions when $\sigma = 1, 2$ and in terms of hyperelliptic functions when $\sigma > 2$ is an integer.

In general, (2.23) shows that the functional G formally acts as a global Lyapunov functional for the flow, a fact that will be made rigorous once sufficient regularity is established to justify the formal manipulations. This will be done in Section 5. The stability of any stationary point v can be examined through the spectrum of $\mathcal{H}(v)$, the Hessian operator of the functional G at the point v, which is given by

$$\begin{aligned}\mathcal{H}(v) &\equiv \begin{pmatrix} \dfrac{\delta^2 G}{\delta v^*\delta v} & \dfrac{\delta^2 G}{\delta v^*\delta v^*} \\ \dfrac{\delta^2 G}{\delta v\delta v} & \dfrac{\delta^2 G}{\delta v\delta v^*}\end{pmatrix} \\ &= \begin{pmatrix} -\Delta + (\sigma+1)|v|^{2\sigma} - R & \sigma|v|^{2\sigma-2}v^2 \\ \sigma|v|^{2\sigma-2}v^{*\,2} & -\Delta + (\sigma+1)|v|^{2\sigma} - R\end{pmatrix}\,.\end{aligned} \tag{2.26}$$

The Hessian is the self-adjoint linear operator associated with the real quadratic form that is the second variation of G at v. Zero is always in the spectrum of $\mathcal{H}(v)$ with multiplicity at least one because $(iv, -iv^*)^T$ is always the null vector associated with changing v by a constant phase. The analysis of the remaining spectrum of $\mathcal{H}(v)$ is generally difficult because of its nonconstant coefficients.

In the case when v is a stationary harmonic wave (2.24) the Hessian operator (2.26) is unitarily similar to the constant coefficient self-adjoint operator

$$\widetilde{\mathcal{H}}(v) = \begin{pmatrix} -\Delta - 2i\xi\cdot\nabla + \sigma|b|^{2\sigma} & \sigma|b|^{2\sigma} \\ \sigma|b|^{2\sigma} & -\Delta + 2i\xi\cdot\nabla + \sigma|b|^{2\sigma} \end{pmatrix}, \tag{2.27}$$

whose spectrum can be analyzed through Fourier analysis. For example, the dimension of the unstable manifold of the circle of stationary harmonic waves (2.24) is given by the number of nonzero $\tilde{\xi} \in 2\pi\mathbb{Z}^d$ such that

$$|\tilde{\xi}\,|^4 + 2\sigma|b|^{2\sigma}|\tilde{\xi}\,|^2 - 4(\xi\cdot\tilde{\xi}\,)^2 < 0\,, \tag{2.28}$$

assuming there are no such $\tilde{\xi}$ for which the left side of this equation vanishes. In particular, the circle of stationary harmonic waves (2.24) will be stable whenever $(\sigma+2)|\xi|^2 < \sigma R$.

When $d = 1$ and $\sigma = 1$ the unstable manifold of each stationary point determined by (2.25) can also be analyzed through the Hessian (2.26) by using the machinery of the integrable NLS equation [**44**]. However, in all other cases our knowledge of the phase portrait for even this simplest of all CGL equations (2.20) is far from complete.

3. The CGL Equation as a Model for Turbulent Phenomena

Much work on the theory of turbulent phenomena has focused on simple fluids, where the phenomena are presumably described by the incompressible Navier-Stokes equations [**5**]. One of the major obstacles in this area of study is that fact that, as of this writing, the existence of smooth, or *classical*, or *strong* solutions to the Navier-Stokes initial value problem in three spatial dimensions has never been demonstrated for arbitrarily large (albeit smooth) initial data, or arbitrarily strong (albeit smooth) applied forces [**14**]. Leray established the existence, although not the uniqueness, of *weak* solutions to the Navier-Stokes equations over half a century ago [**55**], and interpreted the implied loss of regularity in the general case as an indication of the onset of turbulent behavior. This would mean that viscous dissipation in three spatial dimensions is unable to inhibit singularities driven by instabilities in the underlying inviscid system, the Euler equations. There is, however, an obvious logical gap associated with connecting our inability to prove regularity and uniqueness with an actual failure of those properties. Moreover, the necessity of a breakdown of regularity in the Euler equations remains an open area of investigation which is discussed in the lecture by P. Constantin [**12**]. Uniqueness of these weak solutions is not established, and this fact carries its own implications for the question of the self consistency of any hydrodynamic description at all! These issues remain today in an unsatisfactory state of nonresolution.

There are two key attributes of the CGL equation that allow us to seriously consider it as a worthwhile focus for general studies of turbulent behavior. First, in the absence of driving ($R = 0$) it has an inviscid limit that is a Hamiltonian system, namely, when $\mu\nu \neq 0$ it is the nonlinear Schrödinger (NLS) equation

$$i\partial_t u = -\nu\Delta u + \mu|u|^{2\sigma}u\,, \tag{3.1}$$

which is essentially the usual Schrödinger equation of quantum mechanics with its potential V replaced by $\mu|u|^{2\sigma}$. Second, this inviscid limit possesses a self-focusing mechanism capable of provoking spatially localized singularities in the solutions, within a finite time, starting from arbitrarily smooth initial data. This effect can be thought of as a rather violent cascade process whereby energy concentrated in low wavenumber modes is rapidly transported to high wavenumber modes. These two properties are in direct analogy to the problem of incompressible fluid turbulence, at least so far as the conventional wisdom is concerned. The Euler equations constitute the underlying Hamiltonian dynamics of the dissipative Navier-Stokes equations, and instabilities in the inviscid equations can create a cascade of energy between widely separated length scales. We will examine these two attributes in more detail below.

The NLS equation can be naturally recast as a Hamiltonian system in the form

$$i\nu\partial_t u = \frac{\delta H}{\delta u^*}\,, \tag{3.2}$$

where the Hamiltonian H is given by

$$H \equiv \int_{\mathbb{T}^d} \nu^2|\nabla u|^2 + \frac{\mu\nu}{\sigma+1}|u|^{2\sigma+2}\,dx\,. \tag{3.3}$$

Defining the Poisson bracket of any two functionals F and G to be

$$\{F,G\} \equiv \frac{1}{i\nu}\int_{\mathbb{T}^d}\left(\frac{\delta F}{\delta u}\frac{\delta G}{\delta u^*} - \frac{\delta F}{\delta u^*}\frac{\delta G}{\delta u}\right)dx\,, \tag{3.4}$$

the evolution of any functional F under the NLS flow (3.2) is governed by

$$\frac{dF}{dt} = \{F,H\}\,. \tag{3.5}$$

At least formally, the Hamiltonian H is conserved by the NLS flow. Besides H, the NLS flow also generally conserves the mass M and the momentum P, which are given by

$$M \equiv \int_{\mathbb{T}^d}|u|^2\,dx\,, \qquad P \equiv \frac{\nu}{2i}\int_{\mathbb{T}^d}\left(u^*\nabla u - u\nabla u^*\right)dx\,. \tag{3.6}$$

These quantities play an important role in the general study of the NLS equation, but such a digression would take us to far afield. However, it should be noted that the one-dimensional cubic NLS equation ($d = \sigma = 1$) is completely integrable and therefore possesses an infinite number of conserved quantities in involution with respect to the Poisson bracket (3.4) [**82, 83**].

Ginibre and Velo [**36**] showed that the nonlinear Schrödinger equation is a locally well-posed initial-value problem in the Hilbert space $H^1(\mathbb{R}^d)$ provided

$$d < 2 + \frac{2}{\sigma}\,. \tag{3.7}$$

More precisely, if (3.7) is satisfied then for every $\rho > 0$ there exists a time $T(\rho) > 0$ such that for every initial data $u^{\text{in}} \in H^1(\mathbb{R}^d)$ with $\|u^{\text{in}}\|_{H^1} \leq \rho$ there exists a unique $u \in C([0,T], H^1(\mathbb{R}^d))$ satisfying a mild form of the NLS initial-value problem. Moreover, the quantities M, P, and H are all conserved by this solution. Extending this result to $\mathbb{T}^d$ is not direct, and requires some different ideas. Recently, Bourgain has developed new techniques that allow the NLS dynamics over periodic domains to be extended to larger spaces [**8, 9**], but here we shall confine ourselves to H^1 dynamics. We will leave imprecise the exact nature of what it means to be a mild solution of the NLS initial-value problem—it is not important for the following considerations. However, it is important to understand that whenever (3.7) is satisfied, the NLS equation will make sense as a dynamical system in H^1, at least locally in time, with M, P and H conserved.

Condition (3.7) arises from the need to control the contribution of the nonlinear term of the NLS equation to the Hamiltonian H (or equivalently, to control the $L^{2\sigma+2}$ norm of u) with the H^1 norm of u, which we take to be

$$\|u\|_{H^1} \equiv \left(\int_{\mathbb{T}^d} \kappa^2 |u|^2 + |\nabla u|^2 \, dx \right)^{\frac{1}{2}}, \tag{3.8}$$

where $-\kappa^2$ is the largest negative eigenvalue of the Laplacian. This control is achieved through a Gagliardo-Nirenberg type of Sobolev inequality [**2, 32, 68**], which states that there exists a positive nondimensional constant $C(\sigma, d) < \infty$ such that the $L^{2\sigma+2}$ norm of u can be bounded by a geometric average of the L^2 and H^1 norms of u as

$$\begin{aligned} \|u\|_{L^{2\sigma+2}} &= \left(\int_{\mathbb{T}^d} |u|^{2\sigma+2} \, dx \right)^{\frac{1}{2\sigma+2}} \\ &\leq C(\sigma, d) \left(\int_{\mathbb{T}^d} \kappa^2 |u|^2 + |\nabla u|^2 \, dx \right)^{\frac{\theta}{2}} \left(\int_{\mathbb{T}^d} |u|^2 \, dx \right)^{\frac{1-\theta}{2}} \\ &= C(\sigma, d) \, \|u\|_{H^1}^{\theta} \, \|u\|_{L^2}^{1-\theta}, \end{aligned} \tag{3.9}$$

whenever the geometric interpolation parameter θ satisfies

$$\theta \equiv \frac{\sigma d}{2\sigma + 2} < 1 \,. \tag{3.10}$$

The above value of θ is determined by balancing units of length in (3.9), thereby rendering $C(\sigma, d)$ nondimensional, which leads to the relation

$$\frac{1}{2\sigma + 2} d = \frac{\theta}{2}(d-2) + \frac{1-\theta}{2} d \,. \tag{3.11}$$

The inequality in (3.10) is equivalent to condition (3.7).

These local solutions can be extended to be global solutions if one can find an upper bound on $\|u\|_{H^1}$ that is uniform in time. When $\mu\nu > 0$ such a bound follows directly from the estimate

$$\nu^2 \|u\|_{H^1}^2 \leq H + \nu^2 \kappa^2 M \,. \tag{3.12}$$

On the other hand, when $\mu\nu < 0$ the Gagliardo-Nirenberg inequality (3.9) yields the estimate

$$\begin{aligned} H + \nu^2 \kappa^2 M &= \nu^2 \|u\|_{H^1}^2 + \frac{\mu\nu}{\sigma + 1} \|u\|_{L^{2\sigma+2}}^{2\sigma+2} \\ &\geq \nu^2 \|u\|_{H^1}^2 + \frac{\mu\nu}{\sigma+1} C(\sigma, d)^{2\sigma+2} \, \|u\|_{H^1}^{(2\sigma+2)\theta} \, \|u\|_{L^2}^{(2\sigma+2)(1-\theta)}, \end{aligned} \tag{3.13}$$

where θ is given by (3.10). Because $\|u\|_{L^2}$ is conserved, the above estimate gives a uniform upper bound for $\|u\|_{H^1}$ whenever the degree of $\|u\|_{H^1}$ in the second term is less that in the first term, which is the case if and only if $\sigma d < 2$. For the marginal case $\sigma d = 2$, the above estimate gives a uniform upper bound for $\|u\|_{H^1}$ provided $\|u\|_{L^2}$ is small enough.

The above argument shows that necessary conditions for the possible formation of finite-time singularities are $\nu\mu < 0$ and $\sigma d \geq 2$. This is simply understood by considering the NLS equation to be a "mean field" quantum mechanical description of locally interacting particles. Without loss of generality, taking $\nu > 0$, the potential in (3.1) is $V = \mu|u|^{2\sigma}$. When $\mu > 0$ the interaction is repulsive, and consequently there is no tendency for the particles to concentrate; this is referred to as the defocusing case. On the other hand, when $\mu < 0$ the interaction is attractive, providing the so-called self-focusing mechanism, and the particles tend to concentrate; this is referred to as the focusing case. When $\sigma d < 2$ this attraction is not strong enough that it can induce a collapse of the matter. However, as will be argued below, when $\sigma d \geq 2$ this attraction is strong enough that collapse may occur. The term *critical* indicates that the condition for blow-up is satisfied as an equality, $\sigma d = 2$. The terms *supercritical* and *subcritical* indicate $\sigma d > 2$ and $\sigma d < 2$ respectively.

That the self-focusing mechanism can produce finite time blow-up for solutions of the NLS equation when $\sigma d \geq 2$ was first established over $\mathbb{R}^d$ [**38, 81, 84**] by the following argument. Given any $u \in H^1(\mathbb{R}^d)$, consider the vector-valued quantity $\bar{x}$ and scalar-valued quantity I defined by

$$\bar{x} \equiv \frac{1}{M}\int_{\mathbb{R}^d} x\,|u|^2\,dx\,, \qquad I \equiv \int_{\mathbb{R}^d} \tfrac{1}{8}|x-\bar{x}|^2\,|u|^2\,dx\,, \tag{3.14}$$

which, by analogy with classical mechanics, can be interpreted as the center of mass and the moment of inertia of $|u|^2$ respectively. These quantities will be finite for any $u \in H^1(\mathbb{R}^d)$ for which

$$\int_{\mathbb{R}^d} |x|^2\,|u|^2\,dx < \infty\,. \tag{3.15}$$

Moreover, I will be positive for any nonzero $u \in H^1(\mathbb{R}^d)$. Now consider any initial data $u^{\mathrm{in}} \in H^1(\mathbb{R}^d)$ which satisfies (3.15). When (3.7) is satisfied it can be shown that, so long as the corresponding H^1 solution $u(t)$ of the NLS equation exists, condition (3.15) remains satisfied. The quantities $\bar{x}$ and I of (3.14) can then be differentiated and written in terms of the conserved quantities M, P and H defined in (3.6) and (3.3), so that

$$\begin{aligned}
\frac{d\bar{x}}{dt} &= \frac{2P}{M}\,,\\
\frac{dI}{dt} &= \frac{\nu}{4i}\int_{\mathbb{R}^d}(x-\bar{x})\cdot\big(u^*\nabla u - u\nabla u^*\big)\,dx\,,\\
\frac{d^2I}{dt^2} &= \int_{\mathbb{R}^d}\nu^2|\nabla u|^2 + \frac{\sigma d}{2}\frac{\mu\nu}{\sigma+1}|u|^{2\sigma+2}\,dx - \frac{1}{2}\frac{d\bar{x}}{dt}\cdot P\\
&= H - \frac{|P|^2}{M} + \left(\frac{\sigma d}{2}-1\right)\frac{\mu\nu}{\sigma+1}\int_{\mathbb{R}^d}|u|^{2\sigma+2}\,dx\,.
\end{aligned} \tag{3.16}$$

If we assume that $\nu\mu < 0$ and $\sigma d \geq 2$, which are necessary conditions for a possible singularity to develop, then

$$\frac{d^2I}{dt^2} \leq H - \frac{|P|^2}{M}\,. \tag{3.17}$$

This inequality can be integrated to obtain the bound

$$I(t) \leq I(0) + \frac{dI}{dt}(0)\, t + \frac{1}{2}\left(H - \frac{|P|^2}{M}\right) t^2 \,. \tag{3.18}$$

It is clear that if u^{in} is chosen so that $H - |P|^2/M$ is negative then $I(t)$ would have to become zero at a finite time T_*, in contradiction with (3.14), which shows it to be positive. The H^1 solution must therefore develop a singularity no later than the time T_*, an upper bound that could be far larger than the actual blow-up time. By exploiting the local nature of the self-focusing mechanism, this argument has been extended to bounded domains for various kinds of boundary conditions (including periodic) in [**48**] by introducing a cut-off version of the $|x - \bar{x}|^2$ weight in (3.14).

The nature of the NLS singularity has been studied extensively both asymptotically and numerically [**53, 54, 71**], and is understood to behave self-similarly in the supercritical case and nearly so in the critical case. In this sense, our knowledge about the possible breakdown of classical solutions is much better for the NLS equation than for the three-dimensional Euler equations, where it is not even known whether or not singularities develop. Numerical simulations show that the self-focusing mechanism persists in the CGL equation inside the modulationally unstable regime $1 + \nu\mu < 0$ and when $\sigma d \geq 2$ is capable of provoking many spatially localized near NLS-singularities in solutions, starting from arbitrarily smooth initial data. Each near singularity produces a rather violent cascade process whereby energy concentrated in low wavenumber modes is rapidly transported to high wavenumber modes, after which it is damped by the CGL dissipation and then is dispersed in a somewhat less violent inverse cascade process whereby some undissipated energy returns from high wavenumber modes to low wavenumber modes [**39, 75**]. Admittedly the mechanisms by which energy transfer is generated for the CGL equation (self-focusing) is very different than that for the 3-dimensional Euler equations (vortex stretching), and we make no claims as to a direct physical analogy. Self-focusing blow-up in the NLS equation plays a more direct role in describing strong turbulence in plasma (the NLS equation is a limit of the Zakharov equations [**40, 81**]), but dissipative mechanisms in plasma do not directly correspond to the dissipative terms in the CGL equation. Rather, we regard the CGL equation as a convenient nonlinear dissipative partial differential equation displaying intrinsic inviscid instabilities that we may exploit to test and hone our mathematical tools.

Finally, there are several distinct advantages to considering the CGL equation rather than, say, the Navier-Stokes equations or the Zakharov equations. First, there is the obvious point that there are simply fewer dependent variables in the CGL equation, so we may in general expect the analysis to be simpler. Indeed, the basic techniques that have been developed for the study of semilinear parabolic equations [**80**], and the NLS equation over $\mathbb{R}^d$ [**10, 24, 79**] and, just recently, over $\mathbb{T}^d$ [**8, 9**], can be applied to the CGL equation. Second, the self-focusing mechanism and its resulting finite time blow-up for solutions of the NLS equation is quite well understood both physically and mathematically (also see [**54, 61, 62, 71**] and references therein). Third, from a numerical viewpoint it is much faster and more convenient to run simulations in low spatial dimensions, and the self-focusing blow-up behavior is already present in the 1-dimensional NLS equation for $\sigma \geq 2$.

