ON THE DOMAIN OF ANALYTICITY FOR SOLUTIONS OF SECOND ORDER ANALYTIC NONLINEAR DIFFERENTIAL EQUATIONS

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Abstract. The radius of analyticity of periodic analytic functions can be characterized by the decay of their Fourier coefficients. This observation has led to the use of so-called Gevrey norms as a simple way of estimating the time evolution of the spatial radius of analyticity of solutions to parabolic as well as non-parabolic partial differential equations. In this paper we demonstrate, using a simple, explicitly solvable model equation, that estimates on the radius of analyticity obtained by the usual Gevrey class approach do not scale optimally across a family of solutions, nor do they scale optimally as a function of the physical parameters of the equation. We attribute the observed lack of sharpness to a specific embedding inequality, and give a modified definition of the Gevrey norms which is shown to finally yield a sharp estimate on the radius of analyticity.

1. Introduction

In a previous paper [10] we showed that the weakly damped driven nonlinear Schrödinger equation has a real analytic global attractor. Since the proof was based on estimates in so-called Gevrey norms [3]—norms in which bounds on all derivatives of a function are weighted as terms of an exponential sum—we also obtained a sufficient lower bound for the radius of analyticity of the attractor as a function of the parameters of the equation.

It was therefore natural to ask if our inability to find a larger (possibly infinite) radius of analyticity is an artifact of the method, or is indeed an intrinsic property of the equation. We noted that lower bounds on the radius of analyticity of steady state solutions can also be estimated by using Gevrey norms, and found that these bounds exhibit a behavior similar to the attractor estimates. Therefore, if we could show that the result for the steady state solutions is in some sense sharp, then—the steady state solutions being a subset of the attractor—the estimates for the global attractor would have to be sharp, too.

In this paper we analyze the situation using a somewhat simpler "toy" equation,

$$\nu u_{xx} + \lambda u \pm u^3 = 0\,, \tag{1}$$

where u is a real function endowed with periodic (or homogeneous Dirichlet) boundary conditions on the interval $[0,1]$, $\nu > 0$, and λ is an arbitrary real number. In the case where the nonlinearity is preceded by a minus sign, equation (1) describes the steady states of the Chafe–Infante reaction diffusion equation in one spatial

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dimension,

$$u_t = \nu u_{xx} + \lambda u - u^3 \,. \tag{2}$$

Problem (1) can also be thought of as representing real steady states of the complex Ginzburg–Landau, or of the nonlinear Schrödinger equation, where the plus and minus sign correspond to the focusing and defocusing case respectively. On the other hand, we can take the point of view that (1) describes periodic solutions of a nonlinear oscillator. Moreover, the equation can be solved explicitly, which enables us to deduce an exact expression for the radius of (real) analyticity. For these reasons, equation (1) can be studied more transparently than the full steady state problem for the damped driven nonlinear Schrödinger equation [5, 6, 10] which originally motivated this investigation.

Even in this simple case we cannot expect to recover the precise numerical value of the radius of analyticity by using Gevrey class estimates. We therefore resort to studying its scaling behavior in the following asymptotic regimes: The *zero diffusion limit* $\nu \to 0$, the *high energy limit* when (1) is considered as an initial value problem whose periodic solutions are parameterized by a conserved energy, and the *asymptotically linear regime* when $\lambda \to \infty$. We find that in each of these limits the estimates on the radius of analyticity obtained by the usual Gevrey class approach (as, for example, used in [10]) do not scale optimally. We attribute the observed lack of sharpness to a specific embedding inequality, and give an alternative definition of the Gevrey norm which is finally shown to yield sharp estimates on the radius of analyticity.

The estimation techniques we present here are applicable to more general nonlinear elliptic partial differential equations for which explicit solutions are not available. In such cases one first needs to prove existence of solutions in the appropriate class of real analytic functions, for example by using degree theory or the Galerkin method with application of the Brower fixed point theorem before proceeding with the formal arguments as are presented in this paper.

2. Explicit Solutions

It is well known that (1) can be completely solved in terms of elliptic integrals. Our presentation follows Fraenkel [4].

While the sign of the nonlinearity plays a crucial role for the structure of the solution set, the scaling behavior of the radius of analyticity is essentially the same in both cases. Further, the sign of λ plays no significant role in the argument—for simplicity we shall assume that λ is positive.

Every periodic solution must have at least one zero and, hence, be a translation of a solution to (1) endowed with homogeneous Dirichlet boundary conditions $u(0) = u(1) = 0$. This can, for example, be seen from the level sets of the energy integral, which we obtain by multiplying (1) with u_x and integrating once,

$$\nu u_x^2 = c^2 - \lambda u^2 \mp \tfrac{1}{2}\, u^4 \,, \tag{3}$$

where $c^2 = u_x^2(0)$ is the energy of the nonlinear oscillator. The energy will also be used as one of the scaling parameter in our subsequent analysis.

When the last term in (3) is positive (corresponding to the minus sign in the original problem), its right side can be factored as

$$\sqrt{\nu}\, u_x = \pm\sqrt{\tfrac{1}{2}(a^2 - u^2)(b^2 - u^2)}, \tag{4}$$

where

$$a^2 = \lambda + \sqrt{\lambda^2 - 2c^2}\,, \qquad b^2 = \lambda - \sqrt{\lambda^2 - 2c^2}\,. \tag{5}$$

