Length Scales in Solutions of the Complex Ginzburg-Landau Equation

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We generalise and in certain cases improve on previous *a priori* estimates of Sobolev norms of solutions to the generalised complex Ginzburg-Landau equation. A set of dynamic length scales based on ratios of these norms is defined. We are able to derive lower bounds for time averages and long-time limits of these length scales. The bounds scale like the inverses of our L^∞ bounds.

1 Introduction

The concern of this paper is to study the problem of finding natural length scales in solutions of a certain class of parabolic partial differential equations

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that are known to possess a global attractor. Given the rich variety of specific dynamical behaviour which can occur, an important point is to estimate by some means the scale of the smallest spatial structure or feature in solutions evolving from generic initial data. Kolmogorov set the scene with his 1941 scaling argument [18] for three dimensional turbulence. His picture is startlingly simplistic, based on nothing but a scaling invariance of the Navier-Stokes equations and a few heuristic assumptions, but agrees with experimental and numerical data so well that to this day every new theory is judged against it [12,22].

For a broader class of systems which do not necessarily scale in the same way as the Navier-Stokes equations, other properties have to be exploited. For example, the Lyapunov dimension $d_L(\mathcal{A})$ of the global attractor $\mathcal{A}$ for a system has been interpreted as the number of its degrees of freedom (for an introduction, see Temam [28]). In principle these determine the number of grid points that would be needed in a numerical integration to resolve the smallest feature in the solution. Hence one could think of $d_L(\mathcal{A})$, the system length L, the number of spatial dimensions d and the smallest length scale in the solution ℓ being related by

$$d_L(\mathcal{A}) \sim \left(\frac{L}{\ell}\right)^d. \tag{1}$$

This has been studied in some detail for the two-dimensional Navier-Stokes equations [7].

However, the attractor is typically a fractal object which can be embedded in phase space in such a way that its Lyapunov dimension grossly underestimates the dimension of its linear span. This is reflected by the fact that in the process of estimating $d_L(\mathcal{A})$ time averages are taken which, by their very nature, can miss much of the short scale dynamical features. It is therefore desirable to have a different notion of the "smallest feature in a flow" by introducing dynamic length scales that are derived more directly from the equations themselves. We will give one such definition later, and although we do not claim to resolve the issue here, we hope to clarify the merits and demerits of the approach.

As a model problem, we consider the generalised complex Ginzburg-Landau (CGL) equation that describes the evolution of a complex field $u = u(x,t)$ by

$$\frac{\partial u}{\partial t} = R\,u + (1+\mathrm{i}\nu)\,\Delta u - (1+\mathrm{i}\mu)\,|u|^{2\sigma}u, \tag{2}$$

with a nonlinearity of order $2\sigma+1$ on the d-dimensional spatial domain $[0,1]^d$ with periodic boundary conditions. $R>0$ is the driving parameter while μ and ν are real parameters of either sign.

Traditionally, the cubic complex Ginzburg-Landau equation has been studied as a generic amplitude equation near the onset of instabilities in a variety of dissipative systems [24,27]. More recently, it has attracted attention as a model problem for turbulent phenomena, the reasons being twofold. Firstly, in the absence of driving it has an inviscid limit that is a Hamiltonian dynamical system, namely the nonlinear Schrödinger (NLS) equation

$$\mathrm{i}\,\frac{\partial u}{\partial t} = -\nu\,\Delta u + \mu\,|u|^{2\sigma} u. \tag{3}$$

Secondly, when $\sigma d \geq 2$, this inviscid limit possesses instabilities capable of causing finite-time spatially localised singularities starting from arbitrarily smooth initial data. This effect can be thought of as a cascade process whereby energy concentrated in low wavenumbers is rapidly transported to high wavenumbers. These two properties are in direct analogy to the problem of incompressible fluid turbulence. The Euler equations constitute the underlying Hamiltonian dynamics of the dissipative Navier-Stokes equations, and instabilities in the inviscid equations can create a cascade of energy between widely separated length scales. Of course, the mechanisms for energy transfer from long to short length scales is very different in the NLS equation (self-focusing instability) and the three-dimensional Euler equations (vortex stretching), and we make no claims as to a direct physical analogy. Instead, the CGL-NLS pairing can be thought of as a paradigm for the Navier-Stokes and Euler equations. Moreover, the study of the CGL equation is facilitated by its nonlinear dissipative term and the absence of a pressure term as in the Navier Stokes equations, so that there is no commensurately difficult Poisson equation to solve. Nonlinear *and* linear dispersion on the other hand make its dynamics far from being trivial, and—as we shall see later—the major technical difficulties arise indeed from the $\mathrm{i}\nu\Delta u$ term.

The development of spatially localised singularities in solutions of the NLS equation when $\sigma d \geq 2$ is intimately related to the CGL dynamics when ν and μ are large [11]. It is here where we would expect this effect to show itself in much smaller scale structures appearing in the flow. Reference [2] first pointed out that this shows up in Sobolev estimates of the solution, and further identified distinct regions in the $\mu\nu$-parameter plane that might be crucial in understanding the dynamics of the problem. This makes the CGL equation particularly attractive to study new concepts of length scales: One would expect a good length scale to reflect these changes while being bounded from below wherever global smooth solutions exist.

For a more complete review on the role of the CGL equation as a model problem and additional references see [11,20].

The paper is organised as follows: First we present a set of *a priori* estimates for the H^n norms of the solutions to the complex Ginzburg-Landau equation which systematise and in certain cases improve on previous results. Our estimates are formal; we do not address the question of existence and uniqueness of solutions here and refer the interested reader again to [11,20].

Second, we define a set of length scales which are based on the ratio of Sobolev norms in much the same way length scales for the Navier Stokes flow where defined in [3]. The aforementioned estimates then enable us to derive lower bounds for the time average and pointwise bounds for the long-time (time asymptotic) limit of these length scales. We compare these bounds with corresponding estimates of the L^∞ norm.

2 Definitions

It is convenient to introduce the following notation:

$$G_n := \|u\|_{2(\sigma n+1)}^{2(\sigma n+1)} = \int |u|^{2(\sigma n+1)} \, \mathrm{d}x, \tag{4}$$

$$J_n := \|\nabla^n u\|_2^2 = \int |\nabla^n u|^2 \, \mathrm{d}x, \tag{5}$$

$$F_n := J_n + \alpha_n \, G_n \tag{6}$$

where $\alpha_n = 1$ except for $n = 1$ where in the proof of Theorem 15 below a more subtle choice will be made. Here $\nabla^n u$ is a symmetric n-tensor and

$$|\nabla^n u|^2 \equiv \nabla^n u \cdot \nabla^n u^*, \tag{7}$$

the dot denoting the usual tensor inner product that contracts a symmetric k-tensor and a symmetric l-tensor over $\min\{k, l\}$ indices to give a symmetric $|k - l|$-tensor.

Note that in order to achieve dimensional consistency in the CGL equation, u has to scale like $(\text{length})^{-1/\sigma}$ if the spatial variable x has units of (length). Thus G_n and J_n are of the same dimension.

