UNIQUENESS OF SOLUTIONS FOR WEAKLY DEGENERATE CORDIAL VOLTERRA INTEGRAL EQUATIONS

ZYMANTAS DARBENAS AND MARCEL OLIVER

ABSTRACT. We study the question of uniqueness of solutions to cordial Volterra integral equations in the sense of Vainikko (Numer. Funct. Anal. Optim. 30, 2009, pp. 1145–1172) in the case where the kernel (or core) function $\mathcal{K}(\theta) \equiv \mathcal{K}(y/x)$ vanishes on the diagonal $x = y$. When, in addition, $\mathcal{K}$ is sufficiently regular, is strictly positive on $(0,1)$, and $\theta^{-k}\,\mathcal{K}'(\theta)$ is non-increasing for some $k \in \mathbb{R}$, we prove that the solution to the corresponding Volterra integral equation of the first kind is unique in the class of functions which are continuous on the positive real axis and locally integrable at the origin. Alternatively, we obtain uniqueness in the class of locally integrable functions with locally integrable mean. We further discuss a uniqueness-of-continuation problem where the conditions on the kernel need only be satisfied in some neighborhood of the diagonal. We give examples illustrating the necessity of the conditions on the kernel and on the uniqueness class, and sketch the application of the theory in the context of a nonlinear model.

1. Introduction. In this paper, we study the question of uniqueness of solutions to *weakly degenerate cordial Volterra integral equations of the first kind*, abbreviated as WDCVIE, which are integral equations of the form

$$\frac{1}{x}\int_0^x \mathcal{K}\Big(\frac{y}{x}\Big)\, f(y)\,\mathrm{d}y = g(x)\,, \tag{1}$$

where the kernel (or core) function $\mathcal{K}\colon [0,1] \to \mathbb{R}_+$ may be *weakly degenerate* in the sense that $\mathcal{K}(\theta) \sim c\,(1-\theta)^\alpha$ near $\theta = 1$ with constants $c > 0$ and $\alpha \in (0,1)$. Setting $y = x\theta$, we may write (1) in the equivalent

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form

$$\int_0^1 \mathcal{K}(\theta)\, f(x\theta)\, \mathrm{d}\theta = g(x)\,. \tag{2}$$

Formally, (1) is a Volterra integral equation of the first kind. Such equations take the general form

$$\int_0^x \mathcal{H}(x,y)\, f(y)\, \mathrm{d}y = g(x)\,. \tag{3}$$

The classical theory proceeds by differentiating with respect to x. Assuming that g and $\mathcal{H}$ are continuously differentiable and $\mathcal{H}$ is non-degenerate on the diagonal, i.e., $\mathcal{H}(y,y) \neq 0$ for all $y > 0$, equation (3) can be converted into a Volterra integral equation of the second kind. Then, contraction mapping arguments yield existence and uniqueness of a solution in the class of continuous functions; see, e.g., [**1,** Theorem 1.4.1]. In our setting, however, $\mathcal{H}(x,y) = x^{-1}\,\mathcal{K}(y/x)$, so $\mathcal{H}(y,y) = y^{-1}\,\mathcal{K}(1) = 0$. Hence, the equation is *degenerate* and the classical strategy of proof fails.

Vainikko [**7, 8, 9**] substantially generalized the class of admissible kernels, coining the term *cordial Volterra integral equation.* His setting is the following. Let $I = (0,b)$ for some $b \in \mathbb{R}_+$ and let D_I denote the triangular domain

$$D_I = \{(y,s) \colon y \in \overline{I}, s \in [0,y]\}\,. \tag{4}$$

For an integrable function $\mathcal{K} \in L^1([0,1])$ and a continuous function $\mathcal{G} \in C(D_I)$, the left-hand side of (3) is called a *cordial Volterra integral operator* if

$$\mathcal{H}(x,y) = \frac{1}{x}\,\mathcal{K}\Big(\frac{y}{x}\Big)\,\mathcal{G}(x,y)\,, \tag{5}$$

acting on an appropriate class of functions, for instance continuous functions on $\overline{I}$ or essentially bounded measurable functions on $\overline{I}$.

A good survey of what is known about cordial Volterra integral equations can be found in the recent book by Brunner [**1,** Chapter 7, Section 1.4]. The result which, to the best of our knowledge, comes closest to the situation considered here is [**1,** Theorem 7.2.15]. The theorem states that if the kernels $\mathcal{K} \in C^1((0,1))$ and $\mathcal{G} \in C(D_I)$ satisfy

(i) $\mathcal{K} \in L^1([0,1])$ and $\displaystyle\int_0^1 \theta\,(1-\theta)\,\mathcal{K}'(\theta)\,\mathrm{d}\theta < \infty$,

(ii) $\displaystyle\int_0^1 \mathcal{K}(\theta)\,\mathrm{d}\theta > 0$,

(iii) $z\,\mathcal{K}(\theta) + \theta\,\mathcal{K}'(\theta) > 0$ for $\theta \in (0,1)$ and $z < 1$,

(iv) $\partial_x \mathcal{G} \in C(D_I)$ and $\mathcal{G}(y,y) \neq 0$ for $y \in \overline{I}$,

then equation (3) has a unique solution $f \in C(\overline{I})$ for any $g \in C^1(\overline{I})$. In our situation, however, the assumed asymptotic behavior near $\theta = 1$ implies that $\mathcal{K}'(\theta) \to -\infty$ as $\theta \to 1$, so that condition (iii) cannot be satisfied for any value of z. Therefore, this result is also not applicable here.

In this paper, we focus on cordial Volterra operators with $\mathcal{G} \equiv 1$. We believe that the setting may be extended to cases where $\mathcal{G}$ is a more general function. However, as the essential difficulty arises from the weak degeneracy of $\mathcal{K}$ and our main example only requires $\mathcal{G} = 1$, we restrict to this case. Assuming that $\mathcal{K}$ is absolutely continuous on $[0,1]$, strictly positive on $(0,1)$ with $\mathcal{K}(1) = 0$, and that $\theta^{-k}\,\mathcal{K}'(\theta)$ is non-increasing for some $k \in \mathbb{R}$, we prove that the solution to the corresponding Volterra integral equation of the first kind is unique in the class of functions which are continuous on the positive real axis and locally integrable at the origin. Alternatively, we obtain uniqueness in the class of locally integrable functions with locally integrable mean. This class of functions is more general than the setting considered, e.g., in **[7, 10]** who study cordial Volterra integral operators on $C(\overline{I})$.