Assume that $c = u_x(0) > 0$, and let $x_0 > 0$ be the smallest real number with $u_x(x_0) = 0$. Since u is periodic, such a point must exist. We can then integrate (4) on the interval where $0 \le x \le x_0$ and $0 \le u \le b$ and obtain [1, equation 219.00]

$$\begin{aligned} x &= \sqrt{2\nu} \int_0^u \frac{\mathrm{d}v}{\sqrt{(a^2 - v^2)(b^2 - v^2)}} \\ &= \frac{\sqrt{2\nu}}{a}\, \mathrm{sn}^{-1}(\sin\varphi, k)\,, \end{aligned} \tag{6}$$

where, setting $\delta^2 \equiv 1 - 2(c/\lambda)^2$,

$$k^2 = \frac{b^2}{a^2} = \frac{\lambda - \sqrt{\lambda^2 - 2c^2}}{\lambda + \sqrt{\lambda^2 - 2c^2}} = \frac{1-\delta}{1+\delta}\,, \tag{7}$$

$$\sin\varphi = \frac{u}{b} = \frac{u}{\sqrt{\lambda(1-\delta)}}\,. \tag{8}$$

By solving for u, we obtain

$$u(x) = \sqrt{\lambda(1-\delta)}\ \mathrm{sn}\left(\sqrt{\frac{\lambda(1+\delta)}{2\nu}} x, k\right). \tag{9}$$

Since (9) solves equation (1), it is a solution for every $x \in \mathbb{R}$. On the other hand, due to the uniqueness theorem for initial value problems, every solution with $u_x(0) > 0$ can be written in this form. The case $u_x(0) < 0$ is analogous.

The Jacobi elliptic function sn is doubly periodic with real period $4K$ and imaginary period $2\mathrm{i}K'$, where $K = K(k)$ is the complete elliptic integral of the first kind, $K' = K(k')$, and $k' = \sqrt{1-k^2}$ [1]. For u to have period 1, we must hence require that

$$\sqrt{\frac{\lambda(1+\delta)}{2\nu}} = 2nK(k) \tag{10}$$

for some integer n. The poles of sn are located at $\mathrm{i}K' \pmod{2K, 2\mathrm{i}K'}$, so that the radius of analyticity of u is

$$\rho = \sqrt{\frac{2\nu}{\lambda(1+\delta)}}\, K'\,. \tag{11}$$

We now turn to the case when the last term in (3) is negative. The factorization in this case is

$$\sqrt{\nu}\, u_x = \pm\sqrt{\tfrac{1}{2}(a^2 + u^2)(b^2 - u^2)}, \tag{12}$$

where

$$a^2 = \lambda + \sqrt{\lambda^2 + 2c^2}\,, \qquad b^2 = -\lambda + \sqrt{\lambda^2 + 2c^2}\,. \tag{13}$$

As before, we assume that $c = u_x(0) > 0$, and let $x_0 > 0$ be the smallest number with $u_x(x_0) = 0$. We can then integrate (12) on the interval where $0 \le x \le x_0$ and

$0 \le u \le b$ and obtain [1, equation 214.00]

$$x = \sqrt{2\nu} \int_0^u \frac{\mathrm{d}v}{\sqrt{(a^2+v^2)(b^2-v^2)}}$$
$$= \sqrt{\frac{2\nu}{a^2+b^2}}\, \mathrm{sn}^{-1}(\sin\psi, k)\,, \tag{14}$$

where

$$k^2 = \frac{b^2}{a^2+b^2} = \frac{c^2}{\lambda^2+2c^2+\sqrt{\lambda^2+2c^2}}\,, \tag{15}$$

$$\sin\psi = \sqrt{\frac{u^2\,(a^2+b^2)}{b^2\,(a^2+u^2)}}\,. \tag{16}$$

This expression can be solved for u, we find

$$u(x) = M\,\mathrm{sd}(rx, k) \tag{17}$$

with

$$M = \frac{ab}{\sqrt{a^2+b^2}} = \frac{c}{(\lambda^2+2c^2)^{1/4}}\,, \tag{18}$$

$$r = \sqrt{\frac{a^2+b^2}{2\nu}} = \frac{(\lambda^2+2c^2)^{1/4}}{\sqrt{\nu}}\,. \tag{19}$$

We conclude as before that (17) is the unique solution for every $x \in \mathbb{R}$. The function sd is doubly periodic with real period $4K$ and imaginary period $4\mathrm{i}K'$. Thus, for u to have period 1, we must require that $r = 2nK$, or

$$\sqrt{\frac{\lambda}{\nu(1-2k^2)}} = 2nK(k) \tag{20}$$

for some integer n. The poles of sd coincide with the poles of sn, therefore the radius of analyticity of (17) is

$$\rho = \frac{K'}{r} = \frac{\sqrt{\nu}\,K'}{(\lambda^2+2c^2)^{1/4}}\,. \tag{21}$$

3. Limiting Behavior of the Radius of Analyticity

We will investigate the limiting behavior of the radius of analyticity in three different asymptotic regimes. The results of this section will be used as benchmarks for evaluating the Gevrey class estimates on the radius of analyticity of u later on.

We write $f \sim g$ to denote that two quantities f and g have the same asymptotic behavior.

Case 1 (Zero diffusion limit). Let $\nu \to 0$ while keeping $\lambda = O(1)$ and $c = O(1)$. Then K and K' remain bounded, and we see from either (11) or (21) that $n \to \infty$ and therefore

$$\rho \sim \mathrm{const} \cdot \sqrt{\nu}\,. \tag{22}$$

Remark 1. Due to the periodicity condition (10) or (20) respectively, the set of admissible triples (ν, λ, c) is discrete. This does not, however, impact the scaling behavior which concerns us here.

The following two cases are different in that we have to take the dependence of K and K' on λ and c into account. Notice that $a^2 \geq b^2$, so that $0 \leq k^2 \leq \frac{1}{2}$. For such values of k, the elliptic integral $K = K(k)$ is continuous, strictly positive, and bounded [1].