Let $\{S(t)\}_{t\geq 0}$ be the semigroup of operators that describes the time evolution of the solution with initial condition u_0 by

$$u(t) = S(t)u_0 \tag{8}$$

and let $\mathcal{A} \subset H$ denote the attractor as a subset of the phase space H (see Temam [28] for an introduction to these concepts). A compact attractor is known to exists wherever our *a priori* estimates are valid. Then for a functional

$F = F(S(t)u_0)$ we define its time-asymptotic upper bound

$$\overline{F} := \sup_{u_0 \in H} \limsup_{t\to\infty} F(S(t)u_0) \tag{9}$$

and its time average

$$\langle F \rangle := \limsup_{t\to\infty} \sup_{u_0 \in \mathcal{A}} \frac{1}{t} \int_0^t F(S(\tau)u_0)\ \mathrm{d}\tau. \tag{10}$$

The properties of the time average are summarised in

Lemma 1 *(Properties of the time average.)*

(i) $\langle F \rangle \leq \overline{F}$,

(ii) $\langle F\, G \rangle \leq \langle F^p \rangle^{\frac{1}{p}} \langle G^q \rangle^{\frac{1}{q}}$ *if* $\frac{1}{p} + \frac{1}{q} = 1$,

(iii) $\left\langle \dfrac{\mathrm{d}F}{\mathrm{d}t} + G \right\rangle = \langle G \rangle$ *if* $\overline{F} < \infty$.

The proof involves only elementary real analysis and can be found in [25].

Lastly, in this paper c denotes various constants not necessarily equal that do not depend on u and R.

3 Summary of Results

Our main result is the derivation of a new set of differential inequalities for the quantities J_n:

$$\dot{J}_n \leq 2R\, J_n + c\, J_n\, G_m^s - J_n \left(\frac{J_n}{J_r}\right)^{\frac{1}{n-r}} \tag{11}$$

where

$$s = \frac{2\,\sigma}{2 - d\,\sigma + 2\,m\,\sigma}, \qquad \frac{d\,\sigma - 2}{2\,\sigma} < m < \infty. \tag{12}$$

This result elegantly reproduces the results in [2]. Furthermore, it is an essential ingredient for the computation of length scales in Section 7.

A second set of differential inequalities, obtained in much the same way,

$$\dot{J}_n \leq 2R\, J_n + c\left(G_m^{s'} + G_m^{s'_0}\right) - J_n \left(\frac{J_n}{J_r}\right)^{\frac{1}{n-r}} \tag{13}$$

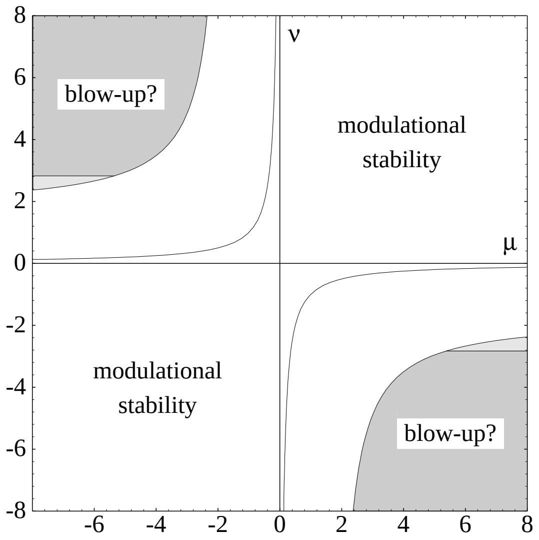

Figure 1. Phase diagram for the supercritical complex Ginzburg-Landau equation (shown here is the case $d = 3$, $\sigma = 1$). Global solutions exist only in a small strip of the shaded region. In the subcritical and critical case this strip extends to the whole parameter plane.

where

$$s' = \frac{2 + 2\,\sigma - d\,\sigma + 2\,n\,\sigma}{2 - d\,\sigma + 2\,m\,\sigma}, \qquad s'_0 = \frac{\sigma\,(n+1) + 1}{\sigma\,m + 1}, \qquad \frac{\sigma\,d - 2}{2\,\sigma} < m \leq n + 1, \tag{14}$$

can be used to obtain estimates which are marginally sharper than the ones obtained via (11). We will summarise these findings in the following. It should be mentioned that the terms soft and hard turbulence have been previously introduced into the CGL context, based on the observation that the L^∞ estimates in the latter regime are much larger and therefore admit more violent dynamics. Numerical studies, although not yet conclusive, have failed to detect a transition across the soft-hard boundary [15–17,21], so that one has to regard this distinction to be at least partially artificial. We will therefore avoid this terminology in most of the paper.

For every σ and d there are three distinct regions in the $\mu\nu$-parameter plane (Figure 1).

(i) The region of modulational stability: This region was first described by Lange and Newell [19]. It entirely includes the first and third quadrant and those parts of the second and fourth quadrant that lie inside the pair of hyperbolae $\mu\nu + 1 = 0$. Inside this region all rotating wave solutions are linearly stable whereas outside the rotating waves become successively unstable as R is increased [10]. Correspondingly, the onset of chaotic dynamics can be observed.

(ii) The region of soft turbulence ("unshaded region"): This is the area outside of modulational stability, bounded by the pair of hyperbolae

$$
-\frac{1+\mu\,\nu}{|\mu-\nu|} < \frac{\sqrt{2\,\sigma+1}}{\sigma} \tag{15}
$$

and restricted to the cases where $d < 2 + 2/\sigma$. In this region one has the *a priori* bound

$$
\overline{J}_n \le c\,R^{\frac{(1+\sigma)\,(2-d\sigma+2n\sigma)}{\sigma(2+2\sigma-d\sigma)}}\,. \tag{16}
$$

(iii) The region of hard turbulence ("shaded region"): Going further out in the $\mu\nu$-plane, estimate (16) fails and has to be replaced with

$$
\overline{J}_n \le c\,R^{\frac{-(d\sigma)+(1+m\sigma)\left(2+2n^2\sigma+n(2+2\sigma-d\sigma)\right)}{(1+n)\sigma(2-d\sigma+2m\sigma)}} \tag{17}
$$

which is a weaker bound and, being subject to the constraints

$$
|1+\mathrm{i}\nu| \le \frac{\sigma\,m+1}{\sigma\,m}, \tag{18}
$$

$$
\frac{\sigma\,d-2}{2\,\sigma} < m \le n+1, \tag{19}
$$

may even fail completely.

In the shaded region it is best to choose m as large as allowed by (18), i.e.

$$
m = \frac{1}{\sigma}\left(\frac{1}{\nu^2} + \sqrt{\frac{1}{\nu^4}+\frac{1}{\nu^2}}\right). \tag{20}
$$

Being a bit more generous, we may take

$$
m = \frac{1}{\sigma\,|\nu|} \tag{21}
$$

instead, keeping the algebra simple without causing any qualitative damage to the estimates.

Unfortunately (19) sets a lower bound on m. Dependent on the quantity σd we have the following three cases:

(i) The subcritical case, $\sigma d < 2$: $m = 0$ is permitted, resulting in estimates which are, although larger than in the unshaded region, of uniform order in R. (Of course it is possible to obtain small, ν-dependent improvements by taking $m > 0$.)

(ii) The critical case, $\sigma d = 2$: Any $m > 0$ subject to (18) is permitted, therefore estimate (16) still holds, but becomes ν-dependent in the exponent.

(iii) The supercritical case, $\sigma d > 2$: The lower bound on m becomes strictly positive, resulting in an upper limit on ν:

$$|\nu| < \frac{2\sqrt{\sigma d - 1}}{\sigma d - 2}. \tag{22}$$

Hence we have regularity only in a strip of finite width in the shaded region. (See Figure 1.)

Table 1 summarises the findings when $\sigma = 1$.

