ANNIHILATORS OF HIGHEST WEIGHT
$\mathfrak{sl}(\infty)$-MODULES

IVAN PENKOV   ALEXEY PETUKHOV

Jacobs University Bremen, Campus   The University of Manchester,
Ring 1, 28759 Bremen, Germany   Oxford Road M13 9PL,
i.penkov@jacobs-university.de   Manchester, UK
on leave from the Institute for
Information Transmission Problems,
Bolshoy Karetniy 19-1, Moscow
127994, Russia

alex--2@yandex.ru

Abstract. We give a criterion for the annihilator in $U(\mathfrak{sl}(\infty))$ of a simple highest weight $\mathfrak{sl}(\infty)$-module to be nonzero. As a consequence we show that, in contrast with the case of $\mathfrak{sl}(n)$, the annihilator in $U(\mathfrak{sl}(\infty))$ of any simple highest weight $\mathfrak{sl}(\infty)$-module is integrable, i.e., coincides with the annihilator of an integrable $\mathfrak{sl}(\infty)$-module. Furthermore, we define the class of ideal Borel subalgebras of $\mathfrak{sl}(\infty)$, and prove that any prime integrable ideal in $U(\mathfrak{sl}(\infty))$ is the annihilator of a simple $\mathfrak{b}^0$-highest weight module, where $\mathfrak{b}^0$ is any fixed ideal Borel subalgebra of $\mathfrak{sl}(\infty)$. This latter result is an analogue of the celebrated Duflo Theorem for primitive ideals.

1. Introduction

The base field is $\mathbb{C}$. If $\mathfrak{g}$ is a semisimple finite-dimensional Lie algebra, the celebrated Duflo Theorem states that any primitive two-sided ideal in the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ (i.e., any annihilator of a simple $U(\mathfrak{g})$-module) is the annihilator of a simple highest weight $\mathfrak{g}$-module.

The purpose of the present paper is to study primitive ideals in the enveloping algebra $U(\mathfrak{sl}(\infty))$ of the infinite-dimensional Lie algebra $\mathfrak{sl}(\infty)$, and in particular to obtain a partial analogue of Duflo's Theorem for $\mathfrak{sl}(\infty)$. Recall that the Lie algebra $\mathfrak{sl}(\infty)$ can be defined in several equivalent ways, for instance as a direct limit $\varinjlim_{n \geq 2} \mathfrak{sl}(n)$ [Ba1, Ba2, DP1].

The study of two-sided ideals in $U(\mathfrak{sl}(\infty))$ has been initiated by A. Zhilinskii [Zh1, Zh2, Zh3], and has been continued in [PP]. Zhilinskii's idea has been to study the joint annihilators of certain systems of $\mathfrak{sl}(n)$-modules for variable $n > 2$, more precisely, the joint annihilators of coherent local systems of finite-dimensional $\mathfrak{sl}(n)$-modules as defined in [Zh1]. Zhilinskii has also provided a classification of coherent local systems [Zh1, Zh2]. We call the ideals introduced by Zhilinskii *integrable* (see Section 5 for the precise definition).

A corollary of the results in [PP] is that the associated "variety" of an arbitrary

ideal in $U(\mathfrak{sl}(\infty))$ coincides with the associated "variety" of some integrable ideal in $U(\mathfrak{sl}(\infty))$. We do not know whether any ideal in $U(\mathfrak{sl}(\infty))$ is integrable, however in the present paper we prove that the annihilator of any highest weight $\mathfrak{sl}(\infty)$-module is an integrable ideal in $U(\mathfrak{sl}(\infty))$.

In order to recall the definition of a highest weight $\mathfrak{sl}(\infty)$-module, we first need to recall the definition of a splitting Borel subalgebra of $\mathfrak{sl}(\infty)$. According to [DP1], a splitting Borel subalgebra is a subalgebra of $\mathfrak{sl}(\infty)$ which can be obtained as a direct limit of $\varinjlim \mathfrak{b}_n$ of Borel subalgebras $\mathfrak{b}_n \subset \mathfrak{sl}(n)$ for a suitable presentation $\mathfrak{sl}(\infty)$ as a direct limit $\varinjlim_{n \geq 2} \mathfrak{sl}(n)$. In contrast with the finite-dimensional case, the splitting Borel subalgebras of $\mathfrak{sl}(\infty)$ are not conjugate by the group of automorphisms of $\mathfrak{sl}(\infty)$; in fact, there are uncountably many conjugacy classes (and even isomorphism classes) of splitting Borel subalgebras of $\mathfrak{sl}(\infty)$. However, a $\mathfrak{b}$-highest weight module is defined as usual as an $\mathfrak{sl}(\infty)$-module generated by a 1-dimensional $\mathfrak{b}$-submodule.

The difference between the structure of ideals in $U(\mathfrak{sl}(\infty))$ and in $U(\mathfrak{g})$ for a finite-dimensional semisimple $\mathfrak{g}$, becomes apparent in the fact that the annihilators in $U(\mathfrak{sl}(\infty))$ of many simple highest weight modules equal to zero. In this paper we give an explicit criterion for a simple $\mathfrak{b}$-highest weight module to have nonzero annihilator. A further central result which we establish is that the annihilator of any $\mathfrak{b}$-highest weight $\mathfrak{sl}(\infty)$-module is integrable.

Our third notable result is an analogue of Duflo's Theorem. We define a special class of splitting Borel subalgebras $\mathfrak{b}^0 \subset \mathfrak{sl}(\infty)$, which we call ideal, and prove that any prime integrable ideal of $U(\mathfrak{sl}(\infty))$ is the annihilator of a simple $\mathfrak{b}^0$-highest weight module for any $\mathfrak{b}^0$. The ideal Borel subalgebras $\mathfrak{b}^0$ have the property that the adjoint representation of $\mathfrak{sl}(\infty)$ is a $\mathfrak{b}^0$-highest weight module.

The paper is structured as follows. In Section 2 we review some well known and some not so well known results about the Lie algebra $\mathfrak{sl}(\infty)$ and its representations. Section 3 contains a precise statement of our main results. The proofs are given in Sections 4, 5 and 6. In Section 7 we characterize simple $\mathfrak{sl}(\infty)$-modules which are determined up to isomorphism by their annihilators in $U(\mathfrak{sl}(\infty))$, under the assumption that the annihilator is integrable.

2. Preliminaries

1. The Lie algebra $\mathfrak{sl}(\infty)$

The superscript * indicates dual space, and $\mathbf{S}^{\cdot}(\cdot)$ and $\Lambda^{\cdot}(\cdot)$ stand respectively for symmetric and exterior algebra. For a Lie algebra $\mathfrak{g}$, $U(\mathfrak{g})$ stands for the universal enveloping algebra of $\mathfrak{g}$. If M is a $\mathfrak{g}$-module, then $\mathrm{Ann}_{U(\mathfrak{g})} M$ denotes

the annihilator of M in $\mathrm{U}(\mathfrak{g})$.

The Lie algebra $\mathfrak{gl}(\infty)$ can be defined as the Lie algebra of matrices $(a_{ij})_{i,j \in \mathbb{Z}_{>0}}$ each of which has at most finitely many nonzero entries. Equivalently, $\mathfrak{gl}(\infty)$ can be defined by giving an explicit basis. Let $\{e_{ij}\}_{i,j \in \mathbb{Z}_{>0}}$ be a basis of a countable-dimensional vector space denoted by $\mathfrak{gl}(\infty)$. Set $\widehat{\mathfrak{h}} := \mathrm{span}\{e_{ii}\}_{i \in \mathbb{Z}_{>0}}$. The structure of a Lie algebra on $\mathfrak{gl}(\infty)$ is given by the formula

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj},$$

where $i, j \in \mathbb{Z}_{>0}$ and δ_{mn} is Kronecker's delta.

Next, one defines $\mathfrak{sl}(\infty)$ as the commutator subalgebra of $\mathfrak{gl}(\infty)$:

$$\mathfrak{sl}(\infty) := [\mathfrak{gl}(\infty), \mathfrak{gl}(\infty)].$$

We set

$$\mathfrak{h} := \widehat{\mathfrak{h}} \cap \mathfrak{sl}(\infty).$$

Clearly, $\widehat{\mathfrak{h}}$ is a maximal commutative subalgebra of $\mathfrak{gl}(\infty)$, and $\mathfrak{h}$ is a maximal commutative subalgebra of $\mathfrak{sl}(\infty)$. Moreover, $\mathfrak{gl}(\infty)$ has the following root decomposition

$$\mathfrak{gl}(\infty) = \widehat{\mathfrak{h}} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{gl}(\infty)^{\alpha},$$

similar to the usual root decomposition of $\mathfrak{gl}(n)$. Here $\Delta = \{\varepsilon_i - \varepsilon_j\}_{i,j \in \mathbb{Z}_{>0}}$ where the system of vectors $\{\varepsilon_j\}_{j \in \mathbb{Z}_{>0}}$ in $\widehat{\mathfrak{h}}^*$ is dual to the basis $\{e_{ii}\}_{i \in \mathbb{Z}_{>0}}$ of $\widehat{\mathfrak{h}}$. The Lie subalgebra $\mathfrak{sl}(\infty)$ inherits this root decomposition:

$$\mathfrak{sl}(\infty) = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{sl}(\infty)^{\alpha},$$

where $\mathfrak{sl}(\infty)^{\alpha} = \mathfrak{gl}(\infty)^{\alpha}$ for $\alpha \in \Delta$.

It is not difficult to prove that any Lie algebra obtained as a direct limit $\varinjlim_{n \geq 2} \mathfrak{sl}(n)$ is isomorphic to $\mathfrak{sl}(\infty)$ as defined above. Moreover, a general definition of a *splitting Cartan subalgebra* $\mathfrak{h}'$ of $\mathfrak{sl}(\infty)$ is as a direct limit of Cartan subalgebras $\mathfrak{h}'_n$ of $\mathfrak{sl}(n)$, where $\mathfrak{sl}(\infty)$ is identified with $\varinjlim_{n \geq 2} \mathfrak{sl}(n)$. Then, as noted in [DPSn], all splitting Cartan subalgebras of $\mathfrak{sl}(\infty)$ are conjugate via the automorphism group $\mathrm{Aut}\,\mathfrak{sl}(\infty)$. This enables us to henceforth restrict ourselves to considering only the fixed splitting Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{sl}(\infty)$ introduced above.

A *splitting Borel subalgebra* of $\mathfrak{sl}(\infty) \cong \varinjlim_{n \geq 2} \mathfrak{sl}(n)$ is defined as a direct limit $\varinjlim_{n \geq 2} \mathfrak{b}_n$ of Borel subalgebras $\mathfrak{b}_n \subset \mathfrak{sl}(n)$, see [DP1]. Since a general splitting Borel subalgebra of $\mathfrak{sl}(\infty)$ is conjugate under $\mathrm{Aut}(\mathfrak{sl}(\infty))$ to a splitting Borel subalgebra containing our fixed splitting Cartan subalgebra $\mathfrak{h} \subset \mathfrak{sl}(\infty)$, in what follows we only consider splitting Borel subalgebras containing $\mathfrak{h}$. The latter Borel subalgebras are given by the following construction. We say that a subset $\Delta^* \subset \Delta$ is a subset *of positive roots* if

(1) for any root $\alpha \in \Delta$, precisely one of α and $-\alpha$ belongs to Δ^*;

(2) $\alpha, \beta \in \Delta^*$ and $\alpha + \beta \in \Delta$ imply $\alpha + \beta \in \Delta^*$.

To any positive subset of roots Δ^* we assign the Borel subalgebra

$$\mathfrak{b}(\Delta^*) := \mathfrak{h} \bigoplus_{\alpha \in \Delta^*} \mathfrak{sl}(\infty)^\alpha$$

of $\mathfrak{sl}(\infty)$, and in this way we obtain all splitting Borel subalgebras of $\mathfrak{sl}(\infty)$ containing $\mathfrak{h}$.

This leads naturally to the observation [DP1] that the splitting Borel subalgebras containing $\mathfrak{h}$ are in one-to-one correspondence with linear orders on $\mathbb{Z}_{>0}$: given such a linear order $\prec$, the corresponding subset of positive roots is $\{\varepsilon_i - \varepsilon_j\}_{i \prec j}$.

It is easy to see that different Borel subalgebras containing $\mathfrak{h}$ do not have to be $\mathrm{Aut}\,\mathfrak{sl}(\infty)$-conjugate, as they simply may not be isomorphic as abstract Lie algebras. Consider, for instance, the following three linear orders on $\mathbb{Z}_{>0}$:

(i) $... \prec 5 \prec 3 \prec 1 \prec 2 \prec 4 \prec 6 \prec ...$,

(ii) $1 \prec 2 \prec 3 \prec ...$,

(iii) $1 \prec 3 \prec 5 \prec ... \prec 2n + 1 \prec ... \prec 2n \prec ... \prec 4 \prec 2$.

The reader can check that the corresponding Borel subalgebras are not isomorphic as Lie algebras.

2. S-notation

Let S be a subset of $\mathbb{Z}_{>0}$. We denote by $\mathfrak{sl}(S)$ the subalgebra of $\mathfrak{sl}(\infty)$ spanned by

$$\{e_{ij}\}_{i,j \in S, i \neq j} \quad \text{and} \quad \{e_{ii} - e_{jj}\}_{i,j \in S}.$$

Then $\mathfrak{sl}(\mathbb{Z}_{>0}) = \mathfrak{sl}(\infty)$.

Set $\mathfrak{h}_S := \mathfrak{h} \cap \mathfrak{sl}(S)$. Note that

(1) if S is finite, then $\mathfrak{sl}(S)$ is isomorphic to $\mathfrak{sl}(n)$ where $n = |S|$ is the cardinality of S, and $\mathfrak{h}_S$ is a Cartan subalgebra of $\mathfrak{sl}(S)$;

(2) if S is infinite, then $\mathfrak{sl}(S)$ is isomorphic to $\mathfrak{sl}(\infty)$, and $\mathfrak{h}_S$ is a splitting Cartan subalgebra of $\mathfrak{sl}(S)$.

Next, we fix a splitting Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$ of $\mathfrak{sl}(\infty)$ and put $\mathfrak{b}_S := \mathfrak{sl}(S) \cap \mathfrak{b}$. We note that

(1) if S is finite, then $\mathfrak{b}_S$ is a Borel subalgebra of $\mathfrak{sl}(S)$,

(2) if S is infinite, then $\mathfrak{b}_S$ is a splitting Borel subalgebra of $\mathfrak{sl}(S)$.

