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DISTRIBUTION-VALUED INITIAL DATA FOR THE COMPLEX GINZBURG-LANDAU EQUATION

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Abstract. The generalized complex Ginzburg-Landau (CGL) equation with a nonlinearity of order $2\sigma + 1$ in d spatial dimensions has a unique local classical solution for distributional initial data in the Sobolev space H^q provided that $q > \frac{d}{2} - \frac{1}{\sigma}$. This result directly corresponds to a theorem for the nonlinear Schrödinger (NLS) equation which has been proved by Cazenave and Weissler in 1990. While the proof in the NLS case relies on Besov space techniques, it is shown here that for the CGL equation, the smoothing properties of the linear semigroup can be used to obtain an almost optimal result by elementary means.

1. Introduction

The generalized complex Ginzburg-Landau (CGL) equation

$$\partial_t u = Ru + (1 + i\nu)\Delta u - (1 + i\mu)|u|^{2\sigma} u \tag{1}$$

describes the evolution of a complex-valued field $u = u(x,t)$ on $\Omega \times [0,T]$ for some maximal time $T > 0$. The parameters μ, ν, σ and R are real numbers, with σ and R positive. To simplify notation, we explicitly consider spatial

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domains $\Omega = \mathbb{T}^d$ for any number of dimensions d only. However, corresponding results hold over other domains without boundary.

It is well known that the CGL equation is locally well-posed for L^p initial data whenever $p > \sigma d$. Generally, the Laplacian dissipation will regularize the solution for positive times. When σ is a positive integer, one can in fact show that the solution becomes real analytic for all positive times [5, 9], so that the following local existence theorem holds.

Theorem 1. *Let $\sigma \geq 1$ be an integer and $p > \sigma d$. Then for every $\rho > 0$ there exists a time $T(\rho) > 0$ such that for every initial data $u^{\mathrm{in}} \in \mathrm{L}^p(\mathbb{T}^d)$ with $\|u^{\mathrm{in}}\|_{\mathrm{L}^p} \leq \rho$ there exists a unique classical solution*

$$u \in \mathrm{C}([0,T]; \mathrm{L}^p(\mathbb{T}^d)) \cap \mathrm{C}^\omega((0,T] \times \mathbb{T}^d) \tag{2}$$

to the complex Ginzburg-Landau initial-value problem (1).

When σ is not an integer, n-fold differentiation of (1) introduces unbounded singularities at the zeros of u whenever $n > 2\sigma$, which restricts the degree of regularity on can expect. More precisely, the following is true.

Theorem 2. *Let $\sigma > 0$ and $p > \sigma d$. Then for every $\rho > 0$ there exists a time $T(\rho) > 0$ such that for every initial data $u^{\mathrm{in}} \in \mathrm{L}^p(\mathbb{T}^d)$ with $\|u^{\mathrm{in}}\|_{\mathrm{L}^p} \leq \rho$ there exists a unique classical solution*

$$u \in \mathrm{C}([0,T]; \mathrm{L}^p(\mathbb{T}^d)) \cap \mathrm{C}((0,T]; \mathrm{C}^{n+2}(\mathbb{T}^d)) \cap \mathrm{C}^{\left[\frac{n}{2}\right]+1}((0,T]; \mathrm{C}^{n-2\left[\frac{n}{2}\right]}(\mathbb{T}^d)) \tag{3}$$

to the complex Ginzburg-Landau initial-value problem (1).

Both theorems are based on contraction mapping arguments and are essentially due to Weissler [11]. Proofs can also be found in [6, 9]. In the next section, we prove an abstract local existence theorem much in the spirit of the above results. Applied to the CGL equation in Section 3, the abstract theorem enables us to actually enlarge the class of initial data for which the statements of Theorems 1 and 2 hold.

The motivation for this further inquiry comes from the observation that in the CGL equation $|u|^{2\sigma}$ scales dimensionally like Δ, so that by formally trading powers for derivatives in the condition $p > \sigma d$, one transforms either of the above theorems into the following conjecture: Given $u^{\mathrm{in}} \in \mathrm{H}^q(\mathbb{T}^d)$ where $q > \frac{d}{2} - \frac{1}{\sigma}$, there exists a unique local classical solution. Our result shows that, under certain restrictions, the conjecture is true.