4. Global Weak Solutions

For many nonlinear evolution equations of mathematical physics, classical solutions are either known to break down or not known to exist after a finite time. The best example of the former case is the formation of shocks in solutions of the compressible Euler equations of gas dynamics, while the best example of the later case is the open problem as to whether temporally global classical solutions exist for the incompressible Navier-Stokes equations of hydrodynamics. One approach around this problem is to enlarge the notion of a solution to that of a so-called *weak solution* by only requiring that the equation be satisfied in a "weak" sense, namely, after integrating it against a member of a prescribed class of test functions. In a seminal paper of 1934, Leray [**55**] proved the existence of a temporally global weak solution to the incompressible Navier-Stokes equations over the whole space $\mathbb{R}^3$ for any initial data with finite energy. Subsequently, the strategy used by Leray has been applied to prove the existence of global weak solutions for a number of other equations, most notably for the Boltzmann equation by DiPerna and Lions [**19**]. This section gives the analogue of this result for the CGL equation (1.1) over $\mathbb{T}^d$, the d-dimensional periodic box.

The technical setup and notation for this section is as follows. Throughout this section it is more convenient to use $\gamma = \sigma + 1$. We consider the evolution of a complex valued field $u = u(t,x)$ governed by the generalized CGL equation

$$\partial_t u = Ru + (1+i\nu)\Delta u - (1+i\mu)|u|^{2(\gamma-1)}u\,, \tag{4.1}$$

with initial condition

$$u(0,x) = u^{\mathrm{in}}(x)\,. \tag{4.2}$$

As before, we assume $R > 0$ and $\gamma > 1$ ($\sigma > 0$), while ν and μ can take any real value. This arbitrariness contrasts sharply with the theory of classical solutions presented in later sections where the admissible parameter values of γ, ν, and μ and the dimension d are interrelated.

Without loss of generality the units of length may be chosen so that

$$\int_{\mathbb{T}^d} dx = 1\,. \tag{4.3}$$

We employ the notation

$$\langle f\rangle = \int_{\mathbb{T}^d} f\,dx\,, \tag{4.4}$$

which by (4.4) denotes the mean value of f over the torus.

In order to state the main result of this section we must first introduce the principal spaces involved. For $1 \le p \le \infty$, the classical Lebesgue L^p-space over $(\mathbb{T}^d, dx)$ will be denoted $L^p(\mathbb{T}^d)$. The Sobolev space of functions in $L^2(\mathbb{T}^d)$ with partial derivatives in $L^2(\mathbb{T}^d)$ will be denoted $H^1(\mathbb{T}^d)$. Given any Banach space $\mathbb{X}$ with norm $\|\cdot\|_{\mathbb{X}}$ and $1 \le p \le \infty$, the space of (equivalence classes of) measurable functions $v = v(t)$ from $[0,\infty)$ into $\mathbb{X}$ such that $\|v\|_{\mathbb{X}} \in L^p([0,T])$ for every $T > 0$ will be denoted $L^p_{\mathrm{loc}}([0,\infty),\mathbb{X})$. And finally $C([0,\infty),\text{w-}L^2(\mathbb{T}^d))$ will denote the space of continuous functions from $[0,\infty)$ into w-$L^2(\mathbb{T}^d)$, which denotes $L^2(\mathbb{T}^d)$ equipped with its weak topology. This means that $v \in C([0,\infty),\text{w-}L^2(\mathbb{T}^d))$ if for every $\psi \in L^2(\mathbb{T}^d)$ the function $t \mapsto \langle\psi^* v(t)\rangle$ is in $C([0,\infty))$ endowed with the usual topology of uniform convergence over compact intervals. We remark that $L^p_{\mathrm{loc}}([0,\infty),\mathbb{X})$ and $C([0,\infty),\text{w-}L^2(\mathbb{T}^d))$ are Frechét spaces rather than Banach spaces. As such, their topologies are completely determined by the class of convergent sequences.

The main result of this section, established in [**25**], is the following global existence theorem for L^2 initial data; it is the CGL analogue of Leray's result for the Navier-Stokes equation.

THEOREM 4.1. *Given $u^{\mathrm{in}} \in L^2(\mathbb{T}^d)$, there exists a function*

$$u \in C([0,\infty),\text{w-}L^2(\mathbb{T}^d)) \cap L^2_{\mathrm{loc}}([0,\infty),H^1(\mathbb{T}^d)) \cap L^{2\gamma}_{\mathrm{loc}}([0,\infty),L^{2\gamma}(\mathbb{T}^d))\,, \tag{4.5}$$

that satisfies the initial condition (4.2) *and the weak form of the CGL equation* (4.1)*:*

$$\begin{aligned} 0 &= \langle \psi^* u(t_2)\rangle - \langle \psi^* u(t_1)\rangle - R\int_{t_1}^{t_2} \langle \psi^* u\rangle\, dt' \\ &\quad + \int_{t_1}^{t_2} \langle (1+i\nu)\nabla\psi^*\cdot\nabla u\rangle\, dt' + \int_{t_1}^{t_2} \langle (1+i\mu)\psi^*|u|^{2(\gamma-1)}u\rangle\, dt'\,, \end{aligned} \tag{4.6}$$

for every $[t_1,t_2]\subset[0,\infty)$ and every test function $\psi\in C^\infty(\mathbb{T}^d)$. Moreover, it satisfies the energy relation

$$\begin{aligned} \tfrac12\|u(t)\|_{L^2}^2 &+ \int_0^t \|\nabla u\|_{L^2}^2\, dt' + \int_0^t \|u\|_{L^{2\gamma}}^{2\gamma}\, dt' \\ &\le \tfrac12\|u^{\mathrm{in}}\|_{L^2}^2 + R\int_0^t \|u\|_{L^2}^2\, dt'\,, \end{aligned} \tag{4.7}$$

for every $t\in[0,\infty)$.

REMARK 4.1. The notion of weak solution given above is stronger in two ways than the notion of weak solution in the sense of distributions. Recall that u is a weak solution of the CGL equation in the sense of distributions when it satisfies the equation one obtains formally by integrating (4.1) over $[0,\infty)\times\mathbb{T}^d$ against a test function $w=w(t,x)\in C_{\mathrm{c}}^\infty([0,\infty)\times\mathbb{T}^d)$ and integrating by parts so as to put all the derivatives onto the test function. It can be seen directly that if u satisfies (4.6) then it satisfies the distribution form of the CGL equation for test functions in the factored form $w=\phi(t)\psi^*(x)$ where $\phi\in C_{\mathrm{c}}^\infty([0,\infty)$ and $\psi\in C^\infty(\mathbb{T}^d)$. Indeed, set $t_1=0$ in (4.6), multily by $\partial_t\phi(t_2)$ and integrate t_2 over $[0,\infty)$, and integrate by parts to put the spatial derivatives onto ψ^*. As linear combinations of factored test functions are dense in $C_{\mathrm{c}}^\infty([0,\infty)\times\mathbb{T}^d)$, it is clear that (4.6) alone implies that u is a weak solution of the CGL equation in the sense of distributions. In addition to (4.6), the weak solutions given in Theorem 4.1 are required to satisfy the energy relation (4.7).

REMARK 4.2. The strategy of the proof follows that introduced by Leray in the context of the Navier-Stokes equations, as well as to that of many other existence proofs for weak solutions of other equations. For this reason, the proof will be given in sufficient detail so as to clearly illustrate this strategy. Roughly, the idea is to construct a sequence of solutions to equations that approximate (4.6), then show the sequence is relatively compact in a topology that is strong enough to allow us pass from the approximate equation to the limit (4.6) for any converging subsequence. This involves striking a balance between the facts that compactness is easier to establish for weaker topologies, while convergence is easier to prove in stronger topologies. Uniqueness can never be asserted by such a compactness argument, but often requires the knowledge of additional regularity of the solution. Since we do not limit ourselves to any particular spatial dimension, or to any particular nonlinearity, the results of this section are equally applicable to the supercritical, critical, and subcritical CGL equations.

PROOF. The proof proceeds in four distinct steps.

STEP 1. *Construct a family of approximate solutions u_ϵ constructed by any method that yields a consistent weak formulation and an energy relation—for example, the Fourier-Galerkin method. Let P_ϵ denote the L^2-orthogonal projection onto the span of all Fourier modes of wave vectors ξ with $|\xi| \le 1/\epsilon$. Define $u_\epsilon^{\text{in}} = P_\epsilon u^{\text{in}}$ and let $u_\epsilon = u_\epsilon(t)$ be the unique solution of the ordinary differential initial-value problem*

$$\begin{aligned}\partial_t u_\epsilon &= R u_\epsilon + (1+i\nu)\Delta u_\epsilon - (1+i\mu)P_\epsilon\big(|u_\epsilon|^{2(\gamma-1)}u_\epsilon\big)\,,\\ u_\epsilon(0) &= u_\epsilon^{\text{in}} \in P_\epsilon L^2(\mathbb{T}^d) \subset C^\infty(\mathbb{T}^d)\,.\end{aligned} \tag{4.8}$$

The regularized initial data $u_\epsilon^{\text{in}}(x)$ converges to u^{in} strongly in $L^2(\mathbb{T}^d)$ as ϵ tends to zero and is chosen so that $\|u_\epsilon^{\text{in}}\|_{L^2} \le \|u^{\text{in}}\|_{L^2}$.

Furthermore, these solutions will satisfy the regularized version of the weak form (4.6) *given by*

$$\begin{aligned}0 = \langle\psi^* u_\epsilon(t_2)\rangle - \langle\psi^* u_\epsilon(t_1)\rangle - R\int_{t_1}^{t_2}\langle\psi^* u_\epsilon\rangle\,dt'\\ + \int_{t_1}^{t_2}\langle(1+i\nu)\nabla\psi^*\cdot\nabla u_\epsilon\rangle\,dt' + \int_{t_1}^{t_2}\langle(1+i\mu)\psi_\epsilon^*|u_\epsilon|^{2(\gamma-1)}u_\epsilon\rangle\,dt'\,,\end{aligned} \tag{4.9}$$

for every $[t_1,t_2] \subset [0,\infty)$ and $\psi \in C^\infty(\mathbb{T}^d)$. Here $\psi_\epsilon \equiv P_\epsilon\psi$ will converge to ψ in C^∞ as ϵ tends to zero. Moreover, these solutions will satisfy the regularized version the energy relation (4.7) *as the equality*

$$\begin{aligned}\tfrac12\|u_\epsilon(t)\|_{L^2}^2 + \int_0^t \|\nabla u_\epsilon\|_{L^2}^2\,dt' + \int_0^t \|u_\epsilon\|_{L^{2\gamma}}^{2\gamma}\,dt'\\ = \tfrac12\|u_\epsilon^{\text{in}}\|_{L^2}^2 + R\int_0^t \|u_\epsilon\|_{L^2}^2\,dt'\,.\end{aligned} \tag{4.10}$$

for every $t \in [0,\infty)$.

Step 1 follows from the standard Picard local existence theory for ordinary differential equations applied to (4.8) as posed in the finite dimensional space $P_\epsilon L^2(\mathbb{T}^d)$, and the fact that (4.10) provides a global L^2 bound on the solutions, ensuring that they are global. Hence, we omit all details of the proof and proceed under the premise of Step 1.

STEP 2. *Show that the sequence u_ϵ is a relatively compact set (has compact closure) in*

$$C([0,\infty),\text{w-}L^2(\mathbb{T}^d)) \wedge \text{w-}L^2_{\text{loc}}([0,\infty),H^1(\mathbb{T}^d)) \wedge \text{w-}L^{2\gamma}_{\text{loc}}([0,\infty),L^{2\gamma}(\mathbb{T}^d))\,. \tag{4.11}$$

REMARK 4.3. Here the notation "$\wedge$" indicates the intersection equipped with the weak topology induced by the inclusion maps; this means that a sequence in the intersection is convergent if and only if it converges in each space separately. The sense of convergence for these spaces individually is as follows. We have $v_n \to v$ in w-$L^2_{\text{loc}}([0,\infty), H^1(\mathbb{T}^d))$ when for every $T>0$ and every $\psi \in L^2_{\text{loc}}([0,\infty), H^1(\mathbb{T}^d))$ we have

$$\int_0^T \langle \psi^* v_n + \nabla\psi^* \cdot \nabla v_n \rangle \, dt \to \int_0^T \langle \psi^* v + \nabla \psi^* \cdot \nabla v \rangle \, dt \,. \tag{4.12}$$

Similarly, we have $v_n \to v$ in w-$L^{2\gamma}_{\text{loc}}([0,\infty), L^{2\gamma}(\mathbb{T}^d))$ when for every $T>0$ and every $\psi \in L^{(2\gamma)^*}_{\text{loc}}([0,\infty), L^{(2\gamma)^*}(\mathbb{T}^d))$ we have

$$\int_0^T \langle \psi^* v_n \rangle \, dt \to \int_0^T \langle \psi^* v \rangle \, dt \,, \tag{4.13}$$

where $(2\gamma)^* = 2\gamma/(2\gamma-1)$. Finally, we have $v_n \to v$ in $C([0,\infty), \text{w-}L^2(\mathbb{T}^d))$ when for every $\psi \in L^2(\mathbb{T}^d)$ we have

$$\langle \psi^* v_n(t) \rangle \to \langle \psi^* v(t) \rangle \,, \tag{4.14}$$

uniformly on compact subsets of $[0,\infty)$.

Proof of Step 2. Equation (4.10), along with $\|u_\epsilon^{\text{in}}\|_{L^2} \le \|u^{\text{in}}\|_{L^2}$, implies that

$$\tfrac{1}{2}\|u_\epsilon(t)\|_{L^2}^2 \le \tfrac{1}{2}\|u^{\text{in}}\|_{L^2}^2 + R\int_0^t \|u_\epsilon\|_{L^2}^2 \, dt' \,, \tag{4.15}$$

from which the Gronwall Lemma then gives

$$\|u_\epsilon(t)\|_{L^2}^2 \le \|u^{\text{in}}\|_{L^2}^2 \, e^{2Rt} \,. \tag{4.16}$$

Inserting this into the right side of the regularized energy relation (4.10) yields the explicit bound, uniform in ϵ:

$$\tfrac{1}{2}\|u_\epsilon(t)\|_{L^2}^2 + \int_0^t \|\nabla u_\epsilon\|_{L^2}^2 \, dt' + \int_0^t \|u_\epsilon\|_{L^{2\gamma}}^{2\gamma} \, dt' \le \tfrac{1}{2}\|u^{\text{in}}\|_{L^2}^2 \, e^{2Rt} \,. \tag{4.17}$$

This bound establishes that the sequence $\{u_\epsilon\}$ is contained in compact sets of both w-$L^2_{\text{loc}}([0,\infty), H^1(\mathbb{T}^d))$ and w-$L^{2\gamma}_{\text{loc}}([0,\infty), L^{2\gamma}(\mathbb{T}^d))$, because norm bounded sets are relatively compact in weak-$*$ topologies, which are the same as the weak topologies on these reflexive spaces. For the same reason, the uniform bound (4.16) also shows that $\{u_\epsilon(t)\}$ is a relatively compact set in w-$L^2(\mathbb{T}^d)$ for every $t \ge 0$.

In order to complete Step 2 it must be shown that $\{u_\epsilon\}$ is a relatively compact set in $C([0,\infty), \text{w-}L^2(\mathbb{T}^d))$. Compactness requires more than just boundedness here because of the strong topology over t. We appeal to the Arzela-Ascoli theorem [**7, 72**] which asserts that $\{u_\epsilon\}$ is a relatively compact set in $C([0,\infty), \text{w-}L^2(\mathbb{T}^d))$ if and only if

(i) $\{u_\epsilon(t)\}$ is a relatively compact set in w-$L^2(\mathbb{T}^d)$ for every $t \ge 0$;
(ii) $\{u_\epsilon\}$ is equicontinuous in $C([0,\infty), \text{w-}L^2(\mathbb{T}^d))$.

As was noted at the end of the last paragraph, condition (i) is satisfied. In order to establish (ii), we must show for every $\psi \in L^2(\mathbb{T}^d)$

(ii′) $\{\langle \psi^* u_\epsilon \rangle\}$ is equicontinuous in $C([0,\infty))$.

This is done by first using the regularized weak form (4.9) of the CGL equation to establish (ii′) for ψ in C^∞ and then using a density argument to extend (ii′) to the general case of ψ in $L^2(\mathbb{T}^d)$ (see [**25**]). ☐

Step 3. *Show the sequence $\{u_\epsilon\}$ is relatively compact in $L^2_{\text{loc}}([0,\infty), L^2(\mathbb{T}^d))$ and $L^{2\gamma-1}_{\text{loc}}([0,\infty), L^{2\gamma-1}(\mathbb{T}^d))$ considered with their usual strong topologies.*

REMARK 4.4. This step is necessary because Step 2 allows us to assert the existence of a weakly convergent subsequence of $\{u_\epsilon\}$, say with a limit u, from which we may conclude only that

$$\int_0^T \|u\|_{L^2}^2\, dt \le \liminf_{\epsilon\to 0} \int_0^T \|u_\epsilon\|_{L^2}^2\, dt\,. \tag{4.18}$$

However, the argument that recovers (4.7) from (4.10) will need

$$\int_0^T \|u\|_{L^2}^2\, dt = \lim_{\epsilon\to 0} \int_0^T \|u_\epsilon\|_{L^2}^2\, dt\,, \tag{4.19}$$

which requires strong L^2 convergence. Similarly, the argument that recovers the nonlinear term in (4.6) from that in (4.9) will require strong $L^{2\gamma-1}$ convergence.

PROOF OF STEP 3. The crucial point is to use the results of Step 2 along with the following embedding lemma, which is essentially due to Leray.

LEMMA 4.2. *The injection*

$$C([0,\infty), \text{w-}L^2(\mathbb{T}^d)) \wedge \text{w-}L^2_{\text{loc}}([0,\infty), H^1(\mathbb{T}^d)) \hookrightarrow L^2_{\text{loc}}([0,\infty), L^2(\mathbb{T}^d)) \tag{4.20}$$

is continuous.