Remark 2 *For $n > 1$ in the shaded region, the estimates given here are sharper than those in [2,11], although the qualitative picture remains the same.*

In Section 7 we define the inverse of a length scale ℓ by

$$\ell^{-1} \sim \mathcal{K}_{n,r} := \left(\frac{J_n + \beta\, 2^n R^{n+1/\sigma}}{J_r + \beta\, 2^r R^{r+1/\sigma}}\right)^{\frac{1}{2(n-r)}}, \tag{23}$$

$$\kappa_{n,r} := \mathcal{K}_{n,r}\Big|_{\beta=0}. \tag{24}$$

We estimate time averages and time asymptotic upper bounds for these quantities and also for $\|u\|_\infty^{2\sigma}$. The results have the ordering

$$\langle\|u\|_\infty^{2\sigma}\rangle \sim \langle\mathcal{K}_{n,r}^2\rangle \le \overline{\|u\|}_\infty^{2\sigma} \sim \overline{\mathcal{K}}_{n,r}^2 \le \overline{\kappa}_{n,r}^2 \tag{25}$$

where $\sim$ denotes "scales like (in the parameter R)".

The significance of this hierarchy is the following: The quantity $\overline{\kappa}_{n,r}^2$ to the

d	region	$\overline{\|\nabla^n u\|}_2^2$	$\langle\|\nabla^n u\|_2^2\rangle$	$\overline{\|u\|}_\infty^2$	$\langle\|u\|_\infty^2\rangle$
1	unshaded	$\frac{2}{3}(2n+1)$	$\frac{2}{3}(2n+1)$	$\frac{4}{3}$	$\frac{4}{3}$
1	shaded	$2n+1$	$2n^\dagger$	2	$\frac{4}{3}$
2	unshaded	$2n$	$2n$	2	2
2	shaded	$\frac{n+n^2+m(1+n+n^2)}{m(1+n)}$	$\frac{-1+n+m(1+n)}{m}^\dagger$	$\frac{1+m}{m}^\dagger$	2
3	unshaded	$4n-2$	$4n-2$	4	4
3	shaded	$\frac{(1+m)(2+n+2n^2)-3}{(2m-1)(1+n)}$	$\frac{2(m+n-2+mn)}{2m-1}^\dagger$	$\frac{2(1+m)}{2m-1}^\dagger$	$\frac{5m-1}{2m-1}^\dagger$

Table 1. R-exponents of the upper bounds on four quantities arising in the study of the cubic CGL equation. The estimate for $d = 3$ in the shaded region is only valid if $\nu < \sqrt{8}$. Entries marked with a † are approximate, see the remark at the end of Section 8.

far right is the one we would like to control. This however is impossible with present methods. We can only gain control by introducing an extra term, i.e. taking $\beta \neq 0$ in (23) and, hence, obtaining the approximate time asymptotic bound $\overline{\mathcal{K}}^2_{n,r}$. Note that this bound can be larger than $\langle \mathcal{K}^2_{n,r} \rangle$, suggesting that it is more sensitive to the dynamics of the equation.

We also find that the $\|u\|_\infty^{2\sigma}$ and $\mathcal{K}^2_{n,r}$ bounds scale almost identically in both the time averaged and time asymptotic case (almost identical in the sense that, from families of estimates on $\|u\|_\infty^{2\sigma}$ and $\mathcal{K}^2_{n,r}$, we can select pairs whose difference in the scaling exponents is arbitrarily small). This is consistent with the following interpretation: The L^∞ norm measures the absolute height of spikes in the solution, and therefore should scale inversely with spatial length scales. However, since we lack information about the spike height relative to some average (we do not have lower bounds for the L^2 norm, for example), this argument is approximate in the same sense as $\mathcal{K}_{n,r}$ is an approximation to $\kappa_{n,r}$.

Finally, we remark that our differential inequalities easily permit the derivation of estimates analogous to the results of Beale, Kato and Majda for the Navier-Stokes equations [1,4].

4 Preliminary Estimates

This section provides a collection of differential and interpolation inequalities that form the basis of the subsequent calculations. The time evolution of the functionals G_n and J_n is obtained in a straightforward way directly from the equation:

Lemma 3 *For the complex Ginzburg-Landau equation, the G_n and J_n obey the following evolution equations:*

$$\frac{\dot{G}_n}{2(\sigma n + 1)} = R\, G_n - G_{n+1} + \mathrm{Re}\left[(1 + \mathrm{i}\nu) \int |u|^{2n\sigma}\, u^*\, \Delta u \;\mathrm{d}x\right], \qquad (26)$$

$$\tfrac{1}{2}\dot{J}_n = R\, J_n - J_{n+1} - \mathrm{Re}\left[(1 + \mathrm{i}\mu) \int \nabla^n (u\,|u|^{2\sigma}) \cdot \nabla^n u^* \;\mathrm{d}x\right]. \qquad (27)$$

In the following, we will derive various bounds on the last term of equation (27). Lemma 4 provides a raw estimate which is sharp in the sense that the exponents of the underlying Gagliardo-Nirenberg interpolation inequalities are kept at their lower margin of validity, resulting in an estimate that is independent of the dimension d. All other estimates will be derived from there by straightforward interpolation and Young's inequalities.

Lemma 4 *For all positive integers d, σ and n the following inequality holds:*

$$\left| \int \nabla^n (u\,|u|^{2\sigma}) \cdot \nabla^n u^* \; \mathrm{d}x \right| \leq c(n,p)\, \|\nabla^n u\|_{2p}^2 \, \|u\|_{2r\sigma}^{2\sigma} \tag{28}$$

where

$$\frac{1}{p} + \frac{1}{r} = 1, \qquad 1 \leq p, r \leq \infty. \tag{29}$$

Remark 5 *The advantage of this result compared to previous estimates is that it permits a free choice of r while keeping the estimate sharp. E.g. Lemma 2 in [2] can be recovered by setting $r = \infty$.*

Proof. Moving the absolute value inside the integral, performing a Leibniz expansion on $\nabla^n(u\,|u|^{2\sigma})$ and then using the triangle inequality one gets

$$\begin{aligned}
\left| \int \nabla^n (u\,|u|^{2\sigma}) \cdot \nabla^n u^* \; \mathrm{d}x \right| &\leq c_1(n) \sum_{\substack{\alpha \in \mathbb{N}^{2\sigma+1} \\ |\alpha| = n}} \int |\nabla^n u| \prod_{j=1}^{2\sigma+1} |\nabla^{\alpha_j} u| \; \mathrm{d}x \\
&\leq \left(\text{by Hölder's inequality with } \textstyle\sum_{j=0}^{2\sigma+1} \frac{1}{p_j} = 1\right) \\
&\leq c_1(n)\, \|\nabla^n u\|_{p_0} \sum_{\substack{\alpha \in \mathbb{N}^{2\sigma+1} \\ |\alpha| = n}} \prod_{j=1}^{2\sigma+1} \|\nabla^{\alpha_j} u\|_{p_j} \\
&\leq \left(\text{interpolating each of the } \|\nabla^{\alpha_j} u\|_{p_j}, \text{ see below}\right) \\
&\leq c_2(n,p_0)\, \|\nabla^n u\|_{p_0} \sum_{\substack{\alpha \in \mathbb{N}^{2\sigma+1} \\ |\alpha| = n}} \prod_{j=1}^{2\sigma+1} \|\nabla^n u\|_{p_0}^{\theta_j} \, \|u\|_{2\sigma p_0/(p_0-2)}^{1-\theta_j} \\
&= \left(\text{using relation 33 and setting } p = p_0/2,\ r = p_0/(p_0-2)\right) \\
&= c_3(n,p)\, \|\nabla^n u\|_{2p}^2 \, \|u\|_{2r\sigma}^{2\sigma}.
\end{aligned} \tag{30}$$

In order to achieve the second step of (30), we set

$$\theta_j := \frac{\alpha_j}{n} \tag{31}$$

which is a valid choice that uniquely determines the p_j to be

$$p_j = \frac{2\,n\,p_0\,\sigma}{-2\,n + n\,p_0 + \alpha_j\,(2 - p_0 + 2\,\sigma)}. \tag{32}$$

