ANALYTICITY OF THE ATTRACTOR AND THE NUMBER OF DETERMINING NODES FOR A WEAKLY DAMPED DRIVEN NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT. The attractor of the weakly damped driven nonlinear Schrödinger equation in one spatial dimension with periodic boundary conditions is known to be smooth whenever the driving term is smooth (O. Goubet, Appl. Anal. **60** (1996), 99–119). We show that the attractor is in fact contained in a subclass of the real analytic functions provided the driving term is real analytic. Our method is based on a characterization of Gevrey classes (and thus of the real analytic functions) in terms of decay of Fourier coefficients. These tools were previously developed in the context of parabolic dissipative equations, and we extend their applicability to the case of weak damping.

As a consequence of having an analytic attractor we prove that any two close enough points in the physical domain form a set of determining nodes for the long-time dynamics of the equation.

1. INTRODUCTION

In this article we prove real-analyticity of the global attractor for the weakly damped driven nonlinear Schrödinger (NLS) equation

$$\partial_t u + (\alpha - \mathrm{i})u + \mathrm{i}\partial_{xx} u + \mathrm{i}|u|^2 u = f \,, \tag{1a}$$

$$u(0) = u^{\mathrm{in}} \,, \tag{1b}$$

posed on the one-dimensional periodic spatial domain $\mathbb{T}$, with positive damping coefficient α and real analytic forcing function $f = f(x)$.

Ghidaglia [12] has proved that this equation is globally well posed and possesses a weak global attractor in the Sobolev space $H^1(\mathbb{T})$. Wang [30] showed that the attractor is in fact a global attractor in the strong H^1 topology. His work relies on a technique introduced by Ball [2] in the context of a damped semilinear wave equation which has also been used by Ghidaglia [13, 14] for the Korteweg–de Vries equation and by Abounouh [1] for a weakly damped driven nonlinear Schrödinger equation in a two-dimensional domain with boundaries. Goubet [15], on the other hand, constructs a function which asymptotically approximates the solution of (1), and is actually contained in any Sobolev space $H^m(\mathbb{T})$. He can therefore assert the existence of a strong attractor by classical methods (see [29], for example). The result can be summarized as follows.

Theorem 1. *If $f \in H^1(\mathbb{T})$, the weakly damped driven nonlinear Schrödinger equation (1) posed in $H^1(\mathbb{T})$ has a global attractor $\mathcal{A}$, i.e., it has a set $\mathcal{A}$ which is compact, invariant, and attracts every bounded set in the strong H^1 topology. Moreover,*

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if $f \in H^m(\mathbb{T})$ for some $m > 1$, $\mathcal{A}$ is also a global attractor in the strong $H^m(\mathbb{T})$ topology. Thus, if $f \in C^\infty(\mathbb{T})$, then $\mathcal{A} \subset C^\infty(\mathbb{T})$.

An essential feature of the proof is the splitting of u into a low wavenumber part $y \equiv P_N u$ and a high wavenumber part $Q_N u$, where P_N denotes the orthogonal projector from L^2 onto the low wavenumber space, $\operatorname{span}\{\exp(\mathrm{i}kx)\colon k \in 2\pi\mathbb{Z}, |k| \leq N\}$, and where $Q_N = I - P_N$. The low wavenumber part is a smooth function by definition, and the high wavenumber part can be approximated asymptotically by the solution z to the equation

$$\partial_t z + (\alpha - \mathrm{i})z + \mathrm{i}\partial_{xx} z + \mathrm{i} Q_N\big(|y+z|^2(y+z)\big) = Q_N f\,, \tag{2a}$$

$$z(t_0) = 0\,, \tag{2b}$$

for $t \geq t_0$. This equation formally arises by projecting (1a) onto the high wavenumber space, but is initialized with zero so that $v \equiv y + z$ is not necessarily a solution to the original nonlinear Schrödinger equation. One can show, however, that for large enough initial time t_0 and high enough cut-off wavenumber N there exists a unique solution

$$z \in C_{\mathrm{bdd}}([t_0,\infty); Q_N H^m(\mathbb{T})) \cap C^1_{\mathrm{bdd}}([t_0,\infty); Q_N H^{m-2}(\mathbb{T})) \tag{3}$$

for every $m \geq 1$, which has H^m bounds independent of u^{in}, and satisfies

$$\lim_{t\to\infty} \|z(t) - Q_N u\|_{H^1} = 0\,. \tag{4}$$

Every orbit u can therefore by approximated by the smooth and bounded function $v = y + z$ as $t \to \infty$. Some further details of this argument will be given in Section 7.

In this paper we show that the argument above holds true in spaces of analytic functions, and conclude that the attractor $\mathcal{A}$ not only is smooth, but is real analytic. Our result relies on a characterization of the space of real analytic functions $C^\omega(\mathbb{T})$ by the exponential decay of the Fourier coefficients. This fact is most conveniently expressed in terms of the operator domains

$$\mathcal{D}(A^r \mathrm{e}^{\tau A}) = \big\{u \in L^2(\mathbb{T})\colon A^r \mathrm{e}^{\tau A} u \in L^2(\mathbb{T})\big\}\,, \tag{5}$$

where $A \equiv \sqrt{-\Delta}$ is a non-negative semi-definite, self-adjoint operator in L^2. We denote the L^2 norm and inner product by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ respectively. Then we have

$$C^\omega(\mathbb{T}) = \bigcup_{\tau>0} \mathcal{D}(A^r \mathrm{e}^{\tau A}) \tag{6}$$

for any $r \geq 0$. This result is useful because the spaces $\mathcal{D}(A^r\mathrm{e}^{\tau A})$ carry the natural norm

$$\|\cdot\|_{\mathcal{D}(A^r \mathrm{e}^{\tau A})} = \|A^r \mathrm{e}^{\tau A}\cdot\| + \|\cdot\|\,, \tag{7}$$

and are thus amenable to Sobolev estimation techniques. Moreover, if a function f is contained in a given class $\mathcal{D}(A^r\mathrm{e}^{\tau A})$, then its radius of analyticity is at least τ.

Equation (6) is a special case of a more general theorem by de La Vallée Poussin [23], characterizing Gevrey classes through the decay of their Fourier coefficients. Estimates in $\mathcal{D}(A\mathrm{e}^{\tau A})$ have first been used by Foias and Temam [9] to present a simple proof of the analyticity of solutions to the Navier–Stokes equations subject to periodic boundary conditions. Promislow [28] extended the method to more general dissipative partial differential equations with polynomial nonlinearities. The estimates can often be simplified by using the norm (7), because for $r > d/2$, d being the spatial dimension of the problem, $\mathcal{D}(A^r e^{\tau A})$ is a topological algebra [6]. For a more complete review see [24].

Using the notation above, we can formulate our main result.

Theorem 2. *Let $r > \frac{3}{2}$, $f \in \mathcal{D}(A^r e^{\sigma A})$ for some $\sigma > 0$, and let $\mathcal{A}$ denote the attractor given by Theorem 1. Then there exists a $\tau \in (0, \sigma]$ such that $\mathcal{A} \in \mathcal{D}(A^r e^{\tau A})$. Moreover, for every $\mathcal{D}(A^r e^{\tau A})$-bounded neighborhood B of $\mathcal{A}$ there exists a $\rho \in (0, \tau]$ such that $\mathcal{A}$ attracts B in the $\mathcal{D}(A^r e^{\rho A})$ norm.*

Due to the characterization (6) of $C^\omega(\mathbb{T})$ we immediately have the following.

Corollary 3. *If $f \in C^\omega(\mathbb{T})$, then $\mathcal{A}$ is a global attractor in $C^\omega(\mathbb{T})$ endowed with the inductive limit topology induced by the family of normed spaces $\{\mathcal{D}(e^{\tau A}) \colon \tau > 0\}$.*

Remark 1. Although the attractor is contained in some regularity class $\mathcal{D}(A^r e^{\tau A})$, we cannot prove that it is globally attracting in this topology due to the non-compactness of the semigroup $\{S(t) \colon S(t)u^{\text{in}} \equiv u(t), 0 \leq t < \infty\}$. However, one can show that $\mathcal{A}$ is a local attractor [17] in $\mathcal{D}(A^r e^{\rho A})$ for every $\rho \in (0, \tau)$.

Remark 2. In this paper we assume that f does not depend on time. None of our formal calculations, however, depends on this assumption, and we can assert that solutions to (1) become asymptotically analytic so long as $f(t)$ remains bounded in some class $\mathcal{D}(A^r e^{\sigma A})$ as $t \to \infty$. On the other hand, the notion of an attractor is more complicated to define when the system is non-autonomous [17, Section 3.7].