This lemma is proved by demonstrating that sequences that converge to zero in both $C([0,\infty), \text{w-}L^2(\mathbb{T}^d))$ and $\text{w-}L^2_{\text{loc}}([0,\infty), H^1(\mathbb{T}^d))$ will also converge in the space $L^2_{\text{loc}}([0,\infty), L^2(\mathbb{T}^d))$. The proof is fairly direct, so we refer to [**25**] for details. The key tool used is the Rellich Theorem, which states that $H^1 \hookrightarrow L^2$ compactly.

Given the lemma, the L^2 portion of the assertion of Step 3 is argued as follows. Step 2 states that $\{u_\epsilon\}$ is a relatively compact set in both $C([0,\infty), \text{w-}L^2(\mathbb{T}^d))$ and $\text{w-}L^2_{\text{loc}}([0,\infty), H^1(\mathbb{T}^d))$, and because the continuous image of a compact set is compact, it follows that $\{u_\epsilon\}$ is a relatively compact set in $L^2_{\text{loc}}([0,\infty), L^2(\mathbb{T}^d))$. Therefore, any subsequence of $\{u_\epsilon\}$ that converges in both $C([0,\infty), \text{w-}L^2(\mathbb{T}^d))$ and $\text{w-}L^2_{\text{loc}}([0,\infty), H^1(\mathbb{T}^d))$ will be strongly convergent in $L^2_{\text{loc}}([0,\infty), L^2(\mathbb{T}^d))$.

The $L^{2\gamma-1}$ portion of the assertion of Step 3 is a direct consequence of the L^2 portion. We consider two cases. First, if $\gamma \le 3/2$ (so that $2\gamma - 1 \le 2$) then the injection

$$L^2_{\text{loc}}([0,\infty), L^2(\mathbb{T}^d)) \hookrightarrow L^{2\gamma-1}_{\text{loc}}([0,\infty), L^{2\gamma-1}(\mathbb{T}^d))\,, \tag{4.21}$$

is continuous, so that strong convergence in $L^2_{\text{loc}}([0,\infty), L^2(\mathbb{T}^d))$ also gives strong convergence in $L^{2\gamma-1}_{\text{loc}}([0,\infty), L^{2\gamma-1}(\mathbb{T}^d))$, as was asserted. Second, if $\gamma > 3/2$ (so that $2\gamma - 1 > 2$) then the strong convergence in $L^2_{\text{loc}}([0,\infty), L^2(\mathbb{T}^d))$ and the weak convergence in $L^{2\gamma}_{\text{loc}}([0,\infty), L^{2\gamma}(\mathbb{T}^d))$ combine in a standard interpolation argument to yield the result [**25**]. $\square$

STEP 4. *Go to the limit. That is, the weak solution u in Theorem 4.1 is identified as the limit of a convergent subsequence of $\{u_\epsilon\}$. The fact that this subsequence converges in the various function spaces is used to verify the weak form* (4.6) *and energy relation* (4.7).

PROOF OF STEP 4. Step 2 ensures that there is a subsequence of $\{u_\epsilon\}$, which we also refer to as $\{u_\epsilon\}$, that simultaneously converges to a limit u in the spaces

$C([0,\infty),\text{w-}L^2(\mathbb{T}^d))$, $\text{w-}L^2_{\text{loc}}([0,\infty),H^1(\mathbb{T}^d))$ and $\text{w-}L^{2\gamma}_{\text{loc}}([0,\infty),L^{2\gamma}(\mathbb{T}^d))$. Thus, as was asserted in (4.5),

$$u \in C([0,\infty),\text{w-}L^2(\mathbb{T}^d)) \cap L^2_{\text{loc}}([0,\infty),H^1(\mathbb{T}^d)) \cap L^{2\gamma}_{\text{loc}}([0,\infty),L^{2\gamma}(\mathbb{T}^d))\,. \tag{4.22}$$

Step 3 then implies the strong convergence of u_ϵ to u in both $L^2_{\text{loc}}([0,\infty),L^2(\mathbb{T}^d))$ and $L^{2\gamma-1}_{\text{loc}}([0,\infty),L^{2\gamma-1}(\mathbb{T}^d))$. All that remains is to show that the limit u satisfies the weak form of the CGL equation (4.6) as well as the energy relation (4.7). Toward this end we check convergence of each term in the respective regularized versions, (4.9) and (4.10).

First consider the regularized weak form of the CGL equation (4.9) for an arbitrary test function $\psi \in C^\infty(\mathbb{T}^d)$ and interval $[t_1,t_2] \subset [0,\infty)$:

$$\begin{aligned} 0 = \langle \psi^* u_\epsilon(t_2)\rangle - \langle \psi^* u_\epsilon(t_1)\rangle - R\int_{t_1}^{t_2} \langle \psi^* u_\epsilon\rangle\,dt' \\ + \int_{t_1}^{t_2} \langle (1+i\nu)\nabla\psi^* \cdot \nabla u_\epsilon\rangle\,dt' + \int_{t_1}^{t_2} \langle (1+i\mu)\psi_\epsilon^* |u_\epsilon|^{2(\gamma-1)} u_\epsilon\rangle\,dt'\,. \end{aligned} \tag{4.23}$$

The convergence of u_ϵ to u in $C([0,\infty),\text{w-}L^2(\mathbb{T}^d))$ means (see (4.14)) that $\langle \psi^* u_\epsilon(t)\rangle$ converges to $\langle \psi^* u(t)\rangle$ uniformly over $[t_1,t_2]$, whereby

$$\langle \psi^* u_\epsilon(t_1)\rangle \to \langle \psi^* u(t_1)\rangle\,, \qquad \langle \psi^* u_\epsilon(t_2)\rangle \to \langle \psi^* u(t_2)\rangle\,, \tag{4.24}$$

$$\int_{t_1}^{t_2} \langle \psi^* u_\epsilon\rangle\,dt' \to \int_{t_1}^{t_2} \langle \psi^* u\rangle\,dt'\,. \tag{4.25}$$

When combined with (4.25), the convergence of u_ϵ in $\text{w-}L^2_{\text{loc}}([0,\infty),H^1(\mathbb{T}^d))$ (see (4.12)) then yields

$$\int_{t_1}^{t_2} \langle (1+i\nu)\nabla\psi^* \cdot \nabla u_\epsilon\rangle\,dt' \to \int_{t_1}^{t_2} \langle (1+i\nu)\nabla\psi^* \cdot \nabla u\rangle\,dt'\,. \tag{4.26}$$

In order to pass to the limit in the nonlinear term of (4.23) we shall exploit the generally useful fact that if $\{v_n\}$ is a weakly convergent sequence in a Banach space $\mathbb{X}$ and $\{f_n\}$ is a strongly convergent sequence in the dual space $\mathbb{X}^*$ then the sequence $\{f_n(v_n)\}$ is convergent in $\mathbb{C}$. Here, the strong convergence of u_ϵ in $L^{2\gamma-1}_{\text{loc}}([0,\infty),L^{2\gamma-1}(\mathbb{T}^d))$ implies the weak convergence of $|u_\epsilon|^{2(\gamma-1)}u_\epsilon$ in $L^1_{\text{loc}}([0,\infty),L^1(\mathbb{T}^d))$ and, hence, in the Banach space $L^1([t_1,t_2]\times\mathbb{T}^d))$. The fact that ψ_ϵ converges to ψ uniformly over $\mathbb{T}^d$ implies that it converges strongly in $L^\infty([t_1,t_2]\times\mathbb{T}^d))$, which is the dual of $L^1([t_1,t_2]\times\mathbb{T}^d))$. It follows that

$$\int_{t_1}^{t_2} \langle (1+i\mu)\psi_\epsilon^* |u_\epsilon|^{2(\gamma-1)} u_\epsilon\rangle\,dt' \to \int_{t_1}^{t_2} \langle (1+i\mu)\psi^* |u|^{2(\gamma-1)} u\rangle\,dt'\,. \tag{4.27}$$

By combining (4.25)–(4.27), one can pass to the limit in each term of (4.23) and conclude that u satisfies the weak form of the CGL equation (4.6).

To recover the energy relation (4.7) consider its regularized version (4.10):

$$\begin{aligned} \tfrac{1}{2}\|u_\epsilon(t)\|_{L^2}^2 + \int_0^t \|\nabla u_\epsilon\|_{L^2}^2\,dt' + \int_0^t \|u_\epsilon\|_{L^{2\gamma}}^{2\gamma}\,dt' \\ = \tfrac{1}{2}\|u_\epsilon^{\text{in}}\|_{L^2}^2 + R\int_0^t \|u_\epsilon\|_{L^2}^2\,dt'\,. \end{aligned} \tag{4.28}$$

First examining the right side, the strong convergence of the initial data in $L^2(\mathbb{T}^d)$ implies

$$\|u_\epsilon^{\mathrm{in}}\|_{L^2}^2 \to \|u^{\mathrm{in}}\|_{L^2}^2\,, \tag{4.29}$$

while the strong convergence of u_ϵ in $L^2_{\mathrm{loc}}([0,\infty), L^2(\mathbb{T}^d))$ implies

$$\int_0^t \|u_\epsilon\|_{L^2}^2\,dt' \to \int_0^t \|u\|_{L^2}^2\,dt'\,. \tag{4.30}$$

The right side of (4.28) is therefore convergent as ϵ tends to zero. Turning to the left side of (4.28), the convergence of u_ϵ in $C([0,\infty), \text{w-}L^2(\mathbb{T}^d))$, together with the general fact that the norm of the weak limit of a sequence is a lower bound for the limit inferior of the norms, yields

$$\|u(t)\|_{L^2}^2 \le \liminf_{\epsilon\to 0} \|u_\epsilon(t)\|_{L^2}^2\,. \tag{4.31}$$

Similarly, the convergence in $\text{w-}L^2_{\mathrm{loc}}([0,\infty), H^1(\mathbb{T}^d))$ and $\text{w-}L^{2\gamma}_{\mathrm{loc}}([0,\infty), L^{2\gamma}(\mathbb{T}^d))$ implies

$$\int_0^t \|u\|_{H^1}^2\,dt' \le \liminf_{\epsilon\to 0} \int_0^t \|u_\epsilon\|_{H^1}^2\,dt'\,, \tag{4.32}$$

$$\int_0^t \|u\|_{L^{2\gamma}}^{2\gamma}\,dt' \le \liminf_{\epsilon\to 0} \int_0^t \|u_\epsilon\|_{L^{2\gamma}}^{2\gamma}\,dt'\,. \tag{4.33}$$

Taken together, (4.30) and (4.32) give

$$\int_0^t \|\nabla u\|_{L^2}^2\,dt' \le \liminf_{\epsilon\to 0} \int_0^t \|\nabla u_\epsilon\|_{L^2}^2\,dt'\,, \tag{4.34}$$

By combining (4.31), (4.33) and (4.34), one obtains a lower bound on the on the limit inferior of (4.28). The energy relation (4.7) is thereby satisfied and Theorem 4.1 is proved. □

REMARK 4.5. The energy inequality (4.7) in Theorem 4.1 can be strengthened slightly. Indeed, it is readily seen that the approximate solutions defined by (4.8) of Step 1 will satisfy the energy relation

$$\begin{aligned} \tfrac{1}{2}\|u_\epsilon(t_2)\|_{L^2}^2 + \int_{t_1}^{t_2} \|\nabla u_\epsilon\|_{L^2}^2\,dt + \int_{t_1}^{t_2} \|u_\epsilon\|_{L^{2\gamma}}^{2\gamma}\,dt& \\ = \tfrac{1}{2}\|u_\epsilon(t_1)\|_{L^2}^2 + R\int_{t_1}^{t_2} \|u_\epsilon\|_{L^2}^2\,dt&\,, \end{aligned} \tag{4.35}$$

for every $[t_1,t_2] \subset [0,\infty)$. The convergence of u_ϵ to u in $L^2_{\mathrm{loc}}([0,\infty), L^2(\mathbb{T}^d))$ implies $\|u_\epsilon\|_{L^2}$ converges to $\|u\|_{L^2}$ in $L^2_{\mathrm{loc}}([0,\infty))$, whereby, upon also employing the Cantor diagonal method, one can pass to a subsequence of u_ϵ such that

$$\|u_\epsilon(t)\|_{L^2} \to \|u(t)\|_{L^2} \qquad \text{for almost every } t \in [0,\infty)\,. \tag{4.36}$$

Call the set of points for which the limit holds E. When t_1 is restricted to lie in E one can argue starting from (4.35) as in (4.28–4.34) to show that u satisfies

$$\begin{aligned} \tfrac{1}{2}\|u(t)\|_{L^2}^2 + \int_{t_1}^{t_2} \|\nabla u\|_{L^2}^2\,dt + \int_{t_1}^{t_2} \|u\|_{L^{2\gamma}}^{2\gamma}\,dt& \\ \le \tfrac{1}{2}\|u(t_1)\|_{L^2}^2 + R\int_{t_1}^{t_2} \|u\|_{L^2}^2\,dt&\,, \end{aligned} \tag{4.37}$$

for every $[t_1, t_2] \subset [0, \infty)$ with $t_1 \in E$.

REMARK 4.6. Replacing $\mathbb{T}^d$ with $\mathbb{R}^d$ and $C^\infty(\mathbb{T}^d)$ with the compactly supported test functions $C_c^\infty(\mathbb{R}^d)$ everywhere in the statement of Theorem 4.1 gives the analogous $\mathbb{R}^d$ result. The strategy for its proof is embodied in the same four steps used above for the periodic case. Indeed, Step 2 and Step 4 have only superficial differences from those above. The Fourier-Galerkin method used in Step 1 to construct approximate classical solutions breaks down over the whole space, but can be replaced by so-called smoothing approximations. Specifically, a contraction mapping argument such as in [**36**], can be used to construct smooth solutions of the initial-value problem

$$
\begin{aligned}
\partial_t u_\epsilon &= R u_\epsilon + (1 + i\nu)\Delta u_\epsilon - (1 + i\mu) j_\epsilon * \left(|j_\epsilon * u_\epsilon|^{2(\gamma-1)} j_\epsilon * u_\epsilon \right), \\
u_\epsilon(0, x) &= u_\epsilon^{\mathrm{in}}(x) = j_\epsilon * u^{\mathrm{in}}(x),
\end{aligned}
\tag{4.38}
$$

where the regularization is achieved through convolution with a smooth, nonnegative mollifier j_ϵ. Finally, details of Step 3 must be modified to reflect differences in the Rellich Theorem for unbounded domains. These technical changes are straightforward, and so are omitted.

REMARK 4.7. One can show the existence of global weak solutions for the defocusing ($\mu\nu > 0$) NLS equation (1.2) with any $\sigma > 0$ for finite-energy initial data using similar arguments, but using the conservation laws of mass and energy in place of the dissipation relation (4.7) to obtain the weak compactness of Step 2.

5. Local Classical Solutions

The solutions found in the last section are too weak to obtain proofs of their uniqueness or regularity. In this section we establish the local existence, uniqueness, and regularity results for classical solutions of the generalized complex Ginzburg-Landau equation (1.1). Our objective here is to provide the basic results necessary to justify the formal manipulations necessary to establish the existence of global solutions in Section 7. More general results might be derived by adapting the important new techniques developed by Bourgain [**8, 9**] for the NLS equation over $\mathbb{T}^d$, however our needs here are less demanding. Indeed, our proofs mostly use standard techniques [**32, 43, 70, 80**] and some detail will be omitted.

Local existence and uniqueness of classical solutions of the generalized complex Ginzburg-Landau equation will be established by using the contraction mapping theorem. In the first part of this section we will describe—first as an abstract result, then applied to the CGL equation—the setting necessary to obtain a so-called mild solution from continuous initial data, and elevate it to a classical solution by a bootstrapping procedure. In the second part of this section the contraction mapping argument is reexamined in order to extend the class of initial data that evolve into local classical solutions.

Consider the evolution of $u = u(t)$ in a Banach space $\mathbb{X}$ to be governed by the abstract initial-value problem

$$
\partial_t u = Lu + N(u), \qquad u(0) = u^{\mathrm{in}} \in \mathbb{X}.
\tag{5.1}
$$

Here the linear operator L is assumed to be the infinitesimal generator of a strongly continuous semigroup $S(t)$ over $\mathbb{X}$ (so that the linear problem with $N = 0$ is well-posed, see [**70**]). The perturbation N is usually a nonlinear map over $\mathbb{X}$.

The well-posedness of the above initial-value problem can be established by a contraction mapping argument after formally recasting (5.1) in its so-called mild formulation

$$u(t) = S(t)u^{\mathrm{in}} + \int_0^t S(t-t')N(u(t'))\,dt' \,. \tag{5.2}$$

In order to employ the contraction mapping theorem we shall assume that the perturbation N is locally Lipschitz as a map from $\mathbb{X}$ into itself. More specifically, this means

(i) $\|N(u)\|_{\mathbb{X}} \le C_{\mathrm{bd}}(\|u\|_{\mathbb{X}})$ for every $u \in \mathbb{X}$,
(ii) $\|N(u_1) - N(u_2)\|_{\mathbb{X}} \le C_{\mathrm{Lip}}(\|u_1\|_{\mathbb{X}}, \|u_2\|_{\mathbb{X}})\,\|u_1 - u_2\|_{\mathbb{X}}$ for every $u_1, u_2 \in \mathbb{X}$,

where $C_{\mathrm{bd}}(\cdot)$ and $C_{\mathrm{Lip}}(\cdot,\cdot)$ are nondecreasing functions of their arguments. We remark that N and L can always be chosen to satisfy $N(0) = 0$ in which case condition (ii) implies (i). Given such a perturbation N, one can prove the following basic result [**70**, page 184].

THEOREM 5.1 (Basic Local Existence). *For every $\rho > 0$ there exists a time $T(\rho) > 0$ such that for every initial data $u^{\mathrm{in}} \in \mathbb{X}$ with $\|u^{\mathrm{in}}\|_{\mathbb{X}} \le \rho$ there exists a unique $u \in C([0,T],\mathbb{X})$ satisfying the mild formulation* (5.2). *In addition, the map from u^{in} to u is a locally Lipschitz function from $\mathbb{X}$ to $C([0,T],\mathbb{X})$.*

Such a u is called a *mild solution* for the initial-value problem (5.1); it provides the starting point for an analysis demonstrating that it is in fact a classical solution.