Now the difference in behavior due to the choice of sign in front of the nonlinear term in (1) will become important. If we choose the minus sign, there is an upper bound on the energy so that it is impossible to let $c \to \infty$. We therefore exclusively treat the case when u is of the form (17), and, for simplicity, set $\nu = 1$. (Note, however, that Case 3 below does not really depend on the sign of the nonlinearity.)

Case 2 (High energy limit). Let $c \to \infty$ while keeping $\lambda = O(1)$ and $\nu = O(1)$. As K is bounded, the periodicity condition (20) implies $c \sim 2n^2 K^2(1/2) \to \infty$, and we find

$$k^2 \to \tfrac{1}{2}\,, \quad k'^2 \to \tfrac{1}{2}\,, \quad M \sim 2^{-\frac{1}{4}}\sqrt{c}\,, \quad r \sim 2^{\frac{1}{4}}\sqrt{c}\,. \tag{23}$$

The limiting behavior of the radius of analyticity is therefore

$$\rho = \frac{K(k')}{r} \sim \frac{K(\frac{1}{2})}{\sqrt{c}}\,. \tag{24}$$

Case 3 (Asymptotically linear regime). Let $\lambda \to \infty$ while keeping $c = (1)$ and $\nu = O(1)$. For simplicity, we take $c = \nu = 1$, so that $\lambda \sim 4n^2 K^2(0) \to \infty$. In this case we find

$$k^2 \sim \frac{1}{2\lambda^2}\,, \quad k' \to 1\,, \quad M \sim \frac{1}{\sqrt{\lambda}}\,, \quad r \sim \sqrt{\lambda}\,. \tag{25}$$

It is known that

$$\lim_{k' \to 1} \bigl(K(k') - \ln \tfrac{4}{k}\bigr) = 0, \tag{26}$$

so that

$$K(k') \sim -\ln k \sim \ln \lambda\,, \tag{27}$$

whence the radius of analyticity scales

$$\rho = \frac{K(k')}{r} \sim \frac{\ln \lambda}{\sqrt{\lambda}}\,. \tag{28}$$

4. Sobolev Norms of the Explicit Solutions

In Section 5 we will derive an estimate of the radius of analyticity in terms of, for example, the H^1 norm of the solution u. In order to compare the exact radius of analyticity with the result of the estimation, we must first compute or estimate the H^1 norm. When the elliptic modulus k is held constant, as in Case 1, a simple scaling argument is sufficient: Both (9) and (17) are of the form $u = M\, f(rx, k)$, i.e., $u_x = rM f_x(rx, k)$, so that

$$\int_0^1 u_x^2 \,\mathrm{d}x = rM \int_0^1 f_x^2(rx, k) \,\mathrm{d}x \leq \mathrm{const} \cdot rM\,. \tag{29}$$

For Cases 2 and 3 we must take the explicit dependence of the Jacobi elliptic functions on k into account. This is most easily accomplished by an exact computation. Note that u satisfies the integrated version of our model, equation (3).

Hence,

$$\nu \int_0^1 u_x^2 \, \mathrm{d}x = c^2 - \lambda \int_0^1 u^2 \, \mathrm{d}x - \tfrac{1}{2} \int_0^1 u^4 \, \mathrm{d}x \,. \tag{30}$$

We find that [1, Section 318]

$$\begin{aligned} \int_0^1 u^2 \, \mathrm{d}x &= M^2 \int_0^1 \mathrm{sd}^2(rx) \, \mathrm{d}x \\ &= \frac{M^2}{r} \frac{1}{k^2 {k'}^2} \bigl(E(x) - {k'}^2 x - k^2 \, \mathrm{sn}\, x \, \mathrm{cd}\, x \bigr) \Big|_0^{2nK} \\ &= \frac{M^2}{2nK} \frac{1}{k^2 {k'}^2} \bigl(2nE - {k'}^2 \, 2nK \bigr) \\ &= M^2 \, \frac{E - {k'}^2 K}{k^2 {k'}^2 K} \,; \end{aligned} \tag{31}$$

$E = E(k)$ is the complete elliptic integral of the second kind. Similarly,

$$\begin{aligned} \int_0^1 u^4 \, \mathrm{d}x &= M^4 \int_0^1 \mathrm{sd}^4(rx) \, \mathrm{d}x \\ &= \frac{M^4}{r} \frac{1}{3k^4 {k'}^4} \bigl(2(2k^2 - 1) \, E(x) - {k'}^2 (2 - 3k^2) \, x \\ &\qquad - k^2 \, \mathrm{sn}\, x \, \mathrm{cd}\, x \, ({k'}^2 \, \mathrm{nd}^2 \, x + 4k^2 - 2) \bigr) \Big|_0^{2nK} \\ &= \frac{M^4}{2nK} \frac{1}{3k^4 {k'}^4} \bigl(2(2k^2 - 1) \, 2nE + {k'}^2 (2 - 3k^2) \, 2nK \bigr) \\ &= M^4 \, \frac{(E - {k'}^2 K)(3k^2 - 2) + k^2 E}{3k^4 {k'}^4 K} \,. \end{aligned} \tag{32}$$

Let us now consider the limiting behavior of these quantities. When $k^2 = \frac{1}{2}$, the Legendre relation [1, equation 110.10] yields

$$2EK - K^2 = \tfrac{\pi}{2} \,. \tag{33}$$

Therefore, when $c \to \infty$ and $k^2 \to \frac{1}{2}$,

$$\int_0^1 u^2 \, \mathrm{d}x \sim c \, \frac{\pi}{\sqrt{2} \, K^2(1/\sqrt{2})} \,, \qquad \int_0^1 u^4 \, \mathrm{d}x \sim \frac{2}{3} \, c^2 \,, \tag{34}$$

and therefore

$$\int_0^1 u_x^2 \, \mathrm{d}x \sim \frac{1}{\nu} \biggl(c^2 - \frac{1}{3} \, c^2 - \lambda c \, \frac{\pi}{\sqrt{2} \, K^2(1/\sqrt{2})} \biggr) \sim \frac{2}{3} \, \frac{c^2}{\nu} \,. \tag{35}$$