This theorem has two important consequences. First, Galerkin approximations of a solution on the attractor converge exponentially fast, in the wavenumber, to the exact solution of (1). For strongly dissipative equations such results have been observed numerically [4, 16, 19] as well as demonstrated analytically [5, 21, 26]. In the case of weak dissipation it is important to stress that one cannot expect exponential convergence for arbitrary initial data. For a computational study of the weakly damped driven nonlinear Schrödinger equation see Jolly *et al.* [18].

Second, the long-time dynamics of the equation is determined by the values of the solution at any two close enough nodes. In general, a set of nodes $\{x_i \colon i \in I\}$ is called *determining* for a dynamical system if any two solutions u_1 and u_2, which satisfy

$$\lim_{t \to \infty} |u_1(x_i, t) - u_2(x_i, t)| = 0 \tag{8}$$

for every $i \in I$, must be asymptotically identical, i.e.

$$\lim_{t \to \infty} \|u_1(t) - u_2(t)\| = 0\,. \tag{9}$$

The concept was introduced by Foias and Temam [10] in the context of the Navier–Stokes equations. The best known estimate on the number of determining nodes, as well as the number of determining modes and volume elements for the two dimensional Navier–Stokes equations is given in [20]. Kukavica [22] has proved that any two close enough nodes are determining for the long-time dynamics of the one dimensional periodic complex Ginzburg–Landau equation, and a similar result was established for the complex Ginzburg–Landau equation on the real line [3]. On the other hand, for the one dimensional periodic Kuramoto–Sivashinsky equation, a forth order parabolic equation, Foias and Kukavica [7] showed that it has a set of four determining nodes.

The paper is laid out as follows. We first motivate Theorem 2 by proving a similar result for the steady state solutions of (1). We then prove some fundamental estimates in the function spaces $\mathcal{D}(A^r e^{\tau A})$: Commutator estimates (Section 3), "specialized" inequalities—inequalities in the high wavenumber space $Q_N\mathcal{D}(A^r e^{\tau A})$ where the dependence of the estimate on the cut-off wavenumber is made explicit (Section 4), and inverse inequalities—estimates on high norms of the low wavenumber component in terms of weaker norms (Section 5). All of these are generalizations of their well known H^m counterparts. In Section 6 we use these estimates to derive an *a priori* bound on z, the solution of the high mode equation (2), in $\mathcal{D}(A^r e^{\tau A})$. Section 7 discusses existence and regularity of the attractor. In the final Section 8 we prove that any two close enough nodes are determining for the dynamics of (1). Our method is similar to the proofs given in [22, 3], but is not as straightforward due to the fact that (1) is only weakly dissipative.

We believe the techniques presented in this paper can be used to prove that the weakly damped driven Korteweg–de Vries equation on a one dimensional periodic domain has an analytic attractor. By analogy with the Kuramoto–Sivashinsky equation [7] which has a set of four determining nodes, we expect to find three determining nodes in the case of the KdV equation. Here the numbers four and three "reflect" the order of the corresponding steady state problems, for which solutions are determined by four and three constants of integration respectively.

2. Regularity of Steady State Solutions

Steady state solutions to the weakly damped driven nonlinear Schrödinger equation (1) satisfy

$$\partial_{xx}u + \beta u + |u|^2 u = -\mathrm{i} f \,, \tag{10}$$

with $\beta \equiv -1 - \mathrm{i}\alpha$. It is relatively easy to prove the equivalent of Theorem 2 for the steady state problem.

Theorem 4. *Let $r > \frac{3}{2}$ and $f \in \mathcal{D}(A^r e^{\sigma A})$ with $\sigma > 0$. Then there exists a $\tau > 0$ such that every solution to the steady state equation* (10) *must be contained in $\mathcal{D}(A^r e^{\tau A})$.*

Proof. Existence of steady state solutions can be proved by Leray–Schauder degree theory. Equation (10) is homotopic to $u = 0$ via

$$u - sA^{-2}\big(\beta u + |u|^2 u + \mathrm{i} f\big) = 0 \tag{11}$$

with $s \in [0, 1]$. Note that the left hand side is a compact perturbation of the identity acting on u. Moreover, one can show that any solution u, being part of the attractor of the time dependent problem, must be contained in some H^r ball which is independent of s. This aspect of the argument depends crucially on the dissipative nature of the equation.

To prove analytic regularity of the steady state solution, we write the following estimate for (10) which is formal, but can be rigorously derived in the limit of a Galerkin approximating sequence:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\tau}\|A^r e^{\tau A} u\|^2 &= 2\,\|A^{r+1/2} e^{\tau A} u\|^2 \\ &= 2\,\|A^{r-3/2} e^{\tau A}\big(\beta u + |u|^2 u + \mathrm{i} f\big)\|^2 \\ &\le K_1 + K_2\,\|A^r e^{\tau A} u\|^6 \,, \end{aligned} \tag{12}$$

where we used the Poincaré and Young inequalities as well as the fact that $\mathcal{D}(A^r e^{\tau A})$ is an algebra [6]. Since every steady state is contained in the attractor, and hence in $\mathcal{D}(A^r e^{\tau A})$ for $\tau = 0$ by Theorem 1, we can let (12) evolve in τ. This proves that $u \in \mathcal{D}(A^r e^{\tau A})$ up to some finite, explicitly estimable $\tau > 0$. □

Remark 3. It is by no means necessary to refer to Theorem 1 at this point, although it gives convenient *a priori* bounds on u. Equivalent results could be obtained by elliptic regularity theory without reference to the time dependent problem.

Remark 4. Theorem 4 is an analytic regularity result for an elliptic equation. Results of this type are usually proved by using the Cauchy–Kowalevski theorem. We believe our method of proof is new, and will work for general semi-linear elliptic equations with analytic nonlinearities on periodic domains.

It is now natural to ask if perhaps the whole attractor is part of an analytic class of regularity. In Sections 6 and 7 we show that this is indeed the case. Vice versa, if the radius of analyticity of the steady state solutions is finite, as the proof of Theorem 4 suggests, then we would have an upper bound on the radius of analyticity τ in Theorem 2. We will argue in a separate note [27] that equations of the type (10) generically develop poles in the complex plane away from the real axis, and demonstrate for an explicitly solvable model problem that the radius of analyticity obtained through Gevrey norm estimates scales optimally with respect to the parameters in the problem.

3. Commutator Estimates in $\mathcal{D}(A^r e^{\tau A})$

In the following three sections we describe the basic ingredients for our main result. The first is a commutator estimate in $\mathcal{D}(A^r e^{\tau A})$. For clarity of presentation, we prove the result for real valued functions, and comment on the complex valued case later.

Lemma 5. *For $r > \frac{3}{2}$, and real valued functions $u \in \mathcal{D}(A^r e^{\tau A})$, $v \in \mathcal{D}(A^{r+1} e^{\tau A})$ and $w \in \mathcal{D}(A^{r+2} e^{\tau A})$, where u and v are assumed to have mean zero, one has the estimate*

$$\begin{aligned} &\left| \left\langle A^r e^{\tau A} u, A^r e^{\tau A}(vw) \right\rangle - \left\langle A^r e^{\tau A} v, A^r e^{\tau A}(uw) \right\rangle \right| \\ &\le C_1 \, \|A^{r-1} u\| \, \|A^r v\| \, \|A^{r+1} w\| + \tau \, C_2 \, \|A^{r-1} e^{\tau A} u\| \, \|A^{r+1} e^{\tau A} v\| \, \|A^{r+2} e^{\tau A} w\| \,, \end{aligned} \tag{13}$$

where C_1 and C_2 do not depend on τ.

Remark 5. Throughout this paper, we denote universal constants by C_i and constants which depend on the data of equation (1), i.e., on α and f, are written K_i. None of these may depend on the initial data u^{in}. The subscripts are intended to indicate the change of constants from one line to the next, but carry no meaning across different sections of this paper.

Remark 6. Note that the right hand side of (13) is not symmetric in u and v. This is necessary for our particular application of the inequality in Section 6.

Remark 7. When u, v, and w are complex valued functions, we can replace the left side of (13) by either

$$\left| \left\langle A^r e^{\tau A} u, A^r e^{\tau A}(\overline{v} w) \right\rangle - \left\langle A^r e^{\tau A} v, A^r e^{\tau A}(\overline{u} w) \right\rangle \right| , \tag{14}$$

or, provided that w is real, by

$$\left|\operatorname{Re}\left\langle A^r \mathrm{e}^{\tau A} u, A^r \mathrm{e}^{\tau A}(vw)\right\rangle - \operatorname{Re}\left\langle A^r \mathrm{e}^{\tau A} v, A^r \mathrm{e}^{\tau A}(uw)\right\rangle\right| . \tag{15}$$

More generally, any expression of this form will work provided it reduces to zero in the case $r = \tau = 0$.