2. An Abstract Local Existence Theorem

In the following, we formulate the problem in an abstract setting of sufficient generality. The methods are mostly standard [8, 10, 11] and some detail will be omitted. The CGL equation is a semilinear parabolic equation of the form

$$\partial_t u = Lu + N(u)\,, \qquad u(0) = u^{\mathrm{in}}\,. \tag{4}$$

If L is formally the infinitesimal generator of a linear semigroup $S(t)$, the equation can be recast in its *mild formulation*

$$u(t) = S(t)u^{\mathrm{in}} + \int_0^t S(t-t')N(u(t'))\,dt'\,. \tag{5}$$

We consider this problem for all initial data u^{in} in a Banach space $\mathbb{X}$ over which the semigroup $S(t)$ is strongly continuous (so that the linear problem with $N = 0$ is well-posed, see [10]). One aims at proving that the solution of (5) lies in a "better" Banach space $\mathbb{Y}$ that is continuously embedded in $\mathbb{X}$.

For a suitable choice of $\mathbb{Y}$ the semigroup will be strongly continuous over $\mathbb{Y}$ and also map $\mathbb{X}$ back into $\mathbb{Y}$, so that near $t = 0$ one has an estimate of the form

$$\|S(t)w\|_{\mathbb{Y}} \le c\,t^{-\alpha}\,\|w\|_{\mathbb{X}} \quad \text{for every } w \in \mathbb{X} \tag{6}$$

with α a suitably chosen nonnegative constant. In addition, the map N will be locally Lipschitz from $\mathbb{Y}$ to $\mathbb{X}$. Without loss of generality we assume that $N(0) = 0$, and for our convenience, we will only consider the case where the Lipschitz condition can be written as

$$\|N(u_1) - N(u_2)\|_{\mathbb{X}} \le c\left(\|u_1\|_{\mathbb{Y}}^{2\sigma} + \|u_2\|_{\mathbb{Y}}^{2\sigma}\right)\|u_1 - u_2\|_{\mathbb{Y}} \tag{7}$$

for some $\sigma \ge 0$. The σ above will be identified with the σ of the CGL nonlinearity (15). This covers many applications, including the CGL equation. Modifications that embrace more general nonlinearities are not hard [8, 10].

We will establish existence in a space $\mathrm{E}^\alpha([0,T];\mathbb{Y})$ for some $T > 0$, which for each $T > 0$ is defined as the completion of $\mathrm{C}([0,T];\mathbb{Y})$ in the norm

$$\|w\|_{\mathrm{E}^\alpha} \equiv \sup_{t\in[0,T]} \left\{t^\alpha\,\|w(t)\|_{\mathbb{Y}}\right\}. \tag{8}$$

Condition (6) implies both that $\mathbb{Y}$ is dense in $\mathbb{X}$ and that $S(t)w \in \mathrm{E}^\alpha([0,T];\mathbb{Y})$ for every $w \in \mathbb{X}$. The following abstract theorem holds.

Theorem 3 (General Local Existence). *Given Banach spaces $\mathbb{X}$ and $\mathbb{Y}$ and a linear semigroup $S(t)$ for which estimates* (6) *and* (7) *hold with exponents α and σ that satisfy*

$$0 \leq (2\sigma+1)\alpha < 1\,, \tag{9}$$

then for every $\rho > 0$ there exists a time $T(\rho) > 0$ such that for every initial data $u^{\mathrm{in}} \in \mathbb{X}$ with $\|u^{\mathrm{in}}\|_{\mathbb{X}} \leq \rho$ there exists a unique

$$u \in \mathrm{C}([0,T];\mathbb{X}) \cap \mathrm{E}^{\alpha}([0,T];\mathbb{Y}) \tag{10}$$

that satisfies the mild formulation (5)*. In addition, the map from u^{in} to u is a locally Lipschitz function from $\mathbb{X}$ to* $\mathrm{C}([0,T];\mathbb{X})$.

Remark 1. In our application, $\mathbb{Y}$ will be a Lebesgue space L^p rather than the domain of a power of L as is commonly done [8].