Similarly, when considering the limit $\lambda \to \infty$ while $c = 1$ we can use the power series expansions of E and K [1] to find that

$$\int_0^1 u^2 \, \mathrm{d}x \sim \frac{1}{2\lambda} \,, \qquad \int_0^1 u^4 \, \mathrm{d}x \sim \frac{2}{3} \,, \tag{36}$$

and therefore

$$\int_0^1 u_x^2 \, \mathrm{d}x \sim \frac{1}{\nu} \biggl(1 - \frac{1}{3} - \lambda \, \frac{1}{2\lambda} \biggr) = \frac{1}{6\nu} \,. \tag{37}$$

5. Estimates in the Usual Gevrey Norm

Following the notation of [8], we set $A = \sqrt{-\partial_{xx}}$ and introduce the "Gevrey norm" $\|A^s \mathrm{e}^{\tau A} \cdot \| + \| \cdot \|$ where s and τ are non-negative real numbers. When $u \in L^2$ and $\|A^s \mathrm{e}^{\tau A} u\|$ is finite for some $\tau > 0$ and $s \geq 0$, then u is real analytic and its radius of analyticity is greater or equal to τ.

In the context of parabolic partial differential equations, τ is usually identified with physical time [3]. Here, we treat τ as an (artificial) time variable, which we let evolve from $\tau = 0$ to some maximal τ^* at which the estimate blows up. This blow-up "time" τ^* will be our lower bound on the radius of analyticity.

We obtain by direct calculation, the Cauchy–Schwarz inequality, and explicit insertion of equation (1), that

$$\begin{aligned} \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}\tau} \|A^s \mathrm{e}^{\tau A} u\|^2 &= \|A^{s+\frac{1}{2}} \mathrm{e}^{\tau A} u\|^2 \\ &\leq \|A^s \mathrm{e}^{\tau A} u\|^{\frac{3}{2}} \, \|A^{s+2} \mathrm{e}^{\tau A} u\|^{\frac{1}{2}} \\ &\leq \|A^s \mathrm{e}^{\tau A} u\|^{\frac{3}{2}} \left(\frac{\lambda}{\nu} \, \|A^s \mathrm{e}^{\tau A} u\| + \frac{1}{\nu} \, \|A^s \mathrm{e}^{\tau A} u^3\| \right)^{\frac{1}{2}} . \end{aligned} \tag{38}$$

In dimension one, $\mathcal{D}(A^s \mathrm{e}^{\tau A})$ is a topological algebra when $s > 1/2$ [2]. By using the concavity of the square root function, we obtain a differential inequality of Riccati type,

$$\sqrt{\nu} \, \frac{\mathrm{d}}{\mathrm{d}\tau} \|A^s \mathrm{e}^{\tau A} u\| \leq \sqrt{\lambda} \, \|A^s \mathrm{e}^{\tau A} u\| + C \, \|A^s \mathrm{e}^{\tau A} u\|^2 \,, \tag{39}$$

where C is some positive constant. Integration in τ yields

$$\|A^s \mathrm{e}^{\tau A} u\| \leq \frac{\sqrt{\lambda}}{C} \, \frac{\exp(\tau \sqrt{\lambda/\nu})}{\dfrac{\sqrt{\lambda}}{C \, \|A^s u\|} + 1 - \exp(\tau \sqrt{\lambda/\nu})} \,. \tag{40}$$

The blow-up "time" of this bound is

$$\tau^* = \sqrt{\frac{\nu}{\lambda}} \, \ln\!\left(\frac{\sqrt{\lambda}}{C \, \|A^s u\|} + 1 \right) . \tag{41}$$

When $\nu \to 0$ or $c \to \infty$ (Case 1 or Case 2), $\|A^s u\| \to \infty$, and we can expand the logarithm about 1 to obtain

$$\tau^* \sim \frac{\sqrt{\nu}}{C \, \|A^s u\|} = \frac{\sqrt{\nu}}{O(M r^s)} \gtrsim \text{const} \cdot \begin{cases} c^{-\frac{1+s}{2}} & \text{for } \nu = 1 \text{ and } c \to \infty \\ \nu^{\frac{1}{2}+\frac{s}{2}} & \text{for } c \text{ fixed and } \nu \to 0 \,. \end{cases} \tag{42}$$

Note that for $s = 0$ and $s = 1$, the scaling $\|A^s u\| = O(M r^s)$ has been proved in Section 4; for $s \in [0, 1]$, it can be obtained by interpolation between these two cases. This relation also holds for $s > 1$, but we do not need it here.

If (42) were true with $s = 0$, we would have recovered the optimal scaling which was derived explicitly in Section 3. The algebra structure of $\mathcal{D}(A^s \mathrm{e}^{\tau A})$, however, restricts us to $s > 1/2$—the estimate is not sharp.

A similar phenomenon can be seen in Case 3 when $c = \nu = 1$ and $\lambda \to \infty$: Now the first term in the argument of the logarithm in (41) is dominant, so that

$$\tau^* \sim \frac{1}{\sqrt{\lambda}} \ln \frac{\sqrt{\lambda}}{\|A^s u\|} \sim \frac{1}{\sqrt{\lambda}} \ln \frac{\lambda}{\lambda^{s/2}} \,. \tag{43}$$

Again, we see a lack of sharpness due to the restriction $s > 1/2$.