Remark 8. By using the algebra properties of $\mathcal{D}(A^{r+1}\mathrm{e}^{\tau A})$ it is clear, for example, that

$$\left|\left\langle A^r \mathrm{e}^{\tau A} u, A^r \mathrm{e}^{\tau A}(vw)\right\rangle\right| \le C \, \|A^{r-1}\mathrm{e}^{\tau A} u\| \, \|A^{r+1}\mathrm{e}^{\tau A} v\| \, \|A^{r+1}\mathrm{e}^{\tau A} w\| . \tag{16}$$

The main assertion of Lemma 5 is that, by considering the commutator rather than performing separate estimates on each of the terms on the left of (13), the multiplicative constant in front of the Gevrey seminorms can be made arbitrarily small by choosing τ small.

Remark 9. The restriction to mean zero functions u and v can be removed, for example, by defining $A = \sqrt{I - \Delta}$. Such a choice leaves the proofs substantially unchanged, but introduces notational complications which we choose to avoid.

Remark 10. A similar estimate was used for obtaining estimates in Gevrey seminorms for equations of inviscid fluid flow [25].

Proof. We denote the Fourier coefficients of u, v and w by u_j, v_k and w_l respectively. Using the self-adjointness of $A^r \mathrm{e}^{\tau A}$, we can formally write for the terms on the left of (13)

$$\begin{aligned} \Xi &\equiv \left\langle A^r \mathrm{e}^{\tau A} u, A^r \mathrm{e}^{\tau A}(vw)\right\rangle - \left\langle A^r \mathrm{e}^{\tau A} v, A^r \mathrm{e}^{\tau A}(uw)\right\rangle \\ &= \left\langle A^{2r} \mathrm{e}^{2\tau A} u, vw\right\rangle - \left\langle A^{2r} \mathrm{e}^{2\tau A} v, uw\right\rangle . \end{aligned} \tag{17}$$

When u, v, and w are real, the Fourier representation of Ξ reads

$$\Xi = 2\pi \sum_{\substack{j,k,l \in 2\pi\mathbb{Z} \\ j+k+l=0}} u_j \, v_k \, w_l \left(|j|^{2r} \mathrm{e}^{2\tau|j|} - |k|^{2r} \mathrm{e}^{2\tau|k|} \right) . \tag{18}$$

In the complex case, assuming that the inner product is given by

$$\langle \phi, \psi \rangle = \int_{\mathbb{T}} \phi \, \overline{\psi} \, \mathrm{d}x , \tag{19}$$

one has for Ξ given by (14)

$$\Xi = 2\pi \sum_{\substack{j,k,l \in 2\pi\mathbb{Z} \\ j+k-l=0}} u_j \, v_k \, \overline{w}_l \left(|j|^{2r} \mathrm{e}^{2\tau|j|} - |k|^{2r} \mathrm{e}^{2\tau|k|} \right) , \tag{20}$$

and for Ξ given by (15), were we must require that w is real so that $\overline{w}_l = w_{-l}$,

$$\begin{aligned} \Xi = \pi & \sum_{\substack{j,k,l \in 2\pi\mathbb{Z} \\ j-k-l=0}} u_j \, \overline{v}_k \, \overline{w}_l \left(|j|^{2r} \mathrm{e}^{2\tau|j|} - |k|^{2r} \mathrm{e}^{2\tau|k|} \right) \\ & + \pi \sum_{\substack{j,k,l \in 2\pi\mathbb{Z} \\ -j+k+l=0}} \overline{u}_j \, v_k \, w_l \left(|j|^{2r} \mathrm{e}^{2\tau|j|} - |k|^{2r} \mathrm{e}^{2\tau|k|} \right) . \end{aligned} \tag{21}$$

We proceed by considering the real valued case only, (20) and (21) can be estimated in the same manner. Note that for positive real numbers ξ and η,

$$\left|\xi^r \mathrm{e}^{s\xi} - \eta^r \mathrm{e}^{s\eta}\right| \le |\xi - \eta| \Big(r\big(\xi^{r-1} + \eta^{r-1}\big) + s(r+1)\big(\xi^r \mathrm{e}^{s\xi} + \eta^r \mathrm{e}^{s\eta}\big) \Big) . \tag{22}$$

This inequality follows directly from the mean value theorem and the fact that $\mathrm{e}^{s\xi} \leq 1 + s\xi\, \mathrm{e}^{s\xi}$. We estimate every term in (18) by its absolute value and identify $\xi = |j|$ and $\eta = |k|$, so that $|\xi - \eta| \leq |j + k| = |l|$, to obtain

$$|\Xi| \leq C_1 \sum_{\substack{j,k,l \in 2\pi\mathbb{Z} \\ j+k+l=0}} |u_j|\,|v_k|\,|w_l|\,|l| \Big(|j|^{2r-1} + |k|^{2r-1} + \tau \big(|j|^{2r} \mathrm{e}^{2\tau|j|} + |k|^{2r} \mathrm{e}^{2\tau|k|} \big) \Big) . \tag{23}$$

Next, note that for real numbers ξ and η

$$|\xi + \eta|^r \leq C(r) \big(|\xi|^r + |\eta|^r \big) , \tag{24}$$

so that

$$\begin{aligned}
|\Xi| &\leq C_2 \sum_{j+k+l=0} |u_j|\,|v_k|\,|w_l|\,|l| \Big(|j|^{r-1} \big(|k|^r + |l|^r \big) + |k|^r \big(|j|^{r-1} + |l|^{r-1} \big) \Big) \\
&\quad + \tau\, C_3 \sum_{j+k+l=0} \Big\{ |u_j|\,|v_k|\,|w_l|\,|l|\, \mathrm{e}^{\tau(|j|+|k|+|l|)} \\
&\qquad \cdot \Big(|j|^{r-1} \big(|k|^{r+1} + |l|^{r+1} \big) + |k|^{r+1} \big(|j|^{r-1} + |l|^{r-1} \big) \Big) \Big\} \\
&\leq C_2 \Big(2\, \|l\, w_l\|_{\ell^1}\, \|A^{r-1} u\|\, \|A^r v\| \\
&\qquad + \|v_k\|_{\ell^1}\, \|A^{r-1} u\|\, \|A^{r+1} w\| + \|u_j\|_{\ell^1}\, \|A^r v\|\, \|A^r w\| \Big) \\
&\quad + \tau\, C_3 \Big(2\, \|l \mathrm{e}^{\tau|l|}\, w_l\|_{\ell^1}\, \|A^{r-1} \mathrm{e}^{\tau A} u\|\, \|A^{r+1} \mathrm{e}^{\tau A} v\| \\
&\qquad + \|\mathrm{e}^{\tau|k|}\, v_k\|_{\ell^1}\, \|A^{r-1} \mathrm{e}^{\tau A} u\|\, \|A^{r+2} \mathrm{e}^{\tau A} w\| \\
&\qquad + \|\mathrm{e}^{\tau|j|}\, u_j\|_{\ell^1}\, \|A^{r+1} \mathrm{e}^{\tau A} v\|\, \|A^r \mathrm{e}^{\tau A} w\| \Big)
\end{aligned} \tag{25}$$

In the second step we have used a "convolution inequality" for positive sequences $a = \{a_j\}$, $b = \{b_k\}$ and $c = \{c_l\}$,

$$\sum_{j+k+l=0} a_j\, b_k\, c_l = \sum_j a_j \sum_k b_k\, c_{-j-k} \leq \|a\|_{\ell^1}\, \|b\|_{\ell^2}\, \|c\|_{\ell^2} . \tag{26}$$

Moreover, for $s > \frac{1}{2}$ and a sequence $a = a_j$ with $a_0 = 0$,

$$\|a\|_{\ell^1} \leq \Big(\sum_{j \neq 0} |j|^{-2s} \Big)^{\frac{1}{2}} \Big(\sum_j |j|^{2s} |a_j|^2 \Big)^{\frac{1}{2}} = C(s)\, \|j^s\, a_j\|_{\ell^2} \tag{27}$$

which, when applied to the ℓ^1 norms on the right of equation (25), yields the statement of the lemma. □

4. Specialized Inequalities for Functions in $Q_N \mathcal{D}(A^r \mathrm{e}^{\tau A})$

The second ingredient for our result are "specialized" inequalities for functions

$$u = \sum_{|j|>N} u_j\, \mathrm{e}^{\mathrm{i}jx} , \qquad v = \sum_{|k|>N} v_k\, \mathrm{e}^{\mathrm{i}kx} , \tag{28}$$

which do not have a low wavenumber component. The word *specialized* will denote that we make explicit the dependence of the inequalities on the cut-off wavenumber N.

First, since

$$\|u\|^2 = \sum_{|j|>N} |u_j|^2 \leq \frac{1}{N^2} \sum_{|j|>N} j^2 \, |u_j|^2 \,, \tag{29}$$

we have the specialized Poincaré inequality

$$\|u\| \leq \frac{1}{N} \, \|Au\| \,. \tag{30}$$

This inequality is sufficient for deriving nonlinear estimates in H^m, because products of functions can be estimated by Hölder inequalities and expressed in terms of H^m norms through standard Sobolev embeddings. For estimates of products in Gevrey seminorms, we also need a specialized algebra inequality.