Let $\mathbb{C}^S$ denote the set of functions from S to $\mathbb{C}$. Clearly, $\mathbb{C}^S$ is a vector space of dimension $|S|$. When $S = \{1, ..., n\}$ we write simply $\mathbb{C}^n$ instead of $\mathbb{C}^{\{1,...,n\}}$. There is a surjective homomorphism from $\mathbb{C}^S$ to $\mathfrak{h}_S^*$:

$$f \mapsto \lambda_f, \quad \lambda_f(e_{ii} - e_{jj}) = f(i) - f(j). \tag{1}$$

For any $f \in \mathbb{C}^S$ we denote by $|f|$ the the cardinality of the image of f. A weight $\lambda \in \mathfrak{h}_S^*$ is $\mathfrak{sl}(S)$-*integral*, or simply *integral*, if $\lambda(e_{ii} - e_{jj}) \in \mathbb{Z}$ for all $i, j \in S$. Respectively, a function $f \in \mathbb{C}^S$ is *integral* if $f(i) - f(j) \in \mathbb{Z}$ for all $i, j \in S$. A function is *almost integral* if there exists a finite subset $F \subset S$ such that $f|_{S \setminus F}$ is integral.

If $\mathfrak{b}_S \supset \mathfrak{h}_S$ is a fixed splitting Borel subalgebra of $\mathfrak{sl}(S)$, then an integral weight $\lambda \in \mathfrak{h}_S^*$ is $\mathfrak{b}_S$-*dominant* if $\lambda(e_{ii} - e_{jj}) \geq 0$ for $i \prec j$ where the order $\prec$ on S is

determined by $\mathfrak{b}_S$. Respectively, an integral function $f \in \mathbb{C}^S$ is $\prec$-*dominant* if $f(i) - f(j) \geq 0$ for $i \prec j$.

Let $\prec$ be a linear order on S, and let $S = S_1 \sqcup ... \sqcup S_t$ be a finite partition of S. We say that the partition $\{S_i\}_{i \leq t}$ is *compatible* with the order $\prec$ if

$$i_0 \prec j_0 \Leftrightarrow i < j$$

for any $i \neq j \leq t$ and any $i_0 \in S_i, j_0 \in S_j$. Finally, we say that $f \in \mathbb{C}^S$ is *locally constant with respect to* $\prec$ if there exists a compatible partition $S_1 \sqcup ... \sqcup S_t$ of S, such that f is constant on S_i for any $i \leq t$.

We call a splitting Borel subalgebra $\mathfrak{b}_S \supset \mathfrak{h}_S$ of $\mathfrak{sl}(S)$ *ideal* if there is a partition $S = S_1 \sqcup S_2 \sqcup S_3$, compatible with the order $\prec$ defined by $\mathfrak{b}_S$, such that

(a) S_1 is countable and $\prec$ restricted to S_1 is isomorphic to the standard order on $\mathbb{Z}_{>0}$,

(b) S_3 is countable and $\prec$ restricted to S_3 is isomorphic to the standard order on $\mathbb{Z}_{<0}$

(S_2 may be empty). Clearly the Borel subalgebra defined by the above order (iii) is ideal, while the Borel subalgebras defined by (i) and (ii) are not ideal.

3. Highest weight $\mathfrak{sl}(S)$-modules

Fix a splitting Borel subalgebra $\mathfrak{b}_S$ of $\mathfrak{sl}(S)$, corresponding to a linear order $\prec$ on S. A *Verma module* is defined as an induced module

$$M_{\mathfrak{b}_S}(f) := \mathrm{U}\big(\mathfrak{sl}(S)\big) \otimes_{\mathrm{U}(\mathfrak{b}_S)} \mathbb{C}_f,$$

where $\mathbb{C}_f$ is a one-dimensional $\mathfrak{b}_S$-module determined by a weight $\lambda_f \in \mathfrak{h}_S^*$. By definition, a $\mathfrak{b}_S$-*highest weight module* is an $\mathfrak{sl}(\infty)$-module isomorphic to a quotient of $M_{\mathfrak{b}_S}(f)$. It is not difficult to prove that $M_{\mathfrak{b}_S}(f)$ has a unique simple quotient $L_{\mathfrak{b}_S}(f)$, see [DP1].

As S and $\mathfrak{b}_S$ are fixed, in the rest of Section 3 we write simply $M(f)$ and $L(f)$ instead of $M_{\mathfrak{b}_S}(f)$ and $L_{\mathfrak{b}_S}(f)$. We fix also a function $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ and a highest weight vector v of $L(f)$. For any subset $S' \subset S$ we denote by $\widehat{L}(f|_{S'})$ the $\mathfrak{sl}(S')$-submodule of $L(f)$ generated by v. Obviously $\widehat{L}(f|_{S'})$ is a quotient of $M(f|_{S'})$, and $L(f|_{S'})$ is the unique simple quotient of $\widehat{L}(f|_{S'})$.

For any finite subset $F \subset S$, let w_F be a fixed highest weight vector in $M(f|_F)$, and let v_F be its image in $L(f|_F)$. Let $F \subset F' \subset S$ be two finite subsets. Then there exists a unique morphism of $\mathfrak{sl}(F)$-modules

$$\psi_{F,F'} : M(f|_F) \to L(f|_{F'})$$

such that $w_F \mapsto v_{F'}$. It is clear that if $F'' \supset F' \supset F$ then $\ker \psi_{F,F''} \subset \ker \psi_{F,F'}$. Since the $\mathfrak{sl}(F)$-module $M(f|_F)$ has finite length, there exists a sufficiently large finite set $\bar{F} \supset F$ such that $\ker \psi_{F,\bar{F}} \subset \ker \psi_{F',F}$ for any finite set $F' \supset F$. We put $\psi_F := \psi_{F,\bar{F}}$.

Proposition 1. *The $\mathfrak{sl}(F)$-module $\widehat{L}(f|_F)$ is isomorphic to the image of ψ_F.*

Proof. Let $F \subset F'$ be two finite subsets of S. There exists a finite subset $\bar{\bar{F}} \subset S$ such that $F \subset \bar{\bar{F}}$, $\ker \psi_{F,\bar{\bar{F}}} = \ker \psi_F$, and $\ker \psi_{F',\bar{\bar{F}}} = \ker \psi_{F'}$. Then $\operatorname{im} \psi_{F,\bar{\bar{F}}}$ is isomorphic to $\operatorname{im} \psi_F$ and is equal to $\mathrm{U}(\mathfrak{sl}(F)) \cdot v_{\bar{\bar{F}}}$, and $\operatorname{im} \psi_{F',\bar{\bar{F}}}$ is isomorphic to $\operatorname{im} \psi_{F'}$ and is equal to $\mathrm{U}(\mathfrak{sl}(F')) \cdot v_{\bar{\bar{F}}}$. This defines an embedding of $\operatorname{im} \psi_F$ to $\operatorname{im} \psi_{F'}$ such that $\psi_F(w_F) \mapsto \psi_{F'}(w_{F'})$.

The limit of the direct system of such morphisms over all finite subsets F of S defines an $\mathfrak{sl}(\infty)$-module $\tilde{L}(f)$. Clearly, the direct limit of the vectors $\psi_F(w_F)$ is a highest weight vector of weight λ_f in $\tilde{L}(f)$. Denote this vector by $\tilde{v}$. We claim that $\tilde{L}(f)$ is isomorphic to $L(f)$. For the proof we provide two $\mathfrak{sl}(\infty)$-morphisms $L(f) \to \tilde{L}(f)$ and $\tilde{L}(f) \to L(f)$ such that $v \mapsto \tilde{v}$ and $\tilde{v} \mapsto v$ respectively.

The morphism $\tilde{L}(f) \to L(f)$ arises from the fact that $\tilde{L}(f)$ is a highest weight module with highest weight λ_f. We may assume that under this morphism $\tilde{v}$ goes to v (in general, $\tilde{v}$ maps to some vector proportional to v). Now we construct a morphism $L(f) \to \tilde{L}(f)$. For any set F we pick $\bar{F}$ as described above and consider the chain

$$M(f|_F) \to \widehat{L}(f|_F) \to \widehat{L}(f|_{\bar{F}}) \to L(f|_{\bar{F}})$$

of $\mathfrak{sl}(F)$-morphisms whose composition is ψ_F. This defines an $\mathfrak{sl}(F)$-morphism $\widehat{L}(f|_F) \to \operatorname{im} \psi_F$. By passing to the direct limit, we obtain the desired $\mathfrak{sl}(\infty)$-morphism $L(f) \to \tilde{L}(f)$.

Since the $\mathfrak{sl}(F)$-submodule of $\tilde{L}(f)$ generated by the image of v in $\tilde{L}(f)$ is isomorphic to $\operatorname{im} \psi_F$, the proposition is proved.

Any compatible partition $S_1 \sqcup ... \sqcup S_t$ of S defines a parabolic subalgebra of $\mathfrak{sl}(S)$: this is the algebra $\mathfrak{p}$ with root decomposition

$$\mathfrak{h}_S \oplus \bigoplus_{i \prec j \text{ or } i \neq j \in S_k \text{ for some } k} \mathbb{C}e_{ij}.$$

We set $\mathfrak{p} = \mathfrak{l} \dotplus \mathfrak{n}$, where

$$\mathfrak{l} := \mathfrak{h}_S + \bigoplus_{i \in S_k} \mathfrak{sl}(S_i), \quad \mathfrak{n} := \bigoplus_{e_{ij} \notin \mathfrak{l}, i \prec j} \mathbb{C}e_{ij}.$$

Set also $\mathfrak{n}^- := \bigoplus_{e_{ij} \notin \mathfrak{l}, j \prec i} \mathbb{C}e_{ij}$.

Proposition 2. *Let $S_1 \sqcup ... \sqcup S_t$ be a compatible partition of S, and $f \in \mathbb{C}^S$ be a function such that*

$$f(k') - f(l') \notin \mathbb{Z}$$

for all $k' \in S_i, l' \in S_l$ where $i < l$. Then $L(f)$ is isomorphic to

$$\mathrm{U}(\mathfrak{sl}(S)) \otimes_{\mathrm{U}(\mathfrak{p})} L(f)^{\mathfrak{n}},$$

where $L(f)^{\mathfrak{n}}$ stands for the $\mathfrak{n}$-invariants of $L(f)$. Moreover, as an $\mathfrak{sl}(S_1) \oplus ... \oplus \mathfrak{sl}(S_t)$-module, $L(f)^{\mathfrak{n}}$ is isomorphic to

$$L(f|_{S_1}) \otimes ... \otimes L(f|_{S_t}).$$

Proof. We set $L_{\mathfrak{p}} := U(\mathfrak{l}) \cdot v$. Standard arguments show that $L_{\mathfrak{p}}$ is a simple $\mathfrak{l}$-module and that $L_{\mathfrak{p}} = L(f)^{\mathfrak{n}}$. Therefore we have a natural surjective $\mathfrak{sl}(S)$-morphism

$$\alpha : U(\mathfrak{sl}(S)) \otimes_{U(\mathfrak{p})} L_{\mathfrak{p}} \to L(f).$$

We claim that α is an isomorphism. For this it suffices to show that

$$\beta : U(\mathfrak{n}^-) \otimes_{\mathbb{C}} L_{\mathfrak{p}} \to L(f) \quad (n \otimes l \mapsto nl)$$

is injective. However, the injectivity of β follows from the fact established above that the natural map $\widehat{L}(f|_F) \to L(f|_{\bar{F}})$ is an injection for any finite subset $F \subset S$.

Next, one notes that the simplicity of $L_{\mathfrak{p}}$ as a $\mathfrak{l}$-module implies its simplicity as an $[\mathfrak{l}, \mathfrak{l}]$-module. This follows from the fact that any $\mathfrak{h}$-weight space of $L_{\mathfrak{p}}$ is also an $(\mathfrak{h} \cap [\mathfrak{l}, \mathfrak{l}])$-weight space. Then, since $L(f|_{S_1}) \otimes ... \otimes L(f|_{S_t})$ and $L_{\mathfrak{p}}$ are simple $([\mathfrak{l}, \mathfrak{l}] \cap \mathfrak{b}_S)$-highest weight $(\mathfrak{sl}(S_1) \oplus ... \oplus \mathfrak{sl}(S_t) = [\mathfrak{l}, \mathfrak{l}])$-modules with the same highest weight, they are isomorphic.

We say that an $\mathfrak{sl}(S)$-module M is *integrable*, if

$$\dim(U(\mathfrak{g}) \cdot m) < \infty$$

for any $m \in M$ and any finite-dimensional Lie subalgebra $\mathfrak{g} \subset \mathfrak{sl}(S)$.

Proposition 3. *Let $S_1 \sqcup ... \sqcup S_t$ be a partition of S compatible with $\prec$. Then the $\mathfrak{sl}(S)$-module $L(f)$ is $\mathfrak{sl}(S_k)$-integrable if and only if $f|_{S_k} \in \mathbb{C}^{S_k}$ is dominant.*

Proof. Assume that $L(f)$ is $\mathfrak{sl}(S_k)$-integrable. Then the $(\mathfrak{sl}(2) \cong \mathfrak{sl}(\{i,j\}))$-module $U(\mathfrak{sl}(\{i,j\})) \cdot v$ is integrable for $i, j \in S_k$. Hence $f(i) - f(j) \in \mathbb{Z}_{\geq 0}$ for $i, j \in S_k$ by a well-known statement about $\mathfrak{sl}(2)$.

Now we wish to prove that if $f|_{S_k}$ is dominant, then $L(f)$ is $\mathfrak{sl}(S_k)$-integrable. Clearly, it suffices to show that $\widehat{L}(f|_F)$ is $\mathfrak{sl}(S_k \cap S)$-integrable for any finite subset $F \subset S$. According to Proposition 1, $\widehat{L}(f|_F) \cong \text{im } \psi_F$. The fact that $\text{im } \psi_F$ is $\mathfrak{sl}(S_k \cap S)$-integrable follows from the well-known fact, concerning modules over finite-dimensional Lie algebras, that, for any finite subset $F' \subset S$, the $\mathfrak{sl}(F')$-module $L(f|_{F'})$ is $\mathfrak{sl}(S_k \cap F')$-integrable.