This observed loss of sharpness can be traced back to the embedding

$$\|u\|_{\ell^1} \leq C(s) \, \|u\|_{H^s} \,, \tag{44}$$

where $\|\cdot\|_{\ell^1}$ denotes the ℓ^1 norm of the Fourier transform of u, see (46) below, which is used in the proof of the algebra inequality for $\mathcal{D}(A^s \mathrm{e}^{\tau A})$ [2]. This inequality becomes increasingly unsaturated—it has a large gap between its left and right hand sides—when u is dominated by contributions from high wavenumbers. (This can be illustrated, for example, by simply setting $u(x) = \exp(\mathrm{i}kx)$, whence the left side of the inequality remains constant while the right side is increasing with k.) This behavior is already present in the usual estimate for the Sobolev algebra H^s,

$$\|uv\|_{H^s} \leq C(s) \, \|u\|_{H^s} \, \|v\|_{H^s} \tag{45}$$

for $s > d/2$, where d is the dimension of the domain.

We therefore seek a replacement of the usual Gevrey norms with a norm that can still be used to characterize the real analytic functions, and, moreover, has an algebra structure which can saturate independent of the dominant wavenumbers. Thus, a generalization of the L^∞ algebra would be a natural choice, or—since it is easily handled by using Fourier methods—the ℓ^1 algebra of the Fourier transform. One would thereby avoid estimates like (44).

6. Generalized Gevrey Norms

We now define the new family of Gevrey type norms. The definition and basic properties are stated for functions defined on the d-dimensional torus $\mathbb{T}^d$, as this does not lead to additional complications. We denote the Fourier transform of $u\colon \mathbb{T}^d \to \mathbb{R}$ by $\hat{u}_k$, and write

$$\|u\|_{\ell^p} = \left(\sum_{k \in 2\pi\mathbb{Z}^d} |\hat{u}_k|^p \right)^{\frac{1}{p}} \tag{46}$$

to denote the ℓ^p norm of the Fourier transform. The generalized Gevrey norms are then defined by

$$\|u\|_{G_\tau(\ell^p)} = \left(\sum_{k \in 2\pi\mathbb{Z}^d} \mathrm{e}^{\tau |k|} \, |\hat{u}_k|^p \right)^{\frac{1}{p}} = \left(\sum_{n=0}^{\infty} \frac{\tau^n}{n!} \|A^{\frac{n}{p}} u\|_{\ell^p}^p \right)^{\frac{1}{p}} , \tag{47}$$

where $\tau > 0$ is to be chosen later, and we set

$$G_\tau(\ell^p) \equiv \{ u \in L^1(\mathbb{T}^d) \colon \|u\|_{G_\tau(\ell^p)} < \infty \} \,. \tag{48}$$

In these spaces we can prove estimates for products of functions which closely resemble the usual Young convolution inequality for sequences in ℓ^p.

Lemma 1. *Let $u \in G_\tau(\ell^q)$ and $v \in G_\tau(\ell^r)$ where $r, q \in [1, \infty)$. Then $uv \in G_\tau(\ell^p)$ with*

$$1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}\,, \tag{49}$$

and

$$\|uv\|_{G_\tau(\ell^p)} \le \|u\|_{G_\tau(\ell^q)} \, \|v\|_{G_\tau(\ell^r)} \,. \tag{50}$$

In particular, $G_\tau(\ell^1)$ is a Banach algebra.

Remark 2. The case $p = q = 2$ and $r = 1$ is the inequality for products in the usual Gevrey norms. In particular, it can be used to show that $\mathcal{D}(A^r e^{\tau A})$ is a topological algebra when $r > d/2$ [2].

Remark 3. In the following we will only use the case $p = d = 1$. However, we present the basic estimate for the general case because it shows how the usual Gevrey norm arises as a special case of (47), and because we believe that choosing p different from 1 and 2 will be useful in the study of other equations. In this context, it is interesting to note that one can prove a set of interpolation inequalities which formally resemble the usual Sobolev embeddings, but with the L^p norm replaced by the $G_\tau(\ell^p)$ norm.

To prove Lemma 1, we need the following auxiliary estimate.

Lemma 2. *Let p, q, and r be as in Lemma 1. Then, for all j, $k \in \mathbb{R}^d$,*

$$|k| \le \frac{|j|}{q} + \frac{|k-j|}{r} + \frac{p-1}{p}\,|k|\,. \tag{51}$$

Proof. Without loss of generality, assume that $r \ge q \ge 1$. Then

$$\left(r\,\frac{1-q}{q} + 1 \right) |k| \le |k| \le |j| + |k-j| \le \frac{r}{q}\,|j| + |k-j|\,. \tag{52}$$

Dividing through by r and using (49), we obtain

$$\frac{|k|}{p} \le \frac{|j|}{q} + \frac{|k-j|}{r}\,. \tag{53}$$

This directly implies (51). □

Proof of Lemma 1. Set $\hat{w}_k = \sum_j \hat{u}_j \hat{v}_{k-j}$ and use the triangle inequality as well as Lemma 2 to obtain

$$\begin{aligned} \|uv\|^p_{G_\tau(\ell^p)} &= \sum_k e^{\tau|k|}\,|\hat{w}_k|^p \\ &\le \sum_k e^{\tau|k|}\,|\hat{w}_k|^{p-1} \sum_j |\hat{u}_j|\,|\hat{v}_{k-j}| \\ &\le \sum_{k,j} e^{\tau|j|/q}\,|\hat{u}_j|\,e^{\tau|k-j|/r}\,|\hat{v}_{k-j}|\,e^{\tau(p-1)|k|/p}\,|\hat{w}_k|^{p-1}\,. \end{aligned} \tag{54}$$