Lemma 6. *For $\tau \geq 0$ and $r > \frac{1}{2}$, $\mathcal{D}(A^r \mathrm{e}^{\tau A})$ is a topological algebra, i.e., any two functions $u, v \in \mathcal{D}(A^r \mathrm{e}^{\tau A})$ satisfy*

$$\|A^r \mathrm{e}^{\tau A}(uv)\| \leq C \, \|A^r \mathrm{e}^{\tau A} u\| \, \|A^r \mathrm{e}^{\tau A} v\| \tag{31}$$

for some constant $C = C(r)$. Moreover, if u and v are contained in $Q_N \mathcal{D}(A^r \mathrm{e}^{\tau A})$, then

$$\|A^r \mathrm{e}^{\tau A}(uv)\| \leq \frac{C}{N^{r-1/2}} \, \|A^r \mathrm{e}^{\tau A} u\| \, \|A^r \mathrm{e}^{\tau A} v\| \,. \tag{32}$$

Proof. The proof is essentially given in [6]. We sketch it briefly for real valued u and v to demonstrate how the dependence on N arises. The proof holds true for the complex valued case with minor notational complications.

As usual, we write the left side of (32) in the Fourier representation and use (24) together with the Young inequality to find that

$$\begin{aligned} \|A^r \mathrm{e}^{\tau A}(uv)\|^2 &= \sum_{n \in 2\pi\mathbb{Z}} \Bigg| \sum_{\substack{j+k=n \\ |j|,|k|>N}} u_j \, v_k \Bigg|^2 n^{2r} \mathrm{e}^{2\tau n} \\ &\leq C(r) \sum_{n \in 2\pi\mathbb{Z}} \Bigg(\sum_{\substack{j+k=n \\ |j|,|k|>N}} |u_j| \, |v_k| |j|^r \mathrm{e}^{\tau(|j|+|k|)} \Bigg)^2 \\ &\quad + C(r) \sum_{n \in 2\pi\mathbb{Z}} \Bigg(\sum_{\substack{j+k=n \\ |j|,|k|>N}} |u_j| \, |v_k| |k|^r \mathrm{e}^{\tau(|j|+|k|)} \Bigg)^2 . \end{aligned} \tag{33}$$

The sums on the right hand side can be estimated by twice applying the convolution inequality for sequences $a = \{a_j\}$ and $b = \{b_k\}$,

$$\|a * b\|_{\ell^2} \leq \|a\|_{\ell^1} \, \|b\|_{\ell^2} \,, \tag{34}$$

so that

$$\begin{aligned} \|A^r \mathrm{e}^{\tau A}(uv)\|^2 \leq C(r) &\Bigg(\sum_{|k|>N} |v_k| \, \mathrm{e}^{\tau|k|} \Bigg)^2 \|A^r \mathrm{e}^{\tau A} u\|^2 \\ &+ C(r) \Bigg(\sum_{|j|>N} |u_j| \, \mathrm{e}^{\tau|j|} \Bigg)^2 \|A^r \mathrm{e}^{\tau A} v\|^2 \,. \end{aligned} \tag{35}$$

As in the proof of Lemma 5 we estimate the ℓ^1 norm by

$$\sum_{|k|>N} |v_k| \, \mathrm{e}^{\tau|k|} \le \left(\sum_{|k|>N} |k|^{-2r} \right)^{\frac{1}{2}} \left(\sum_{|k|>N} |k|^{2r} \mathrm{e}^{2\tau|k|} \, |v_k|^2 \right)^{\frac{1}{2}} = \frac{C_2(r)}{N^{2r-1}} \, \|A^r \mathrm{e}^{\tau A} v\| \, . \tag{36}$$

The corresponding estimate for u then completes the proof. □

5. Inverse Inequalities in $P_N H^m$

The "specialized" inequalities of the previous section allow us to achieve small multiplicative constants in front of estimates on the high wavenumber component z by choosing large cut-off wavenumbers N. This, however, is of use only if the low wavenumber component y is at least bounded in $P_N H^m$ independent of N. Hence we need the following lemma.

Lemma 7. *For $m \ge 1$, every $N_{\max} \in 2\pi\mathbb{N}$, and every bounded set $B \subset H^1(\mathbb{T})$ there exist a time $t_0 = t_0(m, N_{\max}, B)$ and constants K_1 and K_2 independent of t_0, $N_{\max}$, and B, such that for every $N \le N_{\max}$ and $t \ge t_0$,*

$$\|y(t)\|_{H^m} \le K_1 \quad \textit{and} \quad \|\partial_t y(t)\|_{H^{m-2}} \le K_2 \, , \tag{37}$$

where $y = P_N u$, and u solves the weakly damped driven nonlinear Schrödinger equation (1) with $u^{\text{in}} \in B$.

Remark 11. In analogy with (7), we define the H^m norm by

$$\|\cdot\|_{H^m} = \|A^m \cdot\| + \|\cdot\| \, , \tag{38}$$

which is equivalent to the usual H^m norm.

Remark 12. Note that the lemma does not require $u^{\text{in}} \in H^m(\mathbb{T})$.

Proof. We denote the H^m distance between a set $S \subset H^m(\mathbb{T})$ and a point $\phi \in H^m(\mathbb{T})$ by

$$\operatorname{dist}_{H^m}(S, \phi) = \inf_{\theta \in S} \|\theta - \phi\|_{H^m} \, . \tag{39}$$

Then, by the usual inverse inequality,

$$\operatorname{dist}_{H^m}(S, P_N \psi) \le N^{m-1} \operatorname{dist}_{H^1}(S, P_N \psi) \tag{40}$$

for every $\psi \in H^1(\mathbb{T})$.

Now, according to Theorem 1 the attractor is contained in some H^m ball of radius K_3. Therefore, for any solution $u(t)$ of (1) we have

$$\begin{aligned} \|y(t)\|_{H^m} &\le \sup_{\theta \in \mathcal{A}} \|P_N \theta\|_{H^m} + \operatorname{dist}_{H^m}(P_N \mathcal{A}, y(t)) \\ &\le K_3 + N_{\max}^{m-1} \operatorname{dist}_{H^1}(P_N \mathcal{A}, y(t)) \, . \end{aligned} \tag{41}$$

As any trajectory $u(t)$ converges in H^1 to the attractor when $t \to \infty$, we can always choose t_0 large enough so that the right side of (41) is bounded by $K_1 = 2K_3$. Moreover, according to Theorem 1, $\mathcal{A}$ attracts bounded sets. The choice of t_0 can thus be made uniform over all $u^{\text{in}} \in B$.

A similar argument can be made to obtain an H^{m-2} bound on $\partial_t y$. A direct estimate on $\partial_t u$ in (1) and Theorem 1 show that

$$\mathcal{A}_t \equiv \{\partial_t u(t) \colon t \in \mathbb{R}, u^{\text{in}} \in \mathcal{A}\}$$

is bounded in H^{m-2} and attracts all orbits of $\partial_t u(t)$ in H^{-1}. Thus, the equivalent of (41) becomes

$$\begin{aligned} \|\partial_t y(t)\|_{H^{m-2}} &\le \sup_{\theta \in \mathcal{A}_t} \|P_N \theta\|_{H^{m-2}} + \operatorname{dist}_{H^{m-2}}(P_N \mathcal{A}_t, \partial_t y(t)) \\ &\le K_4 + N_{\max}^{m-1} \operatorname{dist}_{H^{-1}}(P_N \mathcal{A}_t, \partial_t y(t)) \,. \end{aligned} \tag{42}$$

The argument can now be concluded as before. □

We can easily extend the "inverse inequalities" into spaces $P_N \mathcal{D}(A^r \mathrm{e}^{\tau A})$.

Corollary 8. *Let m, $N_{\max}$, t_0, K_1 and K_2 be as in Lemma 7. Then for every $N \le N_{\max}$ and $t \ge t_0$,*

$$\|A^m \mathrm{e}^{\tau A} y(t)\| \le K_1 \, \mathrm{e}^{\tau N} \quad \textit{and} \quad \|A^{m-2} \mathrm{e}^{\tau A} \partial_t y(t)\| \le K_2 \, \mathrm{e}^{\tau N} \,. \tag{43}$$

6. Absorbing Ball in $\mathcal{D}(A^r \mathrm{e}^{\tau A})$

Our key result is the following bound on z which, together with Corollary 8, implies the existence of an absorbing ball for the approximate solution $v = y + z$ in $\mathcal{D}(A^r \mathrm{e}^{\tau A})$.