Now it is easy to see that

$$\sum_{j=1}^{2\sigma+1} \theta_j = \frac{|\alpha|}{n} = 1 \tag{33}$$

and

$$\sum_{j=0}^{2\sigma+1} \frac{1}{p_j} = \frac{1}{p_0} + (2\,\sigma+1)\frac{n\,p_0 - 2\,n}{2\,n\,p_0\,\sigma} + |\alpha|\frac{(2-p_0+2\,\sigma)}{2\,n\,p_0\,\sigma} = 1. \tag{34}$$

Lastly, we have to verify that the p_j defined by (32) satisfy $1 \le p_j \le \infty$. By elementary calculus, the denominator of (32) is non-zero and $\partial p_j/\partial\alpha_j$ does not change sign for $\alpha_j \in (0,n)$. Thus it is sufficient to consider the extreme cases $\alpha_j = 0$ where $p_j = 2p_0\sigma/(p_0-2)$ and $\alpha_j = n$ where $p_j = p_0$, for which $1 \le p_j \le \infty$ clearly holds. This retrospectively asserts the validity of Hölder's inequality above.

The next estimate exploits that we have free choice of p in Lemma 4. More precisely, we will set $p = 2 + 2\sigma$ for all σ and d. This choice is crucial for the length scale estimates in Section 7.

Corollary 6 *For all positive integers d, σ and n the following inequality holds:*

$$\left|\int \nabla^n(u\,|u|^{2\sigma})\cdot\nabla^n u^*\ \mathrm{d}x\right| \le \varepsilon\,J_{n+1} + c(\varepsilon)\,J_n\,G_m^s \tag{35}$$

where

$$s = \frac{2\,\sigma}{2-d\,\sigma+2\,m\,\sigma}, \qquad \frac{d\,\sigma-2}{2\,\sigma} < m < \infty. \tag{36}$$

Proof. Using an interpolation inequality with Lemma 4, one gets

$$\begin{aligned}\left|\int \nabla^n(u\,|u|^{2\sigma})\cdot\nabla^n u^*\ \mathrm{d}x\right| &\le c_4\,\|\nabla^{n+1}u\|_2^{d/r}\,\|\nabla^n u\|_2^{2-d/r}\,\|u\|_{2r\sigma}^{2\sigma}\\ &\equiv c_4\,J_{n+1}^{s_1}\,J_n^{1-s_1}\,G_m^{s_2} \qquad (37)\\ &\le \text{(by Young's inequality)}\\ &\le \varepsilon\,J_{n+1} + c_5(\varepsilon)\,\left(J_n^{1-s_1}\,G_m^{s_2}\right)^{\frac{1}{1-s_1}}\\ &\equiv \varepsilon\,J_{n+1} + c_5(\varepsilon)\,J_n\,G_m^s. \qquad (38)\end{aligned}$$

The exponents are

$$s_1 = \frac{d\,\sigma}{2\,(1+m\,\sigma)}, \qquad s_2 = \frac{\sigma}{1+m\,\sigma}.$$

The constraint (36) results from the fact that we need $s_1 < 1$ in order to apply Young's inequality above.

If $d = 1$, the condition $r \geq 1$ from Lemma 4 would be more restrictive than (36), so that for $\frac{1}{2} < r < 1$ we must complement the proof with a different interpolation in order to obtain the result in its full generality. Thus we estimate

$$\begin{aligned}
&\left| \int \nabla^n(u\,|u|^{2\sigma}) \cdot \nabla^n u^* \;\mathrm{d}x \right| \leq c_3 \left\|\nabla^n u\right\|_\infty^2 \left\|u\right\|_{2\sigma}^{2\sigma} \\
&\leq \text{(by interpolation on the last factor)} \\
&\leq c_6 \left\|\nabla^n u\right\|_\infty^{2\lambda} \left\|\nabla^n u\right\|_\infty^{2(1-\lambda)} \left(\left(\left\|\nabla^{n+1} u\right\|_2^{s_3} \left\|u\right\|_{2r\sigma}^{1-s_3} \right)^{2\sigma} + \left\|u\right\|_{2r\sigma}^{2\sigma} \right) \\
&\leq c_7 \left(\left\|\nabla^{n+1} u\right\|_2^{s_4} \left\|u\right\|_{2r\sigma}^{1-s_4} \right)^{2\lambda} \left(\left\|\nabla^{n+1} u\right\|_2 \left\|\nabla^n u\right\|_2 \right)^{1-\lambda} \left(\left\|\nabla^{n+1} u\right\|_2^{s_3} \left\|u\right\|_{2r\sigma}^{1-s_3} \right)^{2\sigma} \\
&\quad + c_8 \left\|\nabla^{n+1} u\right\|_2^{2\lambda} \left(\left\|\nabla^{n+1} u\right\|_2 \left\|\nabla^n u\right\|_2 \right)^{1-\lambda} \left\|u\right\|_{2r\sigma}^{2\sigma}. \qquad (39)
\end{aligned}$$

Note that in the last step we have used a Poincaré inequality on the L^2 norm of the mean-zero function $\nabla^n u$ in order to absorb the additional lower order term—arising in the interpolation of $\|u\|_{2\sigma}$ because u is not mean-zero—into the leading order term. Indeed, as the Gagliardo-Nirenberg exponents s_3 and s_4 are

$$s_3 = \frac{1 - r}{2\,r\,\sigma\,n + r\,\sigma + 1}, \qquad s_4 = \frac{2\,r\,\sigma\,n + 1}{2\,r\,\sigma\,n + r\,\sigma + 1},$$

by a clever choice of $\lambda = 1/r - 1$, equation (39) immediately reduces to equation (37), so that the proof can proceed as before.

Corollary 6 is already sufficient for *a priori* control of all J_n (cf. the remark after Theorem 16). However, with a little extra work one can quantitatively improve the estimates. The key estimate is Corollary 9 (note its similarity with Corollary 6) which we derive in two steps:

Corollary 7 *For all positive integers d, σ and n the following inequality holds:*

$$\left| \int \nabla^n(u\,|u|^{2\sigma}) \cdot \nabla^n u^* \;\mathrm{d}x \right| \leq c(n)\, J_{n+1}^{\frac{n}{n+1}}\, G_{n+1}^{\frac{1}{n+1}}. \qquad (40)$$

Remark 8 *This result recovers Lemma 2 in [25].*

Proof. Using an interpolation inequality with Lemma 4, one gets

$$\begin{aligned}
\left| \int \nabla^n(u\,|u|^{2\sigma}) \cdot \nabla^n u^* \;\mathrm{d}x \right| &\leq c_9 \left(\left\|\nabla^{n+1} u\right\|_2^{\frac{n}{n+1}} \left\|u\right\|_{2(\sigma(n+1)+1)}^{\frac{1}{n+1}} \right)^2 \left\|u\right\|_{2(\sigma(n+1)+1)}^{2\sigma} \\
&= c_9\, J_{n+1}^{\frac{n}{n+1}}\, G_{n+1}^{\frac{1}{n+1}}. \qquad (41)
\end{aligned}$$

Corollary 9 *For all positive integers d, σ and n the following inequality holds:*

$$\left| \int \nabla^n (u\,|u|^{2\sigma}) \cdot \nabla^n u^* \, \mathrm{d}x \right| \le \varepsilon \, J_{n+1} + c(\varepsilon) \; (G_m^s + G_m^{s_0}) \tag{42}$$

where

$$s = \frac{2 + 2\,\sigma - d\,\sigma + 2\,n\,\sigma}{2 - d\,\sigma + 2\,m\,\sigma}, \qquad s_0 = \frac{\sigma\,(n+1)+1}{\sigma\,m+1}. \qquad \frac{\sigma\,d - 2}{2\,\sigma} < m \le n+1, \tag{43}$$