Corollary 4. *The $\mathfrak{sl}(S)$-module $L(f)$ is integrable if and only if $f \in \mathbb{C}^S$ is dominant.*

Corollary 5. *Assume that f is locally constant with respect to a compatible partition $S_1 \sqcup ... \sqcup S_t$ of S. Then $L(f)$ is an integrable $\mathfrak{sl}(S_i)$-module for any $i \leq t$.*

4. Ideals of $U(\mathfrak{sl}(\infty))$

Let I be an ideal of $U(\mathfrak{sl}(\infty))$. Under an ideal we always mean a two-sided ideal. Fix an exhaustion

$$\mathfrak{sl}(2) \hookrightarrow \mathfrak{sl}(3) \hookrightarrow ... \hookrightarrow \mathfrak{sl}(n) \hookrightarrow \mathfrak{sl}(n+1) \hookrightarrow ... \tag{2}$$

of $\mathfrak{sl}(\infty)$. Then $I = \varinjlim_{n \geq 2}(I \cap U(\mathfrak{sl}(n)))$. Set $I_n := I \cap U(\mathfrak{sl}(n))$. To the ideal I_n we assign a closed subvariety $\text{Var } I_n \subset \mathfrak{sl}(n)^*$ as follows. Consider the standard filtration $\{U^{\leq d}\}_{d \in \mathbb{Z}_{\geq 0}}$ on $U(\mathfrak{sl}(n))$. The associated graded algebra

$$\text{gr}U(\mathfrak{sl}(n)) := \oplus_{d \in \mathbb{Z}_{\geq 0}}(U^{\leq d} / U^{\leq d-1})$$

is isomorphic to $\mathbf{S}^{\cdot}(\mathfrak{sl}(n))$. We denote by $\mathrm{Var}\,I_n$ the set of common zeros in $\mathfrak{sl}(n)^*$ of the associated graded ideal $\mathrm{gr}I := \oplus_{d \geq 0}(I \cap \mathrm{U}^{\leq d}/I \cap \mathrm{U}^{\leq d-1})$, and call $\mathrm{Var}\,I_n$ the *associated variety of I_n*. By identifying $\mathfrak{sl}(n)$ and $\mathfrak{sl}(n)^*$ via the Killing form we can assume that $\mathrm{Var}\,I_n \subset \mathfrak{sl}(n)$.

For any positive integer r we introduce the varieties

$$\mathfrak{sl}(n)^{\leq r} := \{x \in \mathfrak{sl}(n) \mid \exists \lambda \in \mathbb{C} \text{ such that } \mathrm{rk}(x - \lambda) \leq r\},$$

where λ is understood as a scalar $n \times n$-matrix. One can easily see that $\mathfrak{sl}(n)^{\leq r}$ is an $SL(n)$-stable subvariety of $\mathfrak{sl}(n)$.

The following theorem reproduces the claim of [PP, Corollary 6.2 b)] for $\mathfrak{sl}(\infty)$.

Theorem 6. *For any nonzero ideal $I \subset \mathrm{U}(\mathfrak{sl}(\infty))$ such that $\mathrm{Var}\,I_n \neq 0$ for some n, there exists a positive integer r such that $\mathrm{Var}\,I_n = \mathfrak{sl}(n)^{\leq r}$ for any $n \geq 2$.*

5. Integrable ideals and coherent local systems

We say that an ideal $I \subset \mathrm{U}(\mathfrak{sl}(\infty))$ is *integrable*, if I is the annihilator of an integrable $\mathfrak{sl}(\infty)$-module. Integrable ideals are closely connected with coherent local systems of modules which we define next.

Let Irr_n denote the set of isomorphism classes of simple finite-dimensional $\mathfrak{sl}(n)$-modules.

Definition 1. *A coherent local system of modules* (further shortened as *c.l.s.*) for $\mathfrak{sl}(\infty) = \varinjlim \mathfrak{sl}(n)$ is a collection of subsets

$$\{Q_n\}_{n \in \mathbb{Z}_{\geq 2}} \subset \prod_{n \in \mathbb{Z}_{\geq 2}} \mathrm{Irr}_n$$

such that $Q_m = \langle Q_n \rangle_m$ for any $n > m$, where $\langle Q_n \rangle_m$ denotes the set of isomorphism classes of all simple $\mathfrak{sl}(m)$-constituents of the $\mathfrak{sl}(n)$-modules from Q_n.

A c.l.s. $\{Q_n\}_{n \in \mathbb{Z}_{>0}}$ is *proper* if $Q_n \neq \mathrm{Irr}_n$ for some n.

A. Zhilinskii [Zh2, Zh3] has classified c.l.s. for $\mathfrak{sl}(\infty)$ and more generally for any locally simple Lie algebra. Moreover, if Q is a c.l.s., then

$$I(Q_m) := \cap_{z \in Q_m} \mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(m))}\, z \subset \cap_{z \in Q_n} \mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(n))}\, z =: I(Q_n)$$

for any $n > m$. Therefore $\cup_n(\cap_{z \in Q_n} \mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(n))}\, z) = \cup_n I(Q_n)$ is an ideal of $\mathrm{U}(\mathfrak{sl}(\infty))$; we denote it by $I(Q)$. It follows from [Zh2, Lemma 1.1.2] that $I(Q)$ is integrable. Moreover, $I(Q) \neq 0$ if Q is proper.

It turns out that Zhilinskii's classification of c.l.s. yields a classification of integrable ideals of $\mathrm{U}(\mathfrak{sl}(\infty))$. In this paper we present only the classification of c.l.s. For the classification of integrable ideals see [PP, Theorem 7.9].

A c.l.s. Q is *irreducible* if $Q \neq Q' \cup Q''$ with $Q' \not\subset Q''$ and $Q'' \not\subset Q'$. The following proposition clarifies the role of the irreducible c.l.s.

Proposition 7. *a) If Q is a proper irreducible c.l.s., then $I(Q)$ is the annihilator of a simple $\mathfrak{sl}(\infty)$-module. In particular, $I(Q)$ is primitive and hence prime.*

b) If I is an integrable prime ideal of $\mathrm{U}(\mathfrak{sl}(\infty))$, then $I = I(Q)$ for an irreducible c.l.s. Q.

Proof. Part a) follows directly from [Zh2, Lemma 1.1.2]. Part b) is a consequence of [PP, Theorem 7.9].

Fix n. The set Irr_n is parametrized by the lattice of integral dominant weights of $\mathfrak{sl}(n)$. Let z_1, z_2 be isomorphism classes of simple $\mathfrak{sl}(n)$-modules with respective highest weights λ_1, λ_2. We denote by $z_1 z_2$ the isomorphism class of the simple module with highest weight $\lambda_1 + \lambda_2$. If $S_1, S_2 \subset \mathrm{Irr}_n$ we set

$$S_1 S_2 := \{z \in \mathrm{Irr}_n \mid z = z_1 z_2 \text{ for some } z_1 \in S_1 \text{ and } z_2 \in S_2\}.$$

If Q', Q'' are c.l.s., we denote by $Q'Q''$ the smallest c.l.s. such that

$$(Q')_n (Q'')_n \subset (Q'Q'')_n.$$

However, by [Zh2] we have an equality

$$(Q')_n (Q'')_n = (Q'Q'')_n.$$

Zhilinskii's classification of c.l.s.. In this subsection we reproduce Zhilinskii's classification of c.l.s. for $\mathfrak{sl}(\infty)$ [Zh1]. Any integrable $\mathfrak{sl}(\infty)$-module M determines a c.l.s. $Q := \{Q_n\}_{n \in \mathbb{Z}_{>0}}$, where

$$Q_n := \{z \in \mathrm{Irr}_n \mid \mathrm{Hom}(z, M) \neq 0\}.$$

We set $\mathrm{cls}(M) := Q$. Moreover, $\mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} M = I(\mathrm{cls}(M))$. We construct an irreducible c.l.s. as the c.l.s. of some explicitly given integrable $\mathfrak{sl}(\infty)$-module.

Let $V(\infty)$ denote a vector space with basis $\{e_i\}_{i \in \mathbb{Z}_{>0}}$. We endow $V(\infty)$ with an action of $\mathfrak{sl}(\infty)$ by putting

$$e_{ij} \cdot e_k = e_i \delta_{jk}, \quad (e_{ii} - e_{jj}) \cdot e_k = e_i \delta_{ik} - e_j \delta_{jk} \quad \text{for } i, j, k \in \mathbb{Z}_{>0}.$$

In this way $V(\infty)$ becomes a simple integrable $\mathfrak{sl}(\infty)$-module, and we call it the *natural $\mathfrak{sl}(\infty)$-module*. By $V(\infty)_*$ we denote the restricted dual to $V(\infty)$, i.e., the $\mathfrak{sl}(\infty)$-submodule of $V(\infty)^*$ spanned by the vectors $\{e_i^*\}_{i \in \mathbb{Z}_{>0}}$ which satisfy

$$e_i^*(e_j) = \delta_{ij}.$$

Any proper irreducible c.l.s. Q for $\mathfrak{sl}(\infty)$ is a product of the following *basic* c.l.s.:

$$\mathcal{E} := \mathrm{cls}(\Lambda^{\cdot} V(\infty)), \quad \mathcal{L}_p := \mathrm{cls}(\Lambda^p V(\infty)), \quad \mathcal{L}_p^\infty := \mathrm{cls}(\mathbf{S}^{\cdot}(V(\infty) \otimes \mathbb{C}^p)),$$
$$\mathcal{R}_q := \mathrm{cls}(\Lambda^q V(\infty)_*), \quad \mathcal{R}_q^\infty := \mathrm{cls}(\mathbf{S}^{\cdot}(V(\infty)_* \otimes \mathbb{C}^q)),$$

where $p, q \in \mathbb{Z}_{\geq 0}$. More precisely, any proper irreducible c.l.s. is expressed uniquely as

$$(\mathcal{L}_v^\infty \mathcal{L}_{v+1}^{x_{v+1}} \mathcal{L}_{v+2}^{x_{v+2}} ... \mathcal{L}_n^{x_n}) \ \mathcal{E}^m \ (\mathcal{R}_w^\infty \mathcal{R}_{w+1}^{z_{w+1}} \mathcal{R}_{w+2}^{z_{w+2}} ... \mathcal{R}_l^{z_l}), \tag{3}$$

where

$$m, n, l, v, w, x_i, z_j \in \mathbb{Z}_{\geq 0}.$$

Here, if $v = 0$ (respectively $w = 0$), then $\mathcal{L}_v^\infty$ (respectively $\mathcal{R}_w^\infty$) is assumed to be the identity (the c.l.s. consisting of the isomorphism class of the trivial 1-dimensional module at all levels). In [Zh2] the above formulas are called the *unique factorization property.*

6. C.l.s. of simple integrable highest weight modules

We start with the following definition.

Definition 2. A c.l.s. Q is *of finite type* if Q_n is finite for any n.

One can easily check that the irreducible c.l.s. of finite type are precisely the c.l.s. of the form (3) with $v = w = 0$.

Let $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ be an integral function. We assume that a linear order on $\mathbb{Z}_{>0}$ is fixed and therefore we use the notation of Section 3 for $S = \mathbb{Z}_{>0}$.

A result of Zhilinskii [Zh1, Lemma 2.3.1] (see also Proposition 12 below) implies that if $|f| = \infty$ then $\mathrm{cls}(L(f))$ is not a proper c.l.s.. If $|f| < \infty$, there are two values $a, b \in \mathbb{C}$ of f such that $a - b \in \mathbb{Z}_{\geq 0}$ is maximal. We set $s := a - b$. For any nonnegative integer $c \leq s$ we denote by d_c the multiplicity of the value $b + c$ of f (note that $d_c \in \mathbb{Z}_{\geq 0} \cup +\infty$). Let p be the smallest integer such that $d_p = +\infty$, and q be the largest integer such that $d_q = +\infty$ (if d_c is finite for all $0 \leq c \leq b - a$, we put $p = q = 0$).

Proposition 8. *a) For a $\prec$-dominant function $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ with $|f| < \infty$, we have*

$$\mathrm{cls}(L(f)) = \mathcal{L}_{d_0}\mathcal{L}_{d_0+d_1}...\mathcal{L}_{d_0+d_1+...+d_{p-1}}\mathcal{E}^{q-p}\mathcal{R}_{d_s}\mathcal{R}_{d_s+d_{s-1}}...\mathcal{R}_{d_s+...+d_{q+1}}. \quad (4)$$

b) A c.l.s. of the form (4) is of finite type.

c) Let $\mathfrak{b}^0$ be a fixed ideal Borel subalgebra of $\mathfrak{sl}(\infty)$. Then any irreducible c.l.s. of finite type is equal to $\mathrm{cls}(L_{\mathfrak{b}^0}(f^0))$ for an appropriate $\mathfrak{b}^0$-dominant function $f^0 \in \mathbb{C}^{\mathbb{Z}_{>0}}$.

Proof. First we prove part a). Recall that $L(f) = \varinjlim_{n \geq 2} L(f|_{\{1,...,n\}})$. Thus the coherent local system $\mathrm{cls}(L(f))$ is determined by the highest weights λ_n of the finite-dimensional $\mathfrak{sl}(n)$-modules $L(f|_{\{1,...,n\}})$. Such local systems have been considered by Zhilinskii [Zh1] and he provides an explicit algorithm which assigns to $\{\lambda_i\}$ a c.l.s. of the form (4). This implies a).

b) It is clear that any c.l.s. of the form (4) is a c.l.s. of finite type.

c) The ideal subalgebra $\mathfrak{b}^0$ defines a partition $S_1 \sqcup S_2 \sqcup S_3$ of $\mathbb{Z}_{>0}$ with fixed order preserving bijections $\mathbb{Z}_{>0} \to S_1$, $\mathbb{Z}_{<0} \to S_3$. We denote the image of $k \in \mathbb{Z}_{>0}$ in S_1 by k^1, and the image of $-k \in \mathbb{Z}_{<0}$ in S_3 by k^3.