REMARK 5.1. If $u \in C([0,T],\mathbb{X})$ is a mild solution of (5.1) then a direct calculation shows that it is also a weak solution of (5.1) in the sense that

$$\begin{aligned} &\langle \psi | u(t_2)\rangle_{\mathbb{X}} - \langle \psi | u(t_1)\rangle_{\mathbb{X}} \\ &\qquad = \int_{t_1}^{t_2} \langle L^*\psi | u(t)\rangle_{\mathbb{X}}\,dt + \int_{t_1}^{t_2} \langle \psi | N(u(t))\rangle_{\mathbb{X}}\,dt\,, \end{aligned} \tag{5.3}$$

for every $0 \le t_1 \le t_2 \le T$ and $\psi \in \mathcal{D}(L^*)$. Here $\langle\,\cdot\,|\,\cdot\,\rangle_{\mathbb{X}}$ is the usual bilinear duality between $\mathbb{X}$ and its dual space $\mathbb{X}^*$, while L^* is the usual dual adjoint of L with domain $\mathcal{D}(L^*)$ dense in $\mathbb{X}^*$.

We now apply this general theory to the initial-value problem for the spatially periodic generalized complex Ginzburg-Landau equation. In that case we choose

$$Lu = (1+i\nu)\Delta u + Ru\,, \qquad N(u) = -(1+i\mu)|u|^{2\sigma}u\,. \tag{5.4}$$

The associated semigroup $S(t)$ acting on $u \in \mathbb{X}$ can be written as a convolution, $S(t)u = G_t * u$, with its Green function $G_t = G_t(x)$ for $t > 0$ given by

$$\begin{aligned} G_t(x) &= \sum_{n\in\mathbb{Z}^d} g_t(x+n)\,, \\ g_t(x) &= \frac{1}{(4\pi(1+i\nu)t)^{d/2}} \exp\left(-\frac{|x|^2}{4(1+i\nu)t} + Rt\right). \end{aligned} \tag{5.5}$$

The integral equation (5.2) recast in terms of this Green function takes the form

$$u(t) = G_t * u^{\mathrm{in}} + \int_0^t G_{t-t'} * N(u(t'))\,dt' \,. \tag{5.6}$$

In order to apply the local existence theorem one only need identify the space $\mathbb{X}$ and verify the conditions (i) and (ii).

For $t > 0$ the Green function (5.5) satisfies the L^1-estimate

$$\|G_t\|_{L^1} \leq \sum_{n\in\mathbb{Z}^d} \int_{\mathbb{T}^d} |g_t(x+n)|\,dx = \int_{\mathbb{R}^d} |g_t(x)|\,dx = (1+\nu^2)^{d/4}\,e^{Rt}\,, \tag{5.7}$$

from which it follows that $S(t)$ is bounded over $L^p(\mathbb{T}^d)$ for every $1 \leq p \leq \infty$ with

$$\|S(t)u\|_{L^p} = \|G_t * u\|_{L^p} \leq \|G_t\|_{L^1}\,\|u\|_{L^p} \leq (1+\nu^2)^{d/4}\,e^{Rt}\,\|u\|_{L^p}\,. \tag{5.8}$$

Moreover, it can be shown [**32**] that $S(t)$ is a strongly continuous semigroup over $C(\mathbb{T}^d)$ and over $L^p(\mathbb{T}^d)$ for every $1 \leq p < \infty$.

REMARK 5.2. The bounded operators $S(t)$ defined by $S(0) = I$ and $S(t)u = G_t * u$ for $t > 0$ form a semigroup over $L^\infty(\mathbb{T}^d)$ that is strongly continuous for every $t > 0$ but only weak-$*$ continuous at $t = 0$. (A sequence $\{v_n\}$ converges to v in the weak-$*$ topology on $L^\infty(\mathbb{T}^d)$ if it does so integrated against arbitrary functions in $L^1(\mathbb{T}^d)$.)

We first appeal to the local existence theorem with $\mathbb{X} = C(\mathbb{T}^d)$. The perturbation N given by (5.4) is clearly locally Lipschitz continuous as a map from $C(\mathbb{T}^d)$ into itself. A direct application of the local existence theorem then yields a unique mild solution $u = u(t)$ of the CGL equation over a time interval $[0,T]$ that depends only on the L^∞ norm of $u^{\rm in}$. This solution is the limit of a sequence $\{u^{(n)}\}$ of successive iterates of (5.6), say that defined by

$$\begin{aligned} u^{(0)}(t) &= G_t * u^{\rm in}\,,\\ u^{(n+1)}(t) &= G_t * u^{\rm in} + \int_0^t G_{t-t'} * N\big(u^{(n)}(t')\big)\,dt'\,, \end{aligned} \tag{5.9}$$

that converge in $C([0,T],C(\mathbb{T}^d))$ for some T chosen sufficiently small that the sequence contracts.

Additional regularity must be demonstrated in order to elevate these mild solutions to classical solutions, specifically they ought to be C^2 in space and C^1 in time, so that the derivatives appearing in the equation are classical. This is done using the following standard bootstrapping argument. We use the regularity of G_t for $t > 0$ to estimate the gradient of the successive iterates (5.9) in the L^∞ norm to show that the sequence $\{u^{(n)}\}$ converges in $C((0,T],C^1(\mathbb{T}^d))$. This is accomplished as follows. Take the gradient of the integral equation (5.9),

$$\nabla u^{(n+1)}(t) = \nabla G_t * u^{\rm in} + \int_0^t \nabla G_{t-t'} * N(u^{(n)}(t'))\,dt'\,, \tag{5.10}$$

and estimate

$$\|\nabla u^{(n+1)}(t)\|_{L^\infty} \leq \|\nabla G_t\|_{L^1}\,\|u^{\rm in}\|_{L^\infty} + \int_0^t \|\nabla G_{t-t'}\|_{L^1}\,\|N(u^{(n)}(t'))\|_{L^\infty}\,dt'\,. \tag{5.11}$$

Then, use the L^1-estimate

$$\begin{aligned} \|\nabla G_t\|_{L^1} &\leq \sum_{n\in\mathbb{Z}^d} \int_{\mathbb{T}^d} |\nabla g_t(x+n)|\, dx \\ &= \int_{\mathbb{R}^d} |\nabla g_t(x)|\, dx = C_d\,(1+\nu^2)^{d/4}\, \frac{e^{Rt}}{\sqrt{t}} \end{aligned} \tag{5.12}$$

where $C_d > 0$ is a constant depending only on the dimension d, to find that

$$\begin{aligned} \|\nabla u^{(n+1)}(t)\|_{L^\infty} &\leq C_d\,(1+\nu^2)^{d/4}\, \frac{e^{Rt}}{\sqrt{t}}\, \|u^{\text{in}}\|_{L^\infty} \\ &\quad + C_d\,(1+\nu^2)^{d/4} \int_0^t \frac{e^{R(t-t')}}{\sqrt{t-t'}}\, \|N(u^{(n)}(t'))\|_{L^\infty}\, dt'\,. \end{aligned} \tag{5.13}$$

This shows that each $\nabla u^{(n)}$ lies in $C((0,T],C(\mathbb{T}^d))$. By invoking the Lipschitz continuity of N, a similar estimate shows that the sequence is Cauchy in $C((0,T],C(\mathbb{T}^d))$, so that $u \in C((0,T],C^1(\mathbb{T}^d))$. If $u^{\text{in}} \in C^1(\mathbb{T}^d)$, it is possible to rewrite the integral equation for ∇u without putting any derivatives on G_t, namely

$$\nabla u(t) = G_t * \nabla u^{\text{in}} + \int_0^t G_{t-t'} * \big(DN(u(t'))\nabla u(t')\big)\, dt'\,, \tag{5.14}$$

where $DN(u)$ represents the derivative of $N(u)$ with respect to u, which, when $N(u)$ is given by (5.4), acts on an arbitrary function w by

$$DN(u)w = -(1+i\mu)\big((\sigma+1)|u|^{2\sigma}w + \sigma|u|^{2\sigma-2}u^2w^*\big)\,. \tag{5.15}$$

Consequently, estimates analogous to (5.13) will not exhibit a singularity at $t = 0$ and hence yield a $u \in C([0,T],C^1(\mathbb{T}^d))$. A repetition of the above regularity argument starting from (5.14) rather than (5.6) then shows that u is in $C((0,T],C^2(\mathbb{T}^d))$. Moreover, because the CGL equation relates the first time derivative to the second space derivative, the solution must also be in $C^1((0,T],C(\mathbb{T}^d))$ and is therefore a classical solution of the CGL equation so long as it is a mild solution. Summarizing, we have proved the following result.

THEOREM 5.2 (Local Classical Solutions for C^0 Initial Data). *For every $\rho > 0$ there exists a time $T(\rho) > 0$ such that for initial data $u^{\text{in}} \in C(\mathbb{T}^d)$ with $\|u^{\text{in}}\|_{L^\infty} \leq \rho$ there exists a unique*

$$u \in C([0,T],C(\mathbb{T}^d)) \cap C((0,T],C^2(\mathbb{T}^d)) \cap C^1((0,T],C(\mathbb{T}^d))\,, \tag{5.16}$$

satisfying the CGL initial-value problem. In addition, the map from u^{in} to u is a locally Lipschitz function from $C(\mathbb{T}^d)$ to $C([0,T],C(\mathbb{T}^d))$. Moreover, for initial data $u^{\text{in}} \in C^2(\mathbb{T}^d)$ one has $u \in C([0,T],C^2(\mathbb{T}^d)) \cap C^1([0,T],C(\mathbb{T}^d))$.

In this generality one can not expect these solutions to be in $C((0,T],C^3(\mathbb{T}^d))$ because unbounded singularities will be introduced at zeros of u upon further differentiation of (5.15). However, additional regularity can be gained in some cases. For example, when σ is a positive integer the nonlinearity is a polynomial in u and u^* and one can freely differentiate (5.15) without introducing any singularities. Whence, continuing with the bootstrapping argument, it can be shown that for every t in $(0,T]$ the solution has at least one more spatial derivative than it had initially. But then repeating this argument implies that the solution is in $C((0,T],C^\infty(\mathbb{T}^d))$. Moreover, because the equation relates temporal derivatives to spatial derivatives, the solution must possess all temporal derivatives too and is therefore a smooth (C^∞) solution of the CGL equation (1.1) so long as it is a mild solution. More precisely, we have proved the following result.

THEOREM 5.3 (Local Smooth Solutions). *Let $\sigma > 0$ be an integer. Then for every $\rho > 0$ there exists a time $T(\rho) > 0$ such that for every initial data $u^{\text{in}} \in C(\mathbb{T}^d)$ with $\|u^{\text{in}}\|_{L^\infty} \leq \rho$ there exists a unique*

$$u \in C([0,T], C(\mathbb{T}^d)) \cap C^\infty((0,T] \times \mathbb{T}^d)\,, \tag{5.17}$$

satisfying the CGL initial-value problem. Moreover, given smooth initial data $u^{\text{in}} \in C^\infty(\mathbb{T}^d)$ then $u \in C^\infty([0,T] \times \mathbb{T}^d)$.

REMARK 5.3. The above result justifies the formal manipulations we will carry out in Section 8, thereby enabling us to focus our attention on the establishment of global uniform Sobolev bounds. Moreover, in Section 6 we will extend Theorem 5.3 to show that when σ is a positive integer these solutions are real analytic.

Even when σ is not a positive integer, we can still advance the bootstrapping argument so long as the differentiation of (5.15) does not introduce unbounded singularities at the zeros of u. The lowest degree of homogeneity for the factors u and u^* appearing in a term of the $(n+1)^{\text{st}}$ derivative of $|u|^{2\sigma}u$ will be $2\sigma - n$, and this can be controlled whenever $\sigma \geq n/2$. In that case the bootstrapping argument will gain an additional n spatial derivatives, showing that the solution is in $C((0,T], C^{n+2}(\mathbb{T}^d))$. This observation then leads to the following.

THEOREM 5.4 (Local C^k Solutions). *Let $\sigma \geq n/2$ for some positive integer n. Then for every $\rho > 0$ there exists a time $T(\rho) > 0$ such that for every initial data $u^{\text{in}} \in C(\mathbb{T}^d)$ with $\|u^{\text{in}}\|_{L^\infty} \leq \rho$ there exists a unique*

$$u \in C([0,T], C(\mathbb{T}^d)) \cap C((0,T], C^{n+2}(\mathbb{T}^d)) \cap C^1((0,T], C^n(\mathbb{T}^d))\,, \tag{5.18}$$

satisfying the CGL initial-value problem. Moreover, for $u^{\text{in}} \in C^{n+2}(\mathbb{T}^d)$ one has $u \in C([0,T], C^{n+2}(\mathbb{T}^d)) \cap C^1([0,T], C^n(\mathbb{T}^d))$.

REMARK 5.4. Of course, we can continue to trade two spatial derivatives for one time derivative. If $n = 2k$ is even this leads to $u \in C^{k+1}([0,T], C(\mathbb{T}^d))$, while if $n = 2k+1$ is odd one finds $u \in C^{k+1}([0,T], C^1(\mathbb{T}^d))$.

REMARK 5.5. Theorem 5.4 will be applied in Section 7 with $n = 1$ in order to justify some of the formal manipulations there, enabling us to focus attention on the question of global existence.

REMARK 5.6. Each of the above results have a version for initial data u^{in} in $L^\infty(\mathbb{T}^d)$. By Remark 5.2 the semigroup $S(t)$ acting on $L^\infty(\mathbb{T}^d)$ is generally not strongly continuous at $t = 0$ but rather is weak-$*$ continuous. The corresponding concession must be made regarding u in the conclusions of the L^∞ versions of the above theorems. For example, in the L^∞ version of Theorem 5.2 one must replace (5.15) with

$$u \in C([0,T], \text{w}^*\text{-}L^\infty(\mathbb{T}^d)) \cap C((0,T], C^2(\mathbb{T}^d)) \cap C^1((0,T], C(\mathbb{T}^d))\,, \tag{5.19}$$

where $\text{w}^*\text{-}L^\infty(\mathbb{T}^d)$ denotes $L^\infty(\mathbb{T}^d)$ endowed with its weak-$*$ topology. Outside of that consideration, the proofs of the L^∞ versions are virtually identical to those given above [**25**].

The basic existence argument given above can be refined to greatly enlarge the class of initial data that evolve into classical solutions for positive times. We recall the mild formulation (5.2),

$$u(t) = S(t)u^{\text{in}} + \int_0^t S(t-t')N(u(t'))\,dt' \,. \tag{5.20}$$

In this formulation it is natural to distinguish three spaces. One aims at proving that for initial data in some space $\mathbb{X}$ the solution is in a "better" space $\mathbb{Y}$ on some interval $(0,T]$. In general, the nonlinearity maps $\mathbb{Y}$ into a third space we shall denote by $\mathbb{Z}$.

For suitable choices of $\mathbb{X}$, $\mathbb{Y}$ and $\mathbb{Z}$ the semigroup will be strongly continuous on $\mathbb{X}$ and also map $\mathbb{X}$ and $\mathbb{Z}$ back into the "better" space $\mathbb{Y}$ with a singularity at $t=0$, so that near $t=0$ one has nonnegative constants α and β such that

$$\|S(t)w\|_{\mathbb{Y}} \le c\,t^{-\alpha}\,\|w\|_{\mathbb{X}} \quad \text{for every } w\in\mathbb{X}\,, \tag{5.21}$$

$$\|S(t)w\|_{\mathbb{Y}} \le c\,t^{-\beta}\,\|w\|_{\mathbb{Z}} \quad \text{for every } w\in\mathbb{Z}\,. \tag{5.22}$$

Again, we will assume that N is locally Lipschitz, now as a map from $\mathbb{Y}$ to $\mathbb{Z}$. Without loss of generality one can assume that $N(0)=0$, and for our convenience we will only consider the case where the Lipschitz condition can be written as

$$\|N(u_1)-N(u_2)\|_{\mathbb{Z}} \le c\left(\|u_1\|_{\mathbb{Y}}^{2\sigma} + \|u_2\|_{\mathbb{Y}}^{2\sigma}\right)\|u_1-u_2\|_{\mathbb{Y}} \tag{5.23}$$

for some $\sigma\ge 0$. The σ above will be identified with the σ of the CGL nonlinearity (5.4). This covers many applications, including the CGL equation. Modifications that embrace more general nonlinearities are not hard [**43, 70**].

THEOREM 5.5 (Extended Local Existence). *Given the above estimates* (5.21), (5.22) *and* (5.23) *where the exponents α, β and σ satisfy*

$$0 \le \beta < 1\,, \tag{5.24}$$

$$0 \le (2\sigma+1)\alpha < 1\,, \tag{5.25}$$

$$\beta + 2\sigma\alpha < 1\,, \tag{5.26}$$

for every $\rho>0$ there exists a time $T(\rho)>0$ such that for every initial data $u^{\text{in}}\in\mathbb{X}$ with $\|u^{\text{in}}\|_{\mathbb{X}}\le\rho$ there exists a unique

$$u \in C([0,T],\mathbb{X})\cap C((0,T],\mathbb{Y}) \tag{5.27}$$

satisfying the mild formulation (5.20). *In addition, the map from u^{in} to u is a locally Lipschitz function from $\mathbb{X}$ to $C([0,T],\mathbb{X})$.*

PROOF. In order to indicate how to set up the contraction mapping argument, we first estimate the fixed point equation (5.20) in the $\mathbb{Y}$ norm,

$$\|u(t)\|_{\mathbb{Y}} \le c\,t^{-\alpha}\,\|u^{\text{in}}\|_{\mathbb{X}} + c\int_0^t (t-t')^{-\beta}\,\|u(t')\|_{\mathbb{Y}}^{2\sigma+1}dt'\,, \tag{5.28}$$

or

(5.29)

$$\begin{aligned} t^{\alpha}\,\|u(t)\|_{\mathbb{Y}} &\le c\,\|u^{\text{in}}\|_{\mathbb{X}} + c\,t^{\alpha}\int_0^t (t-t')^{-\beta}\,t'^{-(2\sigma+1)\alpha}\left(t'^{\alpha}\,\|u(t')\|_{\mathbb{Y}}\right)^{2\sigma+1}dt' \\ &\le c\,\|u^{\text{in}}\|_{\mathbb{X}} + c\sup_{t'\in[0,t]}\left(t'^{\alpha}\,\|u(t')\|_{\mathbb{Y}}\right)^{2\sigma+1} t^{\alpha}\int_0^t (t-t')^{-\beta}\,t'^{-(2\sigma+1)\alpha}\,dt'\,. \end{aligned}$$

This suggests that we might be able to prove contractivity in the space $\mathbb{E}([0,T])$, defined as the completion of $C([0,T],\mathbb{Y})$ in the norm

$$\|w\|_{\mathbb{E}} \equiv \sup_{t\in[0,T]} t^\alpha \, \|w(t)\|_{\mathbb{Y}} \,. \tag{5.30}$$

Indeed, (5.29) shows that provided

$$t^\alpha \int_0^t (t-t')^{-\beta}\, t'^{\,-(2\sigma+1)\alpha}\, dt' \to 0 \quad \text{as } t \to 0\,, \tag{5.31}$$

an assumption that is directly seen to be equivalent to (5.24–5.26), there exists a $T>0$ such that the right side of the fixed point equation (5.20) maps sufficiently large balls with respect to the $\mathbb{E}$ norm into themselves. Invoking the Lipschitz condition (5.23), one can show in a similar way that this mapping contracts under the same assumption (5.31) for some $T>0$. Then by the contraction mapping theorem, there exists a unique solution $u \in \mathbb{E}([0,T]) \subset C((0,T],\mathbb{Y})$ to the fixed point problem (5.20).