Remark 10 *The $G_m^{s_0}$ term is always of lower order, i.e. $s \ge s_0$, and arises because a Gagliardo-Nirenberg inequality is used to interpolate the function u which is not mean-zero. The result can probably be derived in a way more similar to the proof of Corollary 6 where the lower order term can be absorbed into the constant.*

Proof. We continue to interpolate the right side of (41)

$$\begin{aligned} J_{n+1}^{\frac{n}{n+1}} \, G_{n+1}^{\frac{1}{n+1}} &\le c_{10} \, J_{n+1}^{\frac{n}{n+1}} \left(J_{n+1}^{s_5} \, G_m^{s_6} + G_m^{s_0} \right)^{\frac{1}{n+1}} \\ &\le \text{(by Young's inequality)} \\ &\le \varepsilon \, J_{n+1} + c_{11}(\varepsilon) \left(G_m^{\frac{s_6}{1-s_5}} + G_m^{s_0} \right) \\ &\equiv \varepsilon \, J_{n+1} + c_{11}(\varepsilon) \; (G_m^s + G_m^{s_0}) \end{aligned} \tag{44}$$

where

$$\begin{aligned} s_5 &= \frac{d\;(1 - m + n)\;\sigma}{2 + 2\,n + 2\,m\,\sigma - d\,m\,\sigma + 2\,m\,n\,\sigma}, \\ s_6 &= \frac{(1+n)\;(2 + 2\,\sigma - d\,\sigma + 2\,n\,\sigma)}{2 + 2\,n + 2\,m\,\sigma - d\,m\,\sigma + 2\,m\,n\,\sigma}. \end{aligned}$$

Clearly we need $0 \le s_5 < 1$, hence the restriction in the statement of the corollary.

Remark 11 *It is possible to derive corresponding estimates for the last term in the evolution equation for G_n, equation (26). The estimates are relatively straightforward so that we omit them here as they are not needed in the following.*

5 Long-time Bounds for J_n and G_n

In this section we will work our way from proving a simple L^2, a general L^p and a H_1 estimate up to the general H^n *a priori* estimate.

Theorem 12 *For all d and $\sigma > 0$, there holds the* a priori *estimate*

$$\overline{G}_0 \leq R^{1/\sigma}. \tag{45}$$

Proof. Either equation (26) or (27) gives

$$\begin{aligned}\dot{G}_0 &= 2R\,G_0 - 2\,G_1 - 2\,J_1 \qquad (46)\\ &\leq 2R\,G_0 - 2\,G_1 \\ &\leq (\text{by a Hölder inequality } G_0 \leq \|1\|_{\sigma/(\sigma+1)}\,G_1^{1/(\sigma+1)}) \\ &\leq 2R\,G_0 - 2\,G_0^{\sigma+1}. \qquad (47)\end{aligned}$$

Solving this differential inequality gives (45); a method is for example described in the appendix of reference [20].

When $m > 0$, it is possible to find *a priori* bounds on G_m for a restricted range of ν only. This is the content of

Theorem 13 *For all d, μ, $\sigma > 0$, $m > 0$ and provided that*

$$|1 + \mathrm{i}\nu| \leq \frac{\sigma\,m + 1}{\sigma\,m} \tag{48}$$

there holds the a priori *estimate*

$$\overline{G}_m \leq R^{1/\sigma + m}. \tag{49}$$

Proof. From Lemma 3, using a Hölder inequality on G_{m+1} and integration by parts on the last term, one obtains

$$\begin{aligned}\frac{\dot{G}_m}{2(\sigma m + 1)} &\leq R\,G_m - \frac{G_m^{\frac{m+1}{m}}}{G_0^{1/m}} - (\sigma\,m + 1)\int |u|^{2\sigma m}\,|\nabla u|^2\,\mathrm{d}x \\ &\quad - \sigma\,m\ \mathrm{Re}\left[(1 + \mathrm{i}\nu)\int |u|^{2(\sigma m - 1)}\,u^{*2}\,\nabla u \cdot \nabla u\ \mathrm{d}x\right] \\ &\leq R\,G_m - \frac{G_m^{\frac{m+1}{m}}}{G_0^{1/m}} - ((\sigma\,m + 1) - \sigma\,m\,|1 + \mathrm{i}\nu|)\int |u|^{2\sigma m}\,|\nabla u|^2\,\mathrm{d}x.\end{aligned} \tag{50}$$

Hence if $\sigma\,m\,|1 + \mathrm{i}\nu| \leq \sigma\,m + 1$, the last term is non-positive and can be neglected. Noting that $\overline{G}_0 \leq R^{1/\sigma}$, the differential inequality is easily solved.

For $n = 1$ only, the indefinite terms of the $\dot{J}_n$ and $\dot{G}_n$ evolution equations are equal up to a complex constant. Taking the linear combination

$$F_1 := J_1 + \frac{\alpha^2}{\sigma + 1} G_1, \tag{51}$$

optimisation in α allows for partial cancellation of these terms. The result is due to Ginibre [14] who has been able to slightly extend the region of validity in the $\mu\nu$-parameter plane presented in [2,11]. The proof given here is based on the following observation (see also [20]):

Lemma 14 *For $\sigma \geq 0$ the following* a priori *inequalities hold:*

$$\operatorname{Re}\left[\int |u|^{2\sigma} u^* \,\Delta u \;\mathrm{d}x\right] \leq -\frac{\sqrt{2\,\sigma+1}}{\sigma+1}\left|\int |u|^{2\sigma} u^* \,\Delta u \;\mathrm{d}x\right|, \tag{52}$$

$$\left|\operatorname{Im}\left[\int |u|^{2\sigma} u^* \,\Delta u \;\mathrm{d}x\right]\right| \leq \frac{\sigma}{\sigma+1}\left|\int |u|^{2\sigma} u^* \,\Delta u \;\mathrm{d}x\right|, \tag{53}$$

$$-J_2 - \alpha^2 G_2 \leq -2\alpha\left|\int |u|^{2\sigma} u^* \,\Delta u \;\mathrm{d}x\right|. \tag{54}$$

Proof. Integration by parts gives the identity

$$\int |u|^{2\sigma} u^* \,\Delta u \;\mathrm{d}x = -\sigma \int \nabla u \cdot \nabla u \, u^{*2} \,|u|^{2\sigma-2} \;\mathrm{d}x - (\sigma+1) \int |\nabla u|^2 \,|u|^{2\sigma} \;\mathrm{d}x.$$

Noting that $|\nabla u \cdot \nabla u| \leq |\nabla u|^2$, this has the structure

$$z = \sigma\,\lambda\, r \;\mathrm{e}^{\mathrm{i}\theta} - (\sigma+1)\, r \tag{55}$$

with r a positive real number and $0 \leq \lambda \leq 1$. Thus clearly

$$\operatorname{Re} z = \sigma\,\lambda\, r\, \cos\theta - (\sigma+1)\, r \tag{56}$$

and

$$|z|^2 = \sigma^2\,\lambda^2\, r^2 + (\sigma+1)^2\, r^2 - 2\,\sigma(\sigma+1)\,\lambda\, r^2 \,\cos\theta \tag{57}$$

(Law of Cosines). Combining these two equations gives

$$\begin{aligned} \operatorname{Re} z &= -\frac{|z|^2 + (\sigma^2\,(1-\lambda^2) + 2\,\sigma + 1)\, r^2}{2\,(\sigma+1)\, r} \\ &\leq -\frac{|z|^2 + (2\,\sigma+1)\, r^2}{2\,(\sigma+1)\, r}. \end{aligned}$$

Optimisation in r then gives (52). This result plugged into the identity

$$|z|^2 = |\operatorname{Re} z|^2 + |\operatorname{Im} z|^2$$

immediately yields (53). Lastly, estimate (54) is easily obtained by applying the Cauchy-Schwarz inequality and Young's inequality to the expression on its right side.