It is clear that any c.l.s. Q of the form (3) with $v = w = 0$ can be presented in the form (4) for suitable integers $p, q, d_0, d_1, ..., d_{p-1}, d_s, d_{s-1}, ..., d_{q+1}$. Moreover, $\mathrm{cls}(L_{\mathfrak{b}^0}(f)) = Q$ for the $\mathfrak{b}^0$-dominant function $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ defined as follows:

$$f(i) := \begin{cases} p & \text{if } i \in S_2 \text{ or } i = k^1 \text{ for } k > d_0 + ... + d_{p-1}; \\ q & \text{if } i = k^3 \text{ for } k > d_s + ... + d_{q+1}; \\ j & \text{if } i = k^1 \text{ and } d_0 + ... + d_{j-1} < k \leq d_0 + ... + d_j \text{ for } j < p; \\ j & \text{if } i = k^3 \text{ and } d_s + ... + d_{j+1} < k \leq d_s + ... + d_j \text{ for } j > q. \end{cases}$$

3. Statements of Main Results

Theorem 9. *Let $\mathfrak{b} \supset \mathfrak{h}$ be a splitting Borel subalgebra of $\mathfrak{sl}(\infty)$, and $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$. Then $\mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} L_{\mathfrak{b}}(f) \neq 0$ if and only if f is almost integral and locally constant with respect to the linear order defined by $\mathfrak{b}$.*

Theorem 10. *The following conditions on a nonzero ideal I of $\mathrm{U}(\mathfrak{sl}(\infty))$ are equivalent:*

— $I = \mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} L_{\mathfrak{b}}(f)$ for some splitting Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$ and some function $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$;

— I is a prime integrable ideal of $\mathrm{U}(\mathfrak{sl}(\infty))$;

— $I = \mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} L_{\mathfrak{b}^0}(f^0)$ for some $f^0 \in \mathbb{C}^{\mathbb{Z}_{>0}}$, where $\mathfrak{b}^0$ is any fixed ideal Borel subalgebra.

Proposition 11. *If $\mathfrak{b}$ is a nonideal Borel subalgebra then there exists a prime integrable ideal I which does not arise as the annihilator of a simple $\mathfrak{b}$-highest weight $\mathfrak{sl}(\infty)$-module.*

We split the proof of Theorem 9 into two parts:

a) if $\mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} L_{\mathfrak{b}}(f) \neq 0$, then f is almost integral and locally constant;

b) if f is almost integral and locally constant with respect to the order defined by $\mathfrak{b}$, then

$$\mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} L_{\mathfrak{b}}(f) \neq 0.$$

Parts a) and b) of Theorem 9 are proved in Sections 4 and 5 respectively. Theorem 10 and Proposition 11 are proved in Section 6.

4. Proof of Theorem 9 a)

To prove Theorem 9 a), we fix a Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$ of $\mathfrak{sl}(\infty)$ and hence an order $\prec$ on $\mathbb{Z}_{>0}$. Throughout Sections 4 and 5 we suppress the dependence from $\mathfrak{b}$ and $\prec$ in all notation. We set $I(f) := \mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} L(f)$ for any $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$. Sometimes we consider the finite-dimensional Lie algebra $\mathfrak{sl}(n)$. In this case the fixed order $\{1, ..., n\}$ is the standard order, and $I(f) \subset \mathrm{U}(\mathfrak{sl}(n))$ is the annihilator of the simple $\mathfrak{sl}(n)$-module with highest weight λ_f for $f \in \mathbb{C}^n$.

Theorem 9a) follows from Propositions 12, 13 and 14 below.

Proposition 12. *Let $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$. If $I(f)$ is nonzero, then $|f| < \infty$.*

Proposition 13. *Let $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$. If $I(f)$ is nonzero, then f is almost integral.*

Proposition 14. *Let $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$. If $I(f)$ is nonzero, then f is locally constant with respect to $\prec$.*

We prove these propositions consecutively in Sections 1, 4 and 5. Clearly, Proposition 12 follows from Proposition 14, however we require Proposition 12 for the proof of Proposition 14. Propositions 13 and 14 rely on a version of the Robinson-Schensted algorithm which we present in Section 2.

1. Proof of Proposition 12

We start with some notation. Let $n \geq 2$ be a positive integer. For any ideal $I \subset \mathrm{U}(\mathfrak{sl}(n))$ we denote by $\mathrm{gr} I \subset \mathbf{S}^{\cdot}(\mathfrak{sl}(n))$ the associated graded ideal. By $\mathrm{Var}(I) \subset \mathfrak{sl}(n)^*$ we denote the set of zeros of $\mathrm{gr} I$.

The radical ideals of the center $\mathrm{ZU}(\mathfrak{sl}(n))$ of $\mathrm{U}(\mathfrak{sl}(n))$ are in one-to-one correspondence with $\mathfrak{G}_n$-invariant closed subvarieties of $\mathfrak{h}_n^*$, where $\mathfrak{h}_n$ is a fixed Cartan subalgebra of $\mathfrak{sl}(n)$ and $\mathfrak{G}_n$ is the symmetric group on n letters. Let I be an ideal

of $U(\mathfrak{sl}(n))$. Then $ZVar(I)$ denotes the subvariety of $\mathfrak{h}_n^*$ corresponding to the radical of the ideal $I \cap ZU(\mathfrak{sl}(n))$ of $ZU(\mathfrak{sl}(n))$. If $\{I_t\}$ is any collection of ideals in $U(\mathfrak{sl}(n))$, then

$$ZVar(\cap_t I_t) = \overline{\cup_t ZVar(I_t)}, \tag{5}$$

where here and below bar indicates Zariski closure.

Let $\phi : \{1, ..., n\} \to \mathbb{Z}_{>0}$ be an injective map. Slightly abusing notation, we denote by ϕ the induced homomorphism

$$\phi : U(\mathfrak{sl}(n)) \to U(\mathfrak{sl}(\infty)).$$

By $\mathrm{inj}(n)$ we denote the set of injective maps from $\{1, ..., n\}$ to $\mathbb{Z}_{>0}$, and by $\mathrm{inj}_0(n)$ the set of order preserving maps from $\{1, ..., n\}$ to $\mathbb{Z}_{>0}$ with respect to the standard order on $\{1, ..., n\}$ and the order $\prec$ on $\mathbb{Z}_{>0}$.

By $\mathfrak{sl}(\phi)$ we denote $\mathfrak{sl}(\mathrm{im}\,\phi) \subset \mathfrak{sl}(\infty)$. For any $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ we set $f_\phi := f \circ \phi$. Then $M(f_\phi) := M_{\mathfrak{b} \cap \mathfrak{sl}(\phi)}(f_\phi)$ and $L(f_\phi) := L_{\mathfrak{b} \cap \mathfrak{sl}(\phi)}(f_\phi)$ are well defined $(\mathfrak{b} \cap \mathfrak{sl}(\phi))$-highest weight $\mathfrak{sl}(\phi)$-modules. If f is dominant and $\phi \in \mathrm{inj}_0(n)$, then f_ϕ is $(\mathfrak{b} \cap \mathfrak{sl}(\phi))$-dominant.

Let $\phi \in \mathrm{inj}_0(n)$ and $\widetilde{M}(f)$ be any quotient of $M(f)$. It is well known that

$$ZVar(\mathrm{Ann}_{U(\mathfrak{sl}(\phi))} M(f_\phi)) = ZVar(\mathrm{Ann}_{U(\mathfrak{sl}(\phi))} \widetilde{M}(f_\phi)) =$$
$$ZVar(\mathrm{Ann}_{U(\mathfrak{sl}(\phi))} L(f_\phi)) = \mathfrak{G}_n(\rho_n + \lambda_{f_\phi}),$$

where $\rho_n \in \mathfrak{h}_n^*$ is the half-sum of positive roots.

Let $\mathfrak{g}$ be a Lie algebra. *The adjoint group of $\mathfrak{g}$* is the subgroup of $\mathrm{Aut}\,\mathfrak{g}$ generated by the exponents of all nilpotent elements of $\mathfrak{g}$. We denote this group by $\mathrm{Adj}\,\mathfrak{g}$.

Lemma 15. *Let $\phi_1 : \mathfrak{k} \to \mathfrak{g}$ and $\phi_2 : \mathfrak{k} \to \mathfrak{g}$ be two $\mathrm{Adj}\,\mathfrak{g}$-conjugate morphisms of Lie algebras. Let I be a two-sided ideal of $U(\mathfrak{g})$. Then*

$$\phi_1^{-1}(I) = \phi_2^{-1}(I).$$

Proof. The adjoint action of $\mathfrak{g}$ on $U(\mathfrak{g})$ extends uniquely to an action of $\mathrm{Adj}\,\mathfrak{g}$ on $U(\mathfrak{g})$. The ideal I is $\mathfrak{g}$-stable and thus is $\mathrm{Adj}\,\mathfrak{g}$-stable. Let $g \in \mathrm{Adj}\,\mathfrak{g}$ be such that $\phi_1 = g \circ \phi_2$. Then

$$\phi_1^{-1}(g(i)) = \phi_2^{-1}(i)$$

for any $i \in I$. Hence,

$$\phi_1^{-1}(I) = \phi_2^{-1}(I).$$

Proof of Proposition 12. Let $I(f) \neq 0$. Assume to the contrary that there exist $i_1, ..., i_s, ... \in \mathbb{Z}_{>0}$ such that

$$f(i_1), ..., f(i_s), ...$$

are pairwise distinct elements of $\mathbb{C}$. As $I(f) \neq U(\mathfrak{sl}(\infty))$, there exists a positive integer n and an injective map $\phi : \{1, ..., n\} \to \mathbb{Z}_{>0}$ such that

$$I_\phi := I(f) \cap U(\mathfrak{sl}(\phi)) \neq 0,$$

or equivalently

$$U(\mathfrak{sl}(n)) \supset \phi^{-1}(I(f)) = \phi^{-1}(I_\phi) \neq 0. \tag{6}$$

Let $\psi \in \mathrm{inj}(n)$ be another map. Since ϕ and ψ are conjugate via the adjoint group of $\mathfrak{sl}(\infty)$, we have

$$\phi^{-1}(I(f)) = \psi^{-1}(I(f)) \neq 0. \tag{7}$$

This means that $\phi^{-1}(I(f))$ depends on n and f but not on ϕ, and we set

$$I_n := \phi^{-1}(I(f)).$$

Assume now that $\phi \in \mathrm{inj}_0(n)$. Then the highest weight space of the $\mathfrak{sl}(\infty)$-module $L(f)$ generates a highest weight $\mathfrak{sl}(\phi)$-submodule $\widehat{L}(f_\phi)$. Clearly,

$$\mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\phi))} L(f) \subset \mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\phi))} \widehat{L}(f_\phi).$$

Therefore,

$$I_n \subset \cap_{\phi \in \mathrm{inj}_0(n)} \mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(n))} \widehat{L}(f_\phi)$$

and

$$I_n \cap \mathrm{ZU}(\mathfrak{sl}(n)) \subset \cap_{\phi \in \mathrm{inj}_0(n)} (\mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(n))} \widehat{L}(f_\phi) \cap \mathrm{ZU}(\mathfrak{sl}(n))).$$

Hence, according to (5) we have

$$\overline{\cup_{\phi \in \mathrm{inj}_0(n)} \mathfrak{G}_n(\rho_n + \lambda_{f_\phi})} = \mathfrak{G}_n(\rho_n + \overline{\cup_{\phi \in \mathrm{inj}_0(n)} \lambda_{f_\phi}}) \subset \mathrm{ZVar}(I_n).$$

We claim that

$$\overline{\mathfrak{G}_n(\cup_{\phi \in \mathrm{inj}_0(n)} \lambda_{f_\phi})} = \mathfrak{h}_n^*,$$

and thus that

$$\mathrm{ZVar}(I_n) = \mathfrak{h}_n^*. \tag{8}$$

Our claim is equivalent to the equality

$$\overline{\mathfrak{G}_n(\cup_{\phi \in \mathrm{inj}_0(n)} \lambda_{f_\phi})} = \overline{(\cup_{\phi \in \mathrm{inj}(n)} \lambda_{f_\phi})} = \mathfrak{h}_n^*$$

which is implied by the following equality:

$$\overline{(\cup_{\phi \in \mathrm{inj}(n)} f_\phi)} = \mathbb{C}^n. \tag{9}$$

We now prove (9) by induction. The inclusion $\{1, ..., n-j\} \to \{1, ..., n\}$ induces a restriction map

$$res : \mathbb{C}^n \to \mathbb{C}^{n-j}.$$

Denote by $f_\psi *$ the preimage of f_ψ under res for $\psi \in \mathrm{inj}(n-j)$. We will show that

$$f_\psi * \subset \overline{\cup_{\phi \in \mathrm{inj}(n)} f_\phi} \tag{10}$$

for any $j \leq n$ and any map $\psi \in \mathrm{inj}(n-j)$. This holds trivially for $j = 0$. Assume that it also holds for j. Fix $\psi \in \mathrm{inj}(n-j-1)$ and set

$$(\psi \times k)(l) := \begin{cases} \psi(l) & \text{if } l \leq n-j-1 \\ i_k & \text{if } l = n-j \end{cases}.$$

It is clear that there exists $s \in \mathbb{Z}_{\geq 1}$ such that

$$(\psi \times k) \in \mathrm{inj}(n - j)$$

for any $k \in \mathbb{Z}_{\geq s}$. Moreover, $f_{\psi \times k_1} \neq f_{\psi \times k_2}$ for any $k_1 \neq k_2$. Therefore

$$\overline{\cup_{k \in \mathbb{Z}_{\geq s}} f_{\psi \times k}*} = f_\psi *,$$

which yields (10).

For $j = n$, (10) yields $\mathbb{C}^n \subset \overline{\cup_{\phi \in \mathrm{inj}(n)} f_\phi}$, consequently (9) holds. Then (8) holds also, hence

$$I_n \cap \mathrm{ZU}(\mathfrak{sl}(n)) = 0.$$

It is a well known fact that an ideal of $\mathrm{U}(\mathfrak{sl}(n))$ whose intersection with $\mathrm{ZU}(\mathfrak{sl}(n))$ equals zero is the zero ideal [Dix, Proposition 4.2.2]. Therefore, we have a contradiction with (6), and the proof is complete.