Proof. In order to indicate how to set up the contraction mapping argument, we first estimate the fixed-point equation (5) in the $\mathbb{Y}$ norm,

$$\|u(t)\|_{\mathbb{Y}} \leq c\,t^{-\alpha}\,\|u^{\mathrm{in}}\|_{\mathbb{X}} + c\int_0^t (t-t')^{-\alpha}\,\|u(t')\|_{\mathbb{Y}}^{2\sigma+1}\,dt'\,, \tag{11}$$

or

$$\begin{aligned} t^{\alpha}\,\|u(t)\|_{\mathbb{Y}} &\leq c\,\|u^{\mathrm{in}}\|_{\mathbb{X}} + c\,t^{\alpha}\int_0^t (t-t')^{-\alpha}\,t'^{\,-(2\sigma+1)\alpha}\,\Big(t'^{\alpha}\,\|u(t')\|_{\mathbb{Y}}\Big)^{2\sigma+1}\,dt' \\ &\leq c\,\|u^{\mathrm{in}}\|_{\mathbb{X}} + c\sup_{t'\in[0,t]}\big\{t'^{\alpha}\,\|u(t')\|_{\mathbb{Y}}\big\}^{2\sigma+1} t^{\alpha}\int_0^t (t-t')^{-\alpha}\,t'^{\,-(2\sigma+1)\alpha}\,dt'\,. \end{aligned} \tag{12}$$

This suggests we might be able to prove contractivity in a space $\mathrm{E}^{\alpha}([0,T];\mathbb{Y})$ whose norm is defined by (8). Indeed, (12) shows that provided

$$t^{\alpha}\int_0^t (t-t')^{-\alpha}\,t'^{\,-(2\sigma+1)\alpha}\,dt' \to 0 \qquad \text{as } t \to 0\,, \tag{13}$$

an assumption that is directly seen to be equivalent to (9), there exists a $T > 0$ such that the right side of the fixed-point equation (5) maps sufficiently large balls with respect to the E^{α} norm into themselves. Invoking the Lipschitz condition (7), one can show in a similar way that this mapping contracts under the same assumption (13) for some $T > 0$. Then by the contraction mapping theorem, there exists a unique solution $u \in \mathrm{E}^{\alpha}([0,T];\mathbb{Y})$ to the fixed-point problem (5).

Finally, one can show that $u \in \mathrm{C}([0,T];\mathbb{X})$. To this end we only need to check the continuity of u at $t=0$. First subtracting u^{in} from both sides of the mild formulation (5), a direct estimate in $\mathbb{X}$ gives

$$\begin{aligned}\|u(t)-u^{\mathrm{in}}\|_{\mathbb{X}} &\leq \|S(t)u^{\mathrm{in}}-u^{\mathrm{in}}\|_{\mathbb{X}} + c\int_0^t t'^{-(2\sigma+1)\alpha}\left(t'^{\alpha}\,\|u(t')\|_{\mathbb{Y}}\right)^{2\sigma+1} dt' \\ &\leq \|S(t)u^{\mathrm{in}}-u^{\mathrm{in}}\|_{\mathbb{X}} + c\,\|u\|_{\mathrm{E}^\alpha}^{2\sigma+1}\int_0^t t'^{-(2\sigma+1)\alpha} dt'\,.\end{aligned} \tag{14}$$

The first term on the right converges to zero due to the strong continuity of the semigroup over $\mathbb{X}$. Since $\|u\|_{\mathrm{E}^\alpha}$ is bounded, the second term converges to zero provided that $(2\sigma+1)\alpha<1$, a condition already ensured by (9). ☐

3. The CGL Initial-Value Problem

Let us now apply Theorem 3 to the CGL equation where N and L are given by

$$Lu = (1+i\nu)\Delta u + Ru\,, \qquad N(u) = -(1+i\mu)|u|^{2\sigma}u\,. \tag{15}$$

The associated semigroup can be written as a convolution, $S(t)u = G_t * u$, with its Green function $G_t = G_t(x)$ for $t>0$ given by

$$\begin{aligned} G_t(x) &= \sum_{n\in\mathbb{Z}^d} g_t(x+n)\,, \\ g_t(x) &= \frac{1}{(4\pi(1+i\nu)t)^{d/2}}\exp\left(-\frac{|x|^2}{4(1+i\nu)t}+Rt\right).\end{aligned} \tag{16}$$

The integral equation (5) recast in terms of this Green function takes the form

$$u(t) = G_t * u^{\mathrm{in}} + \int_0^t G_{t-t'} * N(u(t'))\,dt'\,. \tag{17}$$

In order to apply Theorem 3 one only need identify spaces $\mathbb{X}$ and $\mathbb{Y}$ for which (6) and (7) hold, and verify condition (9).

