GEVREY REGULARITY FOR THE ATTRACTOR OF A PARTIALLY DISSIPATIVE MODEL OF BÉNARD CONVECTION IN A POROUS MEDIUM

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ABSTRACT. Convective flow though a porous medium can be modeled by Darcy's law—a linear, weakly damped momentum equation—coupled with an advection-diffusion equation for the energy. The solution semigroup for this system is not smoothing, and the solution of the momentum equation does not gain regularity with respect to its initial value in finite time. However, it is known that the semigroup is asymptotically smoothing, so that the system possesses a finite dimensional global attractor as well as exponential attractors. We show that the global attractor is contained in a special Gevrey class of regularity and, in particular, is real analytic. The key idea is the use of a Fourier splitting method to approximate every orbit asymptotically in time by a Gevrey-regular function.

1. INTRODUCTION

We study a model for Bénard convection in a porous medium in which the evolution of fluid velocity $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ and pressure $p = p(\boldsymbol{x}, t)$ is described by Darcy's law, and the modified temperature $\theta = \theta(\boldsymbol{x}, t)$ satisfies an advection-diffusion equation. Momentum and energy equations are coupled via the Boussinesq approximation, so that the system in non-dimensional variables reads

$$\gamma \partial_t \boldsymbol{u} + \boldsymbol{u} + \boldsymbol{\nabla} p - R \boldsymbol{k} \theta = 0 \,, \tag{1a}$$

$$\partial_t \theta - \kappa \Delta \theta + \boldsymbol{u} \cdot \boldsymbol{\nabla} (\theta - x_3) = 0 \,, \tag{1b}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \,, \tag{1c}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}^{\text{in}} \,, \quad \theta(0) = \theta^{\text{in}} \,. \tag{1d}$$

We assume that the fluid is contained in the box $\Omega = [0, L] \times [0, \ell] \times [0, 1]$. It is uniformly heated from below and cooled from above, and isolated at the sides. This corresponds to the boundary conditions

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \qquad \text{on } \partial\Omega \,, \tag{1e}$$

$$\theta = 0 \qquad \text{on } \Gamma_{\text{vert}} \equiv \{\boldsymbol{x} \colon x_3 = 0 \text{ or } x_3 = 1\} \,, \tag{1f}$$

$$\boldsymbol{n} \cdot \boldsymbol{\nabla} \theta = 0 \qquad \text{on } \Gamma_{\text{hor}} \equiv \{\boldsymbol{x} \colon x_1 = 0, x_1 = L, x_2 = 0, \text{ or } x_2 = \ell\} \,. \tag{1g}$$

The nondimensionalized parameters γ, κ and R are the Darcy-Prandtl number, the thermal conductivity of the saturated medium, and the square root of the Rayleigh-Darcy number Ra, respectively; $\boldsymbol{k}$ is a unit vector in x_3 direction and $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{x})$ denotes a vector normal to the boundary $\partial\Omega$ at the point $\boldsymbol{x}$. For a derivation of the equation and discussion of the parameters see, e.g., Beck [1] and Joseph [15].

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When the Darcy-Prandtl number γ is zero, system (1) can be treated as a non-local scalar parabolic partial differential equation and is known to have global real analytic solutions and, in particular, a real analytic global attractor [17].

For non-zero Darcy-Prandtl number, Fabrie [5, 6] has proved existence of global weak and strong solutions. Moreover, the temperature field θ satisfies a maximum principle and has an absorbing ball in every space L^p for finite p. We denote by $\mathbb{H}$ the L^2-completion of

$$\mathbb{D} = \{\boldsymbol{u} \in C_0^\infty(\Omega, \mathbb{R}^3) \colon \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \text{ in } \Omega\}\,, \tag{2}$$

and set

$$M_b = \{\theta \in L^\infty(\Omega) \colon |\theta(\boldsymbol{x})| \leq b \text{ a.e. in } \Omega\}\,. \tag{3}$$

Since divergence free functions in $L^2(\Omega, \mathbb{R}^3)$ have normal traces (see, e.g., [4, 22]), one can write

$$\mathbb{H} = \{\boldsymbol{u} \in L^2(\Omega, \mathbb{R}^3) \colon \boldsymbol{\nabla} \cdot \boldsymbol{u} = 0 \text{ in } \Omega, \boldsymbol{u} \cdot \boldsymbol{n} = 0 \text{ on } \partial\Omega\} \tag{4}$$

We can summarize the part of Fabrie's results which is important for the present work as follows.

Theorem 1 ([7], Proposition 1.1)**.** *For $\boldsymbol{u}^{\text{in}} \in \mathbb{H}$ and $\theta^{\text{in}} \in L^\infty(\Omega)$, system* (1) *has a unique weak solution where*

$$\begin{aligned} &\boldsymbol{u} \in L^\infty([0,\infty); \mathbb{H})\,, \\ &\theta \in L^\infty([0,\infty) \times \Omega) \cap L^2_{\text{loc}}([0,\infty); H^1(\Omega))\,. \end{aligned}$$

If $\theta^{\text{in}} \in M_b$ for some $b \geq 1$, then $\theta(t) \in M_b$ for a.e. $t \geq 0$. Moreover, for every finite p,

$$\limsup_{t\to\infty} \|\theta(t)\|_{L^p} \leq |\Omega|^{1/p}\,. \tag{5}$$

Fabrie and Nicolaenko [7] show that for initial data in $\mathbb{H} \times L^\infty$ there exists a compact, finite dimensional global attractor $\mathcal{A} \subset L^2 \times L^\infty$ which attracts orbits with respect to the $L^2 \times L^2$ metric. Since there are no absorbing balls for θ with respect to the L^∞ norm, $\mathcal{A}$ attracts a subset of the phase space only if it is contained in $\mathbb{H} \times M_b$ for some finite b. Initial data in $(\mathbb{H} \cap H^1(\Omega, \mathbb{R}^3)) \times (M_b \cap H^2(\Omega))$ converges with respect to the $L^2 \times L^2$ metric to a closed compact subset $\mathcal{A}_1 \subset \mathcal{A}$ which is contained in $H^1 \times H^2$. Moreover, the system has exponential attractors.

In this paper, we prove two further results. We show that $\mathcal{A}_1 = \mathcal{A}$, and that $\mathcal{A}$ is contained in a subclass of the real analytic functions:

Theorem 2. *For initial data $(\boldsymbol{u}^{\text{in}}, \theta^{\text{in}}) \in \mathbb{H} \times M_b$ with $b \geq 1$, system* (1) *has a global attractor, i.e., a set $\mathcal{A}$ which is compact, invariant and attracts every every L^2-bounded subset of $\mathbb{H} \times M_b$ in the $L^2 \times L^2$ topology. Moreover, there exists a $\tau > 0$ such that $\mathcal{A}$ consists of real analytic functions on Ω whose radius of analyticity is greater or equal to τ.*

Remark 1. If the initial data is more regular (for example, in $H^r \times H^r$ for $r \geq 2$), then one can also prove convergence in the stronger norm.

Remark 2. Although the theorem asserts that every initial datum $(\boldsymbol{u}^{\text{in}}, \theta^{\text{in}}) \in \mathbb{H} \times L^\infty(\Omega)$ converges in $L^2 \times L^2$ to $\mathcal{A}$, L^2-bounded *sets* for θ are only attracted if their L^∞ norm is uniformly bounded.

Remark 3. We do not expect that the lower bound on the radius of analyticity τ to be arbitrarily large. In other words, our result is qualitatively optimal. For a discussion of the sharpness of the method used here in the context of the damped driven nonlinear Schrödinger equation, see [19].

The strategy for proving Theorem 2 is as follows. Starting with exact solutions $\boldsymbol{u}^{(0)} \equiv \boldsymbol{u}$, $p^{(0)} \equiv p$, and $\theta^{(0)} \equiv \theta$ to the convection system (1), we construct a sequence of asymptotic approximants—functions $\boldsymbol{u}^{(k)}$, $p^{(k)}$, and $\theta^{(k)}$ which converge in L^2 to $\boldsymbol{u}$ and θ as $t \to \infty$—each being more regular than the previous member of the sequence. Note that we can always eliminate the pressure from Darcy's law (1a) by projecting it onto $\mathbb{H}$. With this understanding we can denote a solution to (1) by the tuple $(\boldsymbol{u}, \theta)$; the corresponding results for the pressure can always be reconstructed *a posteriori*.