2. Algorithm for $\mathfrak{sl}(n)$

According to Duflo's Theorem, any primitive ideal of $\mathrm{U}(\mathfrak{sl}(n))$ is the annihilator of some simple highest weight module, i.e., any primitive ideal is of the form $I(f)$ for some $f \in \mathbb{C}^n$. The associated variety of $I(f)$ is the closure of a certain nilpotent coadjoint orbit $\mathcal{O}(f)$ of $\mathfrak{sl}(n)$ [Jo4]. To $\mathcal{O}(f) \subset \mathfrak{sl}(n)^*$ one assigns a partition $p(f)$ of n as follows. One first represents $\mathcal{O}(f)$ by a nilpotent element $x \in \mathfrak{sl}(n)$. Then $p(f)$ is the partition conjugate to the partition arising from the sizes of Jordan blocks of x considered as a linear operator on the natural representation of $\mathfrak{sl}(n)$.

We now describe the algorithm which computes $p(f)$. This is a modification of the Robinson-Schensted algorithm, see [Knu, Theorem A on p. 52].

Let $f \in \mathbb{C}^n$ be a function.

Step 1) Set $f^+ := (f(1), f(2) - 1, ..., f(n) - n + 1)$.

Step 2) Introduce an equivalence relation $\sim$ on $\{1, ..., n\}$:

$$i \sim j \text{ if and only if } f(i) - f(j) \in \mathbb{Z}.$$

Let t be the number of equivalence classes for $\sim$, and let $n_1, ..., n_t$ be the cardinalities of the respective equivalence classes.

Step 3) Consider f^+ as a function $f^+ : \{1, .., n\} \to \mathbb{C}$. The restriction of f^+ to the equivalence classes of Step 2) defines subsequences

$$\mathrm{seq}_1(f^+), \mathrm{seq}_2(f^+), ..., \mathrm{seq}_t(f^+)$$

of respective lengths $n_1, ..., n_t$.

Step 4) Fix i. Note that the elements of $\mathrm{seq}_i(f^+)$ are linearly ordered as their pairwise differences are integers. Since the elements of $\mathrm{seq}_i(f^+)$ are not necessary pairwise distinct, we modify the above linear order by letting $f^+(m) \rhd f^+(k)$ if $m > k$ and $f^+(m) = f^+(k)$. In this way we introduce a new linearly ordered set $\widetilde{\mathrm{seq}_i(f^+)}$ of cardinality n_i.

Step 5) Apply the Robinson-Schensted algorithm to the linearly ordered sets $\widetilde{\mathrm{seq}_i(f^+)}$ from Step 4) to produce partitions p_i of n_i.

Step 6) Consider the partitions $p_1, p_2, ..., p_t$ as a partition $\mathrm{RS}(f)$ of n.

Proposition 16. *Let $f \in \mathbb{C}^n$ be a function. Then $p(f) = \mathrm{RS}(f)$.*

Proof. This statement is contained in the work of A. Joseph, so all we need to do is to translate Joseph's result to the language which we use in this paper. For any $f' \in \mathbb{C}^n$ set $(f')^{\#} := (f'(1), f'(2) + 1, ..., f'(n) + n - 1)$.

We note first that

$$I(f) = I((\mathrm{seq}_1(f^+), ..., \mathrm{seq}_t(f^+))^{\#}). \tag{11}$$

This is a translation of the equality

$$\mathrm{J}(\mathrm{w}_1\mathrm{w}_2\lambda) = \mathrm{J}(\mathrm{w}_2\lambda) \tag{12}$$

for appropriate choices of Weyl group elements $\mathrm{w}_1, \mathrm{w}_2$, as stated at the bottom of the first page of [Jo1] (the equality (12) uses the notation of A. Joseph which is slightly different from ours). Thus we can assume further that

$$f^+ = (\mathrm{seq}_1(f^+), \mathrm{seq}_2(f^+), ..., \mathrm{seq}_t(f^+)).$$

Next, using the well known fact that $p(f)$ is recovered uniquely from $p(\mathrm{seq}_i(f^+))$ for all i, we can suppose that $f^+ = \mathrm{seq}_1(f^+)$, i.e., that f is integral.

In the case when f^+ is regular, i.e. when $f^+(k) \neq f^+(l)$ for $k \neq l$, Joseph states [Jo3, Section 3.3] that $p(f)$ equals the shape of the output of the standard Robinson-Schensted algorithm [Knu, 5.1.4, proof of Theorem A] applied to the unique Weyl group element w such that $\mathrm{w}(f^+)$ is dominant. It is easy to check that this statement is equivalent to the claim that $p(f) = \mathrm{RS}(f)$ in this case.

In the case when f^+ is not regular, following Joseph [Jo1, Section 2.1] we replace f by any function f' such that $(f')^+$ is regular and f belongs to the upper closure $\widehat{F}_{f'}$ of a certain facette $F_{f'}$ containing f' [Jo1, Section 2.1]. In our terminology this means that f' and f satisfy the following conditions:

$$\text{if } f^+(i) > f^+(j), \text{ then } (f')^+(i) > (f')^+(j) \text{ for all } i, j \leq n; \tag{13}$$

$$\text{if } f^+(i) < f^+(j), \text{ then } (f')^+(i) < (f')^+(j) \text{ for all } i, j \leq n; \tag{14}$$

$$\text{if } f^+(i) = f^+(j) \text{ and } i < j, \text{ then } (f')^+(i) > (f')^+(j). \tag{15}$$

Then, according to Joseph, $p(f) = p(f')$ [Jo2, Section 2.4]. A direct checking using (13)-(15) and the above linear order $\lhd$ shows that in this case $p(f') = \mathrm{RS}(f)$.

Example 17. *Let $f = (\sqrt{2} - 1, 5, 9, \sqrt{2} + 3, 5, \sqrt{2} + 4, 7, 7) \in \mathbb{C}^8$.*
1) $f^+ = (\sqrt{2} - 1, 4, 7, \sqrt{2}, 1, \sqrt{2} - 1, 1, 0)$.
2-3) $\underbrace{\mathrm{seq}_1(f^+)} = (\sqrt{2}-1, \sqrt{2}, \sqrt{2}-1)$ $(n_1 = 3)$, $\underbrace{\mathrm{seq}_2(f^+)} = (4, 7, 1, 1, 0)$ $(n_2 = 5)$.
4) $\widetilde{\mathrm{seq}_1(f^+)} = \{(\sqrt{2} - 1)', \sqrt{2}, (\sqrt{2} - 1)''\}, \widetilde{\mathrm{seq}_2(f^+)} = \{4, 7, 1', 1'', 0\}$.
5) Applying the Robinson-Schensted algorithm we have

$$\widetilde{\mathrm{seq}_1(f^+)} \mapsto \begin{array}{|c|c|} \hline \sqrt{2} & (\sqrt{2} - 1)'' \\ \hline (\sqrt{2} - 1)' \\ \cline{1-1} \end{array} \mapsto (2, 1) \quad ,$$

$$\widetilde{\mathrm{seq}_2(f^+)} \mapsto \begin{array}{|c|c|c|} \hline 7 & 1'' & 0 \\ \hline 4 & 1' \\ \cline{1-2} \end{array} \mapsto (3, 2)$$

6) $p(\sqrt{2} - 1, 5, 9, \sqrt{2} + 3, 5, \sqrt{2} + 4, 7, 7) = (2, 1) \cup (3, 2) = (3, 2, 2, 1)$.

3. Rank of a partition

Let, as above, $\mathcal{O}(f) \subset \mathfrak{sl}(n)^*$ be the nilpotent coadjoint orbit of $\mathfrak{sl}(n)$ assigned to a function $f \in \mathbb{C}^n$. For $x \in \mathcal{O}(f)$, the rank of x is independent on x and equals $n - p(f)_{max}$, where $p(f)_{max}$ is the maximal element of the partition $p(f)$. By definition, the integer $p(f)_{max}$ is the *corank* of p.

Lemma 18. *Let $f \in \mathbb{C}^n$. The corank of $p(f)$ equals the length of a longest strictly decreasing subsequence of f^+ such that the difference between any two elements is an integer.*

Proof. It is obvious that the corank of $p(f)$ equals the maximum of coranks of $p_1, ..., p_t$, where $p_1, ..., p_t$ are the partitions defined in Step 5) of Section 2. It is known that for each i the corank of $p_i = p(\widetilde{\text{seq}_i(f^+)})$ equals to the length of a longest strictly decreasing subsequence [Knu, p. 69, Ex. 7] of $\widetilde{\text{seq}_i(f^+)}$. For some i_0 a longest strictly decreasing subsequence of $\widetilde{\text{seq}_{i_0}(f^+)}$ will also be a longest strictly decreasing subsequence of f^+ such that the difference between any two elements is an integer, and the lemma is proved.

4. Proof of Proposition 13

Proposition 13 is implied by the following two lemmas.

Lemma 19. *Let $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$. If $I(f) \neq 0$, there exists $r \in \mathbb{Z}_{\geq 0}$ such that any finite subset $F \subset \mathbb{Z}_{>0}$ has a subset $F' \subset F$ so that $f|_{F'}$ is integral and $|F \backslash F'| \leq r$.*

Lemma 20. *Fix $r \in \mathbb{Z}_{\geq 0}$. If for any finite subset $F \subset \mathbb{Z}_{>0}$ there is $F' \subset F$ so that $f|_{F'}$ is integral and $|F \backslash F'| \leq r$, then there is a finite subset $F \subset \mathbb{Z}_{>0}$ such that $f|_{\mathbb{Z}_{>0} \backslash F}$ is integral and $|F| \leq r$.*

Proof of Lemma 19. Due to the description of the corank of $p(f)$ presented in Lemma 18, Lemma 19 is implied by the following lemma.

Lemma 21. *Fix $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$. If $I(f) \neq 0$, then there exists $r \in \mathbb{Z}_{\geq 0}$ such that $\text{rk}\, p(f|_F) \leq r$ for any finite subset $F \subset \mathbb{Z}_{>0}$.*

Proof. Assuming that $I(f) \neq 0$, pick r as in Theorem 6. Let F be a finite subset of $\mathbb{Z}_{>0}$.

There is a nonzero homomorphism of $\mathfrak{sl}(F)$-modules $M(f|_F) \to L(f)$. Therefore, as $L(f|_F)$ is the unique simple quotient of $M(f|_F)$, $L(f|_F)$ is isomorphic to a subquotient of $L(f)$ considered as an $\mathfrak{sl}(F)$-module. This implies

$$(\text{U}(\mathfrak{sl}(F)) \cap I(f)) \cdot L(f|_F) = 0$$

and

$$\text{Var}(I(f) \cap \text{U}(\mathfrak{sl}(F))) \subset \mathfrak{sl}(F)^{\leq r}.$$

As all elements of $\text{Var}(I(f|_F))$ are nilpotent, we have $\text{rk}\, \mathcal{O}(f|_F) \leq r$, and thus $\text{rk}\, p(f|_F) \leq r$.

Proof of Lemma 20. We reduce the problem to a statement concerning the graph $\Gamma := (\mathbb{Z}_{>0}, E_f)$ attached to the pair $(\mathbb{Z}_{>0}, f)$ in the following way: the vertices of Γ are the elements of $\mathbb{Z}_{>0}$, E_f stands for the edges of Γ, and $i, j \in \mathbb{Z}_{>0}$ are connected by an edge if and only if $f(i) - f(j) \notin \mathbb{Z}$.

Lemma 20 is implied by the following lemma.

Lemma 22. *Let $\Gamma = (S, E)$ be a graph. Assume that there is $r \in \mathbb{Z}_{\geq 0}$ so that any finite subset $F \subset S$ decomposes into two subsets*

$$inf(F) \cup fin(F)$$

with the properties

$$a) \; \Gamma|_{inf(F)} \text{ has no edges}, \quad b) \; |fin(F)| \leq r. \tag{16}$$

Then S decomposes into the union of two subsets $inf(S) \cup fin(S)$ satisfying (16) with F replaced by S.

Proof. In what follows we say that a vertex of S is connected with another vertex if they belong to a common edge. Denote by $S^{>r}$ the set of vertices of S which belong to at least $r + 1$ edges. Respectively, let $S^{\leq r}$ be the set of vertices of S which belong to at most r edges. In addition, denote by $\underline{S}^{\leq r}$ the subset of $S^{\leq r}$ consisting of vertices connected with at least one vertex from $S^{\leq r}$.

We claim that both $S^{>r}$ and $\underline{S}^{\leq r}$ are finite and

$$i) \; |S^{>r}| \leq r, \quad ii) \; |\underline{S}^{\leq r}| \leq r^2. \tag{17}$$

First we show (17) under the assumption that $S^{>r}$ and $\underline{S}^{\leq r}$ are finite. Let $\widetilde{S}^{>r}$ be a finite subset of S such that
1) $S^{>r} \subset \widetilde{S}^{>r}$,
2) any vertex from $S^{>r}$ is connected with at least $r + 1$ vertices form $\widetilde{S}^{>r}$ (such a subset $\widetilde{S}^{>r}$ always exists).
A vertex $i \in inf(\widetilde{S}^{>r})$ can be connected only with vertices from $fin(\widetilde{S}^{>r})$, and hence $i \in S^{<r}$ by (16)a). Therefore,

$$inf(\widetilde{S}^{>r}) \subset S^{\leq r} \cap \widetilde{S}^{>r}.$$

This implies

$$S^{>r} \subset fin(\widetilde{S}^{>r}), \tag{18}$$

and since $|fin(\widetilde{S}^{>r})| < r$ by (16)b), we obtain (17)i).

To prove (17)ii), note that since any vertex of $fin(\underline{S}^{\leq r})$ belongs to at most r edges, the entire set $fin(\underline{S}^{\leq r})$ belongs to at most r^2 edges. As any vertex from $\underline{S}^{\leq r}$ is connected with a vertex from $fin(\underline{S}^{\leq r})$, we obtain (17)ii).

Now we drop the assumption that both $S^{>r}$ and $\underline{S}^{\leq r}$ are finite. Applying the preceding arguments we show that (17) holds if we replace $S^{>r}$ and $\underline{S}^{\leq r}$ by their intersections with any finite subset of S. Thus (17) holds also for $S^{>r}$ and $\underline{S}^{\leq r}$.