We now use the Young convolution inequality for sequences $a = \{a_k\}$, $b = \{b_k\}$, and $c = \{c_k\}$,

$$\left| \sum_k (a * b)_k \, c_k \right| \le \|a\|_{\ell^q} \, \|b\|_{\ell^r} \, \|c\|_{\ell^s}\,, \tag{55}$$

where

$$\frac{1}{q} + \frac{1}{r} + \frac{1}{s} = 2\,. \tag{56}$$

Due to condition (49) we can set $s = p/(p-1)$, so that

$$\|uv\|^p_{G_\tau(\ell^p)} \leq \left(\sum_k \mathrm{e}^{\tau|k|}\, |\hat{u}_k|^q \right)^{\frac{1}{q}} \left(\sum_k \mathrm{e}^{\tau|k|}\, |\hat{v}_k|^r \right)^{\frac{1}{r}} \left(\sum_k \mathrm{e}^{\tau|k|}\, |\hat{w}_k|^p \right)^{\frac{p-1}{p}} . \tag{57}$$

Division by $\|uv\|^{p-1}_{G_\tau(\ell^p)}$ completes the proof. □

We still have to prove that our generalized Gevrey norms indeed characterize the real analytic functions. This is done in two steps. We first prove a growth condition for derivatives of real analytic functions (see, e.g., John [7]).

Lemma 3. *A function $u \in C^\infty(\mathbb{T}^d)$ is real analytic with uniform radius of analyticity ρ, if for every $0 < \theta < \rho\sqrt{d}$ there exists an $M = M(\theta, u)$ such that for every $n \in \mathbb{N}$,*

$$\|A^n u\|_{\ell^p} \leq M\, \frac{n!}{\theta^n}\,. \tag{58}$$

Vice versa, if u is real analytic with uniform radius of analyticity ρ, then an estimate of the form (58) holds for every $\theta \in (0, \rho)$.

Proof. We first consider the case when $p = 2$. To prove that w is analytic, we show that its Taylor series converges uniformly. Using the multi-index notation in which for $\alpha \in \mathbb{N}^d$,

$$|\alpha| \equiv \sum_{i=1}^d \alpha_i\,, \qquad \alpha! \equiv \prod_{i=1}^d \alpha_i!\,, \qquad \xi^\alpha \equiv \prod_{i=1}^d \xi_i^{\alpha_i}\,, \tag{59}$$

we estimate

$$|u(x+\xi)| = \left| \sum_\alpha \frac{\partial^\alpha u(x)}{\alpha!}\, \xi^\alpha \right| \leq \text{const} \cdot \sum_\alpha \frac{\|\partial^\alpha A^m u\|_{\ell^2}}{\alpha!}\, \sigma^{|\alpha|} + |u(x)| \tag{60}$$

for $|\xi_i| \leq \sigma < \rho$ and m large enough. With the Hölder inequality, the polynomial identity, and estimate (58) we find that

$$\begin{aligned} \sum_\alpha \frac{\|\partial^\alpha A^m u\|_{\ell^2}}{\alpha!}\, \sigma^{|\alpha|} &\leq \sum_{n=0}^\infty \sigma^n \left(\sum_{|\alpha|=n} \frac{1}{\alpha!} \right)^{\frac{1}{2}} \left(\sum_{|\alpha|=n} \frac{\|\partial^\alpha A^m u\|^2_{\ell^2}}{\alpha!} \right)^{\frac{1}{2}} \\ &= \sum_{n=0}^\infty \sigma^n \left(\frac{d^n}{n!} \right)^{\frac{1}{2}} \left(\frac{1}{n!}\, \|A^{m+n} u\|^2_{\ell^2} \right)^{\frac{1}{2}} \\ &\leq \frac{M}{\theta^m} \sum_{n=0}^\infty \sigma^n \frac{(n+m)!}{n!} \left(\frac{\sigma\sqrt{d}}{\theta} \right)^n . \end{aligned} \tag{61}$$

This last series—and hence the Taylor series in (60)—converges for $\theta > \sigma\sqrt{d}$.

Vice versa, if u is analytic, we can estimate its derivatives with the Cauchy integral formula and obtain for $0 < \sigma < \rho$,

$$\begin{aligned} \|A^n u\|_{\ell^2}^2 &= \sum_{|\alpha|=n} \frac{n!}{\alpha!} \, \|\partial^\alpha u\|_{\ell^2}^2 \\ &\le \sum_{|\alpha|=n} \frac{n!}{\alpha!} \biggl(M \, \frac{\alpha!}{\sigma^{|\alpha|}} \biggr)^2 \\ &= \biggl(M \, \frac{n!}{\sigma^n} \biggr)^2 \sum_{|\alpha|=n} \frac{\alpha!}{n!} \, . \end{aligned} \tag{62}$$

The last sum grows at most polynomially in n—it does not change the asymptotics of the estimate.

The case when $p \neq 2$ can be derived from the above by noting that $\|u\|_{\ell^q} \le \|u\|_{\ell^p}$ for $q \ge p$, or $\|u\|_{\ell^q} \le c(q,p,d) \, \|(1+A^m)u\|_{\ell^p}$ for $q < p$ and $m > d(p-q)/(pq)$, in which case a translation of the index in (58) by m is required. □

The main characterization result is the following.

Theorem 4. *A function $u \in C^\infty(\mathbb{T}^d)$ is real analytic with uniform radius of analyticity ρ, if $u \in G_\tau(\ell^p)$ for some fixed $p \ge 1$ and every $\tau < \rho p \sqrt{d}$. Vice versa, if u is real analytic with uniform radius of analyticity ρ, it is contained in every $G_\tau(\ell^p)$ with $\tau < \rho p$.*

Remark 4. The factor $\sqrt{d}$ which occurs in the statement of Lemma 3 and Theorem 4 is the constant of equivalency between the Euclidean and the L^1 norm in $\mathbb{R}^d$. It arises from defining $A^2 = -\Delta$, whereby we implicitly use the Euclidean norm in Fourier space. It could be avoided by taking $(Au)\hat{} \equiv (|k_1| + \cdots + |k_d|)\hat{u}$. In this case we would not need to use the Hölder inequality in (61), and both Lemma 3 and Theorem 4 would turn into "if and only if" statements.