The class of kernels considered in this paper includes, in particular, kernels which are locally of root-type behavior near the diagonal, i.e., where $\mathcal{K}(\theta) \sim c\,(1-\theta)^\alpha$ near $\theta = 1$ with constants $c > 0$ and $\alpha \in (0,1)$. Moreover, when the positivity and concavity conditions on the kernel are only satisfied in some neighborhood of the diagonal, we can still formulate a uniqueness-of-continuation problem of the following type: Suppose $f \equiv 0$ on some initial interval $(0, x^\star) \subset I$, is it true that $f \equiv 0$ on I?

Our motivation comes from studying extended solutions to a simplified Keller–Rubinow model for the formation of Liesegang precipitation rings in the fast reaction limit **[2, 3]** where a weakly degenerate integral kernel arises naturally. Existence of solutions can be shown via weak convergence techniques following ideas from Hilhorst *et al.* **[4]**, so that the remaining nontrivial aspect is uniqueness, which we address here. This problem can be formulated naturally as a uniqueness-of-continuation question.

The question of existence of solutions in the setting considered here is subtle and remains open. Direct contraction mapping arguments appear to fail. Due to our results, this observation does not come as a surprise because solutions with a singularity at zero may in general exist, but integrability at zero is essential for selecting a unique solution. Thus, any contraction mapping argument must build a reflection of the uniqueness class into the Banach space on which the problem is posed.

To fix notation, we consider an interval $I = (0, b)$ where either $b \in \mathbb{R}_+$ or $b = \infty$. As we are only concerned with uniqueness, we consider (1) with $g \equiv 0$. Uniqueness of solutions to (1) then amounts to showing that

$$\int_0^1 \mathcal{K}(\theta)\, f(x\theta)\, \mathrm{d}\theta = 0 \tag{6}$$

for all $x \in I$ implies that $f \colon I \to \mathbb{R}$ is necessarily zero within the class of functions considered.

The paper is structured as follows. In Section 2, we introduce shorthand notation for the assumptions on the kernel and prove several auxiliary results. Section 3 contains our main result, Theorem 6, followed by several extensions to the uniqueness class and the class of kernels considered. The final Section 4 contains examples and counterexamples which show that our results are substantially sharp. This section also contains a sketch of the mathematical setup of the Liesegang ring problem which spawned this investigation.

2. Preliminaries.

We begin by introducing shorthand notation for the necessary assumptions on the kernel $\mathcal{K}$.

Definition 1 (Technical conditions)**.** *We say that a kernel $\mathcal{K}$ together with two numbers $k \in \mathbb{R}$ and $a \in [0, 1)$ satisfy technical conditions $\mathcal{TC}(\mathcal{K}, k, a)$ if*

(i) $\mathcal{K} \colon [0, 1] \to \mathbb{R}_+$ *is absolutely continuous on* $[0, 1]$ *and strictly positive on* $(a, 1)$ *with* $\mathcal{K}(1) = 0$.

(ii) *The function* $\theta^{-k}\,\mathcal{K}'(\theta)$ *is non-increasing on* $(a,1)\cap U'$, *where* U' *denotes the subset of* $[0,1]$ *on which* $\mathcal{K}'$ *exists.*

We note that U' is always a set of full measure. The technical assumptions imply the following properties which we will need later.

Lemma 2. *Assume that the triple* $(\mathcal{K},k,a)$ *with* $k\in\mathbb{R}$ *and* $a\in[0,1)$ *satisfies* $\mathcal{TC}(\mathcal{K},k,a)$. *Then there exists* $a^\star\in[a,1)$ *such that*

(i) $\mathcal{K}'<0$ *on* $(a^\star,1)\cap U'$,
(ii) $\min\limits_{[\theta_1,\theta_2]}\mathcal{K}=\min\{\mathcal{K}(\theta_1),\mathcal{K}(\theta_2)\}$ *for each pair* $a\le\theta_1<\theta_2\le 1$,
(iii) $\theta^{-k-1}\,\mathcal{K}(\theta)$ *is non-increasing on* $(a^\star,1)$.

Proof. First, we note that if $\mathcal{K}'(b)<0$ for some $b\in(a,1)\cap U'$, then, due to Definition 1(ii), $\mathcal{K}'(\theta)<0$ for all $\theta\in(b,1)\cap U'$. Second, there exists $b\in(a,1)\cap U'$ such that $\mathcal{K}'(b)<0$, for otherwise

$$\mathcal{K}(\theta)=\mathcal{K}(1)-\int_\theta^1\mathcal{K}'(\sigma)\,\mathrm{d}\sigma\le 0 \tag{7}$$

for all $\theta\in(a,1)$. Thus,

$$a^\star=\inf\{y\in U'\cap(a,1)\colon \mathcal{K}'(y)<0\} \tag{8}$$

is well-defined and satisfies claim (i). Since $\mathcal{K}'\ge 0$ on $(a,a^\star]\cap U'$ when $a^\star$ is given by (8), (ii) is a direct consequence.

To prove (iii), we consider the function $h\colon(a^\star,1)\to\mathbb{R}$,

$$h(\theta)=\ln\mathcal{K}(\theta)-(k+1)\ln\theta\,. \tag{9}$$

Clearly, h is absolutely continuous on each closed subinterval of $(a^\star,1)$. Its derivative, defined on $(a^\star,1)\cap U'$, reads

$$h'(\theta)=\frac{\mathcal{K}'(\theta)}{\mathcal{K}(\theta)}-\frac{k+1}{\theta}\,. \tag{10}$$

Given $b\in(a^\star,1)\cap U'$ we note that for all $\theta\in(b,1)\cap U'$,

$$\mathcal{K}'(\theta)\le\Big(\frac{\theta}{b}\Big)^k\,\mathcal{K}'(b)\,, \tag{11}$$

i.e., $\mathcal{K}'(\theta)$ is bounded uniformly from above by some negative constant. Hence, $h'(\theta) < 0$ a.e. in some neighborhood of $\theta = 1$, so that

$$a_k^\star \equiv \inf\{\theta \in U' \cap (a^\star, 1) \colon h'(\theta) < 0\} < 1\,. \tag{12}$$

Re-defining $a^\star \equiv a_k^\star$, we complete the proof. □

The next lemma is the key technical tool for our main result. It is saying that a sufficiently smooth function which is initially zero and which has a dense set of points where it is non-increasing in a certain global sense must remain zero everywhere. The precise statement is the following.

Lemma 3. *Let $b \in \mathbb{R}_+ \cup \{\infty\}$. Suppose g is continuously differentiable on $I = (0, b)$ and absolutely continuous on the closure of this interval with the properties*

(i) *$g(0) = 0$ and*
(ii) *for every $x_1 \in I$ and for every $x_2 \in (x_1, b)$ there exist $y_1 \in (0, x_1)$ and $y_2 \in (x_1, x_2)$ such that $|g(y_1)| \geq |g(y_2)|$.*

Then $g = 0$.