To finish the proof, we set

$$fin(S) := fin(\widetilde{S}^{>r} \cup \underline{S}^{\leq r}).$$

Then $|fin(S)| \leq r$ by (16)b. The same arguments we used to prove (18) imply

$$S^{>r} \subset fin(\widetilde{S}^{>r} \cup \underline{S}^{\leq r}) := S.$$

Due to the definition of $\underline{S}^{\leq r}$, any vertex from

$$S \backslash (\widetilde{S}^{>r} \cup \underline{S}^{\leq r})$$

can be connected only with vertices from $S^{>r}$. Thus $\Gamma|_{S \backslash fin(S)}$ has no edges, and the proof is complete.

5. Proof of Proposition 14

Lemma 23. *Fix $r \in \mathbb{Z}_{\geq 0}$. Let $f \in \mathbb{C}^{2r+2}$ be an integer valued function such that*

$$f(2i) > f(2i-1) \tag{19}$$

for $1 \leq i \leq r+1$. Then $\operatorname{rk} p(f) > r$.

Proof. Assume $\operatorname{rk} p(f) \leq r$. Then the sequence $f^+ = (f(1), f(2)-1, ..., f(n)-n+1)$ contains a strictly decreasing subsequence seq$'$ of length at least $r+2$. The set $\{1, ..., 2r+2\}$ is the disjoint union of $r+1$ pairs of the form $\{2i, 2i-1\}$, hence for some i both $f(2i-1) - (2i-1) + 1$ and $f(2i) - 2i + 1$ belong to seq$'$. On the other hand,

$$f(2i-1) - (2i-1) + 1 \leq f(2i) - 2i + 1$$

by (19), thus seq$'$ is not strictly decreasing. This contradiction shows that $\operatorname{rk} p(f) > r$. □

Proof of Proposition 14. Assume that $I(f) \neq 0$ and pick r as in Lemma 21. Using Proposition 12 and Proposition 13, we reduce Proposition 14 to the following statement:

If an integer valued function $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ takes finitely many distinct values and there exists $r \in \mathbb{Z}_{\geq 0}$ such that $\operatorname{rk} p(f|_F) \leq r$ for any finite subset $F \subset \mathbb{Z}_{>0}$, then f is locally constant.

We prove this statement by induction on $|f|$. The base of induction $(|f| = 1)$ is trivial.

Assume that the statement holds for $|f| = n \geq 1$, and let f be a function which takes precisely $(n+1)$ values. Let M be the maximal value of f. Say that $i, j \in \mathbb{Z}_{>0}, i \neq j$, are equivalent whenever one of the following conditions hold:
  1) $i \prec j$, $f(i) = f(j) = M$, and $f(s) = M$, for any s, $i \prec s \prec j$;
  2) $i \prec j$, $f(i) < M$, $f(j) < M$, and $f(s) < M$, for any s, $i \prec s \prec j$.
It is easy to see that this this is a well defined equivalence relation on $\mathbb{Z}_{>0}$. There are two possibilities for the respective equivalence classes S_α:
  a) $f(s) = M$ for any $s \in S_\alpha$;
  b) $f(s) < M$ for any $s \in S_\alpha$.
We claim that there exist no more than $r+1$ equivalence classes of type b). Assume to the contrary that $s_0 \prec s_2 \prec ... \prec s_{2r+2}$ are elements from $r+2$ distinct equivalence classes of type b). Then, for any i, $0 \leq i \leq r$, there exists $s_{2i+1} \in S$ such that

$$f(s_{2i+1}) = M \text{ and } s_{2i} \prec s_{2i+1} \prec s_{2i+2}.$$

The restriction of f to the set $F := \{s_0, s_1, ..., s_{2r+2}\}$ satisfies the assumption of Lemma 23. Hence $\operatorname{rk} p(f|_F) > r$, which contradicts the statement of Lemma 21.

Therefore, there are at most $r + 1$ equivalence classes S_α of type b). Any two classes of type a) must be separated by a class of type b), and hence there are at most $r + 2$ equivalence classes of type a). In particular the partition $\sqcup_\alpha S_\alpha = \mathbb{Z}_{>0}$ is finite.

Clearly, f takes at most n values on each S_α. By the induction assumption each S_α admits a compatible partition such that $f|_{S_\alpha}$ is locally constant. Therefore, f is also locally constant.

5. Proof of Theorem 9 b)

Theorem 9b) is a corollary of the following result.

Proposition 24. *Let $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ be a locally constant and almost integral function. Then there is a nonzero integrable ideal I of $\mathrm{U}(\mathfrak{sl}(\infty))$ such that $I \subset I(f)$.*

We will prove a more precise version of this result. Let $S_1 \sqcup ... \sqcup S_t = \mathbb{Z}_{>0}$ be a fixed finite partition of $\mathbb{Z}_{>0}$ compatible with the order $\prec$. Denote by $S_{i_1}, ..., S_{i_x}$ all infinite sets in this partition. By γ we denote the total number of elements in the finite sets of the partition. Let $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ be a function locally constant with respect to the partition $S_1 \sqcup ... \sqcup S_t$. It is easy to see that $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ is almost integral if and only if $f(j) - f(k) \in \mathbb{Z}$ for any $j \in S_{j'}$ and $k \in S_{k'}$ such that both $S_{j'}$ and $S_{k'}$ are infinite. Under the assumption that f is almost integral, we set

$$
\begin{aligned}
\alpha(f) &:= \sum_{1 \leq j < x} \max(0, f(S_{i_{j+1}}) - f(S_{i_j})), \\
\mathrm{A}(f) &:= \sum_{1 \leq j < x} \max(f(S_{i_j}) - f(S_{i_{j+1}}),\ 0),
\end{aligned}
\tag{20}
$$

where $f(S_i)$ is the value of f on any element of S_i (we recall that f is constant on S_i).

The following proposition is a more precise version of Proposition 24 and compares the annihilator of a simple highest weight module with the annihilator of a c.l.s. We will prove it by first establishing a finite-dimensional analogue, namely Proposition 26, and then showing that Proposition 25 actually reduces to this finite-dimensional analogue.

Proposition 25. *Let $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ be a function, locally constant with respect to the partition $S_1 \sqcup ... \sqcup S_t$ of $\mathbb{Z}_{>0}$. Then*

$$
I(\mathcal{L}^\infty_{(\alpha(f)+\gamma)} \mathcal{E}^{\mathrm{A}(f)}) \subset I(f).
$$

Let F be a finite subset of $\mathbb{Z}_{>0}$. Clearly,

$$
(S_1 \cap F) \sqcup ... \sqcup (S_t \cap F)
$$

is a partition of F. We wish to define $\alpha(f')$ and $\mathrm{A}(f')$ by formulas analogous to (20) for any function $f' \in \mathbb{C}^F$ which is locally constant with respect to the partition $(S_1 \cap F) \sqcup ... \sqcup (S_t \cap F)$. For this purpose we denote by S'_1 the first S_{i_j}

for which $S_{i_j} \cap F \neq \emptyset$, by S_2' the second S_{i_j} for which $S_{i_j} \cap F \neq \emptyset$ and so on. Then we define $\alpha(f')$ and $A(f')$ by the respective right-hand sides of (20) applied to the subsets $(S_1' \cap F), (S_2' \cap F), ...$ instead of $S_{i_1}, S_{i_2}, ...$ Finally, $\gamma(F)$ stands for the total number of elements in all intersections $S_i \cap F$ for finite sets S_i. For a large enough F we have $\gamma(F) = \gamma, A(f|_F) = A(f), \alpha(f|_F) = \alpha(f)$.

Proposition 26. *Let $F \subset \mathbb{Z}_{>0}$ be a finite subset with n elements, and $f' \in \mathbb{C}^F$ be a function locally constant with respect to the partition $(S_1 \cap F) \sqcup ... \sqcup (S_t \cap F) = F$. Then*

$$I((\mathcal{L}^\infty_{(\alpha(f')+\gamma(F))}\mathcal{E}^{A(f')})_n) \subset I(f').$$

For the proof of Proposition 26 we need two lemmas (Lemmas 27 and 28 below) and some more notation. In Lemma 27 $f = (f_1, ..., f_n)$ stands for a function $f \in \mathbb{C}^n$. We set $L(f_1, ..., f_n) := L(f)$ and $I(f_1, ..., f_n) := I(f)$ (where the fixed order on $\{1, 2, ..., n\}$ is the standard one). For a fixed nonnegative integer $s < n$ and $z_0 \in \mathbb{C}$, we put:

$$\tilde{f} = (f_1, ..., f_s, z_0, f_{s+1}, ..., f_n) \in \mathbb{C}^{n+1}.$$

If A, B are two subsets of Irr_n, $A \otimes B$ stands for the set of isomorphism classes of all simple constituents of the tensor products $\alpha \otimes \beta$ for $\alpha \in A$ and $\beta \in B$.

Lemma 27. *Let Q_n be a subset of Irr_n such that*

$$I(Q_n) \subset I(f_1, ..., f_s, f_{s+1} - 1, ..., f_n - 1).$$

Then

$$I((\mathcal{L}^\infty_1)_n \otimes Q_n) \subset I(\tilde{f}),$$

$I(Q_n \otimes (\mathcal{L}^\infty_1)_n)$ being an ideal of $\mathrm{U}(\mathfrak{sl}(n))$ and $I(\tilde{f})$ being an ideal of $\mathrm{U}(\mathfrak{sl}(n+1))$.

Proof. Our idea is to replace z_0 by a "generic value". To do this, consider the supplementary Lie algebras

$$\mathfrak{sl}(n+1)[z] := \mathfrak{sl}(n+1) \otimes_{\mathbb{C}} \mathbb{C}[z] \subset \mathfrak{sl}(n+1)(z) := \mathfrak{sl}(n+1) \otimes_{\mathbb{C}} \overline{\mathbb{C}(z)},$$

the larger Lie algebra $\mathfrak{sl}(n+1)(z)$ being finite dimensional and simple over the algebraically closed field $\overline{\mathbb{C}(z)}$. The sequence $\hat{f} := (f_1, ..., f_s, z, f_{s+1}, ..., f_n)$ of elements of $\overline{\mathbb{C}(z)}$ defines a weight $\lambda_{\hat{f}} \in \mathfrak{h}^*_{n+1} \otimes \overline{\mathbb{C}(z)}$.

Applying the equality (11) to $\hat{f}$, we obtain

$$I(\hat{f}) = I(f_1, ...f_s, f_{s+1} - 1, f_{s+2} - 1, ..., f_n - 1, z + n - s).$$

By Proposition 2, we have

$$L(f_1, ...f_s, f_{s+1} - 1, f_{s+2} - 1, ..., f_n - 1, z + n - s) \cong$$
$$\cong \mathrm{U}(\mathfrak{sl}(n+1)(z)) \otimes_{\mathrm{U}(\mathfrak{p})} L(f_1, ...f_s, f_{s+1} - 1, f_{s+2} - 1, ..., f_n - 1, z + n - s)^{\mathfrak{n}},$$

where $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{sl}(n+1)(z)$ with a semisimple part $\mathfrak{sl}(n)(z)$ and nilradical $\mathfrak{n}$. Proposition 2 yields also an isomorphism of $\mathfrak{sl}(n)(z)$-modules

$$L(f_1, ...f_s, f_{s+1} - 1, f_{s+2} - 1, ..., f_n - 1, z + n - s)^{\mathfrak{n}} \cong$$
$$\cong L(f_1, ...f_s, f_{s+1} - 1, f_{s+2} - 1, ..., f_n - 1) \otimes_{\mathbb{C}} \mathbb{C}(z).$$

Therefore we have an isomorphism of $\mathfrak{sl}(n)$-modules

$$L(f_1, ... f_s, f_{s+1} - 1, f_{s+2} - 1, ..., f_n - 1, z + n - s) \cong$$
$$\cong L(f_1, ... f_s, f_{s+1} - 1, f_{s+2} - 1, ..., f_n - 1) \otimes_{\mathbb{C}} \mathbf{S}^{\cdot}(\overline{\mathbb{C}(z)}^n).$$

Hence $L(f_1, ... f_s, f_{s+1} - 1, f_{s+2} - 1, ..., f_n - 1, z + n - s)$ is annihilated by

$$I(Q_n \otimes (\mathcal{L}_1^{\infty})_n),$$

i.e.,

$$I(Q_n \otimes (\mathcal{L}_1^{\infty})_n) \subset I(f_1, ..., f_s, f_{s+1} - 1, f_{s+2} - 2, ..., f_n - 1, z + n - s) = I(\hat{f}).$$

For this reason it suffices to show that

$$I(\hat{f}\,) \cap \mathrm{U}(\mathfrak{sl}(n+1)) \subset I(\tilde{f})$$

for any $z_0 \in \mathbb{C}$.

Let $v_{\hat{f}}$ be a highest weight vector of the $\mathfrak{sl}(n+1)(z)$-module $L(\hat{f})$. Consider the $\mathrm{U}(\mathfrak{sl}(n+1)[z])$-module

$$\mathrm{U}(\mathfrak{sl}(n+1)[z]) \cdot v_{\hat{f}}. \tag{21}$$

Clearly, the action of $\mathfrak{h}_{n+1}$ on the module defined in (21) is semisimple. The $\lambda_{\hat{f}}$-weight space of (21) coincides with $\mathrm{U}(\mathfrak{h}_{n+1} \otimes \mathbb{C}[z]) \cdot v_{\hat{f}}$, and is isomorphic to $\mathbb{C}[z]$ as a $\mathbb{C}[z]$-module. Therefore, the $\lambda_{\hat{f}}$-weight space of the quotient

$$\mathrm{U}(\mathfrak{sl}(n+1)[z]) \cdot v_{\hat{f}}/(z - z_0)\,\mathrm{U}(\mathfrak{sl}(n+1)[z]) \cdot v_{\hat{f}} \tag{22}$$

is one-dimensional. In particular, the quotient (22) is nonzero.

Obviously, (22) is annihilated by

$$I(\hat{f}) \cap \mathrm{U}(\mathfrak{sl}(n+1)). \tag{23}$$

On the other hand, (22) has a highest weight vector of weight $\lambda_{\tilde{f}}$, and thus $L(\tilde{f})$ is annihilated by (23). This is precisely what we have to prove.