Proof. The proof involves the same steps as the corresponding proof for the usual Gevrey norms (see [8] for a review of this case).

We first assume that $u \in G_\tau(\ell^p)$ and estimate

$$\begin{aligned} \|A^n u\|_{\ell^p}^p &= \sum_k |k|^{np} \, |\hat{u}_k|^p \\ &= \biggl(\frac{n!}{(\tau/p)^n} \biggr)^p \sum_k \biggl(\frac{(\tau/p)^n}{n!} \, |k|^n \biggr)^p |\hat{u}_k|^p \\ &\le \biggl(\frac{n!}{(\tau/p)^n} \biggr)^p \sum_k \mathrm{e}^{\tau|k|} \, |\hat{u}_k|^p \\ &= \biggl(\frac{n!}{(\tau/p)^n} \biggr)^p \|u\|_{G_\tau(\ell^p)}^p \, . \end{aligned} \tag{63}$$

Thus, u satisfies an estimate of the form (58) with $\sigma = \tau/p$, and, by Lemma 3, u is real analytic with uniform radius of analyticity ρ.

If, on the other hand, u satisfies (58),

$$\begin{aligned}\|u\|^p_{G_\tau(\ell^p)} &= \sum_{n=0}^\infty \frac{\tau^n}{n!} \, \|A^{n/p} u\|^p_{\ell^p} \\ &\le \sum_{n=0}^\infty \frac{\tau^n}{n!} \, \|A^{[n/p]+1} u\|^p_{\ell^p} \\ &\le M^p \sum_{n=0}^\infty \frac{\tau^n}{n!} \biggl(\frac{\bigl([\frac{n}{p}]+1\bigr)!}{\sigma^{[n/p]+1}} \biggr)^p \,, \end{aligned} \tag{64}$$

where $[\,\cdot\,]$ denotes the "greatest integer less or equal to" function. We apply the Hadamard root test, using Stirling's formula to evaluate the limits of all factorials, to find that the last series in (64) converges for $\tau < \sigma p$, whence $u \in G_\tau(\ell^p)$. $\square$

7. Estimates in the Generalized Gevrey Norm

It is easily checked that the generalized Gevrey norms satisfy

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \|w\|^p_{G_\tau(\ell^p)} = \|A^{\frac{1}{p}} w\|^p_{G_\tau(\ell^p)} \tag{65}$$

for every function w that is sufficiently regular, and every $1 \le p < \infty$. By twice applying this identity, we obtain a second order differential inequality for the solution u of (1) in $G_\tau(\ell^1)$, the algebra case of the norms introduced in the previous section,

$$\frac{\mathrm{d}^2}{\mathrm{d}\tau^2} \|u\|_{G_\tau(\ell^1)} = \|A^2 u\|_{G_\tau(\ell^1)} \le \frac{\lambda}{\nu} \, \|u\|_{G_\tau(\ell^1)} + \frac{1}{\nu} \, \|u\|^3_{G_\tau(\ell^1)} \,. \tag{66}$$

This estimate is formal, but can be made rigorous, for example, by considering the limit of a Galerkin approximating sequence.

The structure of the differential inequality is very similar to the original problem, and can be solved in the same way. We set $y(\tau) = \|u\|_{G_\tau(\ell^1)}$, multiply (66) by $y_\tau \equiv \mathrm{d}y/\mathrm{d}\tau$ and integrate:

$$\nu \, y_\tau^2 \le \gamma^2 + \lambda \, y^2 + \tfrac{1}{2} \, y^4 \,, \tag{67}$$

where $\gamma^2 = \nu \, y_\tau^2(0) - \lambda \, y^2(0) - \frac{1}{2} \, y^4(0)$.

In the limits $\nu \to 0$ or $c \to \infty$, equation (67) has the factorization

$$\sqrt{\nu} \, y_\tau \le \sqrt{\tfrac{1}{2} (y^2 + \rho^2)(y^2 + \overline{\rho}^2)} \,, \tag{68}$$

with

$$\rho^2 = \lambda + \mathrm{i} \sqrt{2\gamma^2 - \lambda^2} \,. \tag{69}$$

We have chosen the positive square root in (68) since y must be increasing. Inequality (68) can now be integrated, and we obtain

$$\begin{aligned}\tau^* &\ge \sqrt{2\nu} \int_{y(0)}^\infty \frac{\mathrm{d}z}{\sqrt{(z^2+\rho^2)(z^2+\overline{\rho}^2)}} \\ &= \sqrt{2\nu} \, g \; \mathrm{cn}^{-1}(\cos\phi, \kappa) \,, \end{aligned} \tag{70}$$

where

$$g = \frac{1}{2\sqrt{\rho\overline{\rho}}} = \frac{1}{2^{3/4}\sqrt{\gamma}}\,,$$
$$\cos\phi = \frac{y^2(0)-\rho\overline{\rho}}{y^2(0)+\rho\overline{\rho}}\,,$$
$$\kappa^2 = \frac{-(\rho-\overline{\rho})^2}{4\rho\overline{\rho}}\,. \tag{71}$$

We must now determine the scaling of

$$\gamma^2 = \nu\, y_\tau^2(0) - \lambda\, y^2(0) - \tfrac{1}{2}\, y^4(0) = \nu\, \|u_x\|_{\ell^1}^2 - \lambda\, \|u\|_{\ell^1}^2 - \tfrac{1}{2}\, \|u\|_{\ell^1}^4\,. \tag{72}$$

As $\nu \to 0$, γ is asymptotically independent of ν, so that the primary ν-dependence in (70) is due to the prefactor $\sqrt{\nu}$. Thus, we immediately obtain the scaling of the exact solution as given in (22).