Lemma 9. *When $r > \frac{3}{2}$, there exist t_0, N, K and $\tau > 0$, independent of u^{in}, such that the solution z of (2) is uniformly bounded in $\mathcal{D}(A^r \mathrm{e}^{\tau A})$. That is, for every $t \ge t_0$,*

$$\|A^r \mathrm{e}^{\tau A} z(t)\| \le K \,. \tag{44}$$

Proof. Choose t_0 such that Lemma 7 holds with $m \ge r + 2$. In the course of the calculation we will find explicitly how large the cut-off wave number N, and thus the $N_{\max}$ of Lemma 7 need to be. In particular, it will be possible to choose $N_{\max}$ independent of t_0, guaranteeing the consistency of the argument.

Our goal is to bound the functional

$$\begin{aligned} J(z) \equiv \|A^r \mathrm{e}^{\tau A} z\|^2 + \|A^{r+1} \mathrm{e}^{\tau A} z\|^2 + 2 \operatorname{Im} \left\langle A^r \mathrm{e}^{\tau A} f, A^r \mathrm{e}^{\tau A} z \right\rangle \\ - \operatorname{Re} \left\langle A^r \mathrm{e}^{\tau A} (2|y|^2 y + 2|y|^2 z + y^2 \overline{z}), A^r \mathrm{e}^{\tau A} z \right\rangle \end{aligned} \tag{45}$$

from above and from below. We consider $y = P_N u$, where u solves the damped driven nonlinear Schrödinger equation (1), as given, and aim to study the solution z of the high wave number equation (2). This functional generalizes the one used by Goubet [15], albeit the choice of the "right" nonlinear term is more subtle here. Our estimates are formal, but can be made rigorous by constructing the solution z as the limit of a Galerkin approximating sequence. On periodic domains this is a routine procedure, for details see [15].

By direct calculation, one finds that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}J(z)+\alpha J(z) &= \operatorname{Re}\left\langle A^r \mathrm{e}^{\tau A} z, A^r \mathrm{e}^{\tau A}\partial_t z\right\rangle + \operatorname{Re}\left\langle A^{r+1}\mathrm{e}^{\tau A} z, A^{r+1}\mathrm{e}^{\tau A}\partial_t z\right\rangle \\ &\quad + \alpha\,\|A^r\mathrm{e}^{\tau A} z\|^2 + \alpha\,\|A^{r+1}\mathrm{e}^{\tau A}z\|^2 \\ &\quad - \operatorname{Re}\left\langle A^r\mathrm{e}^{\tau A}\partial_t\big(|y|^2 y + |y|^2 z + \tfrac{1}{2}y^2\overline{z}\big), A^r\mathrm{e}^{\tau A}z\right\rangle \\ &\quad - \operatorname{Re}\left\langle A^r\mathrm{e}^{\tau A}\big(2|y|^2y+2|y|^2z+y^2\overline{z}\big), A^r\mathrm{e}^{\tau A}(\tfrac{1}{2}\partial_t+\alpha)z\right\rangle \\ &\quad + 2\operatorname{Im}\left\langle A^r\mathrm{e}^{\tau A}f, A^r\mathrm{e}^{\tau A}(\tfrac{1}{2}\partial_t+\alpha)z\right\rangle . \end{aligned} \tag{46}$$

On the other hand, compute the $\mathcal{D}(A^r\mathrm{e}^{\tau A})$ inner product of the high wavenumber equation (2a) with $(\partial_t+\alpha)z$ and take the imaginary part. One obtains

$$\begin{aligned} &\operatorname{Re}\left\langle A^r\mathrm{e}^{\tau A}z, A^r\mathrm{e}^{\tau A}\partial_t z\right\rangle + \operatorname{Re}\left\langle A^{r+1}\mathrm{e}^{\tau A}z, A^{r+1}\mathrm{e}^{\tau A}\partial_t z\right\rangle \\ &+ \alpha\,\|A^r\mathrm{e}^{\tau A}z\|^2 + \alpha\,\|A^{r+1}\mathrm{e}^{\tau A}z\|^2 \\ &- \operatorname{Re}\left\langle A^r\mathrm{e}^{\tau A}\big(|y|^2y+2|y|^2z+y^2\overline{z}+2|z|^2y+z^2\overline{y}+|z|^2z\big), A^r\mathrm{e}^{\tau A}(\partial_t+\alpha)z\right\rangle \\ = &-\operatorname{Im}\left\langle A^r\mathrm{e}^{\tau A}f, A^r\mathrm{e}^{\tau A}(\partial_t+\alpha)z\right\rangle . \end{aligned} \tag{47}$$

Combine (46) and (47) to find

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}J(z)+\alpha J(z) &= \alpha\operatorname{Im}\left\langle A^r\mathrm{e}^{\tau A}f, A^r\mathrm{e}^{\tau A}z\right\rangle \\ &\quad + \operatorname{Re}\left\langle A^r\mathrm{e}^{\tau A}(2|z|^2y+z^2\overline{y}+|z|^2z), A^r\mathrm{e}^{\tau A}(\partial_t+\alpha)z\right\rangle \\ &\quad - \operatorname{Re}\left\langle A^r\mathrm{e}^{\tau A}\big((\partial_t+\alpha)(|y|^2y)+2\operatorname{Re}(y\partial_t\overline{y})z+\overline{z}y\partial_t y\big), A^r\mathrm{e}^{\tau A}z\right\rangle \\ &\quad + \operatorname{Re}\Big(\left\langle A^r\mathrm{e}^{\tau A}(|y|^2z+\tfrac{1}{2}y^2\overline{z}), A^r\mathrm{e}^{\tau A}\partial_t z\right\rangle \\ &\qquad - \left\langle A^r\mathrm{e}^{\tau A}(|y|^2\partial_t z+\tfrac{1}{2}y^2\partial_t\overline{z}), A^r\mathrm{e}^{\tau A}z\right\rangle\Big) \\ &\equiv I_1+I_2+I_3+I_4 . \end{aligned} \tag{48}$$

Next, we derive bounds on I_1–I_4.

A direct application of the Cauchy–Schwarz inequality, the specialized Poincaré inequality (30) and the Young inequality shows that

$$|I_1| \le \frac{1}{N}\,\|A^r\mathrm{e}^{\tau A}f\|\,\|A^{r+1}\mathrm{e}^{\tau A}z\| \le K_1 + \frac{1}{N^2}\,\|A^{r+1}\mathrm{e}^{\tau A}z\|^2 . \tag{49}$$

To estimate I_2 and I_4, note that by equation (2)

$$\begin{aligned}\|A^{r-1}\mathrm{e}^{\tau A}\partial_t z\| &\le (1+\alpha)\,\|A^{r-1}\mathrm{e}^{\tau A}z\| + \|A^{r+1}\mathrm{e}^{\tau A}z\| \\ &\quad + \|A^{r-1}\mathrm{e}^{\tau A}\big(|y+z|^2(y+z)\big)\| + \|A^{r-1}\mathrm{e}^{\tau A}f\| \\ &\le (2+\alpha)\,\|A^{r+1}\mathrm{e}^{\tau A}z\| \\ &\quad + C_1\big(\|A^{r-1}\mathrm{e}^{\tau A}y\|^3 + \|A^{r-1}\mathrm{e}^{\tau A}z\|^3\big) + \|A^{r-1}\mathrm{e}^{\tau A}f\| . \end{aligned} \tag{50}$$

The second inequality is due to the Poincaré inequality, the fact that $\mathcal{D}(A^{r-1}\mathrm{e}^{\tau A})$ is an algebra for $r>\frac{3}{2}$, and the Young inequality. Since f is real analytic, there exists a τ_0 so that $\|A^{r-1}\mathrm{e}^{\tau A}f\|$ is bounded for every $\tau\le\tau_0$. Together with Corollary 8 and the specialized Poincaré inequality (30) we obtain

$$\|A^{r-1}\mathrm{e}^{\tau A}\partial_t z\| \le K_2\,\mathrm{e}^{3\tau N} + (2+\alpha)\,\|A^{r+1}\mathrm{e}^{\tau A}z\| + \frac{C_1}{N^6}\,\|A^{r+1}\mathrm{e}^{\tau A}z\|^3 . \tag{51}$$