The construction proceeds in two parts. In Section 3 we introduce a simple splitting of Darcy's law into a homogeneous equation which takes care of the initial data, and an inhomogeneous equation which is driven. The solution to the inhomogeneous equation is shown to be more regular than the class of admissible initial data and approximates the exact solution exponentially. We then set up a bootstrapping scheme which, in principle, can yield asymptotic approximants with any finite degree of differentiability. We, however, terminate this procedure after a few iterations and use it as a starting point for constructing real analytic approximants in Section 4. This part of the construction is based on the Fourier splitting method of Goubet [12] in which finitely many exact low Fourier modes drive infinitely many approximate high modes.

In the remaining Section 5 we show how the existence of analytic asymptotic approximants implies the regularity of the attractor.

Crucial for our proof is the observation that real analytic functions on the n-torus $\mathbb{T}^n$ are characterized by an exponential decay of their Fourier coefficients. This fact is most conveniently expressed in terms of the operator domains

$$\mathcal{D}(A\mathrm{e}^{\tau A}) = \left\{ f \in L^2(\mathbb{T}^n) \colon A\mathrm{e}^{\tau A} u \in L^2(\mathbb{T}^n) \right\}, \tag{6}$$

where $A \equiv \sqrt{-\Delta}$ is a non-negative semi-definite, self-adjoint operator on $L^2(\mathbb{T}^n)$. We denote the L^2 norm and inner product by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ respectively. It can then be shown that

$$C^\omega(\mathbb{T}) = \bigcup_{\tau > 0} \mathcal{D}(A\mathrm{e}^{\tau A}) \tag{7}$$

for any $r \geq 0$ (see, e.g., [16]). This result is useful because the spaces $\mathcal{D}(A\mathrm{e}^{\tau A})$ are Hilbert spaces and carry the natural norm $\|A\mathrm{e}^{\tau A} \cdot \| + \| \cdot \|$. Estimates in $\mathcal{D}(A\mathrm{e}^{\tau A})$ have first been used by Foias and Temam [10] to present a simple proof of the analyticity of solutions to the Navier–Stokes equations, and have subsequently been applied to many other parabolic as well as hyperbolic and damped-driven equations. Moreover, they have been used on $\mathbb{R}^n$ and on the sphere. For a list of references see [3, 8, 16, 17, 18, 20, 21].

The difficulty here, as in [17], are the boundary conditions. However, in this simple geometry one has an explicit Fourier basis, and thus a modified version of (7) still holds. The details will be given in Section 2.

We finally remark that a similar result can be proved for the weakly damped driven nonlinear Schrödinger equation with periodic boundary conditions in one spatial dimension [18]. While the main difficulties in this paper are the boundary conditions as well as the non-scalar nature of the equations, the nonlinear Schrödinger case requires more subtle "commutator estimates" in $\mathcal{D}(A\mathrm{e}^{\tau A})$ because one cannot take advantage of partial dissipativity.

2. Fourier Decomposition of θ and $\boldsymbol{u}$

We first consider the temperature field θ. Let

$$V = \{\theta \in H^1(\Omega) \colon \theta = 0 \text{ on } \Gamma_{\text{vert}}\}\,, \tag{8a}$$

$$W = \{\theta \in H^2(\Omega) \colon \theta = 0 \text{ on } \Gamma_{\text{vert}}, \boldsymbol{n}\cdot\boldsymbol{\nabla}\theta = 0 \text{ on } \Gamma_{\text{hor}}\}\,. \tag{8b}$$

The operator $A^2 \equiv -\Delta$ is then defined on V with values in V', and on W with values in $L^2(\Omega)$. Moreover, even though the domain Ω is not smooth, one can prove elliptic regularity results which show that $A^2 \colon W \to L^2(\Omega)$ is invertible (see Grisvard [13, p. 147]). In other words, A^{-2} is a compact operator on L^2 and possesses a complete orthogonal set of eigenfunctions, explicitly given by

$$\theta_{ijk}(\boldsymbol{x}) = \sqrt{\frac{8}{\ell L}}\,\cos\left(\frac{i\pi x}{L}\right)\,\cos\left(\frac{j\pi y}{\ell}\right)\,\sin(k\pi z) \tag{9}$$

for $i,j = 0,1,2,\ldots$; $k = 1,2,\ldots$; the corresponding eigenvalues are

$$\lambda_{ijk}^2 = \left(\frac{i\pi}{L}\right)^2 + \left(\frac{j\pi}{\ell}\right)^2 + (k\pi)^2\,. \tag{10}$$

We can use this spectral representation to define arbitrary powers of A^2; the domain $\mathcal{D}(A^r)$ is the closure in H^r of the set

$$\{\theta \in W \colon A^r\theta \in L^2(\Omega)\}\,. \tag{11}$$

More generally, for $r \geq 0$ and $\tau \geq 0$, we can set

$$\mathcal{D}(A^r\mathrm{e}^{\tau A}) = \{\theta \in W \colon A^r\mathrm{e}^{\tau A}\theta \in L^2(\Omega)\}\,, \tag{12}$$

where $A^r\mathrm{e}^{\tau A}$ is defined in terms of the temperature eigenfunctions by

$$A^r\mathrm{e}^{\tau A}\theta_{ijk} \equiv \lambda_{ijk}^r\,\mathrm{e}^{\tau\lambda_{ijk}}\,\theta_{ijk} \tag{13}$$

for $i,j = 0,1,2,\ldots$ and $k = 1,2,\ldots$.

In general, $\mathcal{D}(A^r)$ is a closed subspace of $H^r(\Omega)$ and $\|A^r\cdot\|$ defines a norm on $\mathcal{D}(A^r)$ equivalent to the canonical H^r norm. Note that, in particular, $\mathcal{D}(A^0) = L^2(\Omega)$ and $D(A) = V$.

We now establish the corresponding eigenfunction expansion for the velocity $\boldsymbol{u}$. It turns out that the velocity space $\mathbb{H}$ decomposes into a closed subspace $\mathbb{H}_{\shortparallel}$ which is compatible with the buoyancy forcing in Darcy's law (1a), and its nontrivial complement $\mathbb{H}_\perp$. While any $\boldsymbol{u}^{\text{in}} \in \mathbb{H}$ is admissible initial data for system (1), the asymptotic dynamics is confined to $\mathbb{H}_{\shortparallel}$ (see Remark 7 at the end of Section 3).

We define $\mathbb{H}_{\shortparallel}$ to be the completion in L^2 of finite linear combinations of the orthogonal functions system

$$\boldsymbol{u}_{ijk}(\boldsymbol{x}) = \begin{pmatrix} \dfrac{ik}{L}\sin\left(\dfrac{i\pi x}{L}\right)\cos\left(\dfrac{j\pi y}{\ell}\right)\cos(k\pi z) \\ \dfrac{jk}{\ell}\cos\left(\dfrac{i\pi x}{L}\right)\sin\left(\dfrac{j\pi y}{\ell}\right)\cos(k\pi z) \\ -\left(\dfrac{i^2}{L^2}+\dfrac{j^2}{\ell^2}\right)\cos\left(\dfrac{i\pi x}{L}\right)\cos\left(\dfrac{j\pi y}{\ell}\right)\sin(k\pi z) \end{pmatrix} \tag{14a}$$

for $i, j = 0, 1, 2, \ldots$; $k = 1, 2, \ldots$; and let $\mathbb{H}_{\perp}$ be the corresponding completion of

$$\boldsymbol{v}_{ijk}(\boldsymbol{x}) = \begin{pmatrix} \dfrac{j}{\ell}\sin\left(\dfrac{i\pi x}{L}\right)\cos\left(\dfrac{j\pi y}{\ell}\right)\cos(k\pi z) \\ -\dfrac{i}{L}\cos\left(\dfrac{i\pi x}{L}\right)\sin\left(\dfrac{j\pi y}{\ell}\right)\cos(k\pi z) \\ 0 \end{pmatrix} \tag{14b}$$

for $i, j = 1, 2, \ldots$; $k = 0, 1, 2, \ldots$. Then the following is true.