In the limit $c \to \infty$ with u is given by (17) matters are more complicated. We set $\nu = 1$ for simplicity, and use the explicit Fourier series of the sd function which yields

$$u(x) = M\,\mathrm{sd}(rx) = \frac{2\pi M}{kk'K}\sum_{m=0}^{\infty}(-1)^m\,\frac{q^{m+\frac{1}{2}}}{1+q^{2m+1}}\,\sin\biggl((2m+1)\frac{\pi\, rx}{2K}\biggr)\,, \tag{73}$$

where $q = \exp(-\pi K'/K)$ is the so-called nome [1]. Thus, we find

$$\|u\|_{\ell^1} = \frac{2\pi M}{kk'K}\sum_{m=0}^{\infty}\frac{q^{m+\frac{1}{2}}}{1+q^{2m+1}}\,, \tag{74}$$

$$\|u_x\|_{\ell^1} = \frac{2\pi M}{kk'K}\,\frac{\pi r}{2K}\sum_{m=0}^{\infty}(2m+1)\,\frac{q^{m+\frac{1}{2}}}{1+q^{2m+1}}\,. \tag{75}$$

We first check that indeed $\gamma \to \infty$ in order to justify the factorization (68). This argument is elementary and involves estimating the sums in (74) and (75) in terms of a geometric series; we omit all details. Most importantly, however, (75) implies that

$$\gamma \le \|u_x\|_{\ell^1} \sim \mathrm{const}\cdot Mr\,. \tag{76}$$

The elliptic modulus κ of cn^{-1} in (70) approaches $\mathrm{i}/\sqrt{2}$ as $\gamma \to \infty$, while the argument of cn^{-1} approaches 1. By using the imaginary modulus transformation

$$\mathrm{cn}^{-1}(y,\mathrm{i}k) = \frac{1}{\sqrt{1+k^2}}\;\mathrm{cd}^{-1}\biggl(y, \frac{k}{\sqrt{1+k^2}}\biggr)\,, \tag{77}$$

we see that $\mathrm{cn}^{-1}(\cos\phi,\kappa)$ must also approach a constant and does not contribute to the asymptotic scaling. Thus, we find

$$\tau^* \ge \frac{1}{2^{1/4}\sqrt{\gamma}} \gtrsim \frac{\mathrm{const}}{Mr} \ge \frac{\mathrm{const}}{\sqrt{c}}\,. \tag{78}$$

This also reproduces the scaling of the exact solution: Our new estimates are sharp.

Let us now consider the asymptotically linear case where $c = \nu = 1$ and $\lambda \to \infty$. As can be seen from (74) and (75), $\gamma^2 = O(1) \ll \lambda$, so that the factorization of (67) now takes the form

$$y_\tau \le \sqrt{\tfrac{1}{2}(y^2+\rho^2)(y^2+\sigma^2)}\,, \tag{79}$$

where

$$\rho^2 = \lambda + \sqrt{\lambda^2 - 2\gamma^2}\,, \qquad \sigma^2 = \lambda - \sqrt{\lambda^2 - 2\gamma^2}\,. \tag{80}$$

Integration yields

$$\begin{aligned}
\tau^* &\geq \sqrt{2} \int_{y(0)}^{\infty} \frac{\mathrm{d}z}{\sqrt{(z^2+\rho^2)(z^2+\sigma^2)}} \\
&= \frac{\sqrt{2}}{\rho} \operatorname{tn}^{-1} \left(\frac{\rho}{y(0)}, \kappa \right) \\
&= \frac{\sqrt{2}}{\rho} F(\sin^{-1} \sqrt{\frac{\rho^2}{y^2(0)+\rho^2}}, \kappa)\,,
\end{aligned} \tag{81}$$

where F is the normal elliptic integral of the first kind, and

$$\kappa^2 = \frac{2\sqrt{\lambda^2 - 2\gamma^2}}{\lambda + \sqrt{\lambda^2 - 2\gamma^2}}\,. \tag{82}$$

When $\lambda \to \infty$, $\rho \sim \sqrt{2\lambda}$ so that the argument of $\sin^{-1}$ converges to 1. Thus, in this limit,

$$\tau^* \gtrsim \frac{F(\frac{\pi}{2}, \kappa)}{\sqrt{\lambda}} = \frac{K(\kappa)}{\sqrt{\lambda}} \sim \frac{\ln \lambda}{\sqrt{\lambda}}\,. \tag{83}$$

The last step follows from (26) and the fact that

$$\kappa'^2 = 1 - \kappa^2 = \tfrac{1}{2}\, \gamma^2\, \lambda^{-2} + O(\lambda^{-4})\,. \tag{84}$$

We observe that (83) is identical to the exact scaling of the radius of analyticity in the asymptotically linear regime, and improves upon the estimate obtained by the standard Gevrey norm technique.

8. Discussion

We have defined a family of spaces $G_\tau(\ell^p)$ whose norms can be used to estimate the radius of analyticity for solutions of nonlinear partial differential equations on periodic domains. The new norms have the following advantages: The space $G_\tau(\ell^1)$ is a Banach algebra independent of the size of the domain or of possible restrictions to high wavenumbers. In other words, we always have an algebra inequality with constant one. Further, we can now find estimates to bootstrap from $\hat{u} \in \ell^1$ to $u \in G_\tau(\ell^1)$ rather than, as was previously necessary, from some $u \in H^s$ to $u \in \mathcal{D}(A^s \mathrm{e}^{\tau A})$. In ℓ^1 on the Fourier transform we can find initial "bottom rung" estimates which are independent of the wavenumber of the solution, thereby leading to sharper estimates in the limit when the solution is dominated by high wavenumbers.

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