Remark 4. We note that for every $g \in C([0, b])$, condition (ii) is equivalent to the following. Let

$$S = \{x \in I \colon |g(y)| < |g(x)| \text{ for all } y \in (0, x)\} \tag{13}$$

denote the set of all points where g is larger in absolute value than at any point to the left. Then (ii) is satisfied if and only if for every $x \in S$ in any right neighborhood of x there exists a point $y \in (x, b)$ such that $|g(x)| > |g(y)|$. An example of a function which satisfies this requirement is shown in Figure 1. It is constructed as follows. Take the function $f(x) = x$ on the unit interval. Introduce a recursive equipartition of this interval into pairs of subintervals of equal length. At each new center node, inserted at step $i = 1, 2, \dots$ relative to the graph of f on $[0, 1]$, cut the graph and insert a copy of the function

$$v(x) = \begin{cases} 1 - x & \text{for } x \in [0, 1]\,, \\ x - 1 & \text{for } x \in (1, 2] \end{cases} \tag{14}$$

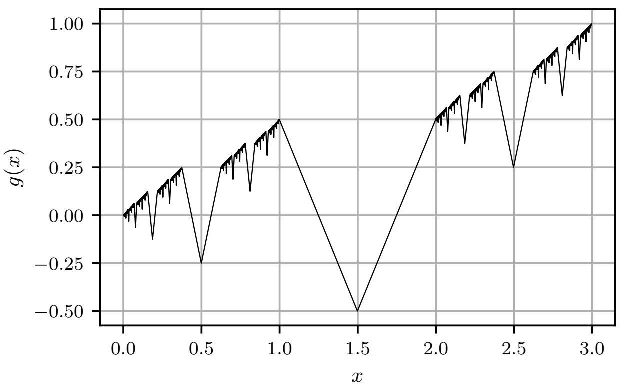

FIGURE 1. Example of a nonzero function which satisfies conditions (i) and (ii) of Lemma 3, but is insufficiently smooth. This example demonstrates that some regularity requirement is necessary.

with width scaled by 2^{1-2i} and height scaled by 2^{1-i}. This defines a function g on the interval $[0,3]$ with $g(0) = 0$ which is continuous and which clearly satisfies the equivalent reformulation of condition (ii). This example shows that continuity is insufficient to conclude that the function is zero under conditions (i) and (ii); at least some uniformity in the modulus of continuity is required as is implied by the differentiability assumption in the statement of Lemma 3.

Proof of Lemma 3. Consider the function $h\colon [0,b] \to \mathbb{R}_+$,

$$h(y) = \max_{z\in[0,y]} |g(z)|\,, \tag{15}$$

set

$$\mathcal{C} = \{y\colon h(y) = |g(y)|\}\,, \tag{16}$$

and define $\mathcal{O} = (0,b) \setminus \mathcal{C}$. Since h and g are continuous, $\mathcal{C}$ is closed and $\mathcal{O}$ is open. We will show that h is non-increasing. By construction, h is constant on each connected component of $\mathcal{O}$. Further, h is absolutely continuous. Indeed, fix $x_1, x_2 \in [0,b]$ with $x_1 < x_2 < \infty$ and select the leftmost $x_l \in \mathcal{C}$ not less than x_1 and the rightmost $x_r \in \mathcal{C}$ not bigger than x_2, if available. If x_l and x_r do not exist, then obviously $[x_1, x_2] \subset \mathcal{O}$ or $h(x_1) = h(x_2)$. Otherwise, $h(x_1) = |g(x_l)|$ and $h(x_2) = |g(x_r)|$, so that

$$h(x_2) - h(x_1) \leq |g(x_r) - g(x_l)| = \left| \int_{x_l}^{x_r} g'(y) \, \mathrm{d}y \right| \leq \int_{x_1}^{x_2} |g'(y)| \, \mathrm{d}y \,. \tag{17}$$

This proves that h is absolutely continuous on $[0, b]$.

We now look at the growth of g and h in $\mathcal{C}$. Fix $z \in \mathcal{C}$ with $z \neq 0$ and $z \neq b$. For simplicity, assume that $g(z) \geq 0$ so that

$$h(z) = g(z) = \int_0^z g'(x) \, \mathrm{d}x \,. \tag{18}$$

The argument which follows is easily adapted to the case $g(z) < 0$. Our goal is to show that $g'(z) = 0$. First, suppose that $g'(z) < 0$. Then, by continuity of g',

$$h(z) = \int_0^z g'(x) \, \mathrm{d}x < \int_0^{z-\varepsilon} g'(x) \, \mathrm{d}x = g(z - \varepsilon) \leq h(z - \varepsilon) \tag{19}$$

for $\varepsilon > 0$ sufficiently small. This is a contradiction as h is non-decreasing. Now suppose that $g'(z) > 0$. In assumption (ii), let $x_1 = z$. Invoking continuity once again, we obtain

$$h(z) = \int_0^z g'(x) \, \mathrm{d}x < \int_0^{y_2} g'(x) \, \mathrm{d}x = g(y_2) \tag{20}$$

for every $y_2 \in (z, z + \varepsilon]$ with $\varepsilon > 0$ sufficiently small. Hence,

$$|g(y_1)| \leq h(z) < g(y_2) \tag{21}$$

for all $y_1 \in [0, z]$ and $y_2 \in (z, z + \varepsilon]$. This statement contradicts assumption (ii) with $x_1 = z$ and $x_2 = z + \varepsilon$. We conclude that the only case possible is $g'(z) = 0$.

Since h is absolutely continuous, it has a weak derivative h' which, by construction, vanishes on $\mathcal{O}$ and satisfies $h' = |g'|$ a.e. on $\mathcal{C}$ (in fact, in the interior of $\mathcal{C}$, this identity is true in the classical sense as $h = |g|$

and g' cannot change sign), so that

$$\begin{aligned} h(x_2) - h(x_1) &= \int_{x_1}^{x_2} h'(x)\,\mathrm{d}x = \int_{[x_1,x_2]\cap\mathcal{C}} h'(x)\,\mathrm{d}x \\ &= \int_{[x_1,x_2]\cap\mathcal{C}} |g'(y)|\,\mathrm{d}y = 0\,. \end{aligned} \tag{22}$$

Thus, $h = 0$ on $[0, b]$ and so is g. □

Remark 5. Lemma 3 remains true under the assumption that g' is *essentially continuous* almost everywhere, i.e.,

$$\lim_{\varepsilon\searrow 0} \operatorname*{ess\,sup}_{[z-\varepsilon,z+\varepsilon]\cap[0,b]} g' = \lim_{\varepsilon\searrow 0} \operatorname*{ess\,inf}_{[z-\varepsilon,z+\varepsilon]\cap[0,b]} g' \quad \text{for a.e. } z \in (0, b)\,. \tag{23}$$

Indeed, inequality (17) is preserved under condition (23), so that h is absolutely continuous. Further, the argument from (19) to (21) remains valid on every point of continuity in sense of (23) within $\mathcal{C}$, i.e. almost everywhere. This suffices to complete the proof as before.

