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On the Barth–Van de Ven–Tyurin–Sato theorem

The Barth–Van de Ven–Tyurin–Sato Theorem claims that any finite rank vector bundle on the complex projective ind-space $\mathbf{P}^\infty$ is isomorphic to a direct sum of line bundles. We establish sufficient conditions on a locally complete linear ind-variety $\mathbf{X}$ which ensure that the same result holds on $\mathbf{X}$. We then exhibit natural classes of locally complete linear ind-varieties which satisfy these sufficient conditions.

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§ 1. Introduction

The Barth–Van de Ven–Tyurin–Sato Theorem claims that any finite rank vector bundle on the complex projective ind-space $\mathbf{P}^\infty$ is isomorphic to a direct sum of line bundles. For rank-two bundles this was established by Barth and Van de Ven in [1], and for finite rank bundles it was proved by Tyurin in [2] and Sato in [3]. This topic was revived in the more recent papers [4], [5], [6], where in particular the case of twisted ind-grassmannians was considered.

In the current paper we consider ind-varieties $\mathbf{X} = \varinjlim X_m$ given by chains of embeddings of smooth complete algebraic varieties

$$X_1 \overset{\phi_1}{\hookrightarrow} X_2 \overset{\phi_2}{\hookrightarrow} \dots \overset{\phi_{m-1}}{\hookrightarrow} X_m \overset{\phi_m}{\hookrightarrow} \dots .$$

We call such ind-varieties *locally complete*. A locally complete ind-variety $\mathbf{X} = \varinjlim X_m$ is *linear* if the map on Picard groups induced by φ_i is a surjection for almost all i. Our main objective is to give a reasonably general sufficient condition for the Barth–Van de Ven–Tyurin–Sato Theorem to hold on a locally complete ind-variety $\mathbf{X}$.

In the linear case, besides the results from the 1970-ies and the important results of Sato [7], [8] in which he considers a case when the Barth–Van de Ven–Tyurin–Sato Theorem no longer holds, some more recent results belong to Donin and Penkov [4]. In particular, it is shown in [4] that the Barth–Van de Ven–Tyurin–Sato Theorem holds on any linear direct limit $\mathbf{G}(\infty) = \varinjlim G(k_m, \mathbb{C}^{n_m})$, where $G(k_m, \mathbb{C}^{n_m})$ denotes the grassmannian of k_m-dimensional subspaces in $\mathbb{C}^{n_m}$, under the assumption that $\lim\limits_{m\to\infty} k_m = \lim\limits_{m\to\infty} (n_m - k_m) = \infty$. It turns out that there is a single isomorphism class of such ind-varieties. Nevertheless, there are other natural homogeneous ind-varieties on which the Barth–Van de Ven–Tyurin–Sato Theorem holds but which have not been considered in the literature. This applies in particular to linear direct limits of isotropic (orthogonal or symplectic) grassmannians, as well as to direct products of such direct limits.

For this reason we formulate a set of abstract conditions on a linear locally complete ind-variety $\mathbf{X}$ which ensure that the Barth–Van de Ven–Tyurin–Sato Theorem (shortly, BVTS Theorem) holds. We then give many examples of ind-varieties $\mathbf{X}$ satisfying these sufficient conditions. An interesting new class of such ind-varieties consists of direct limits $\mathbf{Y}$ of linear sections Y_m of $G(k_m, \mathbb{C}^{n_m})$, where $\varinjlim G(k_m, \mathbb{C}^{n_m}) = \mathbf{G}(\infty)$. Another class of ind-varieties on which the Barth–Van de Ven–Tyurin–Sato Theorem holds are certain ind-varieties of generalized flags, see subsection 6.3.

Probably, there are more general sufficient conditions for the Barth–Van de Ven–Tyurin– Sato Theorem to hold on locally complete ind-varieties. In addition, for non-linear locally complete ind-varieties nothing seems to be known beyond the results of [6]. Therefore, providing a sufficient condition for the Barth–Van de Ven–Tyurin–Sato Theorem to hold on general locally complete ind-varieties remains a project for the future.