Lemma 3 ([6])**.** *$\mathbb{H}_{\shortparallel}$ and $\mathbb{H}_{\perp}$ are orthogonal subspaces, and $\mathbb{H} = \mathbb{H}_{\shortparallel} \oplus \mathbb{H}_{\perp}$. In particular, $\{\boldsymbol{u}_{ijk}\} \cup \{\boldsymbol{v}_{ijk}\}$ is an orthogonal basis for $\mathbb{H}$.*

Proof. The function system $\{\boldsymbol{u}_{ijk}\} \cup \{\boldsymbol{v}_{ijk}\}$ is clearly orthogonal. We also need to prove that these functions are complete in $\mathbb{H}$. Note that every member of the $L^2(\Omega, \mathbb{R}^3)$-complete system

$$\sin\left(\frac{i\pi x}{L}\right)\cos\left(\frac{j\pi y}{\ell}\right)\cos(k\pi z)\,\boldsymbol{i}\,, \tag{15a}$$

$$\cos\left(\frac{i\pi x}{L}\right)\sin\left(\frac{j\pi y}{\ell}\right)\cos(k\pi z)\,\boldsymbol{j}\,, \tag{15b}$$

$$\cos\left(\frac{i\pi x}{L}\right)\cos\left(\frac{j\pi y}{\ell}\right)\sin(k\pi z)\,\boldsymbol{k}\,, \tag{15c}$$

can be uniquely expressed as a linear combination of $\boldsymbol{u}_{ijk}$, $\boldsymbol{v}_{ijk}$, and $\boldsymbol{\nabla} p_{ijk}$, where

$$p_{ijk} = \cos\left(\frac{i\pi x}{L}\right)\cos\left(\frac{j\pi y}{\ell}\right)\cos(k\pi z)\,. \tag{16}$$

Since gradients are in the orthogonal complement of $\mathbb{H}$ (see, e.g., [1] or [22]), we conclude that $\{\boldsymbol{u}_{ijk}\} \cup \{\boldsymbol{v}_{ijk}\}$ is $\mathbb{H}$-complete. ☐

Let $\boldsymbol{P}_{\mathbb{H}}$ denote the Leray orthogonal projection onto $\mathbb{H}$. By expressing $\boldsymbol{P}_{\mathbb{H}}$ in terms of $\boldsymbol{u}_{ijk}$, $\boldsymbol{v}_{ijk}$, and p_{ijk} one can prove the following.

Lemma 4 ([17])**.** $\boldsymbol{P}_{\mathbb{H}}(\boldsymbol{k}\,\cdot)\colon L^2(\Omega) \to \mathbb{H}_{\shortparallel}$.

In other words, by applying $\boldsymbol{P}_{\mathbb{H}}$ to Darcy's law, equation (1a), we find that the contribution of the forcing term $R\boldsymbol{k}\theta$ is confined to $\mathbb{H}_{\shortparallel}$.

We complete the discussion of the functional setting for $\boldsymbol{u}$ by introducing the operator $\boldsymbol{A}^2 \equiv -\Delta$. Clearly, $\boldsymbol{u}_{ijk}$ and $\boldsymbol{v}_{ijk}$ are eigenfunctions of $\boldsymbol{A}^2$; the corresponding eigenvalues are again given by (10). As before, we can define arbitrary powers of $\boldsymbol{A}^2$, and consider $\boldsymbol{A}^r$ as unbounded operators on $\mathbb{H}$ with domain

$$\mathcal{D}(\boldsymbol{A}^r) = \{\boldsymbol{v} \in \mathbb{H}\colon \boldsymbol{A}^r\boldsymbol{v} \in \mathbb{H}\}\,. \tag{17}$$

More generally, for $r \geq 0$ and $\tau \geq 0$, we introduce

$$\mathcal{D}(\boldsymbol{A}^r \mathrm{e}^{\tau \boldsymbol{A}}) = \{\boldsymbol{v} \in \mathbb{H}\colon \boldsymbol{A}^r \mathrm{e}^{\tau \boldsymbol{A}} \boldsymbol{v} \in \mathbb{H}\} \,, \tag{18}$$

where $\boldsymbol{A}^r \mathrm{e}^{\tau \boldsymbol{A}}$ is defined through the velocity eigenfunctions by

$$\boldsymbol{A}^r \mathrm{e}^{\tau \boldsymbol{A}} \boldsymbol{u}_{ijk} \equiv \lambda_{ijk}^r \, \mathrm{e}^{\tau \lambda_{ijk}} \, \boldsymbol{u}_{ijk} \,, \qquad \boldsymbol{A}^r \mathrm{e}^{\tau \boldsymbol{A}} \boldsymbol{v}_{ijk} \equiv \lambda_{ijk}^r \, \mathrm{e}^{\tau \lambda_{ijk}} \, \boldsymbol{v}_{ijk} \tag{19}$$

for all i, j, and k as above.

$\mathcal{D}(\boldsymbol{A}^r)$ is a closed subspace of $H^r(\Omega, \mathbb{R}^3)$ and—since the eigenvalues of $\boldsymbol{A}$ are strictly positive—$\|\boldsymbol{A}^r \cdot\|$ defines a norm on $\mathcal{D}(\boldsymbol{A}^r)$ equivalent to the canonical H^r norm.

Finally, let $\boldsymbol{P}_\lambda$ to be the L^2-orthogonal projection onto the span of eigenfunctions (14) of $\boldsymbol{A}$ corresponding to eigenvalues less or equal to λ, and set $\boldsymbol{Q}_\lambda \equiv I - \boldsymbol{P}_\lambda$. Similarly, P_λ shall be the projector onto the span of eigenfunctions (9) to A corresponding to eigenvalues less or equal to λ, and $Q_\lambda \equiv I - P_\lambda$.

It is important to note that the eigenfunctions (14) and (9) span a subspace of the periodic functions on $[0, 2L] \times [0, 2\ell] \times [0, 2]$ which is invariant under the flow generated by (1). Due to this symmetry, the boundary conditions do not explicitly enter the analysis and we can proceed as if (1) were endowed with periodic boundary conditions.

We now write the characterization of the real analytic functions (7) in terms of the operators $\boldsymbol{A}$ and A defined above. One has

$$C^\omega(\Omega, \mathbb{R}^3) \supset \bigcup_{\tau > 0} \mathcal{D}(\boldsymbol{A} \mathrm{e}^{\tau \boldsymbol{A}}) \,, \tag{20a}$$

and

$$C^\omega(\Omega) \supset \bigcup_{\tau > 0} \mathcal{D}(A \mathrm{e}^{\tau A}) \,. \tag{20b}$$

The proof of (20) depends only on the explicit sequence of eigenvalues, and is therefore identical to the well-known case $\Omega = \mathbb{T}^d$ which, for example, is discussed in [16].

We conclude this section by stating three inequalities which are needed in Section 4. The first is a direct modification of a result by Foias and Temam [10, Lemma 2.1] which has also been used in [17].

Lemma 5. *Let $\boldsymbol{u} \in \mathcal{D}(\boldsymbol{A}^2 \mathrm{e}^{\tau \boldsymbol{A}})$ and $\theta, \varphi \in \mathcal{D}(A^2 \mathrm{e}^{\tau A})$ with $\tau \geq 0$. Then $\boldsymbol{u} \cdot \boldsymbol{\nabla} \theta$ belongs to $\mathcal{D}(\mathrm{e}^{\tau A})$ and, in space dimension $n = 2$ or 3,*

$$\bigl| \bigl\langle A \mathrm{e}^{\tau A} (\boldsymbol{u} \cdot \boldsymbol{\nabla} \theta), A \mathrm{e}^{\tau A} \varphi \bigr\rangle \bigr| \leq c_1 \, \|\boldsymbol{A} \mathrm{e}^{\tau \boldsymbol{A}} \boldsymbol{u}\| \, \|A \mathrm{e}^{\tau A} \theta\|^{\frac{1}{2}} \, \|A^2 \mathrm{e}^{\tau A} \theta\|^{\frac{1}{2}} \, \|A^2 \mathrm{e}^{\tau A} \varphi\| \,. \tag{21}$$

The second is an inverse inequality for functions in the low wave number space $\boldsymbol{P}_\lambda \mathcal{D}(\boldsymbol{A}^r \mathrm{e}^{\tau \boldsymbol{A}})$.