We first remark that it is very problematic to conduct the argument entirely in H^r spaces. The difficulty lies in obtaining a Lipschitz estimate (7), which is easily accomplished only when $\mathbb{X}=\mathbb{Y}\equiv\mathrm{H}^q(\mathbb{T}^d)$ is an algebra, i.e. when $q > d/2$. This case does not however exhaust the conjectured range of admissible values for q.

We next observe that the only interesting case is $q<0$. Namely, when $q > \frac{d}{2}-\frac{1}{\sigma}$ and $q>0$, one can always find a $p>\sigma d$ so that the Sobolev embedding $\mathrm{H}^q \hookrightarrow \mathrm{L}^p$ holds, i.e. that $\frac{1}{p} > \frac{1}{2}-\frac{q}{d}$. Therefore local classical solutions are already guaranteed by Theorems 1 and 2. We shall therefore assume that $u^{\mathrm{in}} \in \mathrm{H}^q(\mathbb{T}^d) \equiv \mathbb{X}$ for $q < 0$, and let $\mathbb{Y} = \mathrm{L}^p(\mathbb{T}^d)$ with p still to be determined. The fact that $S(t)$ is strongly continuous in any such $\mathrm{H}^q(\mathbb{T}^d)$ and $\mathrm{L}^p(\mathbb{T}^d)$ is standard [8, 10].

The first ingredient needed for the application of Theorem 3 is estimate (6) on the linear semigroup, which we derive through the intermediate space L^2. We begin with computing the exponent of the "penalty" incurred by using $S(t)$ as a smoothing operator from L^2 to L^p. For $p \geq 2$ one has

$$\|G_t * w\|_{\mathrm{L}^p} \leq \|G_t\|_{\mathrm{L}^r} \|w\|_{\mathrm{L}^2} \leq \|G_t\|_{\mathrm{L}^1}^{1/r} \|G_t\|_{\mathrm{L}^\infty}^{1-1/r} \|w\|_{\mathrm{L}^2} \tag{18}$$

where

$$1 + \frac{1}{p} = \frac{1}{r} + \frac{1}{2}\,. \tag{19}$$

From the explicit form of the Green function (16) one can directly obtain the L^1 estimate

$$\|G_t\|_{\mathrm{L}^1} \leq \sum_{n \in \mathbb{Z}^d} \int_{\mathbb{T}^d} |g_t(x+n)|\, dx = \int_{\mathbb{R}^d} |g_t(x)|\, dx = (1+\nu^2)^{d/4}\, e^{Rt}\,, \tag{20}$$

and the L^∞ estimate

$$\|G_t\|_{\mathrm{L}^\infty} \leq \frac{(1+\nu^2)^{d/4}\, e^{Rt}}{(4\pi(1+\nu^2)t)^{d/2}} \bigl\{ a + b[(1+\nu^2)t]^{d/2} \bigr\}\,, \tag{21}$$

with positive absolute constants a and b. Hence, as $t \to 0$ one finds that

$$\|G_t\|_{\mathrm{L}^1} = O(t^0)\,, \qquad \|G_t\|_{\mathrm{L}^\infty} = O(t^{-d/2})\,. \tag{22}$$

This used with (18) and (19) gives that for t small,

$$\|S(t)w\|_{\mathrm{L}^p} \leq c\, t^{-\frac{d}{2}\left(\frac{1}{2}-\frac{1}{p}\right)} \|w\|_{\mathrm{L}^2}\,. \tag{23}$$

The exponent of the "penalty" incurred by using $S(t)$ as a smoothing operator from L^2 to L^p is therefore $\frac{d}{2}\bigl(\frac{1}{2} - \frac{1}{p}\bigr)$.