Theorem 15 *For all d, $\sigma > 0$ and provided that*

$$-\frac{1+\mu\,\nu}{|\mu-\nu|} < \frac{\sqrt{2\,\sigma+1}}{\sigma} \tag{58}$$

there holds the a priori *estimate*

$$\overline{F}_1 \leq c(\mu,\nu)\, R^{1+1/\sigma}. \tag{59}$$

Proof. From Lemma 3 one has

$$\begin{aligned}
\tfrac{1}{2}\dot{F}_1 &= R\,(J_1+\alpha^2\,G_1) - J_2 - \alpha^2\,G_2 \\
&\qquad + (1+\alpha^2)\,\mathrm{Re}\left[\int |u|^{2\sigma}\,u^*\,\Delta u\;\mathrm{d}x\right] + (\mu-\alpha^2\,\nu)\,\mathrm{Im}\left[\int |u|^{2\sigma}\,u^*\,\Delta u\;\mathrm{d}x\right] \\
&\leq (\text{by Lemma 14, for some } \kappa\in[0,1]) \\
&\leq R\,(J_1+\alpha^2\,G_1) - (1-\kappa)\,(J_2+\alpha^2\,G_2) \\
&\quad - \left(2\kappa\alpha + (1+\alpha^2)\frac{\sqrt{2\,\sigma+1}}{\sigma+1} - |\mu-\alpha^2\nu|\frac{\sigma}{\sigma+1}\right)\left|\int |u|^{2\sigma}\,u^*\,\Delta u\;\mathrm{d}x\right|.
\end{aligned} \tag{60}$$

The last term can be neglected provided it is possible to choose α such that its coefficient is non-negative. This is always the case if $\mu\nu \geq 0$. On the other hand, when $\mu\nu < 0$, we can consider without loss of generality the case when $\mu > 0$ and $\nu < 0$. We can also restrict the discussion to the limiting case $\kappa = 1$, so that the condition to be met is

$$2\alpha + (1+\alpha^2)\frac{\sqrt{2\,\sigma+1}}{\sigma+1} - (\mu-\alpha^2\nu)\frac{\sigma}{\sigma+1} > 0. \tag{61}$$

Optimisation in α then gives (58). Now the differential inequality (60) reads

$$\begin{aligned}
\tfrac{1}{2}\dot{F}_1 &\leq (\sigma+1)R\,F_1 - (1-\kappa)\,(J_2+\alpha^2\,G_2) \\
&\leq (\text{by interpolation on the last term}) \\
&\leq (\sigma+1)R\,F_1 - (1-\kappa)\,c\,\frac{F_1^2}{G_0}
\end{aligned} \tag{62}$$

which can easily be solved giving (59).

Finally we are able to derive the full J_n result:

Theorem 16 *For the complex Ginzburg-Landau equation with σ, d and n positive integers and for all $m \geq 0$, $r \geq 0$, provided that*

$$\frac{\sigma\, d - 2}{2\,\sigma} < m \leq n + 1, \tag{63}$$

there holds the following a priori *estimate:*

$$\overline{J}_n \leq c(n, \mu)\, \overline{J}_r^{\gamma_1}\, \overline{G}_m^{\gamma_2} + \textit{lower order terms} \tag{64}$$

where

$$\gamma_1 = \frac{1}{1 + n - r}, \qquad \gamma_2 = \frac{(2 + 2\,\sigma - d\,\sigma + 2\,n\,\sigma)\ (n - r)}{(2 - d\,\sigma + 2\,m\,\sigma)\ (1 + n - r)}.$$

Proof. Take equation (27), bound the nonlinear term by Corollary 9 and the Laplacian term by using the interpolation

$$J_n \leq J_{n+1}^{\frac{n-r}{n+1-r}}\, J_r^{\frac{1}{n+1-r}} \tag{65}$$

to obtain

$$\dot{J}_n \leq 2R\, J_n + c_3(n, \mu)\ (G_m^s + G_m^{s_0}) - \frac{J_n^{\frac{n+1-r}{n-r}}}{J_r^{\frac{1}{n-r}}}. \tag{66}$$

Solving this differential inequality gives (64); (63) originates from Corollary 9.

The theorem is applied as follows: In the region where Theorem 15 holds (i.e. in the unshaded region of Figure 1), and provided that $d < 2 + 2/\sigma$, one can take $r = m = 1$, obtaining to leading order

$$\overline{J}_n \leq c\, R^{\frac{(1+\sigma)\,(2-d\sigma+2n\sigma)}{\sigma(2+2\sigma-d\sigma)}} \quad \text{if} \quad -\frac{1 + \mu\,\nu}{|\mu - \nu|} < \frac{\sqrt{2\,\sigma + 1}}{\sigma}. \tag{67}$$

When Theorem 15 fails (i.e. in the shaded region of Figure 1), one has to take $r = 0$ and use the bound on G_m provided by Theorem 13. Thus to leading order

$$\overline{J}_n \leq c\, R^{\frac{-(d\sigma)+(1+m\sigma)\left(2+2n^2\sigma+n(2+2\sigma-d\sigma)\right)}{(1+n)\sigma(2-d\sigma+2m\sigma)}} \quad \text{if} \quad |1 + \mathrm{i}\nu| \leq \frac{\sigma\, m + 1}{\sigma\, m}. \tag{68}$$

Remark 17 *We can easily use Corollary 6 instead of Corollary 9 in the proof of Theorem 16. This will leave the estimate in the unshaded region unchanged,* *however in the shaded region it will increase to*

$$\overline{J}_n \le c\,R^{\frac{2-d\sigma+2m\sigma+2n\sigma+2mn\sigma^2}{\sigma(2-d\sigma+2m\sigma)}}. \tag{69}$$

6 Time Averages

It is interesting to compare time averages of norms of the solution with their corresponding time-asymptotic bounds. Lemma 1 states that the time average of some quantity is always bounded by the corresponding time-asymptotic upper bound, hence

$$\langle J_0 \rangle \le \overline{J}_0 \le R^{1/\sigma}. \tag{70}$$

This result is already the best we can get. For $\langle J_1 \rangle$ however, it is possible to improve on $\overline{J}_1$ and derive a parameter and dimension independent estimate. Time-averaging equation (46),

$$\langle G_1 + J_1 \rangle = \langle R\,G_0 - \tfrac{1}{2}\dot{G}_0 \rangle \tag{71}$$

and with Lemma 1 we obtain

$$\begin{aligned} \langle J_1 \rangle &\le R\,\langle G_0 \rangle \le R^{1+1/\sigma}, \\ \langle G_1 \rangle &\le R\,\langle G_0 \rangle \le R^{1+1/\sigma}. \end{aligned} \tag{72}$$

Unfortunately for $n > 1$ there is not so convenient a way of computing time averages. An easy, although not optimal method is to time-average an interpolation inequality (using Lemma 1 twice),

$$\langle J_n \rangle \le \langle J_s \rangle^{\frac{n-1}{s-1}} \langle J_1 \rangle^{\frac{s-n}{s-1}} \tag{73}$$

$$\le \overline{J}_s^{\frac{n-1}{s-1}} \langle J_1 \rangle^{\frac{s-n}{s-1}}. \tag{74}$$

Time averages for G_n with $n > 1$ can be obtained in a similar way.

