Homework 4 Solutions

1(a) $p(x) = x^3 - 6x^2 + 9x - 2$

As one of the roots is 2, let's divide out the factor $x-2$:

$(x^3 - 6x^2 + 9x - 2) : (x-2) = x^2 - 4x + 1$

$-\,|\,x^3 - 2x^2$

$\quad -4x^2 + 9x$

$-\,|\,-4x^2 + 8x$

$\qquad x - 2$

$-\,|\,x - 2$

$\qquad 0$

The remaining roots are

$$x_\pm = \frac{4 \pm \sqrt{16-4}}{2}$$

$$= 2 \pm \sqrt{3}$$

(b) $p'(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3)$

By inspection, 1 is a root, so let's divide out $x-1$:

$(x^2 - 4x + 3) : (x-1) = x - 3$

$-\,|\,x^2 - x$

$\quad -3x + 3$

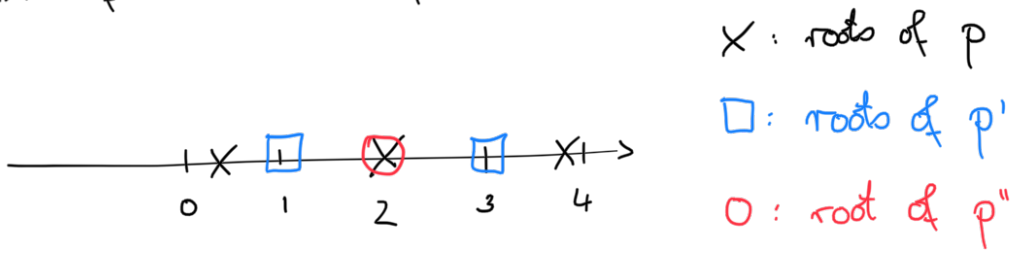
$-\,|\,-3x + 3$

$\qquad 0$

So the second root is $x = 3$

(c) $p''(x) = 6x - 12 \implies x = 2$ is the only root

(d) The pattern of roots is the following:



⇒ Between any two roots of a differentiable function lies one root of its derivative

<u>Note:</u> This is true in general and has been proved as Rolle's Theorem in the last lecture.

2(a) $f(x) = \dfrac{x}{a+x^2}$ $\qquad a = \text{const}$

$$f'(x) = \frac{1\cdot(a+x^2) - 2x\cdot x}{(a+x^2)^2} = \frac{a-x^2}{(a+x^2)^2}$$

(b) $g(t) = \cos(\omega t + \phi)$ $\qquad \omega, \phi$ are constants

$$g'(t) = \underbrace{\omega}_{\text{inner derivative}} \underbrace{\left(-\sin(\omega t + \phi)\right)}_{\text{outer derivative}} = -\omega \sin(\omega t + \phi)$$

(c) $h(s) = \sin(s^3)$

$$h'(s) = 3s^2 \cos(s^3)$$

(d) $j(s) = (\sin s)^3$

$$j'(s) = 3(\sin s)^2 \cos$$

(e) $k(x) = \ln(x^a + x^{-a})$ $\qquad a \neq 0$

$$k'(x) = \left(a x^{a-1} - a x^{-a-1}\right) \frac{1}{x^{a} + x^{-a}}$$

$$\ldots \qquad x^{a} + x^{-a}$$

$$= \frac{a}{x} \; \frac{x^{a} - x^{-a}}{x^{a} + x^{-a}}$$

(f) $\ell(x) = \ln\left(a^{x} + a^{-x}\right) = \ln\left(e^{x\ln a} + e^{-x\ln a}\right), \quad a>0$

$$\ell'(x) = \left(\ln a \, e^{x\ln a} - \ln a \, e^{-x\ln a}\right) \frac{1}{e^{x\ln a} + e^{-x\ln a}}$$

$$= \ln a \; \frac{a^{x} - a^{-x}}{a^{x} + a^{-x}}$$

(g) $u(x) = \exp(bx)$

$$u'(x) = b \exp(bx) = b\, e^{bx}$$

(h) $v(x) = x^{2} e^{x}$

$$v'(x) = 2x\, e^{x} + x^{2} e^{x} = (2+x)x\, e^{x}$$

(i) $w(x) = e^{-x^{2}}$

$$w'(x) = -2x\, e^{-x^{2}}$$

(j) $z(x) = x^{x} = e^{x\ln x}$

$$x \ln x$$

$$z'(x) = \left(\ln x + x \frac{1}{x}\right) e^{x \ln x}$$

$$= (1 + \ln x)\, x^x$$

3. $f(x) = |x|^2$

For $h > 0$, $$\frac{f(0+h) - f(0)}{h} = \frac{h - 0}{h} = 1 \longrightarrow 1 \quad \text{as } h \searrow 0.$$

For $h < 0$, $$\frac{f(0+h) - f(0)}{h} = \frac{-h - 0}{h} = -1 \longrightarrow -1 \quad \text{as } h \nearrow 0$$

Thus, left and right limit do not agree, so the limit

$$\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$$

does not exist, so f is not differentiable at $x = 0$.

4. The inverse function f^{-1} has the property

$$f^{-1}(f(x)) = x$$

Taking the derivative, we find (chain rule!)

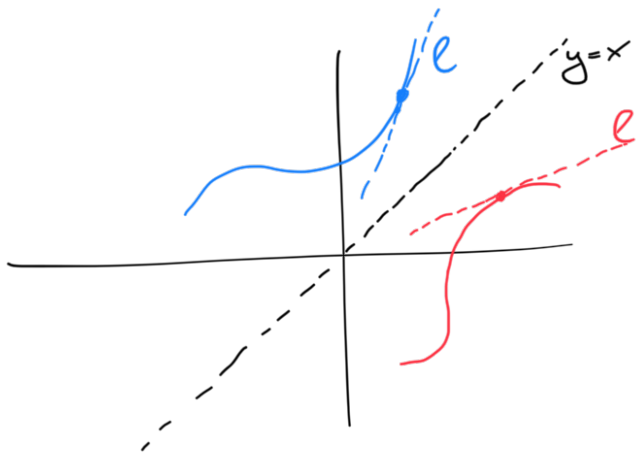
$$(f^{-1})'(f(x))\, f'(x) = 1$$

or, setting $y = f(x)$,

$$(f^{-1})'(y) = \frac{1}{\;} = \frac{1}{\;}$$

$$(f^{-1})'(y) = \frac{1}{f'(x)} = \frac{1}{f'(f^{-1}(y))}.$$

Geometric interpretation: The graph of the inverse function f^{-1} is obtained from the graph of f by mirroring about the line $y = x$. Thus, a tangent line to f^{-1} is obtained from a tangent line of f by the same symmetry:



But mirroring about the line $x = y$ takes slope m of tangent line ℓ to slope $\frac{1}{m}$ of the mirror image ℓ.

(This can be shown using elementary geometry, not required here.)