3. Main results. Our main uniqueness result is Theorem 6 below. The result covers two cases. The first, simpler case has $a = 0$ in the technical condition, Definition 1. This means we have good control on the behavior of the kernel $\mathcal{K}$ on its entire interval of definition. It turns out that most of the difficulty comes from the weakly degenerate local asymptotics near $\theta = 1$. This is used in the second, more subtle case when $a \in (0, 1)$ in the technical conditions. In this case, we only assume control on the kernel in some neighborhood of 1. Without global control on the kernel, all we can prove is a uniqueness-of-continuation result: provided a solution to the WDCVIE (1) on some interval $[0, x^\star]$ for $x^\star > 0$ fixed is given, we show that, under certain conditions, its extension to a larger interval is unique. In our formulation (6), this amounts to assuming that f is known to be zero on some fixed interval $[0, x^\star]$ of positive length. The key steps of the proof are the same in both cases, with few differences which are related to how the respective assumptions enter the key estimates. For this reason, we formulate theorem and proof in a way that covers both cases at once.

A key question throughout this section is the appropriate function class in which uniqueness holds. Initially, we work with functions that are continuous on their open interval of definition $(0, b)$, but may have an integrable singularity at zero. Integrability is crucial, as Example 4 in Section 4 demonstrates. We encode this local integrability condition by requiring membership in a local Lebesgue space, defined in the usual way,

$$L^1_{\mathrm{loc}}([0,b)) = \{f \text{ measurable}\colon \\ f \in L^1((\alpha,\beta)) \text{ for every } 0 \leq \alpha < \beta < b\}\,. \tag{24}$$

When looking at uniqueness-of-continuation, local integrability is trivially satisfied. Moreover, for some admissible kernels the requirement that the WDCVIE makes sense in the first place already imposes local integrability near zero; see the discussion in Remarks 7 and 13. In this situation, local integrability is not a selection criterion between different otherwise admissible solutions, but rather is a basic feature of the equation itself.

The theorem is followed by three corollaries in which we re-formulate the problem in slightly different terms, thereby extending the uniqueness class and the class of admissible kernels.

Theorem 6 (Main theorem)**.** *Assume that the triple $(\mathcal{K}, k, a)$ with $k \in \mathbb{R}$ and $a \in [0,1)$ satisfies $\mathcal{TC}(\mathcal{K}, k, a)$. Let $I = (0,b)$ for some $b \in \mathbb{R}_+ \cup \{\infty\}$. Suppose $f \in C(I) \cap L^1_{\mathrm{loc}}([0,b))$ with, moreover,*

(i) *$\mathcal{K}(0) = 0$ or $\mathcal{K}(0) > 0$ and $k \geq -1$ if $a = 0$, or*
(ii) *there exists $x^\star > 0$ such that $f(x) = 0$ for $x \in [0, x^\star]$ if $a \in (0,1)$,*

satisfies the homogeneous WDCVIE

$$\int_0^1 \mathcal{K}(\theta)\, f(x\theta)\, \mathrm{d}\theta = 0 \tag{25}$$

for all $x \in I$. Then $f \equiv 0$ on I.

Remark 7. When $a = 0$ and $\mathcal{K}(0) > 0$, the integrability requirement $f \in L^1_{\mathrm{loc}}([0,b))$ is already implied by the requirement that f satisfies the WDCVIE in the Lebesgue sense. Indeed, when $\mathcal{K}(0) > 0$, then $\mathcal{K} > 0$ on $[0,1)$ and Lemma 2(ii) implies that $\min_{[0,\theta]} \mathcal{K} = \min\{\mathcal{K}(0), \mathcal{K}(\theta)\}$ for any $\theta \in (0,1)$. So, for each pair $x \in I$ and $z > 1$ such that $xz \in I$, we have $0 < \min\{\mathcal{K}(0), \mathcal{K}(z^{-1})\}$ and

$$\min\{\mathcal{K}(0), \mathcal{K}(z^{-1})\} \int_0^x |f(y)| \,\mathrm{d}y \le \int_0^x \Big| \mathcal{K}\Big(\frac{y}{xz}\Big) f(y) \Big| \,\mathrm{d}y < \infty \,. \tag{26}$$

Thus, the WDCVIE implies local integrability near zero. In this case, local integrability is not a selection criterion to disambiguate between otherwise admissible solutions, but is a core requirement to make sense of the equation.

Proof of Theorem 6. Since $f \in L^1_{\mathrm{loc}}([0,b))$, its anti-derivative

$$g(y) = \int_0^y f(z) \,\mathrm{d}z \tag{27}$$

is absolutely continuous on I and bounded on every interval of the form $[0,x]$ with $x \in (0,b)$. Moreover, since $\mathcal{K}$ is absolutely continuous, $g(y)\,\mathcal{K}'(y/x)$ is integrable and $g(y)\,\mathcal{K}(y/x)$ is absolutely continuous on every such interval, so that, via integration by parts,

$$\frac{1}{x} \int_0^x g(y)\, \mathcal{K}'\Big(\frac{y}{x}\Big) \,\mathrm{d}y = g(y)\, \mathcal{K}\Big(\frac{y}{x}\Big)\Big|_0^x - \int_0^x f(y)\, \mathcal{K}\Big(\frac{y}{x}\Big) \,\mathrm{d}y = 0 \,. \tag{28}$$

Multiplying this equation with x^{k+1}, we infer that

$$x^k \int_0^x g(y)\, \mathcal{K}'\Big(\frac{y}{x}\Big) \,\mathrm{d}y = 0 \,. \tag{29}$$