Lemma 28. *Let F be a finite subset of $\mathbb{Z}_{>0}$ with n elements, and $f' \in \mathbb{C}^F$ be a function locally constant with respect to the partition $(S_1 \cap F) \sqcup ... \sqcup (S_t \cap F) = F$. After identification of $\mathfrak{sl}(F)$ with $\mathfrak{sl}(n)$ we have*

$$I((\mathcal{L}_{(\alpha(f')+\gamma(F))}^{\infty} \mathcal{E}^{\mathrm{A}(f')})_{n-(\gamma(F)+\alpha(f'))}) \subset I(f'),$$

where $I((\mathcal{L}_{(\alpha(f')+\gamma(F))}^{\infty} \mathcal{E}^{\mathrm{A}(f')})_{n-(\gamma(F)+\alpha(f'))}) \subset \mathrm{U}(\mathfrak{sl}(n-(\gamma(F)+\alpha(f'))))$ and $I(f') \subset \mathrm{U}(\mathfrak{sl}(n))$.

Proof. We prove this lemma by induction on $\gamma(F) + \alpha(f')$.

Let $\gamma(F) = \alpha(f') = 0$. Then f' is dominant, $L(f')$ is integrable and the statement of Lemma 27 follows from Lemma 8.

Next, assume that $\gamma(F) + \alpha(f') = k + 1$ and that our statement holds for $\gamma(F) + \alpha(f') \leq k$. Then $\alpha(f') > 0$ or $\gamma(F) > 0$. We consider both possibilities.

Let $\alpha(f') > 0$. Then $f'(S'_j) < f'(S'_{j+1})$ for some j. Denote by s the maximal element of $S'_j \cap F$ (with respect to the order inherited from the order $\prec$). Put

$$F_- := F \backslash s, \quad f'_- := f'|_{F_-} \in \mathbb{C}^{F_-},$$

and note that f'_- is locally constant with respect to the partition

$$(S_1 \cap F_-) \sqcup \dots \sqcup (S_t \cap F_-)$$

of F_-. Moreover, it is easy to see that

$$\gamma(F_-) = \gamma(F), \quad \alpha(f'_-) < \alpha(f') \text{ and } A(f'_-) \leq A(f').$$

Thus we can apply the induction assumption to f'_-, which yields

$$I((\mathcal{L}^\infty_{\gamma(F)+\alpha(f'_-)} \otimes \mathcal{E}^{A(f'_-)})_{n-1-(\gamma(F)+\alpha(f'_-))}) \subset I(f'_-).$$

Applying Lemma 27 to $s, z_0 = f'(s)$, we obtain

$$I((\mathcal{L}^\infty_{\gamma(F)+\alpha(f'_-)+1} \mathcal{E}^{A(f'_-)})_{n-1-(\gamma(F)+\alpha(f'_-))}) \subset I(f'). \tag{24}$$

Since

$$\gamma(F_-) + \alpha(f'_-) + 1 \leq \gamma(F) + \alpha(f') \text{ and}$$
$$n - 1 - (\gamma(F_-) + \alpha(f'_-)) \geq n - (\gamma(F) + \alpha(f')),$$

(24) implies

$$I((\mathcal{L}^\infty_{\gamma(F)+\alpha(f')} \mathcal{E}^{A(f')})_{n-(\gamma(F)+\alpha(f'))}) \subset I(f'),$$

which is precisely what we need to prove.

In the case when $\alpha(f') = 0, \gamma(F) > 0$ we pick s to be the least element of $F \backslash \cup_{j \leq x} S_{i_j}$ with respect to the order inherited from $\prec$. Then we apply the same arguments as above.

Remark 1. It is clear that Lemma 28 applies to an arbitrary linearly ordered finite set F, an arbitrary compatible partition of F, an arbitrary function $f \in \mathbb{C}^F$ locally constant with respect to this partition, and an arbitrary choice of equivalence classes of this partition used to define $\alpha(\cdot), A(\cdot)$ and $\gamma(\cdot)$.

Proof of Proposition 26. Identify F with $\{1, .., n\}$ as ordered sets (the order on F being inherited from the order $\prec$). The function $f' \in \mathbb{C}^F$ becomes

$$f' = (f'_1, ..., f'_n) \in \mathbb{C}^n.$$

Let s be the least element of $S'_1 \cap F$ under the above identification. Put

$$\check{f} := (f'_1, ..., \underbrace{f'_s, f'_s, ..., f'_s}_{(\gamma(F) + \alpha(f') + 1)- \text{ times}}, f'_{s+1}, ..., f'_n).$$

It is clear that $\check{f}$ is locally constant with respect to the partition

$$\check{S}_1 \sqcup \check{S}_2 \sqcup ... = \{1, ..., n + \alpha(f') + \gamma(F)\},$$

which is defined as follows:

(1) $\check{S}_i$ coincides with $(S_i \cap F)$ for $i < j$, where j is defined by the equality $S_j = S'_1$;

(2) $\check{S}_j = (S_j \cap F) \cup \{\check{s}+1, ..., \check{s} + \alpha(f') + \gamma(F)\}$, where $\check{s}$ is the image in $\{1, ..., n\}$ of the last element of $S_j \cap F$;

(3) $\check{S}_i = \{\check{s}_- + \gamma(F) + \alpha(f'), \check{s}_- + \gamma(F) + \alpha(f') + 1, ..., \check{s}_+ + \gamma(F) + \alpha(f') - 1, \check{s}_+ + \gamma(F) + \alpha(f')\}$ for $i > j$, where $\check{s}_-$ and $\check{s}_+$ are is the images in $\{1, ..., n\}$ of the least and the greatest elements of $S_i \cap F$.

Remark 1 enables us to apply Lemma 28 to the function $\check{f}$ and the partition $\check{S}_1 \sqcup \check{S}_2 \sqcup ... = \{1, ..., n + \alpha(f') + \gamma(F)\}$:

$$I((\mathcal{L}^\infty_{\gamma(F) + \alpha(f')} \mathcal{E}^{\mathrm{A}(f'))})_n) \subset I(\check{f}).$$

Finally, since $L(f)$ is an $\mathfrak{sl}(n)$-subquotient of $L(\check{f})$, we have

$$I(\check{f}) \cap \mathrm{U}(\mathfrak{sl}(n)) \subset I(f'),$$

and Proposition 26 is proved.

Proposition 25 follows now from Proposition 26 and the next lemma.

Lemma 29. *Let $I \subset \mathrm{U}(\mathfrak{sl}(\infty))$ be an ideal, and $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ be a function. Then $I \subset I(f)$ if and only if $I_F := I \cap \mathrm{U}(\mathfrak{sl}(F))$ annihilates $L(f|_F)$ for any finite subset F of $\mathbb{Z}_{>0}$.*

Proof. Let $I \subset I(f)$. Denote by v_f a highest weight vector of $L(f)$. If F is a finite set, then $\mathrm{U}(\mathfrak{sl}(F)) \cdot v_f$ is a highest weight $\mathfrak{sl}(F)$-submodule of $L(f)$. Thus $L(f|_F)$ is isomorphic to a subquotient of $L(f)$, and consequently $I_F = I \cap \mathrm{U}(\mathfrak{sl}(F))$ annihilates $L(f|_F)$.

We now prove the converse. Set

$$M(F) := M(f|_F) / (I \cap \mathrm{U}(\mathfrak{sl}(F)) \cdot M(f|_F).$$

As $I \cap \mathrm{U}(\mathfrak{sl}(F))$ annihilates $L(f|_F)$, $M(F)$ is a nonzero highest weight $\mathfrak{sl}(F)$-module. Let $v_f(F)$ be a highest weight vector of $M(F)$. For any finite subsets $F_1 \subset F_2 \subset \mathbb{Z}_{>0}$, there exists a unique morphism of $\mathfrak{sl}(F_1)$-modules

$$\phi_{F_1, F_2} : M(F_1) \to M(F_2)$$

such that $\phi_{F_1, F_2}(v_f(F_1)) = v_f(F_2)$. This defines a direct system of morphisms

$$\{\phi_{F_1, F_2}\}_{F_1 \subset F_2},$$

and we denote its limit by $\tilde{M}(f)$.

By definition, I annihilates the $\mathfrak{sl}(\infty)$-module $\tilde{M}(f)$. Our construction guarantees that $\tilde{M}(f)$ contains a highest vector $v_f := \varinjlim v_{f|_{F_i}}$ of weight λ_f. Thus $L(f)$ is isomorphic to a simple quotient of $\tilde{M}(f)$, which implies $I \subset I(f)$.

6. Proof of Theorem 10 and Proposition 11

Theorem 10 is implied by the following propositions.

Proposition 30. *Let $\mathfrak{b} \supset \mathfrak{h}$ be a splitting Borel subalgebra of $\mathfrak{sl}(\infty)$, and $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ be a function. Then $I = \mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} L_{\mathfrak{b}}(f)$ is a prime integrable ideal of $\mathrm{U}(\mathfrak{sl}(\infty))$.*

Proposition 31. *Let I be a prime integrable ideal of $\mathrm{U}(\mathfrak{sl}(\infty))$ and $\mathfrak{b}^0 \supset \mathfrak{h}$ be an ideal Borel subalgebra of $\mathfrak{sl}(\infty)$. Then $I = \mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} L_{\mathfrak{b}^0}(f^0)$ for some $f^0 \in \mathbb{C}^{\mathbb{Z}_{>0}}$.*

1. Proof of Proposition 30

The annihilator of a simple module is always prime, therefore in order to prove Proposition 30 we have to prove that the ideal $\mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} L_{\mathfrak{b}}(f)$ is integrable for any $\mathfrak{b}$ and any $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$. This is a direct consequence of the following three statements.

Proposition 32. *Let S be an infinite subset of $\mathbb{Z}_{>0}$ and $\phi : \mathbb{Z}_{>0} \to S$ be a fixed bijection. Let I be an ideal of $\mathrm{U}(\mathfrak{sl}(\infty))$. Then the induced isomorphism $\phi : \mathrm{U}(\mathfrak{sl}(\infty)) \to \mathrm{U}(\mathfrak{sl}(S))$ identifies I and $I \cap \mathrm{U}(\mathfrak{sl}(S))$.*

Proof. Fix the exhaustion (2) and assume that $\mathfrak{sl}(n)$ is generated by e_{ij} for $i \neq j, i, j \leq n$. Then $\mathfrak{sl}(S) = \cup_m \mathfrak{sl}(S_m)$, where S_m is the image of $\{1, ..., m\}$ under ϕ. We have

$$I \cap \mathrm{U}(\mathfrak{sl}(S)) = \cup_m (I \cap \mathrm{U}(\mathfrak{sl}(S_m))).$$

Since, for every $n \geq 1$, $\mathfrak{sl}(n)$ is $\mathrm{Adj}\,\mathfrak{sl}(\infty)$-conjugate to $\mathfrak{sl}(S_n)$, Lemma 15 yields

$$\phi^{-1}(I \cap \mathrm{U}(\mathfrak{sl}(S))) = \phi^{-1}(\cup_n (I \cap \mathrm{U}(\mathfrak{sl}(S_n)))) = \cup_n (I \cap \mathrm{U}(\mathfrak{sl}(n))) = I.$$

Corollary 33. *Let M be an $\mathfrak{sl}(\infty)$-module and S be an infinite subset of $\mathbb{Z}_{>0}$. Then $\mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} M$ is an integrable ideal in $\mathrm{U}(\mathfrak{sl}(\infty))$ if and only $\mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(S))} M$ is an integrable ideal of $\mathrm{U}(\mathfrak{sl}(S))$.*

Proposition 34. *Let $\mathfrak{b}$ and f be as in Proposition 30. If $\mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} L_{\mathfrak{b}}(f) \neq 0$, then there exists an infinite subset $S \subset \mathbb{Z}_{>0}$ such that $L_{\mathfrak{b}}(f)$ is an integrable $\mathfrak{sl}(S)$-module.*

Proof. As $\mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} L_{\mathfrak{b}}(f) \neq 0$, f is locally constant relative to some partition $S_1 \sqcup ... \sqcup S_t = \mathbb{Z}_{>0}$, compatible with the order determined by $\mathfrak{b}$. Since S_i is infinite for some i, we apply Proposition 5 to conclude that $L_{\mathfrak{b}}(f)$ is an integrable $\mathfrak{sl}(S_i)$-module.

2. Proof of Proposition 31

Let $\mathfrak{b}^0 \supset \mathfrak{h}$ be a fixed ideal Borel subalgebra of $\mathfrak{sl}(\infty)$. The goal of this section is to show that any integrable ideal is an annihilator of some $\mathfrak{b}^0$-highest weight module of $\mathfrak{sl}(\infty)$, and thus to prove Proposition 31. Due to the fact that an arbitrary irreducible c.l.s. has the form $\mathcal{L}_l^\infty Q \mathcal{R}_r^\infty$ for $l, r \in \mathbb{Z}_{\geq 0}$ and some irreducible c.l.s. of finite type Q, it is enough to prove the following proposition.

Proposition 35. *For any irreducible c.l.s. Q of finite type and any $l, r \in \mathbb{Z}_{\geq 0}$ there exists $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ such that*

$$\text{Ann}_{\text{U}(\mathfrak{sl}(\infty))} L_{\mathfrak{b}^0}(f) = I(\mathcal{L}_l^\infty Q \mathcal{R}_r^\infty).$$

We fix $l, r \in \mathbb{Z}_{\geq 0}$. According to Proposition 32, the ideals $\text{Ann}_{\text{U}(\mathfrak{sl}(\infty))} M$ and $\text{Ann}_{\text{U}(\mathfrak{sl}(S))} M$ can be identified for any $\mathfrak{sl}(\infty)$-module M and any infinite subset S of $\mathbb{Z}_{>0}$. Therefore, Proposition 35 is implied by the following lemma.