Lemma 6. *For $\boldsymbol{u} \in \boldsymbol{P}_\lambda \mathcal{D}(\boldsymbol{A}^r \mathrm{e}^{\tau \boldsymbol{A}})$, one has*

$$\|\boldsymbol{A}^r \mathrm{e}^{\tau \boldsymbol{A}} \boldsymbol{u}\| \leq \mathrm{e}^{\lambda \tau} \, \|\boldsymbol{A}^r \boldsymbol{u}\| \,. \tag{22}$$

A corresponding inequality holds on $P_\lambda \mathcal{D}(A^r \mathrm{e}^{\tau A})$. The proof is straightforward.

Last, we state a “specialized Poincaré inequality” for functions without low wavenumber components.

Lemma 7. *For $\theta \in Q_\lambda V$, one has*

$$\|\theta\| \leq \frac{1}{\lambda} \|A\theta\| . \tag{23}$$

3. Construction of Asymptotic Approximants in $H^1 \times H^2$

In this section we construct a sequence of asymptotic approximants to the solution of (1), and determine their regularity at each step. The estimates follow the work of Fabrie and Nicolaenko [7], the main difference being that their estimates are applied only to exact solutions of (1).

Step 1. *Find an asymptotic approximant $\boldsymbol{u}^{(1)}$ to $\boldsymbol{u}$ which is bounded independent of the initial data in the space $L^\infty([t_0,\infty); L^p(\Omega,\mathbb{R}^3))$ for $1 < p < \infty$, and for t_0 large enough.*

Remark 4. Here, and in the corresponding statements below, we require that the bounds be independent of the initial data. On the other hand, t_0 may be a function of $\boldsymbol{u}^{\text{in}}$, θ^{in}, and p (note, however, that we only need $p = 6$ to proceed with Step 2).

Remark 5. Throughout this paper, we denote universal constants by c_i and constants which depend on the data of equation (1), i.e. on γ, and κ and R, are written K_i. None of these may depend on the initial data. The subscripts are intended to indicate the change of constants from one line to the next, but carry no meaning across different sections.

We split Darcy's law, equation (1a), into two equations. One gets the initial data, the other the driving:

$$\gamma\partial_t \boldsymbol{w} + \boldsymbol{w} + \boldsymbol{\nabla} q = 0\,, \qquad \boldsymbol{w}(t_0) = \boldsymbol{u}(t_0)\,; \tag{24a}$$

$$\gamma\partial_t \boldsymbol{u}^{(1)} + \boldsymbol{u}^{(1)} + \boldsymbol{\nabla} p^{(1)} - R\boldsymbol{k}\theta^{(0)} = 0\,, \qquad \boldsymbol{u}^{(1)}(t_0) = 0\,, \tag{24b}$$

where $\theta^{(0)} = \theta$ and

$$\boldsymbol{\nabla}\cdot\boldsymbol{w} = \boldsymbol{\nabla}\cdot\boldsymbol{u}^{(1)} = 0 \quad \text{in } \Omega\,, \tag{24c}$$

$$\boldsymbol{w}\cdot\boldsymbol{n} = \boldsymbol{u}^{(1)}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega\,. \tag{24d}$$

Note that $\boldsymbol{u}^{(1)} + \boldsymbol{w} = \boldsymbol{u}$, so that the splitting at the first step of the construction is exact. Moreover, the equation for $\boldsymbol{w}$ is linear, so that $\boldsymbol{\nabla} q = 0$ for all times, and $p^{(1)} = p^{(0)} = p$. Similar splittings have been used in other contexts, for example in [2, 14, 23].

It is easy to see that $\boldsymbol{w}$ decays to zero exponentially in time in any spatial norm provided that $\boldsymbol{u}(t_0)$ is finite in this norm. Thus, $\boldsymbol{u}^{(1)}$ is an asymptotic approximant to $\boldsymbol{u}$. Since θ is bounded in L^p, we expect that $\boldsymbol{u}^{(1)}$ is also bounded in L^p. Indeed, by using the Helmholtz decomposition on L^p [11], we can write

$$\gamma\partial_t \boldsymbol{u}^{(1)} + \boldsymbol{u}^{(1)} + R\boldsymbol{P}_{L^p}(\boldsymbol{k}\theta^{(0)}) = 0\,, \tag{25}$$

where $\boldsymbol{P}_{L^p}$ is the L^p-continuous projector onto the completion of $\mathbb{D}$ in $L^p(\Omega,\mathbb{R}^3)$. Note that $\boldsymbol{P}_{L^p}\boldsymbol{u}^{(1)} = \boldsymbol{u}^{(1)}$ since $\boldsymbol{u}^{(1)}$ satisfies divergence and boundary condition (1c) and (1e), respectively. A direct calculation then shows that for every even positive integer p,

$$\gamma\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{u}^{(1)}\|_{L^p} + \|\boldsymbol{u}^{(1)}\|_{L^p} \leq R\,\|\theta^{(0)}\|_{L^p}\,. \tag{26}$$

Due to the asymptotic L^p bound on $\theta = \theta^{(0)}$ and estimate (5), we can choose t_0 sufficiently large so that integration in time implies boundedness of $\boldsymbol{u}^{(1)}$ in $L^\infty([t_0,\infty);L^p(\Omega,\mathbb{R}^3))$.

Remark 6. The case when p is not an even integer can be obtained, for example, by interpolating between even integer cases using the Hölder inequality. In the following, however, we only need the case $p=6$.

Step 2. *Find an asymptotic approximant $\theta^{(1)}$ to θ bounded in $L^\infty([t_0,\infty);H^1(\Omega))$ and in $L^\infty([t_0,\infty);M_1)$.*

Let $\theta^{(1)}$ be defined through

$$\partial_t\theta^{(1)} - \kappa\Delta\theta^{(1)} + \boldsymbol{u}^{(1)}\cdot\boldsymbol{\nabla}(\theta^{(1)} - x_3) = 0\,, \qquad \theta^{(1)}(t_0) = 0\,, \tag{27}$$

subject to boundary conditions (1f) and (1g). The existence of a solution to (27) can be shown by considering a Galerkin approximation based on the eigenfunctions (9). This procedure is standard and we only prove the necessary *a priori* estimates in L^2, H^1 and L^∞. First, integrate (27) against $\theta^{(1)}$, use integration by parts, and the apply then Cauchy–Schwarz inequality to find

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta^{(1)}\|_{L^2}^2 \le -\kappa\,\|\boldsymbol{\nabla}\theta^{(1)}\|_{L^2}^2 + \|\boldsymbol{u}^{(1)}\|_{L^2}\,\|\theta^{(1)}\|_{L^2}\,. \tag{28}$$

Use the Poincaré inequality and integrate. Since $\boldsymbol{u}^{(1)}$ is uniformly bounded in L^2, this proves that $\theta^{(1)} \in L^\infty([t_0,\infty);L^2(\Omega))$.