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§ 2. Linear ind-varieties. Statement of the main result

2.1. The ground field is $\mathbb{C}$. We use the term algebraic variety as a synonym for a reduced Noetherian scheme. If E is a vector bundle (or simply a vector space), E^* stands for the dual bundle (or dual space). We use the standard notation $\mathcal{O}_{\mathbb{P}^n}(a)$ for the line bundle $\mathcal{O}_{\mathbb{P}^n}(-1)^{\otimes -a}$, where $\mathcal{O}_{\mathbb{P}^n}(-1)$ is the tautological bundle on the complex n-dimensional projective space $\mathbb{P}^n$.

Recall that an *ind-variety* is the direct limit $\mathbf{X} = \varinjlim X_m$ of a chain of morphisms of algebraic varieties

$$X_1 \overset{\phi_1}{\to} X_2 \overset{\phi_2}{\to} \dots \overset{\phi_{m-1}}{\to} X_m \overset{\phi_m}{\to} X_{m+1} \overset{\phi_{m+1}}{\to} \dots . \tag{2.1}$$

Note that the direct limit of the chain (2.1) does not change if we replace the sequence $\{X_m\}_{m\geqslant 1}$ by a subsequence $\{X_{i_m}\}_{m\geqslant 1}$, and the morphisms ϕ_m by the compositions $\tilde{\phi}_{i_m} := \phi_{i_{m+1}-1} \circ \dots \circ \phi_{i_m+1} \circ \phi_{i_m}$.

Let $\mathbf{X}$ be the direct limit of (2.1) and $\mathbf{X}'$ be the direct limit of a chain

$$X'_1 \overset{\phi'_1}{\to} X'_2 \overset{\phi'_2}{\to} \dots \overset{\phi'_{m-1}}{\to} X'_m \overset{\phi'_m}{\to} X'_{m+1} \overset{\phi'_{m+1}}{\to} \dots .$$

A *morphism of ind-varieties* $\mathbf{f} : \mathbf{X} \to \mathbf{X}'$ is a map from $\mathbf{X}$ to $\mathbf{X}'$ induced by a collection of morphisms of algebraic varieties $\{f_m : X_m \to Y_{n_m}\}_{m\geqslant 1}$ such that $\tilde{\phi}'_{n_m} \circ f_m = f_{m+1} \circ \phi_m$ for all $m \geqslant 1$. The identity morphism $\mathrm{id}_{\mathbf{X}}$ is a morphism which coincides with the identity as a set-theoretic map from $\mathbf{X}$ to $\mathbf{X}$. A morphism $\mathbf{f}:\mathbf{X}\to\mathbf{X}'$ is an *isomorphism* if there exists a morphism $\mathbf{g}:\mathbf{X}'\to\mathbf{X}$ such that $\mathbf{g}\circ\mathbf{f}=\mathrm{id}_{\mathbf{X}}$ and $\mathbf{f}\circ\mathbf{g}=\mathrm{id}_{\mathbf{X}'}$.

In what follows we only consider chains (2.1) such that X_m are complete algebraic varieties, $\lim\limits_{m\to\infty}(\dim X_m)=\infty$, and the morphisms ϕ_m are embeddings. We call such ind-varieties *locally complete*. Furthermore, we call a morphism $\mathbf{f}:\mathbf{X}=\underset{\to}{\lim}X_m\to\mathbf{X}'=\underset{\to}{\lim}X'_m$ of locally complete ind-varieties an *embedding* if all morphisms $f_m: X_m\to X'_{n_m},\ m\geqslant 1$, are embeddings.

