General Mathematics and Computational Science II

Geometric Transformations – Supplementary Notes

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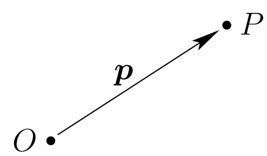
Abstract

These notes provide some additional material supporting Chapter 3 on "Geometric Transformations" from O.A. Ivanov's book "Easy as π".

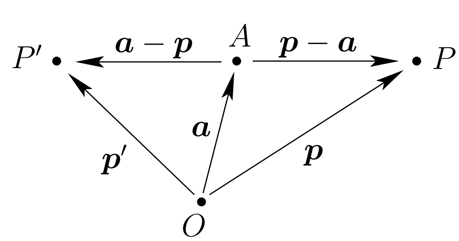
1 Transformations in Cartesian coordinates

In the following, we look more systematically at coordinate expressions for the different geometric transformations. As Ivanov argues [1, p. 37], coordinate-free arguments are often shorter and more elegant—try, for example, to solve Problems 14 and 15 both ways. On the other hand, coordinate expressions often lead to a more direct solution approach; in addition, they are needed for computational algorithms.

Notation. Points are denoted by capital letters, while vectors, which specify a direction and a magnitude or length, are denoted by small boldface letters. In particular, to every point P we associate the coordinate vector $\boldsymbol{p}$ pointing from the origin O to P. In components, we write $\boldsymbol{p} = (p_1, p_2)$.



Recall that H_A denotes the *central reflection* or *point reflection* about the point A. In coordinates, $\boldsymbol{p}' = \boldsymbol{a} - (\boldsymbol{p} - \boldsymbol{a})$, see figure,



so that

$$H_A(\boldsymbol{p}) = 2\boldsymbol{a} - \boldsymbol{p}\,. \tag{1}$$

Moreover, let $\Pi_{\boldsymbol{v}}$ denote the translation by the vector $\boldsymbol{v}$; in coordinates,

$$\Pi_{\boldsymbol{v}}(\boldsymbol{p}) = \boldsymbol{p} + \boldsymbol{v}\,. \tag{2}$$

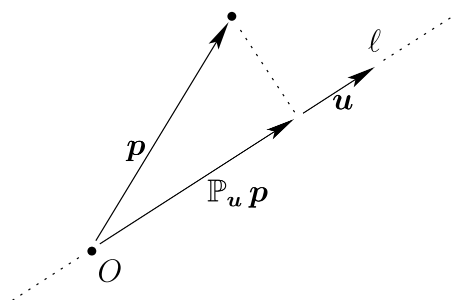
Lemma 1. *$H_B \circ H_A = \Pi_{2(\boldsymbol{b}-\boldsymbol{a})}$.*

Proof. $H_B\, H_A\, \boldsymbol{p} = H_B\,(2\boldsymbol{a} - \boldsymbol{p}) = 2\boldsymbol{b} - (2\boldsymbol{a} - \boldsymbol{p}) = \boldsymbol{p} + 2(\boldsymbol{b} - \boldsymbol{a}) = \Pi_{2(\boldsymbol{b}-\boldsymbol{a})}\, \boldsymbol{p}$. □

Let ℓ denote the line through the origin in the direction of a unit vector $\boldsymbol{u}$. Then, for any vector $\boldsymbol{p}$, the projection of $\boldsymbol{p}$ onto the line ℓ is given the the expression

$$\mathbb{P}_{\boldsymbol{u}}\, \boldsymbol{p} = \boldsymbol{u}\, \boldsymbol{u} \cdot \boldsymbol{p}\,, \tag{3}$$

where $\boldsymbol{u} \cdot \boldsymbol{p} = u_1\, p_2 + u_2\, p_2$ denotes the vector dot product.



Expression (3) is easily proved by elementary trigonometry, recalling that

$$\cos \angle(\boldsymbol{a}, \boldsymbol{b}) = \frac{\boldsymbol{a} \cdot \boldsymbol{b}}{\|\boldsymbol{a}\|\, \|\boldsymbol{b}\|}\,, \tag{4}$$

where $\|\boldsymbol{a}\| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$ denotes the Euclidean length of a vector $\boldsymbol{a}$, and $\cos \angle(\boldsymbol{a}, \boldsymbol{b})$ denotes the cosine of the angle between vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

Let us give a second, analytical argument for (3). The vector $\mathbb{P}_{\boldsymbol{u}}\, \boldsymbol{p}$ describes the point on the line ℓ which has minimal distance to $\boldsymbol{p}$. A general point on the line is given by $t\, \boldsymbol{u}$ for some real number t. We find the projection by minimizing

$$f(t) = \|\boldsymbol{p} - t\, \boldsymbol{u}\|^2\,, \tag{5}$$

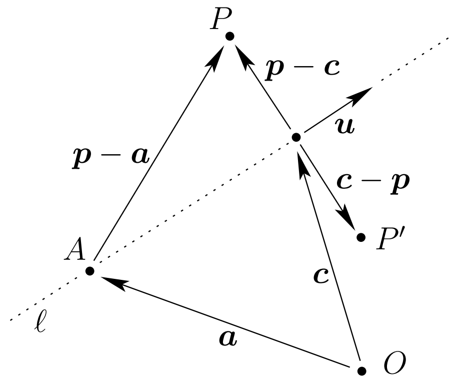
which is nonnegative and quadratic in t. To find the minimum, compute

$$f'(t) = \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{p} - t\, \boldsymbol{u}) \cdot (\boldsymbol{p} - t\, \boldsymbol{u}) = -2\, \boldsymbol{u} \cdot (\boldsymbol{p} - t\, \boldsymbol{u})\,. \tag{6}$$

So f has a critical point when $t = \boldsymbol{u} \cdot \boldsymbol{p}$, which proves (3).