Second, integrate (27) against $\Delta\theta^{(1)}$, so that

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\nabla}\theta^{(1)}\|_{L^2}^2 + \kappa\,\|\Delta\theta^{(1)}\|_{L^2}^2 &\le \left|\left\langle \boldsymbol{u}^{(1)}\cdot\boldsymbol{\nabla}(\theta^{(1)}-x_3), \Delta\theta^{(1)}\right\rangle\right| \\ &\le \|\boldsymbol{u}^{(1)}\|_{L^6}\,\|\boldsymbol{\nabla}\theta^{(1)}\|_{L^3}\,\|\Delta\theta^{(1)}\|_{L^2}^2 + \|\boldsymbol{u}^{(1)}\|_{L^2}\,\|\Delta\theta^{(1)}\|_{L^2}\,. \end{aligned}\tag{29}$$

By using the interpolation inequality $\|\varphi\|_{L^3} \le c\,\|\varphi\|_{L^2}^{1/2}\,\|\varphi\|_{H^1}^{1/2}$, the Young inequality, and finally the interpolation $\|\boldsymbol{\nabla}\varphi\|_{L^2} \le (1+\|\varphi\|_{L^2})^{1/2}\,\|\Delta\varphi\|_{L^2}^{1/2}$, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{\nabla}\theta^{(1)}\|_{L^2}^2 \le -\kappa\,\frac{\|\boldsymbol{\nabla}\theta^{(1)}\|_{L^2}^4}{1+\|\theta^{(1)}\|_{L^2}^2} + c_1\,\|\boldsymbol{u}^{(1)}\|_{L^6}^4\,\|\boldsymbol{\nabla}\theta^{(1)}\|_{L^2}^2 + c_2\,\|\boldsymbol{u}^{(1)}\|_{L^2}^2\,. \tag{30}$$

Since $\boldsymbol{u}^{(1)} \in L^\infty([t_0,\infty);L^6(\Omega,\mathbb{R}^3))$ and $\theta^{(1)} \in L^\infty([t_0,\infty);L^2(\Omega))$, integration in time implies that $\theta^{(1)}$ is also bounded in $L^\infty([t_0,\infty);H^1(\Omega))$.

Third, to prove boundedness in $L^\infty([t_0,\infty);M_1)$, set $\theta^+(\boldsymbol{x}) \equiv \max\{0,\theta^{(1)}(\boldsymbol{x}) - x_3\}$, where the set M_b was defined in (3), and integrate (27) against θ^+:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta^+\|_{L^2}^2 - \kappa\left\langle\Delta\theta^{(1)},\theta^+\right\rangle + \left\langle\boldsymbol{u}^{(1)}\cdot\boldsymbol{\nabla}(\theta^{(1)}-x_3),\theta^+\right\rangle = 0\,. \tag{31}$$

Integrating by parts we see that the third term is zero and the second term gives us

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\theta^+\|_{L^2}^2 = -\kappa\|\boldsymbol{\nabla}\theta^+\|_{L^2}^2\,. \tag{32}$$

As $\theta^+(0) = 0$, the above implies that $\theta^+(t) \equiv 0$, which proves that $\theta^{(1)}$ is globally bounded from above by 1. A similar argument can be made with $\theta^-(\boldsymbol{x}) \equiv \max\{0, x_3 - \theta^{(1)}(\boldsymbol{x})\}$. This method of proving a weak maximum principle has been used, for example, in [9].

We now show that $\theta^{(1)}$ is an asymptotic approximant to θ. Let $\omega \equiv \theta^{(0)} - \theta^{(1)}$. Subtracting (27) from (1b) and integrating against ω we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^2}^2 = -\kappa\|\boldsymbol{\nabla}\omega\|_{L^2}^2 - \langle \boldsymbol{w}\cdot\boldsymbol{\nabla}(\theta^{(0)} - x_3), \omega\rangle - \langle \boldsymbol{u}^{(1)}\cdot\boldsymbol{\nabla}\omega, \omega\rangle\,, \tag{33}$$

where, as before, $\boldsymbol{w} = \boldsymbol{u}^{(0)} - \boldsymbol{u}^{(1)}$. The last term on the right is identically zero. We integrate the second term on the right by parts and use the Hölder, Poincaré and Young inequalities to find that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\omega\|_{L^2}^2 \le -c\,\|\omega\|_{L^2}^2 + \|\boldsymbol{w}\|_{L^2}^2\,\|\theta^{(0)} - x_3\|_{L^\infty}^2\,. \tag{34}$$

Since $\theta^{(0)}$ is bounded and, from the previous step,

$$\lim_{t\to\infty}\|\boldsymbol{w}\|_{L^2} = \lim_{t\to\infty}\|\boldsymbol{u}^{(0)}(t) - \boldsymbol{u}^{(1)}(t)\|_{L^2} = 0\,, \tag{35}$$

we can integrate inequality (34) and find that

$$\lim_{t\to\infty}\|\theta^{(0)}(t) - \theta^{(1)}(t)\|_{L^2} = 0\,. \tag{36}$$

Step 3. *Find an asymptotic approximant $\boldsymbol{u}^{(2)}$ to $\boldsymbol{u}^{(1)}$, and hence to $\boldsymbol{u}$, which is bounded in $L^\infty([t_0,\infty); H^1(\Omega,\mathbb{R}^3))$.*

Let $\boldsymbol{u}^{(2)}$ be a solution of

$$\gamma\partial_t\boldsymbol{u}^{(2)} + \boldsymbol{u}^{(2)} + \boldsymbol{\nabla}p^{(2)} - R\boldsymbol{k}\theta^{(1)} = 0\,, \quad \boldsymbol{u}^{(2)}(t_0) = 0\,, \tag{37a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}^{(2)} = 0 \quad \text{in } \Omega\,, \tag{37b}$$

$$\boldsymbol{u}^{(2)}\cdot\boldsymbol{n} = 0 \quad \text{on } \partial\Omega\,. \tag{37c}$$

Again, one can show the global existence and uniqueness of such a solution, for instance, via a Galerkin approximation based on the eigenfunctions (14).

As Darcy's law is linear, the proof that

$$\lim_{t\to\infty}\|\boldsymbol{u}^{(1)}(t) - \boldsymbol{u}^{(2)}(t)\|_{L^2} = 0\,, \tag{38}$$

and the estimate on the H^1 norm of $\boldsymbol{u}^{(2)}$ are straightforward, based on the previous steps. We omit all details.

Step 4. *Find an asymptotic approximant $\theta^{(2)}$ to $\theta^{(1)}$, and hence to θ, which is bounded in $L^\infty([t_0,\infty); H^2(\Omega))$.*

The construction is parallel to that of Step 2, namely

$$\partial_t\theta^{(2)} - \kappa\Delta\theta^{(2)} + \boldsymbol{u}^{(2)}\cdot\boldsymbol{\nabla}(\theta^{(2)} - x_3) = 0\,, \qquad \theta^{(2)}(t_0) = 0\,, \tag{39}$$

again subject to boundary conditions (1f) and (1g). The H^2 norm of $\theta^{(2)}$ is estimated in the usual way, for details see [7].

We could iterate this procedure *ad infinitum* and construct asymptotic approximants in any space H^r. This would suffice to prove C^∞ regularity of the attractor. However, we will now use a different splitting which will give analytic regularity in a single step.

Remark 7. Lemma 4 applied to (24b) shows that $\boldsymbol{u}^{(1)}$ and any subsequent asymptotic approximant is contained in $\mathbb{H}_{\shortparallel}$. Thus, the velocity component of the attractor $\mathcal{A}$ is also a subset of $\mathbb{H}_{\shortparallel}$.