We now fix a pair $x_1, x_2 \in I$ as follows. When $a = 0$, we fix x_1 arbitrarily, then choose any $x_2 \in (x_1, x_1/a^\star) \cap I$, where $a^\star$ is the constant from Lemma 2. In this case, we understand the computations which follow with the provision that $x^\star = 0$. When $a \in (0,1)$, we choose $x_1 \in (x^\star, x^\star/a^\star)$, then $x_2 \in (x_1, x^\star/a^\star)$. We note that in the second case, the inequality $x_2 < x_1/a^\star$ also holds true by construction. Now define $F \colon \mathbb{R}_+ \to \mathbb{R}$ by

$$F(y) = x_2^{k+1}\, \mathcal{K}\Big(\frac{y}{x_2}\Big) - x_1^{k+1}\, \mathcal{K}\Big(\frac{y}{x_1}\Big) \,, \tag{30}$$

with the understanding that $\mathcal{K}(\theta) = 0$ for $\theta > 1$. F is absolutely continuous on the interval $[0, x_2]$. Hence, F' is defined almost everywhere on $[0, x_2]$; we write U_F to denote its domain of definition. By direct computation,

$$F'(y) = \begin{cases} x_2^k\, \mathcal{K}'\Big(\dfrac{y}{x_2}\Big) - x_1^k\, \mathcal{K}'\Big(\dfrac{y}{x_1}\Big) & \text{for } y \in [0, x_1] \cap U_F\,, \\ x_2^k\, \mathcal{K}'\Big(\dfrac{y}{x_2}\Big) & \text{for } y \in (x_1, x_2) \cap U_F\,. \end{cases} \tag{31}$$

Due to (29),

$$\int_0^{x_2} g(y)\, F'(y)\, \mathrm{d}y = 0\,. \tag{32}$$

Since g is zero on $[0, x^\star]$, this implies

$$\int_{x^\star}^{x_1} g(y)\, F'(y)\, \mathrm{d}y = -\int_{x_1}^{x_2} g(y)\, F'(y)\, \mathrm{d}y\,. \tag{33}$$

When $y \in (x_1, x_2) \cap U_F$, then $a^\star < y/x_2$. Hence, due to (31) and Lemma 2(i), $F'(y)$ is negative and

$$0 > \int_{x_1}^{x_2} F'(y)\, \mathrm{d}y = F(x_2) - F(x_1) = -x_2^{k+1}\, \mathcal{K}\Big(\frac{x_1}{x_2}\Big)\,. \tag{34}$$

When $y \in (0, x_1) \cap U_F$, due to Definition 1(ii),

$$F'(y) = y^k \left(\Big(\frac{y}{x_2}\Big)^{-k} \mathcal{K}'\Big(\frac{y}{x_2}\Big) - \Big(\frac{y}{x_1}\Big)^{-k} \mathcal{K}'\Big(\frac{y}{x_1}\Big) \right) \tag{35}$$

is non-negative, so that

$$0 \le \int_{x^\star}^{x_1} F'(y)\, \mathrm{d}y = x_2^{k+1}\, \mathcal{K}\Big(\frac{x_1}{x_2}\Big) - x_2^{k+1}\, \mathcal{K}\Big(\frac{x^\star}{x_2}\Big) + x_1^{k+1}\, \mathcal{K}\Big(\frac{x^\star}{x_1}\Big)\,. \tag{36}$$

Now the key observation is that the sum of the last two terms on the right is non-positive. Indeed, when $a = 0$, hence $x^\star = 0$, this statement is true by direct inspection given assumption (i) of this theorem; when $a \in (0,1)$, hence $x^\star > 0$, it follows from Lemma 2(iii), which applies as our construction ensures that $x^\star/x_1$ and $x^\star/x_2$ lie in the interval $(a^\star, 1)$. Combining this observation with (34) and (36), we obtain

$$\left| \int_{x^\star}^{x_1} F'(y)\, \mathrm{d}y \right| \le \left| \int_{x_1}^{x_2} F'(y)\, \mathrm{d}y \right|. \tag{37}$$

On the other hand, since F' does not change sign on each of its intervals of definition, we may apply the integral mean value theorem on both sides of (33). It asserts that there exist $y_1 \in (x^\star, x_1)$ and $y_2 \in (x_1, x_2)$ such that

$$g(y_1) \int_{x^\star}^{x_1} F'(y)\,\mathrm{d}y = -g(y_2) \int_{x_1}^{x_2} F'(y)\,\mathrm{d}y\,. \tag{38}$$

Due to (34), the integral on the right is non-zero. Then, comparing (37) and (38), we conclude that $|g(y_1)| \geq |g(y_2)|$.

Summarizing, when $a = 0$, we have shown that for every $x_1 \in I$ and every $x_2 \in (x_1, b)$ there exist $y_1 \in (0, x_1)$ and $y_2 \in (x_1, x_2)$ such that $|g(y_1)| \geq |g(y_2)|$. Lemma 3 applies directly and yields $g = 0$ on $\overline{I}$.

When $a \in (0, 1)$, we have shown that for every $x_1 \in (x^\star, x^\star/a^\star) \cap I$ and every $x_2 \in (x_1, b)$ there exist $y_1 \in (x^\star, x_1)$ and $y_2 \in (x_1, x_2)$ such that $|g(y_1)| \geq |g(y_2)|$. Lemma 3 applies after translating $x^\star$ into the origin and proves that $g = 0$ on $[0, x^\star/a^\star) \cap \overline{I}$. We can now iterate the entire argument with $x^\star$ replaced by $x^\star/a^\star$, exhausting the entire interval I after a finite, when b is finite, or countable, when $b = \infty$ number of iterations.

Since now $g = 0$ on $\overline{I}$, $f = 0$ on I follows from (27). □

Remark 8. Theorem 6 remains valid if the condition $f \in C(I)$ is replaced by assuming essential continuity of f, see Remark 5.

Remark 9. Local integrability does not imply essential continuity. We present one easy example. Mimicking the construction of the Cantor set, we fix $\delta > 0$, start with $C_0 = [0, 1]$, and iteratively construct a sequence of sets C_n for $n \in \mathbb{N}$ by removing a centered piece of length $1/(3 + \delta)^n$ from each of the 2^{n-1} intervals of level C_{n-1}. Let $C = \cap_{n=1}^{\infty} C_n$ and set $f(z) = \mathbb{I}_C(z)$, where $\mathbb{I}_C$ denotes the indicator function of the set C. We show that is not essentially continuous on C in sense of (23). First, note that the measure of $C^c = [0, 1] \setminus C$ is

$$m(C^c) = \sum_{n=1}^{\infty} \frac{2^{n-1}}{(3 + \delta)^n} = \frac{1}{1 + \delta} < 1\,, \tag{39}$$

so $m(C) > 0$. Fix $z \in C$ and $\varepsilon > 0$ small enough. Let $n = \lceil \log_2 \varepsilon \rceil - 1$. The set C_n contains 2^n equal closed intervals; we write $I_{z,\varepsilon} \subset [z - \varepsilon, z + \varepsilon]$

to denote the interval that contains z. Then, on the one hand,

$$m(C \cap I_{z,\varepsilon}) = \frac{1}{2^n}\, m(C) = \frac{\delta}{2^n(1+\delta)} > 0\,, \tag{40}$$

so that $\operatorname{ess\,sup}_{[z-\varepsilon,z+\varepsilon]} f = 1$. On the other hand, $C^c \cap I_{z,\varepsilon}$ contains at least one center interval of the level $n+1$, which implies

$$m(C^c \cap I_{z,\varepsilon}) \geq \frac{1}{(3+\delta)^{n+1}} > 0\,, \tag{41}$$

so that $\operatorname{ess\,inf}_{[z-\varepsilon,z+\varepsilon]} f = 0$.