We write R_ℓ to denote the reflection of the plane about some line ℓ. Let us give two constructions which coincide on the plane, but generalize differently into higher dimensions.

A line ℓ (in any dimension) is uniquely specified by a point A on the line and a direction, expressed by a unit vector $\boldsymbol{u}$. To reflect an arbitrary point P about this line, we first compute the coordinates of the projection of P onto ℓ.



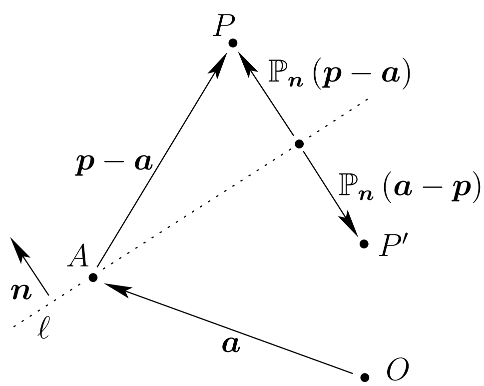
As can be seen from this figure,

$$\boldsymbol{c} = \boldsymbol{a} + \mathbb{P}_{\boldsymbol{u}}\,(\boldsymbol{p} - \boldsymbol{a})\,. \tag{7}$$

Then the coordinates of P', the image of P under the reflection, are given by $\boldsymbol{p}' = 2\boldsymbol{c} - \boldsymbol{p}$, so that

$$R_\ell(\boldsymbol{p}) = 2\boldsymbol{u}\,\boldsymbol{u}\cdot\boldsymbol{p} - \boldsymbol{p} - 2\boldsymbol{u}\,\boldsymbol{u}\cdot\boldsymbol{a} + 2\boldsymbol{a}\,. \tag{8}$$

Alternatively, a line in the plane is be uniquely specified by a point A and a direction normal to the line, expressed by a unit vector $\boldsymbol{n}$. (In d dimension, this construction defines a $(d-1)$-dimensional hyperplane.)



Then, clearly,

$$R_\ell(\boldsymbol{p}) = \boldsymbol{p} - 2\,\mathbb{P}_{\boldsymbol{n}}\,(\boldsymbol{p} - \boldsymbol{a}) = \boldsymbol{p} - 2\boldsymbol{n}\,\boldsymbol{n}\cdot(\boldsymbol{p} - \boldsymbol{a})\,. \tag{9}$$

2 Matrix expressions

Looking at the expressions for central reflections, line reflections, and translations, we notice that they are all linear affine, i.e. of the form $F(\boldsymbol{p}) = \mathsf{M}\boldsymbol{p} + \boldsymbol{b}$ for some matrix M and some vector $\boldsymbol{b}$. Here we adapt the convention that all vectors are read as column vectors and write $\boldsymbol{a}^T$ to denote the transpose of a vector $\boldsymbol{a}$, so that $\boldsymbol{a}\cdot\boldsymbol{b} \equiv \boldsymbol{a}^T\boldsymbol{b}$. We further write I to denote the identity matrix. Then

$$\mathbb{P}_{\boldsymbol{u}}\,\boldsymbol{p} = \boldsymbol{u}\boldsymbol{u}^T\boldsymbol{p} \tag{10}$$

and

$$R_\ell(\boldsymbol{p}) = (2\boldsymbol{u}\boldsymbol{u}^T - \mathsf{I})\,(\boldsymbol{p} - \boldsymbol{a}) + \boldsymbol{a} = (\mathsf{I} - 2\boldsymbol{n}\boldsymbol{n}^T)\,\boldsymbol{p} + 2\boldsymbol{n}\boldsymbol{n}^T\boldsymbol{a} \tag{11}$$

where, as before $\boldsymbol{u}$ denotes some unit vector. Let us now denote the angle between the x-axis and $\boldsymbol{u}$ by ϕ, so that

$$\boldsymbol{u} = \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} . \tag{12}$$

Assuming further that line of reflection passes through the origin, we can take $\boldsymbol{a} = \boldsymbol{0}$ in (11). Hence, the reflection about a line at angle ϕ with the x-axis is represented by multiplication with the matrix

$$\mathsf{R}_\phi = 2\boldsymbol{u}\boldsymbol{u}^T - \mathsf{I} = 2\begin{pmatrix} \cos^2\phi & \cos\phi\,\sin\phi \\ \cos\phi\,\sin\phi & \sin^2\phi \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix} , \tag{13}$$

where the last equality is due to the trigonometric double-angle identities.

Let Φ_α denote the matrix of rotation about the origin through the angle α. It is easy to find the coefficients by elementary trigonometry, see [1]. Alternatively, we may recall that the composition of two reflections about intersecting lines is a rotation about the point of their intersection through an angle equal to twice the angle between them. Taking the line in the direction of $\boldsymbol{u}$ and the x-axis, respectively, this implies that

$$\Phi_{2\phi} = \mathsf{R}_\phi\,\mathsf{R}_0 = \begin{pmatrix} \cos 2\phi & \sin 2\phi \\ \sin 2\phi & -\cos 2\phi \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} \cos 2\phi & -\sin 2\phi \\ \sin 2\phi & \cos 2\phi \end{pmatrix} . \tag{14}$$

Setting $\alpha = 2\phi$, we conclude that

$$\Phi_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} . \tag{15}$$

References

[1] O.A. Ivanov, “Easy as π”, Springer-Verlag, 1998.