With this estimate on $\partial_t z$, we can begin to directly estimate I_2. Due to the self-adjointness of A, the Cauchy–Schwarz inequality and the specialized algebra inequality (32) with $r > \frac{3}{2}$, we have

$$\begin{aligned}
|I_2| &\le \frac{C_2}{N^2} \, \|A^{r+1} \mathrm{e}^{\tau A} z\|^2 \Big(\|A^{r+1} \mathrm{e}^{\tau A} y\| + \frac{1}{N^2} \|A^{r+1} \mathrm{e}^{\tau A} z\| \Big) \|A^{r-1} \mathrm{e}^{\tau A} (\partial_t + \alpha) z\| \\
&\le \frac{C_2}{N^2} \, \|A^{r+1} \mathrm{e}^{\tau A} z\|^2 \Big(K_3 \, \mathrm{e}^{\tau N} + \frac{1}{N^2} \|A^{r+1} \mathrm{e}^{\tau A} z\| \Big) \\
&\quad \cdot \Big(K_2 \, \mathrm{e}^{3\tau N} + (2+\alpha) \, \|A^{r+1} \mathrm{e}^{\tau A} z\| + \frac{C_1}{N^6} \, \|A^{r+1} \mathrm{e}^{\tau A} z\|^3 + \frac{\alpha}{N^2} \, \|A^{r+1} \mathrm{e}^{\tau A} z\| \Big) \\
&\le K_4 \, \mathrm{e}^{6\tau N} + \frac{K_5}{N^4} \, \|A^{r+1} \mathrm{e}^{\tau A} z\|^6 \,. \qquad (52)
\end{aligned}$$

The term I_3 can be estimated in a straightforward way by using the bounds on $\partial_t y$ from Corollary 8:

$$\begin{aligned}
|I_3| &\le C_3 \, \|A^r \mathrm{e}^{\tau A} y\| \, \|A^r \mathrm{e}^{\tau A} (\partial_t + \alpha) y\| \, \|A^r \mathrm{e}^{\tau A} y\| \, \|A^r \mathrm{e}^{\tau A} z\| \\
&\quad + C_3 \, \|A^r \mathrm{e}^{\tau A} z\| \, \|A^r \mathrm{e}^{\tau A} \partial_t y\| \, \|A^r \mathrm{e}^{\tau A} y\| \, \|A^r \mathrm{e}^{\tau A} z\| \\
&\le K_6 \, \mathrm{e}^{4\tau N} + \frac{K_7 \, \mathrm{e}^{6\tau N}}{N^6} \, \|A^{r+1} \mathrm{e}^{\tau A} z\|^6 \,. \qquad (53)
\end{aligned}$$

Finally, we use the commutator estimates from Section 3 on $|I_4|$. By identifying $u = \partial_t z$, $v = z$, $w = y^2$ in (14), and $u = \partial_t z$, $v = z$, $w = |y|^2$ in (15), one finds

$$\begin{aligned}
|I_4| &\le C_4 \, \|A^{r-1} \partial_t z\| \, \|A^r z\| \, \|A^{r+1} y\|^2 \\
&\quad + \tau \, C_5 \, \|A^{r-1} \mathrm{e}^{\tau A} \partial_t z\| \, \|A^{r+1} \mathrm{e}^{\tau A} z\| \, \|A^{r+2} \mathrm{e}^{\tau A} y\|^2 \\
&\le \Big(\frac{C_4}{N} + \tau \, C_5 \Big) \, \|A^{r-1} \mathrm{e}^{\tau A} \partial_t z\| \, \|A^{r+1} \mathrm{e}^{\tau A} z\| \, \|A^{r+2} \mathrm{e}^{\tau A} y\|^2 \,. \qquad (54)
\end{aligned}$$

Using Corollary 8 and estimate (51) we then obtain

$$\begin{aligned}
|I_4| &\le K_8 \, \Big(\tau + \frac{1}{N} \Big) \mathrm{e}^{2\tau N} \, \|A^{r+1} \mathrm{e}^{\tau A} z\| \Big(\mathrm{e}^{3\tau N} + \|A^{r+1} \mathrm{e}^{\tau A} z\| + \frac{1}{N^6} \, \|A^{r+1} \mathrm{e}^{\tau A} z\|^3 \Big) \\
&\le K_9 \, \Big(\tau + \frac{1}{N} \Big) \mathrm{e}^{6\tau N} + C_6 \, \Big(\tau + \frac{1}{N} \Big) \|A^{r+1} \mathrm{e}^{\tau A} z\|^6 \,. \qquad (55)
\end{aligned}$$

Similar estimates can be used to bound $J(z)$ from below,

$$J(z) \ge \|A^{r+1} \mathrm{e}^{\tau A} z\|^2 - \frac{K_{10} \, \mathrm{e}^{2\tau N}}{N^2} \, \|A^{r+1} \mathrm{e}^{\tau A} z\|^2 - K_{11} \, \mathrm{e}^{6\tau N} \,. \qquad (56)$$

Set $\tau = 1/N$ and let N large enough such that

$$\frac{K_{10} \, \mathrm{e}^2}{N^2} \le \frac{1}{2} \,, \qquad \frac{1}{N^2} \le \frac{\alpha}{4} \,. \qquad (57)$$

Then we find from (56) that

$$J(z) \ge \tfrac{1}{2} \, \|A^{r+1} \mathrm{e}^{\tau A} z\|^2 - K_{12} \,, \qquad (58)$$

and with (48), (49), (52), (53) and (55) that

$$\frac{\mathrm{d}}{\mathrm{d}t} J(z) \le K_{13} - \alpha J(z) + \frac{K_{14}}{N} J^3(z) \equiv p(J(z)) \,, \qquad (59a)$$

$$J(z(t_0)) = 0 \,. \qquad (59b)$$

Clearly, $p(J)$ is a cubic polynomial with $p(0) > 0$ and $p(+\infty) = +\infty$. If we can show that, for some N, p has two zeros ζ_1 and ζ_2 on the positive J axis, so that

$p(J) < 0$ for $J \in (\zeta_1, \zeta_2)$, then we must have $J(z(t)) \leq \zeta_1$ for every $t \geq t_0$. An elementary calculation shows that this is indeed the case whenever

$$\frac{K_{14}}{N} < \frac{4}{27} \frac{\alpha^3}{K_{13}^2} \,. \tag{60}$$

In other words, $J(z(t))$, and hence $\|A^{r+1} \mathrm{e}^{\tau A} z(t)\|$, are bounded by a number which depends on α, f, N and τ, but not on time. □

Remark 13. The choice of τ and N are necessarily linked. While N is only a technical device, τ determines the regularity class of the attractor. By combining the conditions on N given by (57) and (60) we find that

$$\tau < \min \left\{ \frac{1}{\mathrm{e}} \frac{1}{\sqrt{2\, K_{10}}}, \frac{\sqrt{\alpha}}{2}, \frac{4}{27} \frac{\alpha^3}{K_{13}^2\, K_{14}} \right\} \,. \tag{61}$$

We do not claim that this estimate is optimal. For example, a slight improvement could be obtained by rewriting the left side of (26) in terms of an integral, which then can be estimated in more general L^p spaces rather than L^1 and L^∞. However, as mentioned in Section 2, we believe fact that τ cannot be arbitrarily large is fundamental.

7. Existence and Regularity of the Attractor

The goal of this section is to sketch the proof of Theorems 1 and 2. Most of the material directly follows the work of Goubet [15], and is included for the reader's convenience.

It is known that the initial value problem (1) is well posed in $H^1(\mathbb{T})$ and that there exists an absorbing ball $\mathcal{B} \subset H^1(\mathbb{T})$ into which all orbits enter in a time uniform over initial data from bounded sets of H^1 (Ghidaglia, [12]). In particular, $u(t) \equiv S(t) u^{\text{in}}$ defines a continuous semigroup $\{S(t) \colon t \geq 0\}$ on $H^1(\mathbb{T})$. The idea of the proof is to decompose $S(t)$ into the sum of two H^1-continuous maps (not necessarily semigroups) $S(t) = S_1(t) + S_2(t)$ where

$$S_1(t) u^{\text{in}} = \begin{cases} y(t) & \text{for } t < t_0(u^{\text{in}}) \\ y(t) + z(t) & \text{for } t \geq t_0(u^{\text{in}}) \end{cases} \tag{62a}$$

and

$$S_2(t) u^{\text{in}} = \begin{cases} Q_N u(t) & \text{for } t < t_0(u^{\text{in}}) \\ Q_N u(t) - z(t) & \text{for } t \geq t_0(u^{\text{in}}) \,, \end{cases} \tag{62b}$$

where $z(t)$ solves the high wavenumber equation (2), N is fixed such that appropriate *a priori* estimates (such as those of the previous section) hold, and $t_0(u^{\text{in}})$ is the supremum over all entry times of $u(t)$ into the absorbing ball $\mathcal{B}$. Then the following are true.

(i) For every bounded $B \subset H^1(\mathbb{T})$ let $t_0^* = \sup\{t_0(u^{\text{in}}) \colon u^{\text{in}} \in B\}$. Then the set

$$\bigcup_{t \geq t_0^*} S_1(t) B \tag{63}$$

is relatively compact in H^1.