Next, we have to find the exponent of the "penalty" incurred by using $S(t)$ as a smoothing operator from H^q to L^2. The convolution $S(t)w = G_t * w$ is best estimated by deriving bounds on its Fourier transform. Specifically, we have

$$\begin{aligned} \|G_t * w\|_{\mathrm{L}^2}^2 &= \sum_{\xi \in 2\pi\mathbb{Z}^d} |\widehat{g}(\xi)\widehat{w}(\xi)|^2 \\ &\leq \sup_{\xi \in 2\pi\mathbb{Z}^d} \bigl\{ (1+|\xi|^2)^{-q}\, |\widehat{g}(\xi)|^2 \bigr\} \sum_{\xi \in 2\pi\mathbb{Z}^d} (1+|\xi|^2)^q\, |\widehat{w}(\xi)|^2\,, \end{aligned} \tag{24}$$

where $\widehat{w}(\xi)$ denotes the Fourier coefficients of w, and the Fourier coefficients of G_t are given by

$$\widehat{g}(\xi) = e^{Rt}\, e^{-(1+i\nu)|\xi|^2 t}\,. \tag{25}$$

The supremum in (24) is easily estimated by optimization with respect to $|\xi|^2$, resulting in

$$\|S(t)w\|_{\mathrm{L}^2}^2 \le \left(\frac{-q}{2et}\right)^{-q} e^{2(R+1)t}\, \|w\|_{\mathrm{H}^q}^2\,. \tag{26}$$

Together with the L^p estimate (23) one finally obtains that for t small,

$$\|S(t)w\|_{\mathrm{L}^p} \le c\, t^{-\frac{d}{2}(\frac{1}{2}-\frac{1}{p})}\, \|S(t/2)w\|_{\mathrm{L}^2} \le c'\, t^{-\alpha}\, \|w\|_{\mathrm{H}^q}\,, \tag{27}$$

whereby estimate (6) is established with

$$\alpha = \frac{d}{2}\left(\frac{1}{2} - \frac{1}{p}\right) - \frac{q}{2}\,. \tag{28}$$

Recalling that q was assumed negative, $p \ge 2$, and $0 \le (2\sigma+1)\alpha < 1$ by (9) of Theorem 3, we find that the semigroup estimate imposes a lower and an upper bound on the possible choices of $1/p$,

$$\frac{1}{2} - \frac{q}{d} - \frac{2}{(2\sigma+1)d} < \frac{1}{p} \le \frac{1}{2}\,. \tag{29}$$

These bounds are consistent if and only if

$$-\frac{2}{2\sigma+1} < q\,. \tag{30}$$

The second ingredient needed for the application of Theorem 3 is the Lipschitz estimate (7), which we derive through the intermediate space L^r where $r = p/(2\sigma+1)$. By straightforward applications of Hölder and Young inequalities one finds

$$\|N(u_1) - N(u_2)\|_{\mathrm{L}^r} \le c\left(\|u_1\|_{\mathrm{L}^p}^{2\sigma} + \|u_2\|_{\mathrm{L}^p}^{2\sigma}\right) \|u_1 - u_2\|_{\mathrm{L}^p}\,. \tag{31}$$

Moreover, the embedding $\mathrm{L}^r \hookrightarrow \mathrm{H}^q$ is continuous whenever $\frac{1}{r} < \frac{1}{2} - \frac{q}{d}$ and $1 \le r \le 2$. We therefore find that (7) is satisfied provided

$$\frac{1}{2} \le \frac{2\sigma+1}{p} \le 1\,, \tag{32a}$$

$$\frac{2\sigma+1}{p} < \frac{1}{2} - \frac{q}{d}\,. \tag{32b}$$

The lower bound of (32a) is consistent with the upper bounds of both (29) and (32b). The lower bound of (29) and the upper bound of (32a) can be simultaneously satisfied if and only if

$$\frac{d}{2} - \frac{1}{\sigma} < q \,, \tag{33}$$

while the lower bound of (29) and the upper bound of (32b) can be simultaneously satisfied if and only if

$$\frac{d}{2} - \frac{d+2}{2\sigma+1} < q \,. \tag{34}$$

Finally, by substituting (33) into the right side of (32b), we find that every p which satisfies (29) and (32) also satisfies

$$p > \sigma d \, (2\sigma + 1) \,. \tag{35}$$

The application of Theorem 3 then yields the following.