A *vector bundle* $\mathbf{E}$ *of rank* $\mathbf{r}\in\mathbb{Z}_{>0}$ on $\mathbf{X}$ is the inverse limit $\underset{\leftarrow}{\lim}E_m$ of an inverse system of vector bundles E_m of rank $\mathbf{r}$ on X_m, i.e., of a system of vector bundles E_m with fixed isomorphisms $\psi_m: E_m\cong\phi_m^*E_{m+1}$; here and below ϕ^* stands for inverse image of vector bundles under a morphism ϕ. Clearly, $\mathbf{E}|_{X_m}\cong E_m,\ m\geqslant 1$. In particular, the *structure sheaf* $\mathcal{O}_{\mathbf{X}}=\underset{\longleftarrow}{\lim}\mathcal{O}_{X_m}$ of an ind-variety $\mathbf{X}$ is well defined. By the *Picard group* $\mathrm{Pic}\mathbf{X}$ we understand the group of isomorphism classes of line bundles on $\mathbf{X}$. Clearly, $\mathrm{Pic}\mathbf{X}$ is the inverse limit $\underset{\longleftarrow}{\lim}\mathrm{Pic}X_m$ of the system of Picard group homomorphisms $\{\phi_m^*:\mathrm{Pic}X_{m+1}\to\mathrm{Pic}X_m\}_{m\geqslant 1}$. In the rest of the paper we automatically assume that all vector bundles considered have finite rank. If $\mathbf{E}$ is a vector bundle on $\mathbf{X}$, $r\mathbf{E}$ stands for the direct sum $\mathbf{E}\oplus\ldots\oplus\mathbf{E}$ of r copies of $\mathbf{E}$. A vector bundle $\mathbf{E}$ is *trivial* if it is isomorphic to $\mathbf{r}\mathcal{O}_{\mathbf{X}},\ \mathbf{r}=\mathrm{rk}\mathbf{E}$.

A *linear ind-variety* is an ind-variety $\mathbf{X}=\underset{\to}{\lim}X_m$ such that, for all large enough $m\geqslant 1$, the induced homomorphisms of Picard groups $\phi_m^*:\mathrm{Pic}X_{m+1}\to\mathrm{Pic}X_m$ are epimorphisms. A typical example of a linear ind-variety is the *projective ind-space* $\mathbf{P}^\infty$ which is the direct limit of a chain of linear embeddings

$$\mathbb{P}^{n_1}\overset{\phi_1}{\hookrightarrow}\mathbb{P}^{n_2}\overset{\phi_2}{\hookrightarrow}\ldots\overset{\phi_{m-1}}{\hookrightarrow}\mathbb{P}^{n_m}\overset{\phi_m}{\hookrightarrow}\ldots,$$

for an arbitrary increasing sequence $\{n_m\}_{m\geqslant 1}$ of nonnegative integers. (It is easy to see that the definition of $\mathbf{P}^\infty$ does not depend, up to isomorphism of ind-varieties, on the choice of the sequence $\{n_m\}_{m\geqslant 1}$ and the embeddings φ_m.) By a *projective ind-subspace* of an ind-variety $\mathbf{X}$ we understand the image of an embedding $\psi:\mathbf{P}^\infty\hookrightarrow\mathbf{X}$.

Another example of a linear ind-variety is the *ind-grassmannian* $\mathbf{G}(\infty)$ which is the direct limit of a chain of linear embeddings

$$G(k_1,\mathbb{C}^{n_1})\overset{\phi_1}{\hookrightarrow}G(k_2,\mathbb{C}^{n_2})\overset{\phi_2}{\hookrightarrow}\ldots\overset{\phi_{m-1}}{\hookrightarrow}G(k_m,\mathbb{C}^{n_m})\overset{\phi_m}{\hookrightarrow}\ldots,$$

where $G(k_m,\mathbb{C}^{n_m})$ is the grassmannian of k_m-dimensional subspaces in an n_m-dimensional vector space and $\lim\limits_{m\to\infty}k_m=\lim\limits_{m\to\infty}(n_m-k_m)=\infty$.