(ii) For every bounded $B \subset H^1(\mathbb{T})$,

$$\limsup_{t \to \infty} \sup_{u^{\text{in}} \in B} \|S_2(t) u^{\text{in}}\|_{H^1} = 0 \,. \tag{64}$$

(iii) Every complete orbit $\{u(t) \colon t \in \mathbb{R}\}$ which is bounded in H^1 is contained in $\mathcal{D}(A^r \mathrm{e}^{\tau A})$.

It is classical that properties (i) and (ii) imply the existence of a compact global attractor $\mathcal{A} \subset H^1$ (see Temam [29, Theorem I.1.1]). Since the attractor is the union of complete orbits, (iii) implies analytic regularity of $\mathcal{A}$, and thus the first claim of Theorem 2.

To verify statement (i) we note that $u(t) \in \mathcal{B}$ for $t \geq t_0^*$. This implies $y(t) = P_N u(t)$ is contained in a compact subset of H^1. Moreover, Goubet [15] has shown that $z(t)$ is bounded in H^2 independent of u^{in}, thus $z(t)$ is also contained in a relatively compact subset of H^1.

The following convergence result for the high wavenumber component is the key to verifying (ii) and (iii).

Lemma 10 ([15], Proposition 3.1.)**.** *For every bounded set $B \subset H^1(\mathbb{T})$ there exists a $t_0 = t_0(B)$ so that for all $t \geq t_0$ and N sufficiently large,*

$$\|z(t) - Q_N u(t)\|_{H^1}^2 \leq K \, \mathrm{e}^{-\alpha(t-t_0)} \,, \tag{65}$$

where $u^{\mathrm{in}} \in B$, and K does not depend on B.

By letting $t \to \infty$, we obtain (ii). When u is a complete bounded orbit, there is no restriction on the choice t_0 in the definition of z, equation (2b). In particular, t_0 can be negative so that we obtain a family of solutions $\{z_{t_0} \colon t_0 \in \mathbb{R}_-\}$ that satisfies

$$\lim_{t_0 \to -\infty} \|z_{t_0}(0) - Q_N u(0)\|_{H^1} = 0 \,. \tag{66}$$

Moreover, $\{z_{t_0}\}$ is a bounded set in $\mathcal{D}(A^r \mathrm{e}^{\tau A})$ by Lemma 9, thus the convergence (66) actually holds in the stronger $\mathcal{D}(A^r \mathrm{e}^{\tau A})$ topology. This implies (iii).

Proof of Lemma 10. Choose t_0 such that $u(t_0) \in \mathcal{B}$. Let $v \equiv y + z$. The difference between the approximate and the exact high wavenumber component, $w = v - u = z - Q_N u$, satisfies the equation

$$\partial_t w + (\alpha - \mathrm{i}) w + \mathrm{i} \partial_{xx} w + \mathrm{i} Q_N \bigl((|u|^2 + |v|^2) w + uv \, \overline{w} \bigr) = 0 \,, \tag{67a}$$

$$w(t_0) = -Q_N u(t_0) \,. \tag{67b}$$

Consider the functional

$$G(w) = \|w\|^2 + \|\partial_x w\|^2 - \int_{\mathbb{T}} (|u|^2 + |v|^2) \, |w|^2 \, \mathrm{d}x - \operatorname{Re} \int_{\mathbb{T}} uv \, \overline{w}^2 \, \mathrm{d}x \,. \tag{68}$$

When N is sufficiently large, the boundedness of u and v in H^1 and the specialized Poincaré inequality (30) imply that $G^{1/2}$ is equivalent to the H^1 norm so that, for example,

$$\tfrac{1}{2} \, \|w\|_{H^1}^2 \leq G(w) \leq \tfrac{3}{2} \, \|w\|_{H^1}^2 \,. \tag{69}$$

By multiplying (67a) with $(\partial_t + \alpha) w$, taking the imaginary part, and integrating in time, we find that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} G(w) + 2\alpha G(w) &= - \int_{\mathbb{T}} \partial_t (|u|^2 + |v|^2) \, |w|^2 \, \mathrm{d}x - \operatorname{Re} \int_{\mathbb{T}} \partial_t (uv) \, \overline{w}^2 \, \mathrm{d}x \\ &\leq \frac{C_1}{\sqrt{N}} \bigl(\|\partial_t u\|_{H^{-1}} + \|\partial_t v\|_{H^{-1}} \bigr) \bigl(\|u\|_{H^1} + \|v\|_{H^1} \bigr) \|w\|_{H^1}^2 \\ &\leq \frac{K_1}{\sqrt{N}} \, G(w) \,. \end{aligned} \tag{70}$$

The first inequality is due to the Cauchy–Schwarz inequality and the specialized algebra inequality (32) with $\tau = 0$, the second inequality is a consequence of having an absorbing ball in H^1. When $N > K_1^2/\alpha^2$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} G(w) \leq -\alpha G(w)\,. \tag{71}$$

This directly implies the statement of the lemma. □

Remark 14. To give a complete self-contained proof of Theorems 1 and 2, one in fact needs Goubet's *a priori* estimates on z up to the Sobolev space H^4. We cannot refer to Section 6 to claim such bounds on z because our result relies, through Lemma 7, on Theorem 1 already proven for $m < 4$.

We now turn to proving the second claim of Theorem 2, namely that $\mathcal{A}$ is a local attractor in some class $\mathcal{D}(A^r \mathrm{e}^{\tau A})$ which may depend on the initial data.

Let $f \in \mathcal{D}(A^r \mathrm{e}^{\sigma A})$, B be a bounded subset of $\mathcal{D}(A^r \mathrm{e}^{\sigma A})$ which contains $\mathcal{A}$ with $r > \frac{3}{2}$ and $\sigma > 0$ fixed, and $u^{\mathrm{in}} \in B$. It is known that

$$M \equiv \sup_{t \geq 0} \sup_{u^{\mathrm{in}} \in B} \|u(t)\|_{H^m} \tag{72}$$

is finite for fixed m and B (see [12] for $m = 2$). Replace (2b) by

$$z(0) = Q_N u^{\mathrm{in}}\,, \tag{73}$$

so that (2) becomes an equation for the exact high wavenumber component of u. Repeat the formal computations of Section 6 for the so modified equation. All references to Lemma 7 for bounds on derivatives of y can be replaced by references to (72), while an equivalent of Corollary 8 still holds. The constants K_1 to K_{14} will therefore depend on α, f and B. Finally, $N = 1/\tau$ needs to be chosen large enough so that the polynomial p, defined in (59a), not only has two positive zeros, but also that the largest zero ζ_2 is larger than the diameter of B.

This proves the existence of $\mathcal{D}(A^r \mathrm{e}^{\tau A})$-bounded sets which absorb subsets of B. Let $\mathcal{A}_B$ denote the maximal invariant set with this property. Since the embedding $\mathcal{D}(A^r \mathrm{e}^{\tau A}) \hookrightarrow \mathcal{D}(A^r \mathrm{e}^{\rho A})$ is compact for every $\rho \in (0, \tau)$, the set $\mathcal{A}_B$ is compact in $\mathcal{D}(A^r \mathrm{e}^{\rho A})$ and is therefore a local attractor.

Since $\mathcal{A}$ is the maximal compact invariant set in H^1, and $\mathcal{D}(A^r \mathrm{e}^{\rho A}) \hookrightarrow H^1$, we must have $\mathcal{A}_B \subset \mathcal{A}$. On the other hand, $\mathcal{A}$ is contained in B, and $\mathcal{A}_B$ attracts B. This implies $\mathcal{A}_B = \mathcal{A}$ and completes the proof of Theorem 2.

8. Determining Nodes

The analytic regularity of the attractor allows us to prove the following result on the number of determining nodes.

Theorem 11. *There exists a $\delta > 0$ such that for every pair of nodes $a, b \in \mathbb{T}$ with $|a - b| \leq \delta$ and all solutions u_1 and u_2 to the weakly damped driven nonlinear Schrödinger equation* (1) *with H^1 initial data, satisfying*

$$\lim_{t \to \infty} |u_1(x,t) - u_2(x,t)| = 0\,, \qquad x \in \{a, b\}\,, \tag{74}$$

one has

$$\lim_{t \to \infty} \|u_1(t) - u_2(t)\|_{H^1} = 0\,. \tag{75}$$

The proof of this theorem, which is inspired by the work of Kukavica [22], relies on two elementary facts which we state beforehand.