Finally one can show that $u \in C([0,T],\mathbb{X})$. To this end we only need to check the continuity of u at $t=0$. First subtracting $u^{\rm in}$ from both sides of the mild formulation (5.20), a direct estimate in $\mathbb{X}$, assuming a continuous embedding $\mathbb{Z} \hookrightarrow \mathbb{X}$, gives

$$\begin{aligned} \|u(t)-u^{\rm in}\|_{\mathbb{X}} &\le \|S(t)u^{\rm in}-u^{\rm in}\|_{\mathbb{X}} + c\int_0^t t'^{\,-(2\sigma+1)\alpha} \left(t'^{\alpha}\,\|u(t')\|_{\mathbb{Y}}\right)^{2\sigma+1} dt' \\ &\le \|S(t)u^{\rm in}-u^{\rm in}\|_{\mathbb{X}} + c\,\|u\|_{\mathbb{E}}^{2\sigma+1} \int_0^t t'^{\,-(2\sigma+1)\alpha} dt' \,. \end{aligned} \tag{5.32}$$

The first term on the right converges to zero due to the strong $\mathbb{X}$ continuity of the linear semigroup. Since $\|u\|_{\mathbb{E}}$ is bounded, the second term converges to zero provided that $(2\sigma+1)\alpha<1$, a condition already contained in (5.31). □

REMARK 5.7. Strictly speaking, here we have proved existence in a smaller space, namely $\mathbb{E}([0,T])$, than that asserted in (5.27).

Let us now apply Theorem 5.5 to the CGL equation where, as before, N and L are given by (5.4) and the associated linear semigroup by (5.5). We first consider the case where the initial data is in L^p, i.e. $\mathbb{X} = L^p(\mathbb{T}^d)$. We also take $\mathbb{Z} = L^p(\mathbb{T}^d)$ and let $\mathbb{Y} = L^r(\mathbb{T}^d)$ where r is still to be determined. It takes some relatively straightforward applications of Hölder and Young inequalities to verify that the Lipschitz condition (5.23) is satisfied whenever

$$r \ge (2\sigma+1)p\,. \tag{5.33}$$

In order to compute the exponent α of (5.21), the exponent of the "penalty" incurred by using $S(t)$ as a smoothing operator from L^p to L^r, we estimate

$$\|G_t * w\|_{L^r} \le \|G_t\|_{L^q}\,\|w\|_{L^p} \le \|G_t\|_{L^1}^{1/q}\,\|G_t\|_{L^\infty}^{1-1/q}\,\|w\|_{L^p} \tag{5.34}$$

where

$$1+\frac{1}{r} = \frac{1}{p}+\frac{1}{q}\,. \tag{5.35}$$

By using the L^1 estimate (5.7) and the L^∞ estimate

$$\|G_t\|_{L^\infty} \le \frac{(1+\nu^2)^{d/4}\, e^{Rt}}{(4\pi(1+\nu^2)t)^{d/2}} \left\{a + b[(1+\nu^2)t]^{d/2}\right\}, \tag{5.36}$$

for positive absolute constants a and b, one finds that for small t

$$\|G_t\|_{L^1} = O(t^0)\,, \qquad \|G_t\|_{L^\infty} = O(t^{-d/2})\,. \tag{5.37}$$

This used with (5.34) and (5.35) gives

$$\|S(t)w\|_{L^r} \le c\, t^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{r})} \|w\|_{L^p}\,, \tag{5.38}$$

i.e. $\alpha = \beta = \frac{d}{2}(\frac{1}{p} - \frac{1}{r})$. Then Theorem 5.5, first applied in the marginal case $r = (2\sigma + 1)p$, asserts the existence of a time $T > 0$ and a unique local solution

$$u \in C([0,T], L^p(\mathbb{T}^d)) \cap C((0,T], L^r(\mathbb{T}^d)) \tag{5.39}$$

subject to the constraint $(2\sigma + 1)\alpha < 1$, i.e. $\sigma d < p$.

Since $u(t) \in L^r(\mathbb{T}^d)$ for all $t > 0$, one can re-apply the argument to show that

$$u \in C((0,T], L^{r_m}(\mathbb{T}^d)) \quad \text{where } r_m = (2\sigma + 1)^m p \tag{5.40}$$

for every $m \in \mathbb{N}$. Finally, for some $r_m > (\sigma + \frac{1}{2})d$ we can take $\mathbb{X} = \mathbb{Z} = L^{r_m}(\mathbb{T}^d)$ and $\mathbb{Y} = C(\mathbb{T}^d)$ endowed with the supremum norm to obtain $u \in C((0,T], C(\mathbb{T}^d))$. An application of Theorem 5.2 then yields the following.

THEOREM 5.6 (Local Classical Solutions for L^p Initial Data). *If p satisfies*

$$1 \le p < \infty\,, \qquad \textit{and} \qquad \sigma d < p\,, \tag{5.41}$$

then for every $\rho > 0$ there exists a time $T(\rho) > 0$ such that for every initial data $u^{\text{in}} \in L^p(\mathbb{T}^d)$ with $\|u^{\text{in}}\|_{L^p} \le \rho$ there exists a unique

$$u \in C([0,T], L^p(\mathbb{T}^d)) \cap C((0,T], C^2(\mathbb{T}^d)) \cap C^1((0,T], C(\mathbb{T}^d))\,, \tag{5.42}$$

satisfying the CGL initial-value problem. In addition, the map from u^{in} to u is a locally Lipschitz function from $L^p(\mathbb{T}^d)$ to $C([0,T], L^p(\mathbb{T}^d))$.

REMARK 5.8. For $p \ge 2$ the uniqueness of such classical solutions can be extended to hold within the class of weak solutions of Theorem 4.1. In other words, so long as a weak solution of the CGL equation is classical then it is unique. What is striking here though is that in subcritical cases ($\sigma d < 2$) Theorem 5.6 asserts the existence of solutions with L^p initial data for some $p < 2$—solutions which therefore lie outside the class of weak solutions of Theorem 4.1. In fact, as we will show below, for the subcritical case $\sigma = d = 1$ one can use the same technique to establish the existence of solutions for initial data in classes of distributions $H^q(\mathbb{T})$ for $-\frac{1}{2} < q < 0$.

REMARK 5.9. This result does more than extend to L^p the class of initial data that evolve into classical solutions for positive times. More importantly, it estimates the time interval over which any classical solution will exist in terms of its L^p norm. This means that in order to prove that any classical solution is global in time it suffices to control its L^p norm, where p satisfies (5.41). In Section 7 we present such temporally global *a priori* L^p bounds for an interval of ν values depending only on σ and d, but not μ. In that case one has global classical solutions (Theorem 7.1).

REMARK 5.10. The proof given here differs from that in [**80**] in that the space $\mathbb{E}([0,T])$ used here for the contraction mapping argument is larger than the space used there, but can be abstracted from [**80**] with minor modifications. Essentially the same methods lead to similar results for the CGL equation over $\mathbb{R}^d$. More sophisticated techniques lead to similar results even in the dissipationless case of the NLS equation, both over $\mathbb{R}^d$ [**10, 24, 79**] and over $\mathbb{T}$ where Bourgain [**8, 9**] has developed an $L^2(\mathbb{T})$ theory for $\sigma < 2$.

REMARK 5.11. An important special case is when $p = 2\sigma + 2$, in which case the above theorem yields local classical solutions whenever

$$d < 2 + \frac{2}{\sigma}\,. \tag{5.43}$$

This condition coincides with condition (3.7) for the existence and uniqueness of local solutions for the NLS equation in $H^1(\mathbb{T}^d)$. In Section 7 we present temporally global *a priori* $L^{2\sigma+2}$ bounds for a region of the $\mu\nu$-plane depending on σ, but not d. In that case, whenever (5.43) is satisfied one has global classical solutions (Theorem 7.2).

In the following, we will use Theorem 5.5 in its full generality to prove existence of a local solution starting from H^q initial data. The Sobolev spaces $H^q(\mathbb{T}^d)$ can be defined for any $q \in \mathbb{R}$. To do so we will use $\widehat{w}(\xi)$ to denote the Fourier coefficients of a function $w \in L^2(\mathbb{T}^d)$, so that

$$w(x) = \sum_{\xi \in 2\pi\mathbb{Z}^d} \widehat{w}(\xi)\, e^{i\xi\cdot x}\,, \qquad \widehat{w}(\xi) = \int_{\mathbb{T}^d} e^{-i\xi\cdot x}\, w(x)\, dx\,. \tag{5.44}$$

The H^q norm of a function w can then be defined [**2**] as

$$\|w\|_{H^q} \equiv \left(\sum_{\xi \in 2\pi\mathbb{Z}^d} (1 + |\xi|^2)^q\, |\widehat{w}(\xi)|^2 \right)^{\frac{1}{2}}\,. \tag{5.45}$$

The existence result we are going to prove is relatively technical and will not be used in any of the subsequent sections, so that it may be skipped in a first reading. The motivation for this further inquiry comes from the observation that in the CGL equation $|u|^{2\sigma}$ scales dimensionally like Δ so that by formally trading powers for derivatives in condition (5.41), one transforms Theorem 5.6 into the following conjecture. Given $u^{\text{in}} \in H^q(\mathbb{T}^d)$ where $q > \frac{d}{2} - \frac{1}{\sigma}$, there exists a unique local classical solution. We will show under certain restrictions that this is indeed the case [**56**].

We first remark that it is very problematic to conduct the argument entirely in H^q spaces. The difficulty lies in obtaining a Lipschitz estimate (5.23), which is easily accomplished only when $\mathbb{Y} = \mathbb{Z} \equiv H^q(\mathbb{T}^d)$ is an algebra, i.e. when $q > d/2$. (See [**2**] for a review of Sobolev algebras.) In this case the basic local existence result, Theorem 5.1, immediately assures the existence of a local mild solution. This approach however does clearly not exhaust the conjectured range of admissible values for q.

Secondly, we observe that the only interesting case is $q < 0$. Namely, when $q > \frac{d}{2} - \frac{1}{\sigma}$ and $q > 0$, one can always find a $p > \sigma d$ so that the Sobolev embedding $H^q \hookrightarrow L^p$ holds, i.e. that $\frac{1}{p} > \frac{1}{2} - \frac{q}{d}$. Therefore in this case local classical solutions are already guaranteed by Theorem 5.5.

Assume therefore that $u^{\text{in}} \in H^q(\mathbb{T}^d) \equiv \mathbb{X}$ for $q < 0$. Take the two other spaces as in the L^p argument, i.e. $\mathbb{Y} = L^r(\mathbb{T}^d)$ and $\mathbb{Z} = L^p(\mathbb{T}^d)$. While $\beta = \frac{d}{2}(\frac{1}{p} - \frac{1}{r})$ and the restriction $r \geq (2\sigma + 1)p$ are as before, it still remains to find the exponent α of the "penalty" for using $S(t)$ as a smoothing operator from H^q to L^r. This can be done by going via the intermediate space L^2, i.e. by using (5.38) to infer that for $r \geq 2$,

$$\|S(2t)w\|_{L^r} \leq c\, t^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{r})} \|S(t)w\|_{L^2}\,. \tag{5.46}$$

The convolution $S(t)w = G_t * w$ is best estimated by deriving bounds on its Fourier transform. Specifically, we have

$$\begin{aligned} \|G_t * w\|_{L^2}^2 &= \sum_{\xi\in 2\pi\mathbb{Z}^d} |\widehat{g}(\xi)\widehat{w}(\xi)|^2 \\ &\leq \sup_{\xi\in 2\pi\mathbb{Z}^d} \left\{(1+|\xi|^2)^{-q}\,|\widehat{g}(\xi)|^2\right\} \sum_{\xi\in 2\pi\mathbb{Z}^d} (1+|\xi|^2)^q\,|\widehat{w}(\xi)|^2\,, \end{aligned} \tag{5.47}$$

where $\widehat{w}(\xi)$ denotes the Fourier coefficients of w and the Fourier coefficients of G_t are given by

$$\widehat{g}(\xi) = e^{Rt}\, e^{-(1+i\nu)|\xi|^2 t}\,. \tag{5.48}$$

The supremum in (5.47) is easily estimated by simple optimization with respect to $|\xi|^2$, resulting in

$$\|G_t * w\|_{L^2}^2 \leq \left(\frac{-q}{2et}\right)^{-q} e^{2(R+1)t}\,\|w\|_{H^q}^2\,. \tag{5.49}$$

For sufficiently small t one has altogether

$$\|S(t)w\|_{L^r} \leq c\, t^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{r})-\frac{-q}{2}}\,\|w\|_{H^q}\,. \tag{5.50}$$

Again, we take the marginal case of the Lipschitz condition, $r = (2\sigma+1)p$. Then condition $\beta + 2\sigma\alpha < 1$ of Theorem 5.5 is satisfied if and only if

$$q > \frac{d}{2} - \frac{1}{\sigma}\,. \tag{5.51}$$

We still need to check the other two conditions of that theorem. Condition (5.25) imposes an upper bound on r, namely

$$r < \frac{2(2\sigma+1)d}{(2\sigma+1)(d-2q)-4}\,. \tag{5.52}$$

Similarly, (5.24) imposes a lower bound on r,

$$r > \sigma d\,. \tag{5.53}$$

Note first that given (5.51), the upper and lower bounds on r can be satisfied simultaneously. For technical reasons, we also have two additional lower bounds on r. In the derivation of (5.50) it was necessary to assume that $r \geq 2$, and we must also require $p \geq 1$, i.e. $r \geq 2\sigma + 1$. These combined with (5.52) translate into

$$q > -\frac{2}{2\sigma+1} \tag{5.54}$$

and

$$q > \frac{d}{2} - \frac{d+2}{2\sigma+1} \tag{5.55}$$

respectively. Then finally Theorem 5.5 yields a unique mild solution $u \in C((0,T],\mathbb{Y})$ for some $T > 0$. Moreover, since $u(t) \in L^r(\mathbb{T}^d)$ for all $t \in (0,T]$, further regularity can be achieved by a direct application of Theorem 5.6. One thus has the following.

THEOREM 5.7 (Local Classical Solutions for H^q Initial Data). *Provided q satisfies* (5.54), (5.55) *and*

$$q > \frac{d}{2} - \frac{1}{\sigma} \tag{5.56}$$

then for every $\rho > 0$ there exists a time $T(\rho) > 0$ such that for every initial data $u^{\mathrm{in}} \in H^q(\mathbb{T}^d)$ with $\|u^{\mathrm{in}}\|_{H^q} \leq \rho$ there exists a unique

$$u \in C([0,T], H^q(\mathbb{T}^d)) \cap C((0,T], C^2(\mathbb{T}^d)) \cap C^1((0,T], C(\mathbb{T}^d)), \tag{5.57}$$

satisfying the CGL initial-value problem. In addition, the map from u^{in} to u is a locally Lipschitz function from $H^q(\mathbb{T}^d)$ to $C([0,T], H^q(\mathbb{T}^d))$.

REMARK 5.12. We believe conditions (5.54) and (5.55) are technical and can be removed by more sophisticated techniques. Moreover, by following [**10**], the use of Besov spaces should allow one to obtain the marginal case when (5.56) is satisfied as an equality. Note also that provided $\sigma \geq \frac{1}{2}$ and $\sigma d \geq 1$—thus in particular when σ is an integer—condition (5.56) dominates and we have proved local existence for the conjectured class of initial data.

REMARK 5.13. The results in this section extend to $\mathbb{R}^d$ with only minor modifications in their proofs.

6. Analyticity of Solutions

Let us now discuss Gevrey class regularity which is a stronger notion than the C^∞ regularity we have encountered before. It not only asserts that all derivatives of the solution u are bounded, but also that these bounds depend on the order of the derivatives in some prescribed way. Gevrey [**34**] used this notion as a setting in which to extend Cauchy-Kowalevski existence arguments to classes of functions that are not necessarily analytic (for a review of the analytic case see e.g. [**45**]). In fact, they are special cases of the *quasianalytic* classes which had been introduced earlier by Hadamard [**41**]. La Vallée Poussin [**52**] showed that, among the quasianalytic functions, the Gevrey classes are characterized by an exponential decay of their Fourier coefficients (see also [**49**]). In turn, this characterization has recently proved useful for showing that the solutions of various nonlinear partial differential equations are analytic. Foias and Temam [**30**] developed this technique in the context of the two-dimensional incompressible Navier-Stokes equations, thereby simplifying earlier work of Kahane [**47**] that had been more in the style of Gevrey. Doelman and Titi [**22**] subsequently applied the technique to the cubic CGL equation, and Ferrari and Titi [**27**] extended the results to a large class of analytic nonlinear parabolic equations. In particular, the authors of the last reference have exploited the fact that certain subclasses are normed algebras, thus making the estimation of polynomial nonlinearities a triviality. This is the strategy we will adopt because of its brevity and elegance.