4. Fourier Splitting

Step 5. *Find $\tau > 0$ and asymptotic approximants $\boldsymbol{u}^{(3)}$ and $\theta^{(3)}$ which are bounded in $L^\infty([t_0,\infty);\mathcal{D}(\boldsymbol{A}\mathrm{e}^{\tau\boldsymbol{A}}))$ and $L^\infty([t_0,\infty);\mathcal{D}(A\mathrm{e}^{\tau A}))$ respectively.*

We split the asymptotic approximants found in Step 4 into the low wavenumber components $\bar{\boldsymbol{u}} \equiv \boldsymbol{P}_\lambda \boldsymbol{u}^{(2)}$ and $\bar\theta \equiv P_\lambda \theta^{(2)}$, which are trigonometric polynomials and therefore smooth, and the high wavenumber parts $\boldsymbol{Q}_\lambda \boldsymbol{u}^{(2)}$ and $Q_\lambda\theta^{(2)}$. The cut-off wavenumber λ is yet to be determined. Our goal is to show that the high wavenumber components can be approximated asymptotically by the solutions $\hat{\boldsymbol{u}}$, $\hat\theta$, respectively, of the system

$$\gamma\partial_t\hat{\boldsymbol{u}} + \hat{\boldsymbol{u}} + \boldsymbol{Q}_\lambda\boldsymbol{\nabla}\hat p - R\boldsymbol{k}\hat\theta = 0\,, \tag{40a}$$

$$\partial_t\hat\theta - \kappa\Delta\hat\theta + Q_\lambda\big((\bar{\boldsymbol{u}}+\hat{\boldsymbol{u}})\cdot\boldsymbol{\nabla}(\bar\theta+\hat\theta-x_3)\big) = 0\,, \tag{40b}$$

$$\boldsymbol{\nabla}\cdot\hat{\boldsymbol{u}} = 0\,, \tag{40c}$$

$$\hat{\boldsymbol{u}}(t_0) = 0\,, \quad \hat\theta(t_0) = 0\,, \tag{40d}$$

where $t \ge t_0$ and $\hat{\boldsymbol{u}}$, $\hat\theta$ are also subject to the boundary conditions (1e–1g). This system formally arises by projecting (1) onto the high wavenumber space, but is initialized with zero at time t_0, which is as in Step 1. Notice that the system is driven by the low wavenumber components of the *previous* asymptotic approximants. The global existence of a solution to (40) which satisfies the boundary conditions can easily be shown via a Galerkin approximation with respect to the bases (14) and (9) as soon as the *a priori* estimates below are established. Our task is to prove that $\boldsymbol{u}^{(3)} \equiv \bar{\boldsymbol{u}}+\hat{\boldsymbol{u}}$ and $\theta^{(3)} \equiv \bar\theta+\hat\theta$ remain in an analytic function class for all times, and are asymptotic approximants to $\boldsymbol{u}$ and θ.

In the following we think of τ as a fixed number to be chosen later. Take the $\mathcal{D}(\boldsymbol{A}\mathrm{e}^{\tau\boldsymbol{A}})$ inner product of equation (40a) with $\hat{\boldsymbol{u}}$ to find

$$\begin{aligned}\gamma\frac{\mathrm{d}}{\mathrm{d}t}\|\boldsymbol{A}\mathrm{e}^{\tau\boldsymbol{A}}\hat{\boldsymbol{u}}\|^2 &= -2\,\|\boldsymbol{A}\mathrm{e}^{\tau\boldsymbol{A}}\hat{\boldsymbol{u}}\|^2 - 2R\left\langle A\mathrm{e}^{\tau A}\hat\theta, \boldsymbol{k}\cdot\boldsymbol{A}\mathrm{e}^{\tau\boldsymbol{A}}\hat{\boldsymbol{u}}\right\rangle \\ &\le -\|\boldsymbol{A}\mathrm{e}^{\tau\boldsymbol{A}}\hat{\boldsymbol{u}}\|^2 + R^2\,\|A\mathrm{e}^{\tau A}\hat\theta\|^2\,.\end{aligned} \tag{41}$$

The corresponding operation on the equation (40b) for $\hat\theta$ gives

$$\frac12\frac{\mathrm{d}}{\mathrm{d}t}\|A\mathrm{e}^{\tau A}\hat\theta\|^2 = -\kappa\,\|A^2\mathrm{e}^{\tau A}\hat\theta\|^2 - \left\langle A\mathrm{e}^{\tau A}\big(\boldsymbol{u}^{(3)}\cdot\boldsymbol{\nabla}(\bar\theta - x_3 + \hat\theta)\big), A\mathrm{e}^{\tau A}\hat\theta\right\rangle. \tag{42}$$

The *a priori* bounds derived in the previous section in conjunction with Lemma 6 imply that

$$\|\boldsymbol{A}\mathrm{e}^{\tau\boldsymbol{A}}\bar{\boldsymbol{u}}\| = \|\boldsymbol{A}\mathrm{e}^{\tau\boldsymbol{A}}\boldsymbol{P}_\lambda\boldsymbol{u}^{(2)}\| \le \mathrm{e}^{\lambda\tau}\,\|\boldsymbol{A}\boldsymbol{u}^{(2)}\| \le \mathrm{e}^{\lambda\tau}\,K_1\,, \tag{43}$$

and

$$\|A^2\mathrm{e}^{\tau A}\bar\theta\| = \|A^2\mathrm{e}^{\tau A}P_\lambda\theta^{(2)}\| \le \mathrm{e}^{\lambda\tau}\,\|A^2\theta^{(2)}\| \le \mathrm{e}^{\lambda\tau}\,K_2\,. \tag{44}$$

We can now use Lemma 5 and the Young inequality to estimate the last term of (42), thus obtaining

$$\begin{aligned}\left|\left\langle A\mathrm{e}^{\tau A}(\boldsymbol{u}^{(3)}\cdot\boldsymbol{\nabla}\bar\theta), A\mathrm{e}^{\tau A}\hat\theta\right\rangle\right| &\le c_1\,\|\boldsymbol{A}\mathrm{e}^{\tau\boldsymbol{A}}\boldsymbol{u}^{(3)}\|\,\|A\mathrm{e}^{\tau A}\bar\theta\|^{\frac12}\,\|A^2\mathrm{e}^{\tau A}\bar\theta\|^{\frac12}\,\|A^2\mathrm{e}^{\tau A}\hat\theta\| \\ &\le c_2\left(\|\boldsymbol{A}\mathrm{e}^{\tau\boldsymbol{A}}\bar{\boldsymbol{u}}\|^2 + \|\boldsymbol{A}\mathrm{e}^{\tau\boldsymbol{A}}\hat{\boldsymbol{u}}\|^2\right)\|A\mathrm{e}^{\tau A}\bar\theta\|\,\|A^2\mathrm{e}^{\tau A}\bar\theta\| + \tfrac14\,\|A^2\mathrm{e}^{\tau A}\hat\theta\|^2 \\ &\le K_3\,\mathrm{e}^{4\lambda\tau} + K_4\,\mathrm{e}^{2\lambda\tau}\|\boldsymbol{A}\mathrm{e}^{\tau\boldsymbol{A}}\hat{\boldsymbol{u}}\|^2 + \tfrac14\,\|A^2\mathrm{e}^{\tau A}\hat\theta\|^2\,,\end{aligned} \tag{45}$$

and

$$\begin{aligned}\big|\big\langle A\mathrm{e}^{\tau A}(\boldsymbol{u}^{(3)}\cdot\boldsymbol{\nabla}\hat\theta), A\mathrm{e}^{\tau A}\hat\theta\big\rangle\big| &\le c_1\,\|\boldsymbol{A}\mathrm{e}^{\tau\boldsymbol{A}}\boldsymbol{u}^{(3)}\|\,\|A\mathrm{e}^{\tau A}\hat\theta\|^{\frac12}\,\|A^2\mathrm{e}^{\tau A}\hat\theta\|^{\frac32}\\ &\le c_3\left(\|\boldsymbol{A}\mathrm{e}^{\tau\boldsymbol{A}}\bar{\boldsymbol{u}}\|^4+\|\boldsymbol{A}\mathrm{e}^{\tau\boldsymbol{A}}\hat{\boldsymbol{u}}\|^4\right)\|A\mathrm{e}^{\tau A}\hat\theta\|^2+\tfrac14\,\|A^2\mathrm{e}^{\tau A}\hat\theta\|^2\\ &\le K_5\,\mathrm{e}^{4\lambda\tau}\,\|A\mathrm{e}^{\tau A}\hat\theta\|^2+c_3\,\|\boldsymbol{A}\mathrm{e}^{\tau\boldsymbol{A}}\hat{\boldsymbol{u}}\|^4\,\|A\mathrm{e}^{\tau A}\hat\theta\|^2+\tfrac14\,\|A^2\mathrm{e}^{\tau A}\hat\theta\|^2\,. \end{aligned}\tag{46}$$