Lemma 12. *For $u \in H^1(\mathbb{T})$ one has the inequality*

$$\int_a^b |u|^2 \, \mathrm{d}x \leq 2(b-a) \, u^2(a) + 2(b-a)^2 \int_a^b |\partial_x u|^2 \, \mathrm{d}x \,. \tag{76}$$

Proof. The proof is a straightforward application of the Fundamental Theorem of Calculus, the Young and the Cauchy–Schwarz inequality. □

Lemma 13. *Let g, h, j, and G be sufficiently smooth (e.g. continuously differentiable) functions of time which satisfy the differential inequality*

$$\dot{G} + \alpha G \leq g + h \, \dot{\ell} \,. \tag{77}$$

Assume that $G(t)$ is non-negative,

$$\lim_{t \to \infty} g(t) = \lim_{t \to \infty} \ell(t) = 0 \tag{78a}$$

and

$$\limsup_{t \to \infty} \bigl(|h(t)| + |\dot{h}(t)| \bigr) < \infty \,. \tag{78b}$$

Then

$$\lim_{t \to \infty} G(t) = 0 \,. \tag{79}$$

Remark 15. The result remains valid if the right side of (77) is replaced by a sum of terms of this form.

Proof. Integrating (77) with respect to time followed by integration by parts on the right side gives

$$\begin{aligned} G(t) &\leq G(0) \, \mathrm{e}^{-\alpha t} + \int_0^t \mathrm{e}^{\alpha(t'-t)} \bigl(g(t') + h(t') \, \dot{\ell}(t') \bigr) \mathrm{d}t' \\ &= \bigl(G(0) - h(0) \, \ell(0) \bigr) \mathrm{e}^{-\alpha t} + h(t) \, \ell(t) \\ &\quad + \int_0^t \mathrm{e}^{\alpha(t'-t)} \bigl(g(t') - \dot{h}(t') \, \ell(t') - \alpha \, h(t') \, \ell(t') \bigr) \mathrm{d}t' \,. \end{aligned} \tag{80}$$

The first two terms on the right clearly converge to zero as $t \to \infty$. The last term has the form

$$\int_0^t \mathrm{e}^{\alpha(t'-t)} F(t') \, \mathrm{d}t' \leq \sup_{t' \in [0,t/2]} |F(t')| \int_0^{t/2} \mathrm{e}^{\alpha(t'-t)} \, \mathrm{d}t' + \sup_{t' \geq t/2} |F(t')| \int_{t/2}^t \mathrm{e}^{\alpha(t'-t)} \, \mathrm{d}t' \,, \tag{81}$$

where $F(t) \to 0$ as $t \to \infty$. Hence this term also converges to zero. □

Proof of Theorem 11. Rather than, as is usually done, directly comparing u_1 and u_2 which only have H^1 regularity, we compare the smooth approximate solutions v_1 and v_2—defined as before by $v_i = y_i + z_i$, $i \in \{1,2\}$, where $y_i = P_N u_i$, and z_i solves the corresponding high mode equation (2). As v_i converges to u_i in H^1, v_1 and v_2 also satisfy property (74). As before, we assume that t_0 in (2b) is large enough so that u_1 and u_2 are already contained in some absorbing ball $\mathcal{B} \subset H^1(\mathbb{T})$ for all $t \geq t_0$.

Define $w = v_2 - v_1$. Then w satisfies the equation

$$\partial_t w + (\alpha - \mathrm{i})w + \mathrm{i}\partial_{xx} w + \mathrm{i}\big((|v_1|^2 + |v_2|^2)w + v_1 v_2 \,\overline{w}\big) = F\,, \tag{82a}$$

$$w(t_0) = y_2(t_0) - y_1(t_0)\,. \tag{82b}$$

This equation is similar to (67) but for some extra terms on the right which arise because v_1 and v_2 are not exact solutions of equation (1). These terms, given by

$$\begin{aligned} F = \mathrm{i}P_N\big(&|v_2|^2 + |u_2|^2)(v_2 - u_2) + v_2 u_2\,(\overline{v}_2 - \overline{u}_2)\big) \\ &- \mathrm{i}P_N\big(|v_1|^2 + |u_1|^2)(v_1 - u_1) + v_1 u_1\,(\overline{v}_1 - \overline{u}_1)\big)\,, \end{aligned} \tag{83}$$

are bounded and converge to zero in H^1 due to Lemma 10. We formally proceed as in the proof of Lemma 10, but instead of integrating over the torus, we integrate over the interval $[a, b]$. Thus the G-functional reads

$$G(w) = \int_a^b |w|^2 \,\mathrm{d}x + \int_a^b |\partial_x w|^2 \,\mathrm{d}x - \int_a^b \big(|v_1|^2 + |v_2|^2\big)|w|^2 \,\mathrm{d}x - \operatorname{Re}\int_a^b v_1 v_2\, \overline{w}^2 \,\mathrm{d}x\,. \tag{84}$$

Lemma 12 now plays the role of the specialized Poincaré inequality. Since v_1 and v_2 are uniformly bounded in L^∞, we can take δ sufficiently small to obtain the equivalence of $G^{1/2}$ and the H^1 norm on $[a, b]$,

$$\tfrac{1}{2}\,\|w\|^2_{H^1([a,b])} \le G(w) \le \tfrac{3}{2}\,\|w\|^2_{H^1([a,b])}\,. \tag{85}$$

Moreover, the same formal calculation as before leads to an evolution equation for G which now contains boundary terms stemming from integration by parts on $[a, b]$:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}G(w) + 2\alpha G(w) = &- \int_a^b \partial_t\big(|v_1|^2 + |v_2|^2\big)|w|^2 \,\mathrm{d}x - \operatorname{Re}\int_a^b \partial_t(v_1 v_2)\, \overline{w}^2 \,\mathrm{d}x \\ &- 2\operatorname{Im}\int_a^b F\,(\partial_t + \alpha)\overline{w}\,\mathrm{d}x + 2\operatorname{Re}(\partial_x w\,\partial_t \overline{w})\Big|_a^b + 2\alpha\operatorname{Re}(\partial_x w\,\overline{w})\Big|_a^b\,. \end{aligned} \tag{86}$$

Since v_1 and v_2 are bounded in $\mathcal{D}(A^r \mathrm{e}^{\tau A})$, one can easily show that they have bounded time and space derivatives of any order. We can therefore estimate the first two integrals on the right by

$$\begin{aligned} \bigg|\int_a^b \partial_t\big(|v_1|^2 + |v_2|^2\big)|w|^2 \,\mathrm{d}x\bigg| + \bigg|\int_a^b \partial_t(v_1 v_2)\, \overline{w}^2 \,\mathrm{d}x\bigg| &\le K_1 \int_a^b |w|^2 \,\mathrm{d}x \\ &\le K_1\,\delta\, w^2(a) + 4K_1\,\delta^2\, G(w) \\ &\le K_2\, w^2(a) + \alpha\, G(w)\,. \end{aligned} \tag{87}$$

The second inequality is due to Lemma 12 and the final inequality holds provided $\delta^2 < \alpha/(4K_1)$.

Similarly, the third integral on the right of (86) is bounded by

$$\bigg|\int_a^b F\,(\partial_t + \alpha)\overline{w}\,\mathrm{d}x\bigg| \le K_3\,\|F\|_{H^1}\,. \tag{88}$$

Combining these estimates, we observe that $G = G(w)$ satisfies an inequality of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}G + \alpha G \le g + h\,\dot{\ell}\Big|_{x=a}^{x=b}\,. \tag{89}$$

The functions

$$g = K_2\, w^2(a) + K_3\, \|F\|_{H^1} + K_4\, |w(b) - w(a)|$$

and

$$\ell = \operatorname{Re} w\,,$$

the latter evaluated at $x \in \{a,b\}$, are decaying functions of time, and

$$h = 2\operatorname{Re}(\partial_x w)$$

has a bounded time derivative due to the regularity of v_1 and v_2. Thus, by Lemma 13 and Remark 15, we have

$$\lim_{t\to\infty} \|v_2(t) - v_1(t)\|^2_{H^1([a,b])} \leq 2 \lim_{t\to\infty} G(w) = 0\,. \tag{90}$$

Moreover, v_1 and v_2 are real analytic functions that converge in $H^1(\mathbb{T})$—and therefore pointwise—to the real analytic attractor. The rate of convergence is uniform in space because both functions are bounded in H^2. In other words, w converges pointwise and uniformly towards a set of functions that are real analytic as well as identically zero on $[a,b]$. The only such function is the zero function. This implies $v_1 - v_2 \to 0$, and we recall that $v_1 - u_1 \to 0$ and $v_2 - u_2 \to 0$ in H^1 as $t \to \infty$. We conclude that $u_1 - u_2 \to 0$ in H^1, i.e., statement (75). □

Remark 16. Goubet [15] has also shown that orbits on the attractor are completely determined by their low wavenumber part, provided that the cut-off is chosen large enough. One can show, parallel to the proof of Lemma 10, that these N modes are actually determining in the sense that

$$\lim_{t\to\infty} \|P_N u_1(t) - P_N u_2(t)\|_{L^2} = 0 \tag{91}$$

implies

$$\lim_{t\to\infty} \|u_1(t) - u_2(t)\|_{H^1} = 0\,. \tag{92}$$

This result does not depend on analyticity of f and $\mathcal{A}$.

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