A function $w \in C^\infty(\mathbb{T}^d)$ is said to be of *Gevrey class s* for some $s > 0$ if there exist constants $\rho > 0$ and $M < \infty$ such that for every $x \in \mathbb{T}^d$ and every $\alpha \in \mathbb{N}^d$ one has

$$|\partial^\alpha w(x)| \leq M \left(\frac{\alpha!}{\rho^{|\alpha|}} \right)^s. \tag{6.1}$$

Here we employ the usual multi-index notation in which

$$|\alpha| \equiv \sum_{j=1}^{d} \alpha_j\,, \qquad \alpha! \equiv \prod_{j=1}^{d} \alpha_j!\,, \qquad \partial^\alpha \equiv \prod_{j=1}^{d} \partial_{x_j}^{\alpha_j}\,. \tag{6.2}$$

The set of all functions of Gevrey class s is a vector space, denoted $G^s(\mathbb{T}^d)$. It is closed under multiplication and differentiation. Moreover, the composition of two functions of Gevrey class s is again of class s.

It is classical that $G^1(\mathbb{T}^d)$ is the space of real analytic functions $C^\omega(\mathbb{T}^d)$; a proof can be found, for example, in [**45**, page 65]. For $0 < s < 1$ the class $G^s(\mathbb{T}^d)$ is subclass of the analytic functions, while for $1 < s < \infty$ it contains the analytic functions. In fact, one has a hierarchy of spaces such that $0 < s_1 < s_2 < \infty$ implies the proper containments

$$G^{s_1}(\mathbb{T}^d) \subset G^{s_2}(\mathbb{T}^d) \subset C^\infty(\mathbb{T}^d)\,. \tag{6.3}$$

Moreover, the union of the $G^s(\mathbb{T}^d)$ does not exhaust $C^\infty(\mathbb{T}^d)$ because there are quasianalytic functions that are not members of a Gevrey class [**52**].

In what follows it will be more convenient to characterize Gevrey classes in terms of the fractional Sobolev spaces $H^r(\mathbb{T}^d)$ (as defined on page 168 for every $r \in \mathbb{R}$) with $r \geq 0$, rather than uniform bounds such as (6.1).

LEMMA 6.1. *Given $s > 0$ and $r \geq 0$. Then $w \in G^s(\mathbb{T}^d)$ if and only if there are constants ρ, $M \in (0,\infty)$ that may depend on r, s and w such that for every $n \in \mathbb{N}$ one has*

$$\|\nabla^n w\|_{H^r} = \left(\sum_{\xi \in 2\pi\mathbb{Z}^d} (1+|\xi|^2)^r\, |\xi|^{2n}\, |\widehat{w}(\xi)|^2 \right)^{\frac{1}{2}} \leq M \left(\frac{n!}{\rho^n} \right)^s\,. \tag{6.4}$$

The proof is achieved by a direct application of the Sobolev embedding theorem [**2**] and shall be left to the reader.

This lemma enables us to characterize functions in G^s in terms of the decay of their Fourier coefficients, a result of the type first due to La Vallée Poussin [**52, 59**], although this attribution has been obscured in the recent literature. The construction here uses the operator $A = \sqrt{-\Delta}$ that, like $-\Delta$ itself, is nonnegative and self-adjoint so that arbitrary powers can be defined by spectral theory. For each $s \in (0,\infty)$ we define a family, parameterized by τ, of normed spaces

$$\mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^d)) \equiv \left\{ w \in H^r(\mathbb{T}^d) \,:\, \left\| e^{\tau A^{1/s}} w \right\|_{H^r} < \infty \right\}. \tag{6.5}$$

The functions in any such space have Fourier coefficients that decay faster than $\exp(-\tau|\xi|^{1/s})$. The next theorem recovers the Gevrey class $G^s(\mathbb{T}^d)$ as the union of all such classes. We believe the proof is new.

THEOREM 6.2. *For any $s > 0$ and $r \geq 0$,*

$$G^s(\mathbb{T}^d) = \bigcup_{\tau > 0} \mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^d))\,. \tag{6.6}$$

REMARK 6.1. The use of the more general H^r rather than simply L^2 as the base space does not complicate the structure of the proof, but it is advantageous for the arguments that will follow.

PROOF. Let $w \in \mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^d))$ for some $\tau > 0$ and let $\rho = \tau/s$. Then

$$\begin{aligned} \|\nabla^n w\|_{H^r}^2 &= \left(\frac{n!}{\rho^n}\right)^{2s} \sum_\xi (1+|\xi|^2)^r \left(\frac{\rho^n |\xi|^{n/s}}{n!}\right)^{2s} |\widehat{w}(\xi)|^2 \\ &\le \left(\frac{n!}{\rho^n}\right)^{2s} \sum_\xi (1+|\xi|^2)^r \, e^{2s\rho|\xi|^{1/s}} \, |\widehat{w}(\xi)|^2 \\ &= \left(\frac{n!}{\rho^n}\right)^{2s} \left\|e^{\tau A^{1/s}} w\right\|_{H^r}^2 . \end{aligned} \tag{6.7}$$

By setting $M = \|e^{\tau A^{1/s}} w\|_{H^r}$ we obtain (6.4), whereby $w \in G^s(\mathbb{T}^d)$.

On the other hand, let $w \in G^s(\mathbb{T}^d)$. For an arbitrary $\tau \ge 0$ one has

$$\begin{aligned} \left\|e^{\tau A^{1/s}} w\right\|_{H^r}^2 &\equiv \sum_\xi (1+|\xi|^2)^r \, e^{2\tau|\xi|^{1/s}} \, |\widehat{w}(\xi)|^2 \\ &= \sum_{m=0}^\infty \frac{(2\tau)^m}{m!} \sum_\xi (1+|\xi|^2)^r \, |\xi|^{\frac{m}{s}} \, |\widehat{w}(\xi)|^2 . \end{aligned} \tag{6.8}$$

Now let ρ and M be such that (6.4) is satisfied. By interpolating (6.4) between $n = 0$ and any integer n such that $m/s \le 2n$, the inner sum appearing in (6.8) can be bounded as

$$\sum_\xi (1+|\xi|^2)^r \, |\xi|^{\frac{m}{s}} \, |\widehat{w}(\xi)|^2 \le M^2 \frac{(n!)^{\frac{m}{n}}}{\rho^m} . \tag{6.9}$$

This bound is best if we choose $n = n_m \equiv [m/(2s)] + 1$, where $[\,\cdot\,]$ denotes the "greatest integer less than" function. Upon making this choice and applying the result in (6.8), one arrives at the bound

$$\left\|e^{\tau A^{1/s}} w\right\|_{H^r}^2 \le \sum_{m=0}^\infty \frac{(2\tau)^m}{m!} M^2 \frac{(n_m!)^{\frac{m}{n_m}}}{\rho^m} = M^2 \sum_{m=0}^\infty \left(\frac{2\tau}{\rho}\right)^m \frac{(n_m!)^{\frac{m}{n_m}}}{m!} . \tag{6.10}$$

By making use of the Stirling formula [**1**] in the form

$$\lim_{n\to\infty} \frac{e}{n} (n!)^{\frac{1}{n}} = 1 , \tag{6.11}$$

the limit of the m^{th} root of the m^{th} term in the last series of (6.10) can be evaluated as

$$\lim_{m\to\infty} \frac{2\tau}{\rho} \frac{(n_m!)^{\frac{1}{n_m}}}{(m!)^{\frac{1}{m}}} = \frac{2\tau}{\rho} \lim_{m\to\infty} \frac{n_m}{m} = \frac{2\tau}{\rho} \frac{1}{2s} = \frac{\tau}{s\rho} . \tag{6.12}$$

Hence, by the Hadamard root test, the series in (6.10) converges for every $\tau < \rho s$, whence $w \in \mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^d))$. ☐

REMARK 6.2. The above proof gives a sharp relationship between the τ of (6.5) and the ρ of (6.4). Had we been less careful in our choice of the n_m used in (6.10) then this relationship would have been missed.

REMARK 6.3. The theorem also holds over $\mathbb{R}^d$. The proof can proceed in the same way, using Fourier integrals in place of Fourier sums.

One reason Gevrey classes are useful in the context of nonlinear partial differential equations is that each G^s is closed under multiplication. It would be particularly useful if this property extended to each of the approximating (normed!) spaces $\mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^d))$. The following theorem states that under certain conditions this is indeed the case.

THEOREM 6.3. *If $s \geq 1$, $\tau \geq 0$, and $r > d/2$ then $\mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^d))$ is a Banach algebra. This means that it is closed under multiplication and that there exists a finite constant $C(r,d)$ such that any two functions v and w in $\mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^d))$ satisfy the inequality*

$$\left\|e^{\tau A^{1/s}}(vw)\right\|_{H^r} \leq C(r,d) \left\|e^{\tau A^{1/s}}v\right\|_{H^r} \left\|e^{\tau A^{1/s}}w\right\|_{H^r} . \tag{6.13}$$

The proof is a direct extension of the usual proof that $H^r(\mathbb{T}^d)$ is a Banach algebra when $r > d/2$ [**2**], a result that is recovered above by setting $\tau = 0$. A proof of this theorem for the case when $s = 1$ is given in [**27**]; this proof is easily generalized for any $s > 1$.

The spaces $\mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^d))$ are naturally suited for the application to parabolic equations: τ will be identified with time so that $\mathcal{D}(e^{\tau A^{1/s}} : H^r(\mathbb{T}^d))$ evolves from being identical to H^r at $\tau = 0$ to being a subset of Gevrey class s in an arbitrarily short time. So if one can show that for some $T > 0$ the solutions to the equation with H^r initial data are in $\mathcal{D}(e^{tA^{1/s}} : H^r(\mathbb{T}^d))$ for every $t \in (0,T]$ then one has proved Gevrey regularity of class s over that interval.

Let us now proceed to show that the solution u of the CGL equation is of Gevrey class 1—in other words, is real analytic. To do this we employ the time dependent norm $\|w\|_t \equiv \|e^{tA}w\|_{L^2}$. As the L^2 norm of u is already controlled by (7.2), it is sufficient to derive a bound on the seminorm $\|A^r u\|_t$. A direct calculation gives

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|A^r u(t)\|_t^2 &= \mathrm{Re}\left(\int A^{r+1}e^{tA}u\, A^r e^{tA}u^*\, dx\right) \\
&\quad + \mathrm{Re}\left(\int A^r e^{tA}\partial_t u\, A^r e^{tA}u^*\, dx\right) \\
&= \|A^{r+1/2}u\|_t^2 + R\,\|A^r u\|_t^2 - \|A^{r+1}u\|_t^2 \\
&\quad - \mathrm{Re}\left((1+i\mu)\int A^r e^{tA}(|u|^{2\sigma}u)\, A^r e^{tA}u^*\, dx\right) .
\end{aligned} \tag{6.14}$$

The first term on the right is easily estimated by using

$$\|A^{1/2}v\|_{L^2}^2 = (v, Av)_{L^2} \leq \|v\|_{L^2}\,\|Av\|_{L^2} \leq \tfrac{1}{4}\|v\|_{L^2}^2 + \|Av\|_{L^2}^2 , \tag{6.15}$$

with $v = A^r e^{tA}u$. By using the Cauchy-Schwarz inequality on the integral in the last term of (6.14), one finds that it is bounded by

$$\|A^r\big(u^{\sigma+1}(u^*)^\sigma\big)\|_t\,\|A^r u^*\|_t \leq c\,\|A^r u\|_t^{2\sigma+2} . \tag{6.16}$$

It is here that we use the fact that $\mathcal{D}(e^{tA} : H^r(\mathbb{T}^d))$ is an algebra. Then finally

$$\frac{1}{2}\frac{d}{dt}\|A^r u(t)\|_t^2 \leq (R + \tfrac{1}{4})\,\|A^r u(t)\|_t^2 + c\,\|A^r u(t)\|_t^{2\sigma+2} . \tag{6.17}$$

This differential inequality is easily integrated and shows that $\|A^r u(t)\|_t$ remains finite on some interval $[0,T]$. Given that σ is an integer, the local existence theorems of the last section guarantee that $u(t) \in H^r(\mathbb{T}^d)$ for any r and $t > 0$ as long as it exists. Hence by initializing (6.17) with $\|u(t)\|_{H^r} = \|A^r u(t)\|_0$ for some $r > d/2$ and every such t one can conclude the following.

THEOREM 6.4. *Provided that σ is a positive integer, a local classical solution is real analytic in x at any positive t as long as it exists.*

REMARK 6.4. In order to achieve global bounds with the method presented here, it is necessary to have *a priori* H^r control for some $r > d/2$. The method used originally by Doelman and Titi [**22**] requires only *a priori* H^1 control independent of d. However, that method requires a separate and fairly involved estimate on the nonlinear term because $\mathcal{D}(e^{tA}\colon H^1(\mathbb{T}^d))$ is not an algebra for $d > 1$. See Theorem 8.4 for the statement of the global result.

7. Global Classical Solutions

The task of this section is to show that there are regions of the $\mu\nu$-plane where we can elevate the local classical solutions found in Section 5 to global classical solutions. In these regions the CGL equation defines a dynamical system and we can investigate the properties of its global attractor. The values of μ and ν for which we establish this regularity will depend on the spatial dimension d and the degree of nonlinearity σ. The results of the last sections were very general—for example, we have the same local existence and regularity results for either sign of the real part of the coefficient of the nonlinear term—and assert substantially little more than the fact that the problem is well-posed. The estimates and results in this section, however, will depend much more delicately on the structure and dynamics of the CGL equation. They rely not only on the signs of the dissipative terms, but on the relative signs of the coefficients of the dispersive terms as well.

As was pointed out in the final remarks of Section 5, it suffices to obtain global control of any L^p norm where $p > \sigma d$. Such bounds can be obtained by computing the time evolution of the L^p norm, say, and then perform estimates on the various terms in order to obtain closed differential inequalities. This shall be first demonstrated for the L^2 norm which is simplest to control. A direct calculation and subsequent application of the Hölder inequality yields

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int |u|^2\,dx &= R\int |u|^2\,dx - \int |u|^{2\sigma+2}\,dx - \int |\nabla u|^2\,dx \\ &\le R\int |u|^2\,dx - \left(\int |u|^2\,dx\right)^{\sigma+1} . \end{aligned} \tag{7.1}$$

In the appendix it is shown that for $\sigma > 0$ this differential inequality implies the L^2 norm of u satisfies

$$\|u(t)\|_{L^2} < \left(\frac{R}{1 - e^{-2\sigma R t}}\right)^{\frac{1}{2\sigma}} . \tag{7.2}$$

This estimate is independent of the initial data u^{in} and decays to $R^{1/2\sigma}$ as $t \to \infty$. Hence for times bounded strictly away from 0, the L^2 norm of u is bounded uniformly in both time and initial condition. Whenever $\sigma d < 2$, Theorem 5.6 then implies that the CGL equation has global classical solutions for all initial data in $L^2(\mathbb{T}^d)$. For $\sigma d \ge 2$, one must control more than the L^2 norm.

Directly computing the evolution of the L^p norm gives

$$\frac{1}{p}\frac{d}{dt}\int |u|^p\,dx = R\int |u|^p\,dx - \int |u|^{2\sigma+p}\,dx + \mathrm{Re}\Big(\int |u|^{p-2}u^*\Delta u\,dx\Big) - \nu\,\mathrm{Im}\Big(\int |u|^{p-2}u^*\Delta u\,dx\Big)\,. \tag{7.3}$$

The net contribution of the last two terms on the right side will be nonpositive for $p \geq 2$ provided

$$|\nu| \leq \frac{2\sqrt{p-1}}{p-2}\,. \tag{7.4}$$

This fact can be established many ways—either using Sobolev bounds as in [**3**], quadratic forms as in [**25**], or, as will be done here, as a consequence of Lemma 7.3. Upon supposing (7.4) is satisfied, the last two terms on the right of (7.3) can be neglected while the remaining terms can be treated through an application of a Hölder inequality analogous to the one for the L^2 estimate (7.1) thereby yielding the inequality

$$\frac{1}{p}\frac{d}{dt}\int |u|^p\,dx \leq R\int |u|^p\,dx - \Big(\int |u|^p\,dx\Big)^{\frac{p+2\sigma}{p}}\,. \tag{7.5}$$

The appendix again gives a uniform bound, identical to (7.2),

$$\|u(t)\|_{L^p} < \left(\frac{R}{1-e^{-2\sigma Rt}}\right)^{\frac{1}{2\sigma}}\,. \tag{7.6}$$

The existence of global classical solutions is then guaranteed by Theorem 5.6 in the subcritical case ($\sigma d < 2$)—as mentioned before—by taking $p = 2$. In the critical case ($\sigma d = 2$) one can still satisfy $\sigma d < p$ and (7.4) simultaneously by choosing p sufficiently close to 2. In the supercritical case ($\sigma d > 2$) however, the best we can do is to choose p sufficiently close to σd, making (7.4) a restriction on the possible values of ν for which the existence of global classical solutions can be established. This is summarized in the following.

THEOREM 7.1. *For $\sigma > 0$, the generalized CGL equation with C^2 initial data has unique global classical solutions provided either that $\sigma d \leq 2$ or that $\sigma d > 2$ and ν satisfies*

$$|\nu| < \frac{2\sqrt{\sigma d-1}}{\sigma d-2}\,. \tag{7.7}$$

Thus, in the supercritical case the CGL equation has global classical solutions only when ν lies in the strip around the μ-axis of the $\mu\nu$-plane given by (7.7) while μ may take on any parameter value.

REMARK 7.1. When global classical solutions exist, estimate (7.6) shows that the attractor is confined to the closed ball of radius $R^{1/(2\sigma)}$ in L^p provided p satisfies (7.4). Furthermore, estimate (7.6) is independent of the initial condition. This means that in an arbitrarily short time all solutions enter some ball of finite radius.

REMARK 7.2. The basic differential inequalities (7.1) on the L^2 norm of u and (7.5) on the L^p norm are sharp for the spatially homogeneous solutions (2.10) of the CGL equation. Moreover, their proofs use essentially the boundedness of the domain $\mathbb{T}^d$ and do not extend to $\mathbb{R}^d$. However, a bound on the L^p norm of u over

$\mathbb{R}^d$ that is exponentially growing in time can be obtained provided (7.4) is satisfied by neglecting the last three terms on the right of (7.3). Such a bound is enough to infer that Theorem 7.1, as stated, holds over $\mathbb{R}^d$.