7 Length Scales

The key idea is to connect the dynamic length scale ℓ with the number of Fourier modes N that significantly contribute to the solution at a given instant of time ("relevant modes") by

$$\ell(t) \sim \left(\frac{V}{N(t)}\right)^{\frac{1}{d}} \tag{75}$$

where V denotes the volume of the d-dimensional domain. As in references [3,9] we obtain N as the number of lowest modes whose contribution to the L^∞ norm of the solution balances the contribution of the (infinitely many) higher modes. This is done as follows. Let κ be a positive number and f a sufficiently smooth periodic function on $[0,1]^d$ with the Fourier series expansion

$$f(x) = \sum_{k\in 2\pi\mathbb{Z}^d} \hat{f}_k \, \mathrm{e}^{\mathrm{i}kx} = \left(\sum_{\|k\|\le\kappa} + \sum_{\|k\|>\kappa} \right) \hat{f}_k \, \mathrm{e}^{\mathrm{i}kx} \tag{76}$$

where $\|k\| \equiv (k_1^2 + \cdots + k_d^2)^{1/2}$. Then for $s > d/2$ we can estimate the L^∞ norm of f by

$$\begin{aligned}\|f\|_\infty &\le \sum_{\|k\|\le\kappa} |\hat{f}_k| + \sum_{\|k\|>\kappa} \|k\|^{-s}\, \|k\|^{s}\, |\hat{f}_k| \\ &\le \text{(by the Cauchy-Schwarz inequality)} \\ &\le \left(\sum_{\|k\|\le\kappa} 1^2 \right)^{\frac{1}{2}} \left(\sum_{k\in 2\pi\mathbb{Z}^d} |\hat{f}_k|^2 \right)^{\frac{1}{2}} + \left(\sum_{\|k\|>\kappa} \|k\|^{-2s} \right)^{\frac{1}{2}} \left(\sum_{k\in 2\pi\mathbb{Z}^d} \|k\|^{2s}\, |\hat{f}_k|^2 \right)^{\frac{1}{2}}.\end{aligned} \tag{77}$$

The first and third term in this expression can be computed up to corrections of lower order in κ by using integral approximations of the sums. The second term is just the L^2 norm of f and the last term is equivalent to $\|\nabla^s f\|_2$, so that altogether we obtain

$$\|f\|_\infty \le c_1\, \kappa^{d/2}\, \|f\|_2 + c_2\, \kappa^{d/2-s}\, \|\nabla^s f\|_2. \tag{78}$$

The first term in (78) originates from the low modes where $\|k\| \le \kappa$, the second from the high modes. They have equal influence when

$$\kappa^s = c_3\, \frac{\|\nabla^s f\|_2}{\|f\|_2}. \tag{79}$$

With this choice of κ inserted back into (78), we have actually proved Agmon's inequality. However the point here is to consider κ to be the dynamic cut-off wavenumber. Moreover, we take $f = \nabla^r u$ and $n = s + r$ which leads to

$$N^{-d}(t) \sim \kappa_{n,r}(t) := \left(\frac{J_n}{J_r} \right)^{\frac{1}{2(n-r)}}. \tag{80}$$

Note that the free parameters n and r parametrise a family of approximations to κ rather than correspond to a family of different scales in the flow. We now hope for the following:

(i) $\ell^{-1} \sim \kappa_{n,r}(t)$ remains bounded for $t \to \infty$,
(ii) the bounds are qualitatively independent of n and r.

A careful analysis of the differential inequalities, however, shows that within the framework of Sobolev estimates it is not possible to derive time asymptotic bounds for $\kappa_{n,r}$. Orbits of the full CGL equation can get arbitrarily close to zero and near zero J_r can decrease faster than J_n resulting in the possibility of $\kappa_{n,r}$ becoming arbitrarily large.

We resort to the following *ad hoc* fix:

$$\mathcal{J}_n := J_n + \beta\, 2^n R^{n+1/\sigma}, \tag{81}$$

$$\mathcal{K}_{n,r} := \left(\frac{\mathcal{J}_n}{\mathcal{J}_r}\right)^{\frac{1}{2(n-r)}}. \tag{82}$$

By adding an extra term to the J_n, one avoids the small denominator problem of the $\kappa_{n,r}$. This perturbation is small unless the solution is small in the H^n norm for some intervals of time. Since we will derive bounds on $\mathcal{K}_{n,r}$ which are independent of n and r, these bounds will accurately characterise large events such as spikes in the solution. The particular form of the perturbation was chosen to be consistent with the dimensional scaling of time asymptotic upper bounds on J_n.

We will first estimate the time average of $\mathcal{K}_{n,r}^2$ and then state the corresponding time asymptotic bounds.

By Lemma 3 with Corollary 6 we obtain

$$\begin{aligned}\dot{\mathcal{J}}_n &\le 2R\, J_n + c_1\, J_n\, G_m^s - J_{n+1}\\ &\le 2R\, \mathcal{J}_n + c_1\, \mathcal{J}_n\, G_m^s - \mathcal{J}_{n+1}\\ &\le 2R\, \mathcal{J}_n + c_1\, \mathcal{J}_n\, G_m^s - \mathcal{J}_n \left(\frac{\mathcal{J}_n}{\mathcal{J}_r}\right)^{\frac{1}{n-r}}.\end{aligned} \tag{83}$$

Dividing by $\mathcal{J}_n$ and taking the time average gives

$$\left\langle \frac{\dot{\mathcal{J}}_n}{\mathcal{J}_n} + \mathcal{K}_{n,r}^2 \right\rangle \le 2R + c_1\, \langle G_m^s \rangle \tag{84}$$

where the time derivative on the left side vanishes by Lemma 1 plus the fact that $\mathcal{J}_n$ is bounded from below, so that to leading order in R

$$\langle \mathcal{K}_{n,r}^2 \rangle \le c_1\, \langle G_m^s \rangle\,. \tag{85}$$

The highest G_m for which we have a parameter independent bound is G_1. Therefore it is best to fix $m = 1$ which allows in the subcritical and critical case the estimate

$$
\begin{aligned}
\langle \mathcal{K}_{n,r}^2 \rangle &\leq c_1 \left\langle G_1^{1+\frac{d\sigma-2}{2+2\sigma-d\sigma}} \right\rangle \\
&\leq c_1 \left\langle G_1 \right\rangle^{1+\frac{d\sigma-2}{2+2\sigma-d\sigma}} \\
&\leq c_2 \, R^{\frac{2(1+\sigma)}{2+2\sigma-d\sigma}}. \qquad (86)
\end{aligned}
$$

In the supercritical case we fix $s = 1$ instead to avoid the restriction $d < 2 + 2/\sigma$:

$$
\langle \mathcal{K}_{n,r}^2 \rangle \leq c_1 \left\langle G_{1+\frac{d\sigma-2}{2\sigma}} \right\rangle. \qquad (87)
$$

This choice for s preserves the special role of the critical case when $d\sigma = 2$. Other choices are possible, depending on which time average on the right side is most advantageous to use. Note that in the critical case both (86) and (87) reduce to

$$
\langle \mathcal{K}_{n,r}^2 \rangle \leq c_1 \left\langle G_1 \right\rangle \leq c_3 \, R^{1+1/\sigma}. \qquad (88)
$$

All these estimates are independent of n and r. Furthermore the exponent is uniform in μ and ν for both the subcritical and the critical case while it becomes ν-dependent for the supercritical CGL.

