Remark on the evaluation of integrals involving the Dirac δ-function:

On the final exam, we evaluated the integral

$$\int_{-\infty}^{\infty} x^2 \delta(x^3)\, dx = \int_{-\infty}^{\infty} \frac{1}{3}\delta(u)\, du = \frac{1}{3}.$$

Two seemingly paradoxical issues arise:

(a) Writing $u = x^3$, we obtain $dx = \frac{1}{3}x^{-2}\, du$

which is undefined at $x=0 \Leftrightarrow u=0$.

(b) Since x^3 is zero at $x=0$, why does the left-hand integral not evaluate to 0 (as is incorrectly claimed e.g. by Mathematica)?

<u>For (a)</u>: There are two ways to resolve this issue:

(i) The transformation <u>theorem</u> for integrals really says

$$\int_a^b g'(x)\, f(g(x))\, dx = \int_{g^{-1}(a)}^{g^{-1}(b)} f(u)\, du, \qquad (*)$$

provided that g is invertible.
So when you apply it, there is no division involved at all. Writing $dx = \frac{du}{3x^2}$ is simply mnemonic notation but cannot be taken literally.

(ii) By the arguments in point ② below, it can be shown that

$$\int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = \lim_{x \to 0} f(x)$$

which makes sense even if f is not defined at $x=0$ but the right-hand limit exists.

For (b): Again, there are two ways to approach the issue which are equivalent under suitable technical assumption.

① The algebraic approach (which is what we have done in class):

We define $\int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = f(0)$, $\left(\text{or} = \lim_{x \to 0} f(x)\right)$ (**)

taking the notation to be purely mnemonic (i.e., this is not an integral as introduced in Calculus, but a pure algebraic rule which tells you how to compute with it.)

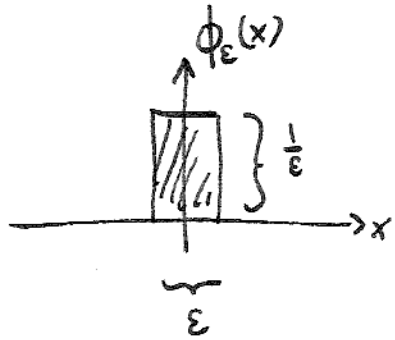
Then the transformation rule (*) is added to this definition again as a purely algebraic rule which does not require proof.

From this perspective, the δ-function is just a linear map from the vector space of continuous functions into $\mathbb{R}$ with properties (*), (**).

② The analytic approach:

We introduce the function

$$\phi_\varepsilon(x) = \begin{cases} 0 & \text{if } |x| > \frac{\varepsilon}{2} \\ \frac{1}{\varepsilon} & \text{if } |x| \le \frac{\varepsilon}{2} \end{cases}$$



(note that $\int_{-\infty}^{\infty} \phi_\varepsilon(x)\, dx = 1$ for all $\varepsilon > 0$.)

Then we define

$$\int_{-\infty}^{\infty} f(x)\, \delta(x)\, dx = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} f(x)\, \phi_\varepsilon(x)\, dx .$$

The following are not hard to show, but require some basic analysis:

(i) If f is continuous, we recover the algebraic definitions made in ①, so ① and ② are consistent.

(ii) The definition does not depend on the particular choice of ϕ_ε so long as it is normalized and "concentrates" in a certain sense as $\varepsilon \to 0$.

Concretely here:

$$\int_{-\infty}^{\infty} x^2\, \phi_\varepsilon(x^3)\, dx = \int_{-\sqrt[3]{\frac{\varepsilon}{2}}}^{+\sqrt[3]{\frac{\varepsilon}{2}}} x^2 \frac{1}{\varepsilon}\, dx = \frac{1}{\varepsilon}\, \frac{1}{3} x^3 \Big|_{-\sqrt[3]{\frac{\varepsilon}{2}}}^{\sqrt[3]{\frac{\varepsilon}{2}}} = \frac{1}{\varepsilon}\, \frac{1}{3} \left(\frac{\varepsilon}{2} - \left(-\frac{\varepsilon}{2}\right)\right)$$

$$= \frac{1}{3}$$