Theorem 4. *Let q satisfy the constraints*

$$q > \frac{d}{2} - \frac{1}{\sigma} \,, \tag{36a}$$

$$q > -\frac{2}{2\sigma+1} \,, \tag{36b}$$

$$q > \frac{d}{2} - \frac{d+2}{2\sigma+1} \,. \tag{36c}$$

Then for some $p > \sigma d$ one has that for every $\rho > 0$ there exists a time $T(\rho) > 0$ such that for every initial data $u^{\mathrm{in}} \in \mathrm{H}^q(\mathbb{T}^d)$ with $\|u^{\mathrm{in}}\|_{\mathrm{H}^q} \leq \rho$ there exists a unique mild solution u to the CGL initial-value problem (17) *with*

$$u \in \mathrm{C}([0,T];\mathrm{H}^q(\mathbb{T}^d)) \cap \mathrm{E}^\alpha([0,T];\mathrm{L}^p(\mathbb{T}^d)) \,, \tag{37}$$

where α is given by (28)*. In addition, the map from u^{in} to u is a locally Lipschitz function from $\mathrm{H}^q(\mathbb{T}^d)$ to $\mathrm{C}([0,T];\mathrm{H}^q(\mathbb{T}^d))$.*

Remark 2. The assertion $p > \sigma d$ in Theorem 4 enables further regularity to be achieved by subsequently applying Theorem 1 or 2.

Remark 3. Theorem 3 extends the class of initial data for which the CGL initial value problem is well-posed beyond that asserted by Theorem 2 whenever $\sigma d < 2$. This is often refered to as the subcritical regime [9]. Its boundary is shown as the dotted line in Figure 1, which also illustrates the regions of the d-σ plane in which each of the constraints in (36) is operative.

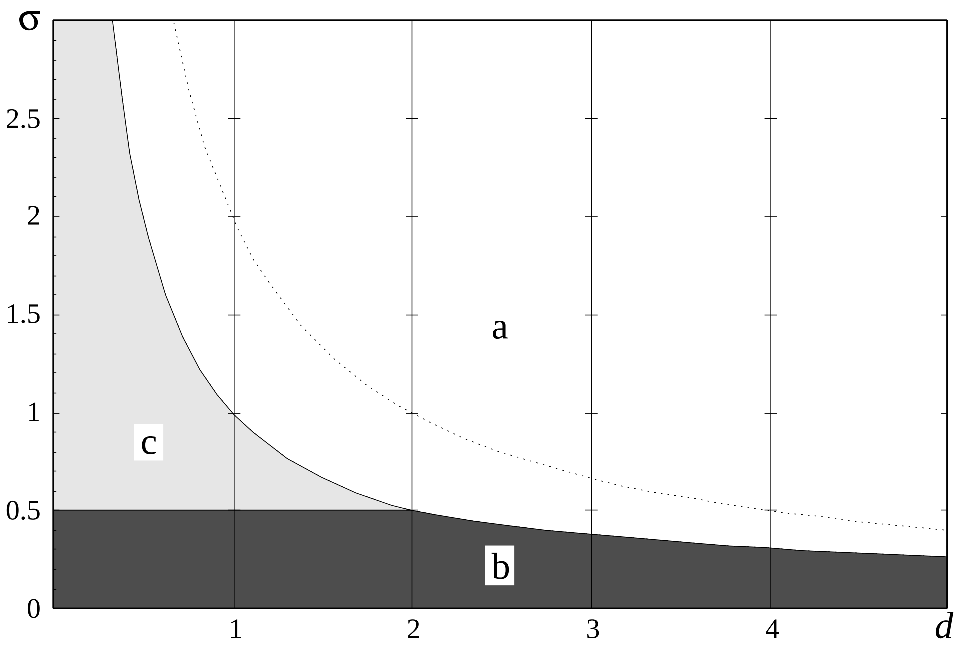

FIGURE 1. The constraints (36) of Theorem 4 are each operative in the correspondingly labeled region of the d-σ plane. Although d is confined to integer values, any positive σ is admissible. The region below the dotted hyperbola $\sigma = 2/d$ is where negative q are possible, and where Theorem 3 improves on Theorem 2.

Remark 4. We believe conditions (36b) and (36c) are technical and can be removed by more sophisticated techniques. Moreover, by following [4], the use of Besov spaces should allow one to obtain the marginal case when (36a) is satisfied as an equality. Note also that for $\sigma \geq \frac{1}{2}$ and $\sigma d \geq 1$—thus in particular when σ is an integer—condition (36a) dominates and we have proved local existence for the conjectured class of initial data. More general results might be derived by adapting the important new techniques developed by Bourgain [2, 3] for the NLS equation over $\mathbb{T}^d$, however our needs here are less demanding.

Remark 5. The results extend to other domains without boundary, for example to $\mathbb{R}^d$, with only minor modifications in their proofs so long as Fourier representations of the Green function and the H^q norms are available.

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