Note that all the constants are independent of β, so that one can choose $\beta > 0$ arbitrarily small. This however does not imply that $\langle \kappa_{n,r}^2 \rangle$ is bounded.

The best we can do for estimating time asymptotic bounds is

$$
\overline{\mathcal{K}}_{n,r}^2 \leq \left(\frac{\overline{\mathcal{J}}_n}{\beta \, 2^r R^{r+1/\sigma}} \right)^{\frac{1}{n-r}}. \qquad (89)
$$

For large n, using the explicit bounds on J_n as given by (67) and (68), one obtains the scaling

$$
\overline{\mathcal{K}}_{n,r}^2 \sim \begin{cases} \dfrac{c_5}{\beta} \, R^{\frac{2+2\sigma}{2+2\sigma-d\sigma}} & \text{in the unshaded region,} \\ \dfrac{c_6}{\beta} \, R^{\frac{2+2m\sigma}{2-d\sigma+2m\sigma}} & \text{in the shaded region.} \end{cases} \qquad (90)
$$

8 L^∞ Estimates

To achieve dimensional consistency, one would like to compute the time average of the L^∞ norm to the power 2σ. This is done as follows:

$$
\begin{aligned}
\left\langle \|u\|_\infty^{2\sigma} \right\rangle &\leq c_1 \left\langle J_n^{r_1}\, G_p^{r_2} \right\rangle \\
&\leq c_1 \left\langle \left(\frac{\mathcal{J}_n}{\mathcal{J}_p}\right)^{r_1} \mathcal{J}_p^{r_1}\, G_p^{r_2} \right\rangle \\
&\leq \text{(by Hölder's inequality)} \\
&\leq c_1 \left\langle \left(\frac{\mathcal{J}_n}{\mathcal{J}_p}\right)^{\frac{1}{n-p}} \right\rangle^{r_1(n-p)} \left\langle \left(\mathcal{J}_p^{r_1}\, G_p^{r_2}\right)^{\frac{1}{1-r_1(n-p)}} \right\rangle^{1-r_1(n-p)} \\
&\leq c_1 \left\langle \mathcal{K}_{n,p}^2 \right\rangle^{r_3} \left\langle \mathcal{F}_p^{r_4} \right\rangle^{r_5}
\end{aligned}
\tag{91}
$$

where

$$\mathcal{F}_n := J_n + G_n + \beta\, 2^n R^{n+1/\sigma}.$$

The exponents in this calculation are

$$
\begin{aligned}
r_1 &= \frac{2\, d\, \sigma}{4\, n - 2\, d\, p\, \sigma + 4\, n\, p\, \sigma}, \qquad & r_2 &= \frac{(d - 2\, n)\ \sigma}{-2\, n + d\, p\, \sigma - 2\, n\, p\, \sigma}, \\
r_3 &= \frac{d\ (n - p)\ \sigma}{2\, n - d\, p\, \sigma + 2\, n\, p\, \sigma}, \qquad & r_4 &= \frac{2\, \sigma}{2 - d\, \sigma + 2\, p\, \sigma}, \\
r_5 &= \frac{n\ (-2 + d\, \sigma - 2\, p\, \sigma)}{-2\, n + d\, p\, \sigma - 2\, n\, p\, \sigma}.
\end{aligned}
$$

Pursuing the same strategy as in Section 7, in the subcritical and critical case where $r_4 \leq 1$, we force $p = 1$ giving

$$r_3 = \frac{d\ (-1 + n)\ \sigma}{2\, n - d\, \sigma + 2\, n\, \sigma}, \qquad r_4 = \frac{2\, \sigma}{2 + 2\, \sigma - d\, \sigma}, \qquad r_5 = \frac{n\ (-2 - 2\, \sigma + d\, \sigma)}{-2\, n + d\, \sigma - 2\, n\, \sigma}.$$

Hence $r_3 + r_5 = 1$ and with the first part of estimate (86)

$$\left\langle \|u\|_\infty^{2\sigma} \right\rangle \leq c_2 \left\langle \mathcal{F}_1 \right\rangle^{1 + \frac{d\sigma - 2}{2 + 2\sigma - d\sigma}}. \tag{92}$$

In the critical and supercritical case, we force $r_4 = 1$ so that

$$
\begin{aligned}
r_3 &= \frac{d\ (-2 + 2\, \sigma + d\, \sigma - 2\, n\, \sigma)}{-2\, d + 2\, d\, \sigma + d^2\, \sigma - 4\, n\, \sigma - 2\, d\, n\, \sigma}, \qquad & p &= \frac{-2 + 2\, \sigma + d\, \sigma}{2\, \sigma}, \\
r_5 &= \frac{4\, n\, \sigma}{2\, d - 2\, d\, \sigma - d^2\, \sigma + 4\, n\, \sigma + 2\, d\, n\, \sigma}.
\end{aligned}
$$

Hence $r_3 + r_5 = 1$ and with estimate (87)

$$\left\langle \|u\|_\infty^{2\sigma} \right\rangle \leq c_3 \left\langle \mathcal{F}_{1+\frac{d\sigma-2}{2\sigma}} \right\rangle . \tag{93}$$

Noting that time averages of $\mathcal{F}_n$ scale like time averages of G_n, upon comparing these two estimates with (86) and (87), one finds that $\langle \|u\|_\infty^{2\sigma} \rangle$ and $\left\langle \mathcal{K}_{n,r}^2 \right\rangle$ scale identically in R.

In order to get a time-asymptotic upper bound for the L^∞ norm, one can use the interpolation

$$\|u\|_\infty^{2\sigma} \leq c\, J_n^{r_6}\, G_m^{r_7} + \text{lower order term}. \tag{94}$$

Using Theorem 16 to bound J_n and Theorem 13 or 15 respectively to bound G_m, for large n one obtains the scaling (90) again.

We conclude that the $\mathcal{K}_{n,r}^2$ and $\|u\|_\infty^{2\sigma}$ bounds scale identically in R in both the time averaged and time asymptotic case (here in the limit when n is large).

Remark 18 *The entries in Table 1 use interpolations like* (74) *and* (94). *Whenever a limit $n \to \infty$ or $s \to \infty$ has been taken, the entry is marked with a dagger †. These limits are not strictly valid as in general the prefactor diverges, but allow us to present the essential features of the scalings more compactly.*

9 Conclusions

The Sobolev estimates on solutions of the complex Ginzburg-Landau equation are now—we believe—in a very satisfactory state. Such techniques also allow us to find upper bounds for the functionals $\langle \mathcal{K}_{n,r}^2 \rangle$ and $\overline{\mathcal{K}}_{n,r}^2$, but not for $\langle \kappa_{n,r}^2 \rangle$ or $\overline{\kappa}_{n,r}^2$ (these quantities are defined in Section 7). We believe that the difficulties in estimating the latter pair are fundamental rather than technical. Therefore one cannot expect a decision on which functional is most appropriate from within the framework of Sobolev estimates. Another limitation of the present technology is that it does not answer the question whether the admissible small scales are realised because it cannot provide lower bounds on our inverse length scales. Hence we see interesting potential for work along the following lines:

(i) Numerical investigation of the scalings of the above functionals: How sharp are the estimates, i.e. does the dynamics actually come close to realising the upper bounds obtained by the techniques presented here? Is there any qualitative correlation with the observed dynamics?

(ii) Comparison with heuristic scaling theories of fluid turbulence [13,23], see also [3,9] for analytic results on the Navier-Stokes equations.
(iii) Comparison with numerical and analytical results for some partial differential equation for which $\overline{\kappa}^2_{n,r}$ can be shown to be bounded.
(iv) Are there functionals that bridge the gap between $\mathcal{K}_{n,r}$ and $\kappa_{n,r}$?

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