By applying these two estimates to the right side of (42), and setting $\tau = 1/\lambda$, we then obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|A\mathrm{e}^{\tau A}\hat\theta\|^2 &\le -\lambda^2\,\kappa\,\|A\mathrm{e}^{\tau A}\hat\theta\|^2\\ &\quad + K_6\left(1+\|A\mathrm{e}^{\tau A}\hat\theta\|^2+\|\boldsymbol{A}\mathrm{e}^{\tau\boldsymbol{A}}\hat{\boldsymbol{u}}\|^2+\|\boldsymbol{A}\mathrm{e}^{\tau\boldsymbol{A}}\hat{\boldsymbol{u}}\|^4\,\|A\mathrm{e}^{\tau A}\hat\theta\|\right).\end{aligned}\tag{47}$$

We set $y \equiv \|\boldsymbol{A}\mathrm{e}^{\tau\boldsymbol{A}}\hat{\boldsymbol{u}}\|^2$ and $z \equiv \|A\mathrm{e}^{\tau A}\hat\theta\|^2$, so that the system of differential inequalities (41) and (47) can be written in the more legible form

$$\gamma\dot y \le -y + R^2 z\,, \tag{48a}$$

$$\dot z \le -\lambda^2\kappa\, z + K_6\big(1+y+z+y^2 z\big)\,. \tag{48b}$$

Since $y(t_0) = 0$ and $z(t_0) = 0$, we can define

$$T = \sup\{t_1 \ge t_0 \colon 0 \le y(t) \le 4R^2 \text{ and } 0 \le z(t) \le 2 \text{ for } t\in[t_0,t_1]\}\,. \tag{49}$$

Clearly, $T > 0$. We will proceed to show that in fact $T = \infty$ when λ is sufficiently large. On the interval $[t_0, T)$ we have

$$\dot y \le -\frac{1}{\gamma}y + \frac{2R^2}{\gamma}\,, \tag{50a}$$

$$\dot z \le -\lambda^2\kappa\, z + K_6\big(1+4R^2+2+32R^2\big) \equiv -\lambda^2\kappa\, z + K_7\,. \tag{50b}$$

These inequalities are decoupled and can be easily integrated. For $t_0 \le t < T$, we find

$$y(t) \le 2R^2\big(1-\mathrm{e}^{-(t-t_0)/\gamma}\big) \le 2R^2\,, \tag{51a}$$

$$z(t) \le \frac{K_7}{\lambda^2\kappa}\big(1-\mathrm{e}^{-\lambda^2\kappa(t-t_0)}\big) \le \frac{K_7}{\lambda^2\kappa}\,. \tag{51b}$$

If we choose λ large enough, we can make the right side of (51b) equal to one. On the other hand, if T is finite then either

$$\limsup_{t\nearrow T} y(t) = 4R^2 \tag{52a}$$

or

$$\limsup_{t\nearrow T} z(t) = 2\,. \tag{52b}$$

This contradicts (51), and therefore $T = \infty$. We conclude that $\|\boldsymbol{A}\mathrm{e}^{\tau\boldsymbol{A}}\hat{\boldsymbol{u}}\|$ and $\|A\mathrm{e}^{\tau A}\hat\theta\|$ are uniformly bounded for $t \ge t_0$.

We now prove that $\boldsymbol{u}^{(3)}$ and $\theta^{(3)}$ are indeed asymptotic approximants to $\boldsymbol{u}$ and θ. Since we know that the low modes converge by virtue of their construction in

Section 3, it is sufficient to prove L^2 convergence for the high modes. The differences $\boldsymbol{w} \equiv \boldsymbol{Q}_\lambda \boldsymbol{u} - \hat{\boldsymbol{u}}$ and $\omega \equiv Q_\lambda \theta - \hat{\theta}$ satisfy the equations

$$\gamma \partial_t \boldsymbol{w} + \boldsymbol{w} + \boldsymbol{\nabla} q - R\boldsymbol{k}\omega = 0 \,, \tag{53a}$$

$$\partial_t \omega - \kappa \Delta \omega + Q_\lambda \big(\boldsymbol{w} \cdot \boldsymbol{\nabla}(\theta - x_3) + \boldsymbol{u}^{(3)} \cdot \boldsymbol{\nabla}(P_\lambda \theta - \bar{\theta} + \omega) \big) = 0 \,, \tag{53b}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{w} = 0 \,, \tag{53c}$$

and are subject to boundary conditions of the form (1e)–(1g). We estimate the L^2 norms of $\boldsymbol{w}$ and ω as in th previous section,

$$\begin{aligned} \gamma \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\boldsymbol{w}\|_{L^2}^2 &= -\|\boldsymbol{w}\|_{L^2}^2 + R \, \langle \boldsymbol{k} \cdot \boldsymbol{w}, \omega \rangle \\ &\le -\|\boldsymbol{w}\|_{L^2}^2 + R \, \|\boldsymbol{w}\|_{L^2} \, \|\omega\|_{L^2} \,, \end{aligned} \tag{54a}$$

and

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\omega\|_{L^2}^2 &= -\kappa \, \|\boldsymbol{\nabla}\omega\|_{L^2}^2 - \big\langle \boldsymbol{w} \cdot \boldsymbol{\nabla}(\theta - x_3), \omega \big\rangle - \big\langle \boldsymbol{u}^{(3)} \cdot \boldsymbol{\nabla}(P_\lambda \theta - \bar{\theta}), \omega \big\rangle \\ &\le -\frac{\kappa\,\lambda^2}{2} \|\omega\|_{L^2}^2 - \frac{\kappa}{2} \|\boldsymbol{\nabla}\omega\|_{L^2}^2 \\ &\quad + \|\boldsymbol{w}\|_{L^2} \, \|\boldsymbol{\nabla}\omega\|_{L^2} \, \|\theta - x_3\|_{L^\infty} + \|\boldsymbol{u}^{(3)}\|_{L^\infty} \, \|\boldsymbol{\nabla}\omega\|_{L^2} \, \|P_\lambda \theta - \bar{\theta}\|_{L^2} \\ &\le -\frac{\kappa\,\lambda^2}{2} \|\omega\|_{L^2}^2 + K_8 \, \|\boldsymbol{w}\|_{L^2}^2 + K_9 \, \|P_\lambda \theta - \bar{\theta}\|_{L^2}^2 \,. \end{aligned} \tag{54b}$$

Note that the first line of (54b) was achieved through integration by parts on the linear as well as on the nonlinear terms. The system of differential inequalities can be written symbolically in the form

$$\gamma \dot{y} \le -y + R^2 z \,, \tag{55a}$$

$$\dot{z} \le -\sigma z + K y + f \,, \tag{55b}$$

where $y = \|\boldsymbol{w}\|_{L^2}^2$, $z = \|\omega\|_{L^2}^2$, $f = 2K_9 \, \|P_\lambda \theta - \bar{\theta}\|_{L^2}^2$, $\sigma = \lambda^2 \kappa$, and $K = 2K_8$.

Multiply the first equation with y/γ, the second with z and add both equations. Then apply the Young inequality, i.e.