Remark 10. Let $\mathcal{G}$ be another kernel such that $\mathcal{G}(\theta) = \theta^n\, \mathcal{K}(\theta)$ with $\mathcal{K}$ satisfying $\mathcal{TC}(\mathcal{K}, k, a)$ for some $k \in \mathbb{R}$ and $a \in [0,1)$. Assume that

$$\int_0^1 \mathcal{G}(\theta)\, h(x\theta)\, \mathrm{d}\theta = 0 \tag{42}$$

for all $x \in I$. Then Theorem 6 implies that $h = 0$ on I within the class of functions where $f(z) = z^n\, h(z)$ conforms to the the continuity and integrability conditions in the theorem, as

$$\int_0^1 \mathcal{K}(\theta) f(x\theta)\, \mathrm{d}\theta = x^n \int_0^1 \mathcal{G}(\theta)\, h(x\theta)\, \mathrm{d}\theta = 0\,. \tag{43}$$

The essentially continuous functions, as Remark 9 shows, are in some sense a small class of functions, so that we investigate other means of enlarging the uniqueness class in which a statement of the form of Theorem 6 is true. Here and in the following, denote the average of a function $h \in L^1_{\text{loc}}([0,b))$ by

$$\bar{h}(y) = \frac{1}{y} \int_0^y h(z)\, \mathrm{d}z\,. \tag{44}$$

Corollary 11. *The statement of Theorem 6 remains valid if the continuity requirement on f is relaxed as follows:*

(i) $f \in \{h \in L^1_{\text{loc}}([0,b)) \colon \bar{h} \in L^1_{\text{loc}}([0,b))\}$ *if* $a = 0$,
(ii) $f \in \{h \in L^1_{\text{loc}}([0,b)) \colon \operatorname{supp} h \subset [x^\star, b)\}$ *for some* $x^\star \in (0,b)$ *if* $a \in (0,1)$.

Remark 12. The integrability requirement on $\bar{f}$ in part (i) is not implied by the assumption $f \in L^1_{\text{loc}}([0,b))$. For example, $f(x) = 1/(x \ln^2 x)$ is integrable on $[0, \mathrm{e}^{-1}]$ with $\bar{f}(x) = -1/(x \ln x)$. However, $\bar{f}$ is not integrable on $[0, \mathrm{e}^{-1}]$.

Remark 13. As in the statement of Theorem 6, the integrability requirement $f \in L^1_{\text{loc}}([0,b))$ is already implied by the requirement that f satisfies the WDCVIE in the Lebesgue sense in the following cases. When $a = 0$ and $\mathcal{K}(0) > 0$, we can apply the argument from Remark 7 to prove local integrability. When $a \in (0,1)$, we can proceed similarly. By the requirement $f = 0$ on $[0, x^\star)$, applying Lemma 2(i) and (ii), we observe that for any $x > x^\star$ with $x^\star/x > a^\star$, $0 < \min\{\mathcal{K}(x^\star/x), \mathcal{K}(\frac{1}{2}(x^\star/x+1))\}$ and

$$\begin{aligned}(45)\qquad \min\{\mathcal{K}(x^\star/x), \mathcal{K}(\tfrac{1}{2}(x^\star/x+1))\} &\int_{x^\star}^{\frac{1}{2}(x^\star+x)} |f(y)| \, \mathrm{d}y \\ &\leq \int_0^x \Bigl| \mathcal{K}\Bigl(\frac{y}{x}\Bigr) \, f(y) \Bigr| \, \mathrm{d}y < \infty \, .\end{aligned}$$

This argument can be iterated to prove integrability on the entire domain. In these cases, local integrability is not a selection criterion to disambiguate between otherwise admissible solutions, but is a core requirement to make sense of the equation.

Proof of Corollary 11. To proceed, we need $\bar{f}$ to be locally integrable near zero. In case (i), this is explicitly assumed. In case (ii), this follows from $f \in L^1_{\text{loc}}([0,b))$ and $f(x) = 0$ on the interval $[0, x^\star]$. Without distinguishing the two cases further, we observe that

$$\begin{aligned}(46)\qquad \int_0^1 \mathcal{K}(\theta) \, \bar{f}(x\theta) \, \mathrm{d}\theta &= \frac{1}{x} \int_0^1 \mathcal{K}(\theta) \int_0^x f(y\theta) \, \mathrm{d}y \, \mathrm{d}\theta \\ &= \frac{1}{x} \int_0^x \int_0^1 \mathcal{K}(\theta) \, f(y\theta) \, \mathrm{d}\theta \, \mathrm{d}y = 0 \, ,\end{aligned}$$

where, in the first equality, we have changed variables $z = y\theta$ and in the second equality, we changed the order of integration; the Fubini–Tonelli theorem applies since $x \int_0^1 |\mathcal{K}(\theta) \, \bar{f}(x\theta)| \, \mathrm{d}\theta$ is finite. Thus, $\bar{f}$ satisfies the assumptions of Theorem 6, which implies that $\bar{f} = 0$ on I. This implies $f = 0$ as well. □

Corollary 14. *In the case $a = 0$, the statement of Theorem 6 remains valid if f is assumed to be of class $L^\infty(I)$.*

Proof. Note that $\bar{f}$ is bounded and continuous on I, so that the argument of Corollary 11 applies. □

The integrability condition on $\bar{f}$ is essential here. For some other kernels this condition may be relaxed as follows.

Corollary 15. *Let $I = (0, b)$ for some $b \in \mathbb{R}_+ \cup \{\infty\}$. Assume that the kernel $\mathcal{K}$ satisfies $\mathcal{TC}(\mathcal{K}, k, 0)$ for some $k \geq -1$. Let $\mathcal{G}(\theta) = \theta^\alpha \, \mathcal{K}(\theta)$ for some $\alpha > 0$. If $f \in L^1_{\mathrm{loc}}([0, b))$ and satisfies*

$$\int_0^1 \mathcal{G}(\theta) \, f(x\theta) \, \mathrm{d}\theta = 0 \quad \text{for all } x \in I \,, \tag{47}$$

then $f \equiv 0$.