Lemma 36. *For any irreducible c.l.s. Q of finite type and any $l, r \in \mathbb{Z}_{\geq 0}$, there exist $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ and an infinite subset $S \subset \mathbb{Z}_{>0}$ such that the $\mathfrak{sl}(\infty)$-module $L_{\mathfrak{b}^0}(f)$ is integrable as an $\mathfrak{sl}(S)$-module and the c.l.s. for $\mathfrak{sl}(S)$ of $L_{\mathfrak{b}^0}(f)$ equals to $\mathcal{L}_l^\infty Q \mathcal{R}_r^\infty$.*

We now prove Lemma 36 by pointing out a concrete set S for which the claim of the lemma holds. We recall that the ideal Borel subalgebra $\mathfrak{b}^0$ defines a partition $S_1 \sqcup S_2 \sqcup S_3$ of $\mathbb{Z}_{>0}$. Let F_l be the set consisting of the first l elements of S_1. As an ordered set F_l is isomorphic to $\{1, ..., l\}$ with the standard order. Let F_r be set consisting of the the last r elements of S_3. As an ordered set F_r is isomorphic to $\{-r, ..., -1\}$ with the standard order. Put

$$S := \mathbb{Z}_{>0} \backslash (F_l \sqcup F_r).$$

Note that $\mathfrak{b}_S^0 := \mathfrak{b}^0 \cap \mathfrak{sl}(S)$ is an ideal Borel subalgebra of $\mathfrak{sl}(S)$. Therefore, Proposition 8 c) asserts that, for any c.l.s. Q of finite type, there is a $\mathfrak{b}_S^0$-dominant function $f^0 \in \mathbb{C}^S$ such that $Q = \text{cls}(L_{\mathfrak{b}_S^0}(f))$. For this reason Lemma 36 is a direct corollary of the following lemma.

Lemma 37. *Let $f \in \mathbb{C}^{\mathbb{Z}_{>0}}$ satisfy the conditions*
    1) $f|_S \in \mathbb{C}^S$ is $\mathfrak{b}_S^0$-dominant,
    2) $f(i) - f(j) \notin \mathbb{Z}$ for any $i \neq j, j \in F_l$,
    3) $f(i) - f(j) \notin \mathbb{Z}$ for any $i \neq j, j \in F_r$.
Then the c.l.s. of the $\mathfrak{sl}(S)$-module $L_{\mathfrak{b}^0}(f)$ is equal to $\mathcal{L}_l^\infty \text{cls}(L_{\mathfrak{b}_S^0}(f|_S)) \mathcal{R}_r^\infty$.

Proof. By Proposition 2,

$$L_{\mathfrak{b}^0}(f) \cong \text{U}(\mathfrak{sl}(\infty)) \otimes_{\text{U}(\mathfrak{p})} L_{\mathfrak{b}^0}(f)^{\mathfrak{n}},$$

where $L_{\mathfrak{b}^0}(f)^{\mathfrak{n}} \cong L_{\mathfrak{b}_S^0}(f|_S)$ as an $\mathfrak{sl}(S)$-module. Hence, there is an isomorphism of $\mathfrak{sl}(S)$-modules

$$L_{\mathfrak{b}^0}(f) \cong \mathbf{S}^\cdot(\mathfrak{sl}(\infty)/\mathfrak{p}) \otimes_{\mathbb{C}} L_{\mathfrak{b}_S^0}(f|_S). \tag{25}$$

Furthermore, there is an isomorphism of $\mathfrak{sl}(S)$-modules

$$\mathfrak{sl}(\infty)/\mathfrak{p} \cong (V(S) \otimes \mathbb{C}^l \oplus \mathbb{C}^{\frac{l(l-1)}{2}}) \oplus (V(S)_* \otimes \mathbb{C}^r \oplus \mathbb{C}^{\frac{r(r-1)}{2}}) \oplus \mathbb{C}^{rl}, \tag{26}$$

where $V(S)$ is the natural $\mathfrak{sl}(S)$-module and $\mathbb{C}$ stands for the one-dimensional trivial $\mathfrak{sl}(S)$-module. Thus,

$$\mathbf{S}^\cdot(\mathfrak{sl}(\infty)/\mathfrak{p}) \cong \mathbf{S}^\cdot(V(S) \otimes \mathbb{C}^l) \otimes \mathbf{S}^\cdot(V(S)_* \otimes \mathbb{C}^r) \otimes \mathbf{S}^\cdot(\mathbb{C}^{\frac{l(l-1)}{2}} \oplus \mathbb{C}^{\frac{r(r-1)}{2}} \oplus \mathbb{C}^{rl}). \tag{27}$$

The c.l.s. of $\mathbf{S}^\cdot(V(S) \otimes \mathbb{C}^l) = \mathbf{S}^\cdot(V(S))^{\otimes l}$ coincides with $\mathcal{L}_l^\infty$, and the c.l.s. of $\mathbf{S}^\cdot(V(S)_* \otimes \mathbb{C}^r) = \mathbf{S}^\cdot(V(S)_*)^{\otimes r}$ coincides with $\mathcal{R}_r^\infty$. Hence the c.l.s. of $\mathbf{S}^\cdot(\mathfrak{sl}(\infty)/\mathfrak{p})$ as an $\mathfrak{sl}(S)$-module coincides with $\mathcal{L}_l^\infty \mathcal{R}_r^\infty$, and the proof is complete.

Example 38. *Consider the fixed exhaustion (2) of $\mathfrak{sl}(\infty)$. Note that there is a canonical injection of $\mathfrak{sl}(i)$-modules $\mathbf{S}^i V_i \to \mathbf{S}^{i+1} V_{i+1}$, where V_i and V_{i+1} are respectively the natural representations of $\mathfrak{sl}(i)$ and $\mathfrak{sl}(i+1)$. The direct limit $D := \varinjlim_i \mathbf{S}^i V_i$ is a simple integrable $\mathfrak{sl}(\infty)$-module which is multiplicity free as an $\mathfrak{h}$-module. The module D has no highest weight with respect to any splitting Borel subalgebra $\mathfrak{b}$ [DP1]. The c.l.s. corresponding to D equals $\mathcal{L}_1^\infty$, and in particular has infinite type. Lemma 37 implies that $\mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} D$ equals to the annihilator of a simple nonintegrable highest weight module. Indeed, let $\mathfrak{b}^0$ be the ideal Borel subalgebra corresponding to the order (*iii*) in Section 1 and let f be the function*

$$f(1) = \alpha \notin \mathbb{Z}, \quad f(n) = 0, \quad n > 1.$$

Then $\mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} D = \mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} L_{\mathfrak{b}^0}(f)$. This example illustrates the role of simple integrable non-highest weight modules in Theorem 10: the annihilators of such simple modules arise as annihilators of simple nonintegrable highest weight modules.

3. Proof of Proposition 11

It remains to prove Proposition 11.

Proof of Proposition 11. We say that an ideal I of $\mathrm{U}(\mathfrak{sl}(\infty))$ is *of locally finite codimension* if $I \cap \mathrm{U}(\mathfrak{g})$ has finite codimension in $\mathrm{U}(\mathfrak{g})$ for any finite-dimensional subalgebra $\mathfrak{g} \subset \mathfrak{sl}(\infty)$. It is easy to see that such ideals have the following remarkable properties:

(i) the map $Q \mapsto I(Q)$ identifies the set of c.l.s. of finite type with the set of ideals of locally finite codimension;

(ii) if an $\mathfrak{sl}(\infty)$-module M is annihilated by an ideal $I \subset \mathrm{U}(\mathfrak{sl}(\infty))$ of locally finite codimension, then M is integrable.

Using the properties (i) and (ii) one observes that if $\mathfrak{b}$ is a Borel subalgebra, such that for any prime ideal I of locally finite codimension there exists $f \in \mathbb{C}^{\mathbb{Z}>0}$ with $I = \mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} L_{\mathfrak{b}}(f)$, then $\mathfrak{b}$ is ideal. Indeed, Proposition 8a) gives an explicit expression of $\mathrm{cls}(L_{\mathfrak{b}}(f))$ in terms of f. The requirement that this procedure allows for every c.l.s. of finite type to appear in the right-hand side of (4) forces the existence of a $\prec$-compatible decomposition $\mathbb{Z}_{>0} = F \sqcup S \sqcup F'$, where F and F' are arbitrary finite sets. Clearly, this is equivalent to the requirement that $\mathfrak{b}$ is ideal.

7. On simple $\mathfrak{sl}(\infty)$-modules determined up to isomorphism by their annihilators

It is a remarkable fact that if $\mathfrak{g}$ is finite dimensional and semisimple, then a simple $\mathfrak{g}$-module M is determined up to isomorphism by its annihilator in $\mathrm{U}(\mathfrak{g})$ if and only if M is finite dimensional. We now provide an analogue of this fact for $\mathfrak{sl}(\infty)$.

Recall that a *simple tensor module* of $\mathfrak{sl}(\infty)$ is a simple submodule of the tensor algebra

$$\mathrm{T}^{\cdot}(V(\infty) \oplus V(\infty)_*)$$

[DPS, PS]. It is easy to check that, for any fixed ideal Borel subalgebra $\mathfrak{b}^0$, the simple tensor modules are precisely the highest weight modules $L_{\mathfrak{b}^0}(f)$ such that f can be chosen to be 0 almost everywhere (recall that the isomorphism class of a module $L_{\mathfrak{b}^0}(f)$ recovers f up to an additive constant).

Proposition 39. *Let M be a simple $\mathfrak{sl}(\infty)$-module which is determined up to isomorphism by its annihilator $I = \mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} M$. If I is integrable, then M is isomorphic to a simple tensor module.*

Proof. If I is not of locally finite codimension, then Lemma 37 shows that our assumption on M is contradictory as the function f from Lemma 37 is not determined uniquely by I up to an additive constant by I. In other words, if I is not of locally finite codimension, then Lemma 37 implies that there exist $f_1, f_2 \in \mathbb{C}^{\mathbb{Z}^{>0}}$ such that

$$\mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} L_{\mathfrak{b}^0}(f_1) = \mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} L_{\mathfrak{b}^0}(f_2) = I$$

but $L_{\mathfrak{b}^0}(f_1) \not\cong L_{\mathfrak{b}^0}(f_2)$.

Assume now that I has locally finite codimension. Then $I = I(Q)$ for an irreducible c.l.s. of finite type Q, and by Proposition 8 c) M is isomorphic to $L_{\mathfrak{b}^0}(f^0)$ for some ideal Borel subalgebra $\mathfrak{b}^0$ and $\mathfrak{b}^0$-dominant function f^0. Moreover, as I is clearly fixed under the group $\tilde{G} := \{g \in \mathrm{Aut}_{\mathbb{C}} V(\infty) \mid g^*(V(\infty)_*) = V(\infty)_*\}$ considered as a group of automorphisms of $\mathrm{U}(\mathfrak{sl}(\infty))$, it follows that M is invariant under $\tilde{G}$. Now Theorems 3.4 and 4.2 in [DPS] imply that $L_{\mathfrak{b}}(f)$ is a simple tensor module.

It remains to show that a simple tensor $\mathfrak{sl}(\infty)$-module M is determined up to isomorphism by its annihilator $\mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} M$. If M' is a simple $\mathfrak{sl}(\infty)$-module with $\mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} M' = \mathrm{Ann}_{\mathrm{U}(\mathfrak{sl}(\infty))} M = I$, then the fact that I has locally finite codimension implies that M' is integrable and that the c.l.s. of M' coincides with the c.l.s. of I, i.e., $\mathrm{cls}(M) = \mathrm{cls}(M')$. Furthermore, a careful analysis (carried out in detail in A. Sava's master's thesis [S]) shows that M' is a highest weight $\mathfrak{sl}(\infty)$-module with respect to the ideal Borel subalgebra given by the order (iii) of Section 1, and that the highest weight of M equals the highest weight of M'. This of course implies that $M' \cong M$.

References

[Ba1]   A. A. Baranov, *Complex finitary simple Lie algebras*, Arch. Math. **71** (1998), 1–6.

[Ba2]   A. A. Baranov, *Finitary simple Lie algebras*, Journ. of Algebra **219** (1999), 299–329.

[DPS]   E. Dan-Cohen, I. Penkov, V. Serganova, *A Koszul category of representations of finitary Lie algebras*, preprint 2011, arXiv:1105.3407.

[DPSn] E. Dan-Cohen, I. Penkov, N. Snyder, *Cartan subalgebras of root-reductive Lie algebras*, Journ. of Algebra **308** (2007), 583–611.

[DP1]   I. Dimitrov, I. Penkov, *Weight Modules of Direct limit Lie Algebras*, IMRN **199 no 5** (1999), 223–249.

[DP2]   I. Dimitrov, I. Penkov, *Locally semisimple and maximal subalgebras of the finitary Lie algebras* $\mathfrak{gl}(\infty), \mathfrak{sl}(\infty), \mathfrak{so}(\infty),$ *and* $\mathfrak{sp}(\infty)$, Journ. of Algebra **322** (2009), 2069–2081.

[Dix]   Jacques Dixmier, *Algébres enveloppantes* (French), Gauthier-Villars, 1974.

[Jo1]   A. Joseph, *Sur la classification des idéaux primitifs dans l'algébre enveloppante de sl(n+1, C)* (French), C. R. Acad. Sci. Paris Sér. A-B **287 no 5** (1978), 303–306.

[Jo2]   A. Joseph, *Dixmier's problem for Verma and principal series submodules*, J. London Math. Soc. (2) **20** (1979), 193–204.

[Jo3]   A. Joseph, *Towards the Jantzen conjecture. III*, Compositio Math. **42** (1980/81), 23–30.

[Jo4]   A. Joseph, *On the associated variety of the primitive ideal*, Journ. of Algebra **88** (1984), 238–278.

[Knu]   D. E. Knuth, *The art of computer programming. Volume 3. Sorting and searching*, Addison-Wesley Series in Computer Science and Information Processing, Addison-Wesley, 1973.

[PS]   I. Penkov, V. Serganova, *Tensor representations of Mackey Lie algebras and their dense subalgebras*, Developments and Retrospectives in Lie Theory: Algebraic Methods, Developments in Mathematics, vol. **38**, Springer Verlag, 2014, 291–330.

[PP]   I. Penkov, A. Petukhov, *On ideals in the enveloping algebra of a locally simple Lie algebra*, IMRN, 2014, doi:10.1093/imrn/rnu085.

[S]   A. Sava, *Annihilators of simple tensor modules, master's thesis*, Jacobs University Bremen, 2012, arXiv: 1201.3829.

[Zh1]   A. G. Zhilinskii, *Coherent systems of representations of inductive families of simple complex Lie algebras* (Russian), preprint of Academy of Belarussian SSR, ser. **38 (438)**, Minsk, 1990.

[Zh2]   A. G. Zhilinskii, *Coherent finite-type systems of inductive families of non-diagonal inclusions* (Russian), Dokl. Acad. Nauk Belarusi **36:1** (1992), 9–13, 92.

[Zh3]   A. G. Zhilinskii, *On the lattice of ideals in the universal enveloping algebra of a diagonal Lie algebra*, preprint, Minsk, 2011.