An alternative to directly controlling the L^p norm is to rather first directly control the H^1 norm and then control L^p through Sobolev estimates. This is however subject to the constraint

$$1 \le p < \begin{cases} \infty & \text{for } d = 1 \text{ or } 2\,, \\ \dfrac{2d}{d-2} & \text{for } d \ge 3\,. \end{cases} \tag{7.8}$$

Given that we are also constrained by $\sigma d < p$, this restricts us to those cases where

$$d < 2 + \frac{2}{\sigma}\,, \tag{7.9}$$

which is incidentally the condition for local well-posedness of the underlying NLS problem, equation (3.7).

An estimate on the H^1 norm can be derived in a very similar way. As the L^2 norm is already controlled by (7.2), it suffices to control the L^2 norm of ∇u. To do so we assume $\sigma \ge \frac{1}{2}$ so that Theorem 5.4 gives enough regularity of the solution to take the gradient of the CGL equation. A direct calculation then shows

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int |\nabla u|^2\,dx &= R\int |\nabla u|^2\,dx - \int |\Delta u|^2\,dx \\ &\quad + \mathrm{Re}\Big(\int |u|^{2\sigma} u^* \Delta u\,dx\Big) + \mu\ \mathrm{Im}\Big(\int |u|^{2\sigma} u^* \Delta u\,dx\Big)\,. \end{aligned} \tag{7.10}$$

The last two terms on the right have the same form as those on the right side of (7.3) upon identifying p with $2\sigma + 2$ and ν with $-\mu$. Upon treating them in the same way, their net contribution will be nonpositive provided (compare with (7.4))

$$|\mu| \le \frac{\sqrt{2\sigma+1}}{\sigma}\,. \tag{7.11}$$

In that case, these terms can be neglected, and an elementary interpolation inequality applied to the second term (i.e. simply integrate by parts and use the Cauchy-Schwarz inequality) leads to

$$\frac{1}{2}\frac{d}{dt}\int |\nabla u|^2\,dx \le R\int |\nabla u|^2\,dx - \frac{\Big(\int |\nabla u|^2\,dx\Big)^2}{\int |u|^2\,dx}\,. \tag{7.12}$$

Upon using the L^2 bound for u provided by (7.2), this differential inequality gives uniform upper bounds on the L^2 norm of ∇u by following the procedure in the appendix. Indeed, even in this case the differential inequality (7.12) can be integrated exactly to obtain

$$\|\nabla u(t)\|_{L^2}^2 < \Phi(\sigma, Rt)\left(\frac{R}{1-e^{-2\sigma Rt}}\right)^{\frac{\sigma+1}{\sigma}}\,, \tag{7.13}$$

where Φ is given in terms of the hypergeometric function F (e.g. see [**1**]) by

$$\Phi(\sigma, Rt) = \frac{2(1+\sigma)}{F(1,1,2+\frac{1}{\sigma};1-e^{-2\sigma Rt})}\,. \tag{7.14}$$

The function $\Phi(\sigma, Rt)$ is monotonically decreasing in t and one can easily find that $\Phi(\sigma,0) = 2(1+\sigma)$ and $\Phi(\sigma,\infty) = 2$. Thus (7.13) provides a uniform bound on $\|\nabla u\|_{L^2}$ analogous to the L^p estimate, equation (7.6). Then Theorem 5.6 shows the existence of global classical solutions in a strip around the ν-axis of the parameter plane (7.11) similar to the strip around the μ-axis (7.7) from Theorem 7.1. In the case when $p = 2+2\sigma$, these strips are completely symmetric, and both subject to the restriction $d < 2+2/\sigma$.

However, for no $p > 2$ does the union of these strips extend to all of the modulationally stable region of the underlying NLS equation where the energy functional is positive definite and hence no finite-time singularities can occur. As the CGL equation is a dissipative perturbation of the NLS equation (disregarding for the moment the linear growth proportional to R), we expect to have regularity at least in this region. And indeed, a "perturbation" of the NLS energy functional,

$$F = \int |\nabla u|^2 + \frac{\beta^2}{\sigma+1}|u|^{2\sigma+2}\, dx\,, \tag{7.15}$$

where $\beta > 0$ is to be chosen later, will help to considerably improve the region for which global existence of classical solutions can be shown. The technical reason why this particular combination yields better results than separate estimates on the H^1 and $L^{2+2\sigma}$ norms is that the problematic terms in the H^1 and $L^{2+2\sigma}$ estimates have the same functional dependence on the solution u and therefore taking the linear combination (7.15) and optimization in β provides for partial cancelation of these terms. This is reflected in the following result, the present form of which is due to Ginibre [**37**] and slightly improves upon the region of validity given in [**3, 25**].

THEOREM 7.2. *For $\sigma \geq \frac{1}{2}$ and $d < 2+2/\sigma$, the generalized CGL equation with C^2 initial data has unique global classical solutions if the dispersive parameters lie in the region of the $\mu\nu$-plane, bounded by hyperbolae,*

$$-\frac{1+\mu\nu}{|\mu-\nu|} \leq \frac{\sqrt{2\sigma+1}}{\sigma}\,. \tag{7.16}$$

This region of the $\mu\nu$-plane completely contains the first and third quadrants where the underlying NLS equation is modulationally stable as well as the strips found from the previous separate H^1 and $L^{2+2\sigma}$ bounds (this by the very form of F). In addition, it contains the whole CGL modulationally stable region $1+\mu\nu \geq 0$ and a bit more depending on the degree of nonlinearity. Indeed, for $d = 1,2$ it reduces to the modulationally stable region in the limit of a high degree of nonlinearity ($\sigma \to \infty$). However, part of the strip (7.7) obtained from a direct L^p bound lies outside this region when $p > 2+2\sigma$. This is summarized in Table 7.1 for integer values of σ.

Figure 7.1 illustrates the supercritical case $d = 3$, $\sigma = 1$; the corresponding picture for any case where $2 < \sigma d < 2+2\sigma$ is similar. The inner pair of hyperbolae bounds the modulationally stable region $1+\mu\nu \geq 0$ while the outer pair limits the region where global classical solutions can be obtained from Theorem 7.2. The little unshaded strip outside the hyperbolic boundaries indicates the region where global classical solutions that can still be obtained by using the direct L^p result, Theorem 7.1.

REMARK 7.3. So far we have allowed for non-integer values of σ which is sufficient to obtain global classical solutions that are C^2 in the spatial variables. For

	$\sigma = 1$	$\sigma = 2$	$\sigma \geq 3$
$d = 1$	no restriction	no restriction	$\lvert\nu\rvert < \dfrac{2\sqrt{\sigma-1}}{\sigma-2}$ or $-\dfrac{1+\mu\nu}{\lvert\mu-\nu\rvert} \leq \dfrac{\sqrt{2\sigma+1}}{\sigma}$
$d = 2$	no restriction	$\lvert\nu\rvert < \sqrt{3}$ or $-\dfrac{1+\mu\nu}{\lvert\mu-\nu\rvert} \leq \dfrac{\sqrt{5}}{2}$	$\lvert\nu\rvert < \dfrac{\sqrt{2\sigma-1}}{\sigma-1}$ or $-\dfrac{1+\mu\nu}{\lvert\mu-\nu\rvert} \leq \dfrac{\sqrt{2\sigma+1}}{\sigma}$
$d = 3$	$\lvert\nu\rvert < \sqrt{8}$ or $-\dfrac{1+\mu\nu}{\lvert\mu-\nu\rvert} \leq \sqrt{3}$	$\lvert\nu\rvert < \dfrac{\sqrt{5}}{2}$	$\lvert\nu\rvert < \dfrac{2\sqrt{3\sigma-1}}{3\sigma-2}$
$d \geq 4$	$\lvert\nu\rvert < \dfrac{2\sqrt{d-1}}{d-2}$	$\lvert\nu\rvert < \dfrac{\sqrt{2d-1}}{d-1}$	$\lvert\nu\rvert < \dfrac{2\sqrt{\sigma d-1}}{\sigma d-2}$

TABLE 7.1. Sufficient restrictions for the existence of global classical solutions to the complex Ginzburg-Landau equation. The bounds on $|\nu|$ and the "no restriction" entries are a result of Theorem 7.1 while the hyperbolic boundaries arise from Theorem 7.2. The case where $d = 3$ and $\sigma = 1$ is illustrated in Figure 7.1.

$\sigma \geq n/2$ one can even get C^{n+2} solutions from Theorem 5.4, but for global smooth (i.e. C^∞) solutions σ needs to be integer. This case will the the focus of Section 8 where we derive explicit global bounds for all spatial derivatives.

In order to prove the results of this section, it is useful to have the following bounds on nonlinear functionals that occur throughout the proofs. They were first used by Ginibre [**37**] to obtain the improved hyperbolic boundaries (7.16).

LEMMA 7.3. *For $p \geq 2$ and a function $u \in H^2(\mathbb{T}^d) \cap L^{2p-2}(\mathbb{T}^d)$ the following inequalities hold:*

$$\operatorname{Re}\left(\int |u|^{p-2} u^* \Delta u \, dx\right) \leq -\frac{2\sqrt{p-1}}{p} \left| \int |u|^{p-2} u^* \Delta u \, dx \right| , \tag{7.17}$$

$$\left|\operatorname{Im}\left(\int |u|^{p-2} u^* \Delta u \, dx\right)\right| \leq \frac{p-2}{p} \left| \int |u|^{p-2} u^* \Delta u \, dx \right| . \tag{7.18}$$

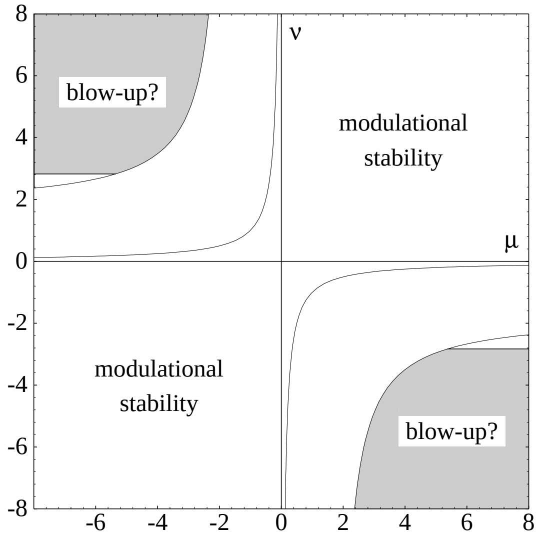

FIGURE 7.1. Parameter plane for the supercritical complex Ginzburg-Landau equation (shown here is the case $d = 3$, $\sigma = 1$). Global existence of smooth solutions is established in the unshaded region only. The inner hyperbolae are the boundaries of the modulationally stable region $1 + \mu\nu \geq 0$. The region within the outer pair of hyperbolae is a result of Theorem 7.1 while the small unshaded strip outside this region is a result of Theorem 7.2. In the subcritical and critical case ($\sigma d \leq 2$) this strip extends to the whole parameter plane.

PROOF. Examine the identity, obtained through integration by parts,

$$
\begin{aligned}
2\int |u|^{p-2}u^*\Delta u\,dx &= -p\int |u|^{p-2}|\nabla u|^2\,dx \\
&\qquad - (p-2)\int |u|^{p-4}u^{*2}\nabla u\cdot\nabla u\,dx\,.
\end{aligned}
\tag{7.19}
$$

For any nonconstant u the first term on the right side is negative while the second term is smaller in magnitude by a factor no greater than $(p-2)/p$, whereby it follows that

$$
\int |u|^{p-2}u^*\Delta u\,dx = -\left|\int |u|^{p-2}u^*\Delta u\,dx\right|\exp(i\theta)\,, \tag{7.20}
$$

for some θ in $|\theta| < \pi/2$ such that $|\sin\theta| \leq (p-2)/p$, or equivalently, such that $\cos\theta \geq 2\sqrt{p-1}/p$. By (7.20), these trigonometric bounds on θ translate into the bounds (7.17) and (7.18). □

This result is more than enough to provide the missing links in the derivations of the differential inequalities (7.5) and (7.12) and it also enables us to prove Theorem 7.2.

PROOF OF THEOREM 7.2. We compute the time evolution of F by taking the appropriate linear combination of (7.3) with $p = 2\sigma + 2$ and (7.10):

(7.21)
$$\frac{1}{2}\frac{dF}{dt} = R\int |\nabla u|^2 + \beta^2 |u|^{2\sigma+2}\,dx - \int |\Delta u|^2 + \beta^2 |u|^{4\sigma+2}\,dx \\ + (1+\beta^2)\,\mathrm{Re}\Big(\int |u|^{2\sigma} u^* \Delta u\,dx\Big) + (\mu - \beta^2\nu)\,\mathrm{Im}\Big(\int |u|^{2\sigma} u^* \Delta u\,dx\Big)\,.$$

The benefits of this linear combination are immediately obvious, noting that when μ and ν have the same sign, one can always find a β that makes the coefficient of the last term zero. In order to obtain an optimal result, we bound the last two terms by Lemma 7.3. In addition to that, one can use a fraction κ of the negative definite term to help control the last two terms. This is best done through the estimate

$$-\int |\Delta u|^2 + \beta^2 |u|^{4\sigma+2}\,dx \le -2\beta \left|\int |u|^{2\sigma} u^* \Delta u\,dx\right|\,. \tag{7.22}$$

Altogether, one obtains the differential inequality

(7.23)
$$\frac{1}{2}\frac{dF}{dt} = R\int |\nabla u|^2 + \beta^2 |u|^{2\sigma+2}\,dx - (1-\kappa)\int |\Delta u|^2 + \beta^2 |u|^{4\sigma+2}\,dx \\ - \Big(2\kappa\beta + (1+\beta^2)\frac{\sqrt{2\sigma+1}}{\sigma+1} - |\mu - \beta^2\nu|\frac{\sigma}{\sigma+1}\Big)\left|\int |u|^{2\sigma} u^* \Delta u\,dx\right|\,.$$

The last term can be neglected provided it is possible to choose β such that its coefficient is nonnegative. This is always the case if $\mu\nu \ge 0$. On the other hand, when $\mu\nu < 0$ we can without loss of generality consider the case where $\mu > 0$ and $\nu < 0$. Taking $\kappa = 1$, this results in the condition

$$2\beta + (1+\beta^2)\frac{\sqrt{2\sigma+1}}{\sigma+1} - (\mu - \beta^2\nu)\frac{\sigma}{\sigma+1} \ge 0\,. \tag{7.24}$$

Optimization in β then results in the restriction to the hyperbolic parameter region (7.16). As we have used up all of the negative definite term, one obtains a bound on F that is exponentially growing in time, sufficient for Theorem 5.6 to assert the existence of global classical solutions. □

REMARK 7.4. For those μ and ν for which (7.16) holds as a strict inequality, we can keep $\kappa < 1$ and obtain uniform bounds on the H^1 and $L^{2\sigma+2}$ norms by closing the differential inequality (7.23) as follows:

(7.25)
$$F^2 \le \Big(1 + \frac{\beta^2}{(\sigma+1)^2}\Big)\Big(\int |\nabla u|^2\,dx\Big)^2 + \Big(\beta^2 + \frac{\beta^4}{(\sigma+1)^2}\Big)\Big(\int |u|^{2\sigma+2}\,dx\Big)^2 \\ \le \Big(1 + \frac{\beta^2}{(\sigma+1)^2}\Big)\int |u|^2\,dx \int |\Delta u|^2 + \beta^2 |u|^{4\sigma+2}\,dx\,.$$

In the second step we have used integration by parts and Cauchy-Schwarz on the first and the Cauchy-Schwarz inequality on the second term. Hence, provided β can be chosen so that (7.24) holds as a strict inequality, F will satisfy the differential inequality

$$\frac{1}{2}\frac{dF}{dt} \leq (\sigma+1)RF - (1-\kappa)\frac{(\sigma+1)^2}{(\sigma+1)^2+\beta^2}\frac{F^2}{\displaystyle\int |u|^2\,dx}\,. \tag{7.26}$$

Upon again using the L^2 bound provided by (7.2), this differential inequality may be integrated as in the appendix to obtain the uniform bound

$$F < \frac{(\sigma+1)^2+\beta^2}{(1-\kappa)(\sigma+1)}\left(\frac{R}{1-e^{-2\sigma Rt}}\right)^{\frac{\sigma+1}{\sigma}}\,. \tag{7.27}$$

This bound has the same scaling in R as $t\to\infty$ as the H^1 bound (7.13). Note that β and κ are subject to the constraint that the coefficient of the last term of (7.23) be nonnegative and, hence, are not independent. The task of optimizing the choice of β and κ in the prefactor of (7.27) is fairly complicated and gives little additional information. However, we conclude by observing that this prefactor blows up as κ approaches unity, which will be the case as (μ,ν) approaches the hyperbolic boundaries of the region (7.16).

8. Global Sobolev Bounds

This section is devoted to the derivation of global bounds on all spatial derivatives of the global smooth solutions found in the last section. As a consequence we will infer that the CGL equation possesses a compact global attractor when considered as a dynamical system over $H^1(\mathbb{T}^d)$. It will be assumed throughout that σ is a positive integer, so that the local existence of C^∞ solutions is ensured by Theorem 5.3, i.e. that the global classical solutions are in fact $C^\infty((0,\infty)\times\mathbb{T}^d)$. This does not yet give us uniform global bounds on norms of the derivatives (although some bounds might be derived by a subtle analysis of the bootstrapping arguments of Section 5). Here we explain how to directly control the norms of the Sobolev spaces $H^n(\mathbb{T}^d)$ and how to use those bounds to obtain global estimates on the analytic regularity of the solution.

Bartuccelli *et al.* [**3**] first estimated the H^n norms of solutions to the cubic CGL equation by constructing a "ladder"—a sequence of nonlinear functionals, each of which controls successively higher derivatives of u, and that are connected by differential inequalities. The method was subsequently applied to the generalized CGL equation [**25**] and the two-dimensional incompressible Navier-Stokes equation [**23**]. Recent work has shown that it is possible to find a differential inequality that yields global H^n control directly from any of the global L^p estimates of Section 7, i.e. for any n without the need for intermediate rungs and functionals [**4**]. This form of the estimate is presented below.