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} (y^2 + z^2) &\le -\frac{y^2}{\gamma} - \sigma z^2 + \frac{R^2}{\gamma} yz + K yz + f(t) z \\ &\le -\frac{1}{2\gamma} y^2 - \big(\frac{\sigma}{2} - \frac{R^4}{\gamma} - \gamma K^2 \big) z^2 + \frac{f^2(t)}{2\sigma} \,. \end{aligned} \tag{56}$$

Now choose $\sigma = \lambda^2 \kappa$ large enough so that

$$\frac{\sigma}{2} - \frac{R^4}{\gamma} - \gamma K^2 \ge \frac{1}{2\gamma} \,. \tag{57}$$

We then obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} (y^2 + z^2) \le -\frac{1}{\gamma} (y^2 + z^2) + \frac{f^2(t)}{\sigma} \,. \tag{58}$$

Integration in time gives

$$y^2(t) + z^2(t) \le \big(y^2(t_0) + z^2(t_0) \big) \mathrm{e}^{-(t - t_0)/\gamma} + \frac{1}{\sigma} \int_{t_0}^{t} \mathrm{e}^{-(t - t')/\gamma} \, f^2(t') \, \mathrm{d}t' \,. \tag{59}$$

If we break up the range of integration in the remaining integral into two equal pieces, setting $t_1 = (t - t_0)/2$, we can estimate

$$
\begin{aligned}
&\int_{t_0}^{t} \mathrm{e}^{-(t-t')/\gamma} f^2(t') \, \mathrm{d}t' \\
&\quad \leq \sup_{t' \in [t_0, t_1]} f^2(t') \int_{t_0}^{t_1} \mathrm{e}^{(t'-t)/\gamma} \, \mathrm{d}t' + \sup_{t' \in [t_1, t]} f^2(t') \int_{t_1}^{t} \mathrm{e}^{(t'-t)/\gamma} \, \mathrm{d}t' \\
&\quad \leq \gamma \sup_{t' \geq t_0} f^2(t') \, \mathrm{e}^{(t-t_1)/\gamma} + \gamma \sup_{t' \geq t_1} f^2(t') \, .
\end{aligned}
\tag{60}
$$

Since

$$
\lim_{t \to \infty} f(t) = \lim_{t \to \infty} K_9 \, \| P_\lambda \theta - \bar{\theta} \|_{L^2}^2 = 0 \, , \tag{61}
$$

we find that

$$
\lim_{t \to \infty} y(t) = \lim_{t \to \infty} \| \boldsymbol{Q}_\lambda \boldsymbol{u} - \hat{\boldsymbol{u}} \|^2 = 0 \, , \tag{62}
$$

and

$$
\lim_{t \to \infty} z(t) = \lim_{t \to \infty} \| Q_\lambda \theta - \hat{\theta} \|^2 = 0 \, , \tag{63}
$$

This proves that $\theta^{(3)}$ is an asymptotic approximant to θ, and $\boldsymbol{u}^{(3)}$ is an asymptotic approximant to $\boldsymbol{u}$. Moreover, the rate of attraction can be estimated in terms of the parameters of the equation.

Remark 8. It is not possible to use the Fourier splitting method in place of the simpler splitting used in Section 3, because Fourier truncations in general do not converge in L^∞, while the L^∞ bound for the temperature θ plays an essential role in our proof.

5. Existence and Regularity of the Attractor

We now explain how the existence of analytic asymptotic approximants implies the existence of a global attractor in the same class of regularity. Our exposition follows references [12, 18].

Let $S(t)$ denote the solution semigroup for (1) with $\mathbb{H} \times L^\infty(\Omega)$ initial data, i.e., $S(t)(\boldsymbol{u}^{\mathrm{in}}, \theta^{\mathrm{in}}) = (\boldsymbol{u}(t), \theta(t))$. We decompose $S(t)$ into the sum of two L^2-continuous maps (not necessarily semigroups) $S(t) = S_1(t) + S_2(t)$ where

$$
S_1(t)(\boldsymbol{u}^{\mathrm{in}}, \theta^{\mathrm{in}}) = \begin{cases} (\bar{\boldsymbol{u}}, \bar{\theta}) & \text{for } t < t_0(\boldsymbol{u}^{\mathrm{in}}, \theta^{\mathrm{in}}) \\ (\boldsymbol{u}^{(3)}, \theta^{(3)}) & \text{for } t \geq t_0(\boldsymbol{u}^{\mathrm{in}}, \theta^{\mathrm{in}}) \end{cases} \tag{64a}
$$

and

$$
S_2(t)(\boldsymbol{u}^{\mathrm{in}}, \theta^{\mathrm{in}}) = \begin{cases} (\boldsymbol{Q}_\lambda \boldsymbol{u}(t), Q_\lambda \theta(t)) & \text{for } t < t_0(\boldsymbol{u}^{\mathrm{in}}, \theta^{\mathrm{in}}) \\ (\boldsymbol{Q}_\lambda \boldsymbol{u}(t) - \hat{\boldsymbol{u}}, Q_\lambda \theta(t) - \hat{\theta}) & \text{for } t \geq t_0(\boldsymbol{u}^{\mathrm{in}}, \theta^{\mathrm{in}}) \, , \end{cases} \tag{64b}
$$

where $\hat{\boldsymbol{u}}$ and $\hat{\theta}$ solve the high wavenumber equation (40) and $t_0(\boldsymbol{u}^{\mathrm{in}}, \theta^{\mathrm{in}})$ is the supremum over all entry times of $(\boldsymbol{u}(t), \theta(t))$ into some fixed absorbing ball $\mathcal{B} \subset \mathbb{H} \times L^6(\Omega)$. The existence of such a ball is guaranteed by Theorem 1.

For initial data $(\boldsymbol{u}^{\mathrm{in}}, \theta^{\mathrm{in}}) \in \mathbb{H} \times M_b$ where $b \geq 1$, existence of a compact global attractor with respect to the $L^2 \times L^2$ metric follows readily from an abstract Theorem by Temam [23, Theorem I.1.1]. It is not difficult to see that its two requirements on the splitting of the semigroup are satisfied:

(i) For every L^2-bounded $B \subset \mathbb{H} \times M_b$ let $t_0^* = \sup\{t_0(\boldsymbol{u}^{\rm in}, \theta^{\rm in}) \colon (\boldsymbol{u}^{\rm in}, \theta^{\rm in}) \in B\}$. Then the set

$$\bigcup_{t \geq t_0^*} S_1(t) B \tag{65}$$

is relatively compact in $\mathbb{H} \times L^2(\Omega)$.

(ii) For every L^2-bounded $B \subset \mathbb{H} \times M_b$,

$$\limsup_{t \to \infty} \sup_{(\boldsymbol{u}^{\rm in}, \theta^{\rm in}) \in B} \| S_2(t)(\boldsymbol{u}^{\rm in}, \theta^{\rm in}) \|_{L^2} = 0 \,. \tag{66}$$

The regularity of $\mathcal{A}$ is proved as follows. The attractor is the union of complete, L^2-bounded orbits [23]. I.e., for an arbitrary $(\boldsymbol{u}^{\rm in}, \theta^{\rm in}) \in \mathcal{A}$ there exists an orbit $\{(\boldsymbol{u}(t), \theta(t)) \colon t \in \mathbb{R}\} \subset \mathcal{B}$ such that $\boldsymbol{u}(0) = \boldsymbol{u}^{\rm in}$ and $\theta(0) = \theta^{\rm in}$. Since the orbit is bounded, there is no restriction on t_0—in particular, t_0 can be negative. We can therefore use the procedure of the last two sections to construct a family of asymptotic approximants $\{(\boldsymbol{u}_{t_0}^{(3)}, \theta_{t_0}^{(3)}) \colon t_0 \in \mathbb{R}_-\}$ where each member is initialized at time t_0. As the rate of convergence of the asymptotic approximants depends only on $\mathcal{B}$ and the parameters in the equation, we clearly have

$$\lim_{t_0 \to -\infty} \| (\boldsymbol{u}_{t_0}^{(3)}(0), \theta_{t_0}^{(3)}(0)) - (\boldsymbol{u}^{\rm in}, \theta^{\rm in}) \|_{L^2} = 0 \,. \tag{67}$$

On the other hand, the family $\{(\boldsymbol{u}_{t_0}^{(3)}, \theta_{t_0}^{(3)}) \colon t_0 \in \mathbb{R}_-\}$ is uniformly bounded in $\mathcal{D}(\boldsymbol{A} \mathrm{e}^{\tau \boldsymbol{A}}) \times \mathcal{D}(A \mathrm{e}^{\tau A})$. This implies $(\boldsymbol{u}^{\rm in}, \theta^{\rm in}) \in \mathcal{D}(\boldsymbol{A} \mathrm{e}^{\tau \boldsymbol{A}}) \times \mathcal{D}(A \mathrm{e}^{\tau A})$, and thus the analytic regularity of $\mathcal{A}$.

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