Proof. The function

$$h(y) = y^{\alpha - 1} \int_0^y f(z) \, \mathrm{d}z \tag{48}$$

is continuous on I and locally integrable on $[0, b)$. Indeed, for any $b^\star \in (0, b)$ and all $y \in (0, b^\star)$,

$$|h(y)| \leq y^{\alpha - 1} \int_0^{b^\star} |f(z)| \, \mathrm{d}z \,, \tag{49}$$

where the right side is integrable on $(0, b^\star)$. As in the proof of Corollary 11, we compute

$$\begin{aligned} \int_0^1 \mathcal{K}(\theta) \, h(x\theta) \, \mathrm{d}\theta &= x^{\alpha - 1} \int_0^1 \theta^\alpha \, \mathcal{K}(\theta) \int_0^x f(y\theta) \, \mathrm{d}y \, \mathrm{d}\theta \\ &= x^{\alpha - 1} \int_0^x \int_0^1 \mathcal{G}(\theta) \, f(y\theta) \, \mathrm{d}\theta \, \mathrm{d}y = 0 \,. \end{aligned} \tag{50}$$

Thus, by Theorem 6, $h = 0$ on I. This implies $f = 0$. □

4. Examples and counterexamples. In the following we give examples of kernels which satisfy suitable technical conditions. We also show that when the technical condition cannot be satisfied, the homogeneous WDCVIE may have a bounded continuous nontrivial solution; vice versa, when the kernel satisfies a technical condition, nontrivial solutions may still exist outside of the stated class of solutions. Finally, we sketch the context in which a homogeneous WDCVIE arises in a simplified version of the Keller–Rubinow model for chemical precipitation bands.

Example 1 (Concave kernels). Any absolutely continuous concave kernel satisfies $\mathcal{TC}(\mathcal{K}, 0, 0)$. Theorem 6 and Corollary 11 apply and restrict existence of nontrivial solutions to the homogeneous WDCVIE.

Example 2 (Uniqueness in the convex case). Consider the kernel $\mathcal{K}(\theta) = 1 - \theta^k$ for $k \in (0, 1)$. $\mathcal{K}$ is convex as $\mathcal{K}'(\theta) = -k\,\theta^{k-1}$ and $\mathcal{K}''(\theta) = -k\,(k-1)\,\theta^{k-2} > 0$. Nevertheless, $\mathcal{K}$ satisfies $\mathcal{TC}(\mathcal{K}, k-1, 0)$, hence the same theorems apply.

In the next two examples, for simplicity, we resort to kernels which are not weakly degenerate in the sense that $\mathcal{K}'(\theta) \to -\infty$ as $\theta \to 1$, but they are fully covered by the theorems and serve to illustrate the scope and limitations of our results. We begin by constructing a convex kernel which fails to be $\mathcal{TC}(\mathcal{K}, k, 0)$ for any $k \geq -1$. We show that the corresponding homogeneous WDCVIE has many non-trivial solutions.

Example 3 (Non-uniqueness due to a convex corner). Consider any positive k_1 and k_2 such that $k_1 > 5\,k_2$. Let

$$\mathcal{K}(\theta) = \begin{cases} \dfrac{k_1 + k_2}{2} - k_1\,\theta & \text{for } \theta \in [0, \frac{1}{2}]\,, \\ k_2 - k_2\,\theta & \text{for } \theta \in [\frac{1}{2}, 1]\,. \end{cases} \tag{51}$$

Clearly, $\mathcal{K}$ is convex but fails to be $\mathcal{TC}(\mathcal{K}, k, 0)$ for any $k \in \mathbb{R}$ as

$$\lim_{\theta \nearrow \frac{1}{2}} \frac{\mathcal{K}'(\theta)}{\theta^k} = -2^k\,k_1 < -2^k\,k_2 = \lim_{\theta \searrow \frac{1}{2}} \frac{\mathcal{K}'(\theta)}{\theta^k}\,. \tag{52}$$

Now take any continuous function f on $[1, 2]$ with $f(1) = f(2) = 0$. Extend its domain of definition recursively to $(0, \infty)$ via

$$f\Big(\frac{x}{2}\Big) = -\frac{4\,k_2}{k_1 - k_2}\,f(x)\,. \tag{53}$$

Noting that

$$\max_{x\in[2^{-n-1},2^{-n}]} |f(x)| = \frac{4\,k_2}{k_1-k_2} \max_{x\in[2^{-n},2^{-n+1}]} |f(x)| \,, \tag{54}$$

further extend f to a continuous function on $[0,\infty)$ by setting $f(0)=0$. Obviously, f is locally integrable near $x=0$, so that

$$F(x) = \int_0^x \mathcal{K}\Big(\frac{y}{x}\Big)\, f(y)\,\mathrm{d}y \tag{55}$$

is well-defined. Differentiating (see, e.g., **[6]** for differentiation under the integral sign), we obtain

$$\begin{aligned} F'(x) &= -\frac{1}{x^2}\int_0^x \mathcal{K}'\Big(\frac{y}{x}\Big)\, y\, f(y)\,\mathrm{d}y \\ &= \frac{k_1}{x^2}\int_0^{x/2} f(y)\,y\,\mathrm{d}y + \frac{k_2}{x^2}\int_{x/2}^x f(y)\,y\,\mathrm{d}y \,. \end{aligned} \tag{56}$$

Further, by the fundamental theorem of calculus and (53),

$$(x^2 F'(x))' = \frac{k_1}{2}\, f\Big(\frac{x}{2}\Big)\,\frac{x}{2} + k_2\,\Big(f(x)\,x - \frac{1}{2}\,f\Big(\frac{x}{2}\Big)\,\frac{x}{2}\Big) = 0\,. \tag{57}$$

Hence, $x^2\,F'(x) = C$ and, for any $x>0$,

$$F(x) = F(1) + C\,\Big(1 - \frac{1}{x_1}\Big)\,. \tag{58}$$

As $F(0)=0$, we must have $C=0$, hence $F\equiv 0$. This proves that f is a solution to the homogeneous WDCVIE; clearly, there are uncountably many of such solutions.

The next example demonstrates a case where the kernel is concave, hence is $\mathcal{TC}(\mathcal{K},0,0)$, yet there is a non-integrable solution to the WDCVIE. In other words, there exist nontrivial solutions that lie outside of the uniqueness classes stated in our theorems.