As the L^2 norm of u is already controlled by (7.2), it is sufficient to estimate the L^2 norm of $\nabla^n u$. Here $\nabla^n u$ is considered to be a symmetric n-tensor and

$$|\nabla^n u|^2 \equiv \nabla^n u^* \cdot \nabla^n u\,, \tag{8.1}$$

where the dot denotes the usual tensor inner product which acts on a symmetric k-tensor and a symmetric l-tensor by contracting over $\min\{k,l\}$ indices to give a symmetric $|k-l|$-tensor.

A direct calculation shows that

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|\nabla^n u\|_{L^2}^2 &= R\,\|\nabla^n u\|_{L^2}^2 - \|\nabla^{n+1}u\|_{L^2}^2 \\
&\quad - \operatorname{Re}\biggl((1+i\mu)\int \nabla^n(|u|^{2\sigma}u)\cdot\nabla^n u^*\,dx\biggr)\,.
\end{aligned}
\tag{8.2}
$$

The second term on the right is easily estimated using the Sobolev inequality

$$
\|\nabla^n u\|_{L^2} \le \|\nabla^{n+1}u\|_{L^2}^{\frac{n}{n+1}}\,\|u\|_{L^2}^{\frac{1}{n+1}}\,. \tag{8.3}
$$

More work is required to achieve a bound on the nonlinear term. The estimate we will use is

$$
\left|\int \nabla^n(|u|^{2\sigma}u)\cdot\nabla^n u^*\,dx\right| \le \delta\,\|\nabla^{n+1}u\|_{L^2}^2 + c(\delta)\,\|\nabla^n u\|_{L^2}^2\,\|u\|_{L^p}^{\frac{2\sigma p}{p-\sigma d}}\,. \tag{8.4}
$$

In this formula (which will be proved later as Lemma 8.3) we can choose $\delta > 0$ and $p > \sigma d$ to suit our purpose. In particular, by setting $\delta = |1+i\mu|^{-1}$ we can cancel the first term with half of the dissipative term of equation (8.2) to obtain

$$
\frac{d}{dt}\|\nabla^n u\|_{L^2}^2 \le \left(2R + c\,\|u\|_{L^p}^{\frac{2\sigma p}{p-\sigma d}}\right)\|\nabla^n u\|_{L^2}^2 - \frac{\|\nabla^n u\|_{L^2}^{2(n+1)/n}}{\|u\|_{L^2}^{2/n}}\,. \tag{8.5}
$$

One can use the *a priori* L^2 bound (7.2) to control the denominator of the last term. It is then clear that global L^p control with $p > \sigma d$—note that this is precisely the condition for local regularity from L^p data—is sufficient to achieve global H^n control for any n. Explicit bounds can be obtained by following the procedure in the appendix, although in general it will not be possible to integrate the differential inequality exactly. Hence we state the result in its simplest form.

THEOREM 8.1. *For all positive integers d, σ and n and for every $p > \sigma d$, when μ and ν lie in the region of the parameter plane where uniform L^p bounds on solutions of the CGL equation with L^p initial data exist, these solutions are global and smooth for all positive times with their H^n norm bounded uniformly for all initial data and $t \to \infty$, satisfying the differential inequality* (8.5).

REMARK 8.1. Estimate (8.5) does not give the best scaling in all cases, although known improvements are slight. For a more detailed discussion see [**4**].

REMARK 8.2. Nonlinear dissipation enters estimate (8.5) only implicitly via the L^2 and possibly the L^p bound. The explicit dependence on the sign of the nonlinear term has been lost in step (8.4). This in particular shows that by the same method global H^n bounds, although not necessarily uniform in time, can be obtained for equations of the type

$$
\partial_t u = Ru + (1+i\nu)\,\Delta u - i\mu\,|u|^{2\sigma}u\,. \tag{8.6}
$$

REMARK 8.3. Theorem 8.1 shows that the CGL equation possesses a compact global attractor when considered as a dynamical system over any $H^n(\mathbb{T}^d)$. (For purposes of comparison, it is natural to consider both the NLS and CGL equations as dynamical systems over $H^1(\mathbb{T}^d)$.) Indeed, because $H^{n+1}(\mathbb{T}^d)$ is compactly embedded in $H^n(\mathbb{T}^d)$, the existence of an absorbing ball in $H^{n+1}(\mathbb{T}^d)$, which is inferred from (8.5) and necessarily contains the attractor, implies the compactness of this ball in $H^n(\mathbb{T}^d)$. Furthermore, by the general theory developed by Mallet-Paret [**58**] and Mañé [**60**] the attractor has finite Hausdorff dimension and can be locally embedded in finite dimensional manifolds. This theory is reviewed in [**42**]. Explicit upper bounds for the Hausdorff dimension of the attractor were derived for $d = 1$ by Doering, Gibbon, Holm and Nicolaenko [**24**], for $d = 1$, 2 by Ghidaglia and Héron [**35**] and for $d = 3$ by Bartuccelli *et al.* [**3**]. The techniques are based on the estimation of global Lyapunov exponents which are discussed in [**13, 14, 78**].

Remark 8.4. In some cases the attractor can be globally embedded in an *inertial manifold*, a notion introduced by Foias, Sell and Temam [**29**]. An inertial manifold is a finite dimensional, exponentially attracting, positively invariant manifold that contains the global attractor—for a general discussion see [**29, 15, 16**] and references therein. For $d = 1, 2$ the CGL equation is known to possess an inertial manifold while for $d \geq 3$ this is still an open question. Moreover, its dimension is bounded by the attractor dimension from below, while upper bounds can be proved following [**16, 24**]. This procedure relies on finding a large enough gap in the spectrum of the Laplacian, leading to a much larger estimate on the dimension of the manifold when $d = 2$ and failing for $d = 3$ as there are no arbitrarily large gaps in the spectrum. By contrast, it is not clear whether the Navier-Stokes equations in two spatial dimensions possess an inertial manifold. This question remains an active area of research.

To prove estimate (8.4), we first derive a preliminary estimate in which we impose that the Gagliardo-Nirenberg inequalities be used at their margin of validity, thus rendering the estimate independent of the dimension d. This result will then be used in two different ways to complete the proof.

Lemma 8.2. *For all positive integers d, σ and n the following inequality holds:*

$$\left| \int \nabla^n (|u|^{2\sigma} u) \cdot \nabla^n u^* \, dx \right| \leq c(n, q) \, \|\nabla^n u\|^2_{L^{2q}} \, \|u\|^{2\sigma}_{L^{2r\sigma}} \,, \tag{8.7}$$

where

$$\frac{1}{q} + \frac{1}{r} = 1 \,, \qquad 1 \leq q, r \leq \infty \,. \tag{8.8}$$

Remark 8.5. The advantage of this result, derived in [**69, 4**], in comparison with earlier estimates [**3, 25**] is that it permits a free choice of r while keeping the estimate independent of d.

Proof. By moving the absolute value inside the integral, performing a Leibniz expansion on $\nabla^n(|u|^{2\sigma}u)$, and then using the triangle inequality, one obtains

$$\left| \int \nabla^n (|u|^{2\sigma} u) \cdot \nabla^n u^* \, dx \right| \leq \sum_{\substack{\alpha \in \mathbb{N}^{2\sigma+1} \\ |\alpha| = n}} \frac{n!}{\alpha!} \int |\nabla^n u| \prod_{j=1}^{2\sigma+1} |\nabla^{\alpha_j} u| \, dx \,. \tag{8.9}$$

Here we have again employed the multi-index notation in which $\alpha!$ and $|\alpha|$ are defined as in (6.2). For a given multi-index α with $|\alpha| = n$ define θ_j and q_j by

$$\theta_j = \frac{\alpha_j}{n} \,, \qquad \frac{1}{q_j} = \theta_j \frac{1}{2q} + (1 - \theta_j) \frac{1}{2\sigma r} \,. \tag{8.10}$$

These satisfy

$$\sum_{j=1}^{2\sigma+1} \theta_j = \frac{|\alpha|}{n} = 1 \,, \qquad \frac{1}{2q} + \sum_{j=1}^{2\sigma+1} \frac{1}{q_j} = \frac{1}{q} + \frac{1}{r} = 1 \,. \tag{8.11}$$

Then the corresponding term on the right side of (8.9) can be estimated with a Hölder inequality followed by a Gagliardo-Nirenberg interpolation on each of the factors:

$$
\begin{aligned}
\int |\nabla^n u| \prod_{j=1}^{2\sigma+1} |\nabla^{\alpha_j} u| \, dx &\leq \|\nabla^n u\|_{L^{2q}} \prod_{j=1}^{2\sigma+1} \|\nabla^{\alpha_j} u\|_{L^{q_j}} \\
&\leq \|\nabla^n u\|_{L^{2q}} \prod_{j=1}^{2\sigma+1} c_j \, \|\nabla^n u\|_{L^{2q}}^{\theta_j} \, \|u\|_{L^{2\sigma r}}^{1-\theta_j} \\
&= c \, \|\nabla^n u\|_{L^{2q}}^2 \|u\|_{L^{2\sigma r}}^{2\sigma} .
\end{aligned}
\tag{8.12}
$$

Finally, by setting this into (8.9), we obtain the stated result. □

This result will now be used to establish estimate (8.4), thereby completing the proof of Theorem 8.1.

LEMMA 8.3. *For all positive integers d, σ and n and for every $p > \sigma d$ and every $\delta > 0$, one has the estimate*

$$
\left| \int \nabla^n (|u|^{2\sigma} u) \cdot \nabla^n u^* \, dx \right| \leq \delta \, \|\nabla^{n+1} u\|_{L^2}^2 + c(\delta) \, \|\nabla^n u\|_{L^2}^2 \, \|u\|_{L^p}^{\frac{2\sigma p}{p-\sigma d}} . \tag{8.13}
$$

PROOF. Estimate the right side of (8.7) with a Gagliardo-Nirenberg inequality, followed by Young's inequality:

$$
\begin{aligned}
\left| \int \nabla^n (|u|^{2\sigma} u) \cdot \nabla^n u^* \, dx \right| &\leq c_1 \, \|\nabla^{n+1} u\|_{L^2}^{d/r} \, \|\nabla^n u\|_{L^2}^{2-d/r} \, \|u\|_{L^{2\sigma r}}^{2\sigma} \\
&\leq \delta \, \|\nabla^{n+1} u\|_{L^2}^2 + c_2(\delta) \, \|\nabla^n u\|_{L^2}^2 \, \|u\|_{L^{2\sigma r}}^{\frac{4\sigma}{2-d/r}} .
\end{aligned}
\tag{8.14}
$$

For the inequalities to be valid, we must have $2 - d/r > 0$. By setting $p = 2\sigma r$, we obtain the statement of Lemma 8.3.

If $d = 1$ however, the condition $r \geq 1$ from Lemma 8.2 would restrict the possible choices of p even further, so that for $\frac{1}{2} < r < 1$ we must complement the proof with a different interpolation in order to obtain the result in its full generality. Thus we estimate

$$
\begin{aligned}
\left| \int \partial^n (|u|^{2\sigma} u) \partial^n u^* \, dx \right| &\leq c_3 \, \|\partial^n u\|_{L^\infty}^{2\lambda} \, \|\partial^n u\|_{L^\infty}^{2(1-\lambda)} \, \|u\|_{L^{2\sigma}}^{2\sigma} \\
&\leq c_4 \Big(\|\partial^{n+1} u\|_{L^2}^a \, \|u\|_{L^{2\sigma r}}^{1-a} \Big)^{2\lambda} \\
&\quad \times \Big(\|\partial^{n+1} u\|_{L^2}^{1/2} \, \|\partial^n u\|_{L^2}^{1/2} \Big)^{2(1-\lambda)} \\
&\quad \times \Big(\|\partial^{n+1} u\|_{L^2}^b \, \|u\|_{L^{2\sigma r}}^{1-b} \Big)^{2\sigma} ,
\end{aligned}
\tag{8.15}
$$

where a and b are Gagliardo-Nirenberg exponents computed in the usual way. By the choice of $\lambda = 1/r - 1$, this equation reduces exactly to (8.14) so that the proof can proceed as before. □

REMARK 8.6. The suggestive grouping of terms in equation (8.15) does not show a lower order term that has to be included in the estimate of $\|u\|_{L^{2\sigma}}$ because u is not a mean-zero function. However, in this case it is possible to absorb the extra term into the constant c_4. For details see [**4**].

REMARK 8.7. The ladder inequalities introduced in [**3**] and generalized in [**25**] use functionals of the form

$$F_{n,m} = \int_{\mathbb{T}^d} |\nabla^n u|^{2m} + \alpha_{n,m} |u|^{2mn(\sigma+1)} \, dx \,, \tag{8.16}$$

the choice of which is motivated by the observation that $|u|^{2\sigma}$ scales dimensionally like Δ in the CGL equation. Terms like $\nabla^n u$ and $|u|^{n\sigma} u$ have the same weight under this dimensional scaling. This however does not give any advantage over the direct approach as presented here.

REMARK 8.8. We remark that although the implications of Theorem 8.1 add nothing so far as the existence of global smooth solutions is concerned, the explicit estimates that have been derived are useful. Some of these estimates are discussed in detail in references [**3, 4**].

Finally we will use the global Sobolev bounds of this section to initialize the local Gevrey class estimate of Section 6. Recall that the differential inequality (6.17) shows that for every t for which $u(t) \in H^r(\mathbb{T}^d)$ there exists a positive finite time $T_{\max}$ such that for every $T \in [0, T_{\max}]$

$$\|A^r u(t+T)\|_T \leq 2\, \|A^r u(t)\|_0 \leq 2\, \|u(t)\|_{H^r} \,. \tag{8.17}$$

We know from Theorem 8.1 that $\|u\|_{H^r}$ is bounded uniformly over all initial data. One therefore has the following.

THEOREM 8.4. *For positive integers d, σ and for every $r > d/2$ and $p > \sigma d$, when μ and ν lie in the region of the parameter plane where uniform L^p bounds on solutions of the CGL equation with L^p initial data exist, these solutions are global and real analytic for all positive times. Furthermore, there is a $T_{\max}$ such that the $\mathcal{D}(e^{TA} \colon H^r(\mathbb{T}^d))$ norm of $u(t)$ is bounded uniformly over all initial data, for every $T \in [0, T_{\max}]$ and for every time $t \geq T$.*

Theorem 8.4 implies that the Fourier modes of a solution to the CGL equation decay exponentially with the wavenumber ξ with a rate of decay which is uniform over large times. Doelman [**21**] first observed that the Fourier coefficients decay faster than any algebraic order. The exponential decay rate was established later in [**22**] by using methods similar to the ones presented in Section 6. This is consistent with numerical observations as well as with theoretical results concerning notions of finite-dimensionality which indicate that CGL dynamics is relatively low dimensional.

From the computational viewpoint the exponential decay of the Fourier coefficients means that simple Fourier-Galerkin truncation works well. The CGL dynamics is captured by a manageable number of Fourier modes [**22, 26**]. Furthermore, more sophisticated nonlinear Galerkin methods offer no advantage over Fourier-Galerkin truncation [**46**], a fact that is intrinsically linked to analytic regularity and the exponential decay of the Fourier coefficients.

A general question of current interest is whether the solution of a dynamical system can be determined by the time history of a finite set of data. The notion of *determining modes* was introduced by Foias and Prodi [**28**] in the context of the two dimensional incompressible Navier-Stokes equations. A set of Fourier modes is called determining if the knowledge of the time history of the corresponding Fourier coefficients completely specifies the solution. In other words, if the determining modes of any two solutions agree then the solutions are identical. This idea was subsequently extended to more general *determining degrees of freedom*—for example, the solution evaluated at nodes or averaged over cells [**31, 11**]. It is important to note that the dynamics of the full equation is not necessarily represented by the restricted dynamics on a set of determining modes. Hence, a low number of determining degrees of freedom has no direct implication for computational considerations. However, when an inertial manifold exists (as for CGL when $d = 1, 2$), a finite number of Fourier modes usually serve as its coordinates. The infinite dimensional dynamics of the remaining Fourier modes is therefore slaved by the dynamics of a finite dimensional set. In such cases one can generally establish bounds on the number of determining degrees of freedom in terms of the dimension of the inertial manifold [**11**]. Such bounds can be far from optimal. Indeed, the CGL equation can exhibit a very low number of determining degrees of freedom. Kukavica [**50**] showed that for $d = \sigma = 1$ the periodic CGL has sets of determining nodes consisting of only two points. His proof readily extends to the case where σ is any positive integer. Analytic regularity is a key ingredient in the argument. There is now a considerable effort to extend and sharpen these results.

Appendix. On the Integration of Differential Inequalities

Throughout Sections 7 and 8 we infer uniform bounds for any nonnegative quantity $F(t)$ that satisfies a differential inequality of the general form

$$\frac{dF}{dt} \leq a(t)F - b(t)F^{1+s}, \tag{A.1}$$

where $s > 0$ and $b(t) > 0$. Following Bernoulli, introduce the variable $Y = F^{-s}$ into (A.1) and obtain the linear differential inequality

$$\frac{dY}{dt} \geq -s\,a(t)\,Y + s\,b(t)\,. \tag{A.2}$$

Now multiply this equation by $\exp\big(\int_{t'}^{t} s\,a(t'')\,dt''\big)$ to obtain

$$\frac{d}{dt}\left(Y \exp\left(\int_{t'}^{t} s\,a(t'')\,dt''\right)\right) \geq s\,b(t)\,\exp\left(\int_{t'}^{t} s\,a(t'')\,dt''\right). \tag{A.3}$$

Integration of (A.3) between 0 and t results in

$$Y(t) \geq Y(0)\,\exp\left(-\int_0^t s\,a(t')\,dt'\right) + \int_0^t s\,b(t')\,\exp\left(-\int_{t'}^{t} s\,a(t'')\,dt''\right)dt'\,. \tag{A.4}$$

The uniform upper bounds on $F(t)$ follow directly by neglecting the $Y(0)$ term, which effectively means taking the supremum over all initial conditions $F(0)$:

$$F(t) < \left(\int_0^t s\,b(t')\,\exp\left(-\int_{t'}^{t} s\,a(t'')\,dt''\right)dt'\right)^{-1/s}. \tag{A.5}$$

This bound is satisfied by any nonnegative quantity $F(t)$ that satisfies the differential inequality (A.1), no matter what its initial value happens to be.

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