Example 4 (Non-integrable solution). Let $\mathcal{K}$ be defined by (51) with $k_1=1$ and $k_2=-1$. Clearly, $\mathcal{K}$ is concave. The construction of the solution follows (53): we take a non-zero continuous function f on $[1,2]$

with $f(1) = f(2) = 0$ and

$$\int_1^2 f(y)\, dy = 0 \tag{59}$$

and extend it recursively to $(0, \infty)$ via

$$f\Big(\frac{x}{2}\Big) = 2\, f(x)\,. \tag{60}$$

First, note that

$$\int_{2^i}^{2^{i+1}} |f(x)|\, \mathrm{d}x = \int_{2^{i-1}}^{2^i} |f(x)|\, \mathrm{d}x\,. \tag{61}$$

So f is not locally integrable near $y = 0$ so that it is not subject to any of our theorems. Next, we show that f solves the homogeneous WDCVIE. Let $\mathcal{G}(\theta) = \theta^{-1}\, \mathcal{K}(\theta)$, so $\mathcal{G}$ is absolutely continuous and

$$\mathcal{G}'(\theta) = \begin{cases} 0 & \text{for } \theta \in [0, \frac{1}{2})\,, \\ -\theta^{-2} & \text{for } \theta \in (\frac{1}{2}, 1]\,. \end{cases} \tag{62}$$

We note that $y\, f(y)$ is bounded. Indeed, for all $i \in \mathbb{Z}$

$$\max_{y \in [2^{i-1}, 2^i]} |y\, f(y)| \le 2^i \max_{y \in [2^{i-1}, 2^i]} |f(y)| = 2 \max_{y \in [1,2]} |f(y)|\,. \tag{63}$$

Therefore,

$$F(x) = \int_0^x y\, f(y)\, \mathcal{G}\Big(\frac{y}{x}\Big)\, \mathrm{d}y \tag{64}$$

is differentiable with

$$F'(x) = \int_{x/2}^x f(y)\, \mathrm{d}y\,. \tag{65}$$

Fix $x \in (0, \infty)$ and select $n \in \mathbb{Z}$ such that $x/2 < 2^n \le x$. Then, due to

(60), we obtain

$$
\begin{aligned}
(-1)^n F'(x) &= \biggl(\int_{\frac{x}{2}}^{2^n} + \int_{2^n}^{x} \biggr) f(y) \, \mathrm{d}y \\
&= \biggl(\int_{x}^{2^{n+1}} + \int_{2^n}^{x} \biggr) f(y) \, \mathrm{d}y = \int_{2^n}^{2^{n+1}} f(y) \, \mathrm{d}y \\
&= \int_1^2 f(y) \, \mathrm{d}y = 0 \,.
\end{aligned}
$$

Hence, F is constant. Since $y\,f(y)$ and $\mathcal{G}$ are bounded, $\lim_{x \searrow 0} F(x) = 0$, so $F \equiv 0$. Finally,

$$
\int_0^x f(y) \, \mathcal{K}\Bigl(\frac{y}{x}\Bigr) \, \mathrm{d}y = \frac{1}{x} \int_0^x y \, f(y) \, \mathcal{G}\Bigl(\frac{y}{x}\Bigr) \, \mathrm{d}y = 0 \tag{66}
$$

for every $x > 0$. We conclude that f is a nonintegrable function, continuous on $(0, \infty)$, which solves the homogeneous WDCVIE.

Example 5 (Simplified Keller–Rubinow model for Liesegang precipitation rings). Our motivating example is the study of a simplified version of the Keller–Rubinow model for Liesegang precipitation rings. For the model setup, we refer the reader to **[2, 3, 4, 5]**. Mathematically, the problem can be stated as follows. Find $\omega \in C([0,b))$ which satisfies the integral equation

$$
\omega(x) = \Gamma - x^2 \int_0^1 K(\theta) \, H(\omega(x\theta)) \, \mathrm{d}\theta \,, \tag{67}
$$

where Γ is a positive constant, H denotes the Heaviside function and K is a kernel, continuous on $[0,1]$ and C^2 on $[0,1)$, with the following properties:

(i) $K(0) = K'(0) = 0$,
(ii) $K(\theta) \sim c\sqrt{1-\theta}$ as $\theta \to 1$ for some $c > 0$,
(iii) $K(\theta)$ is non-negative and unimodal, i.e., there exists $\theta^\star \in (0,1)$ such that $K''(\theta) > 0$ for $\theta \in (0, \theta^\star)$ and $K''(\theta) < 0$ for $\theta \in (\theta^\star, 1)$.

In **[3]**, we show that solutions to (67) necessarily have a finite interval $[0, x^\star)$ of existence. They can be continued past $x^\star$, the point of breakdown, as *extended solutions*, namely pairs (ω, ρ) where $\omega \in$

$C([0,\infty))$ and which satisfy

$$\omega(x) = \Gamma - x^2 \int_0^1 K(\theta)\,\rho(x\theta)\,\mathrm{d}\theta \tag{68}$$

and where ρ takes values from the Heaviside graph, i.e.,

$$\rho(y) \in H(\omega(y)) = \begin{cases} 0 & \text{if } \omega(y) < 0\,, \\ [0,1] & \text{if } \omega(y) = 0\,, \\ 1 & \text{otherwise}\,. \end{cases} \tag{69}$$

Extended solutions can be shown to exist. Uniqueness on the interval $[0, x^\star)$ is elementary, but uniqueness past $x^\star$ is not obvious.

We say that (ω, ρ) is a *regularly extended solution* if ω is identically zero on some interval $[x^\star, b)$ past the point of breakdown. We observe that the question of uniqueness of regularly extended solutions is clearly of the type covered by Theorem 6 with $a > 0$, i.e., it is a problem of uniqueness-of-continuation type. Properties (ii) and (iii) imply $\mathcal{TC}(\mathcal{K}, 0, a)$ for some $a \in (0,1)$, so that Theorem 6 implies that regularly extended solutions to the model as stated above are indeed unique. Further, any (regularly or non-regularly) extended solution is only determined by ω, i.e., given ω, the corresponding function ρ is unique.

We further remark that **[2]** discusses some concrete kernels representing concrete choices of parameters in the modeling context. Numerical verification shows that for each of these kernels, there exists a choice of $\alpha > 0$ such that the assumptions of Corollary 15 are satisfied, thus asserting uniqueness of the abstract linear WDCVIE with the given kernel.

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Email address, Z. Darbenas: **z.darbenas@jacobs-university.de**

Email address, M. Oliver: **m.oliver@jacobs-university.de**

School of Engineering and Science, Jacobs University, 28759 Bremen, Germany