2.2. Let $\mathbf{X}=\underset{\to}{\lim}X_m$ be a linear ind-variety such that there is a finite or countable set $\Theta_{\mathbf{X}}$ and a collection $\{\mathbf{L}_i=\underset{\leftarrow}{\lim}L_{im}\}_{i\in\Theta_{\mathbf{X}}}$ of nontrivial line bundles on $\mathbf{X}$ such that, for any m, $L_{im}\simeq\mathcal{O}_{X_m}$ for all but finitely many indices $i_1(m),\ldots,i_{j(m)}(m)$, and the images of $L_{i_1(m)m},\ldots,L_{i_{j(m)}(m)m}$ in $\mathrm{Pic}X_m$ form a basis of $\mathrm{Pic}X_m$ which is assumed to be a free abelian group. It is clear that in this case $\mathrm{Pic}\mathbf{X}$ is isomorphic to a direct product of infinite cyclic groups with generators the images of $\mathbf{L}_i$.

We denote by $\underset{i\in\Theta_{\mathbf{X}}}{\otimes}\mathbf{L}_i^{\otimes a_i}$ the line bundle on $\mathbf{X}$ whose restriction to X_m equals $\otimes_i L_{im}^{\otimes a_i} = L_{i_1(m)m}^{\otimes a_1}\otimes ... \otimes L_{i_{j(m)}(m)m}^{\otimes a_{j(m)}}$. We say that $\mathbf{X}$ *satisfies the property* L if, in addition to the above condition, $H^1(X_m, \otimes_i L_{im}^{\otimes a_i}) = 0$ for any $m \geqslant 1$ if some a_i is negative.

Let $\mathbf{X}$ satisfy the property L. For a given $i \in \Theta_{\mathbf{X}}$, a smooth rational curve $C \simeq \mathbb{P}^1$ on $\mathbf{X}$ is a *projective line of the i-th family on* $\mathbf{X}$ (or simply, a *line of the i-th family*), if

$$\mathbf{L}_j|_C \cong \mathcal{O}_{\mathbb{P}^1}(\delta_{ij}) \quad for \quad j \in \Theta_{\mathbf{X}}. \tag{2.2}$$

By $\mathbf{B}_i$ we denote the set of all projective lines of the i-th family on $\mathbf{X}$. It has a natural structure of an ind-variety: $\mathbf{B}_i = \underset{\to}{\lim} B_{im}$, where $B_{im} := \{C \in \mathbf{B}_i \mid C \subset X_m\}$ for $m \geqslant 1$. For any point $x \in \mathbf{X}$ the subset $\mathbf{B}_i(x) = \{C \in \mathbf{B}_i \mid C \ni x\}$ inherits an induced structure of an ind-variety.

Assume that $\mathbf{X}$ satisfies the property L. Then we say that $\mathbf{X}$ *satisfies the property* A if for any $i \in \Theta_{\mathbf{X}}$ there is an ind-variety $\mathbf{\Pi}_i$ which is a direct limit of a chain of emeddings $\{... \overset{\pi_{i,m-1}}{\hookrightarrow} \Pi_{im} \overset{\pi_{im}}{\hookrightarrow} \Pi_{i,m+1} \overset{\pi_{i,m+1}}{\hookrightarrow} ...\}$ where the points of Π_{im} are projective subspaces $\mathbb{P}^{n_m}$ of B_{im}, together with linear embeddings $\mathbb{P}^{n_m} \hookrightarrow \mathbb{P}^{n_{m+1}} = \pi_{im}(\mathbb{P}^{n_m})$ induced by the embeddings $B_{im} \hookrightarrow B_{i,m+1}$, so that each point of $\mathbf{\Pi}_i$ is considered as a projective ind-subspace $\mathbf{P}^\infty = \underset{\to}{\lim}\mathbb{P}^{n_m}$ of $\mathbf{B}_i$, and for any $x \in \mathbf{X}$ the following conditions hold:

(A.i) for each $m \geqslant 1$ such that $x \in X_m$, each nontrivial sheaf L_{im} defines a morphism $\psi_{im} : X_m \to \mathbb{P}^{r_{im}} := \mathbb{P}(H^0(X_m, L_{im})^*)$ which maps the family of lines $B_{im}(x)$ isomorphically to a subfamily of lines in $\mathbb{P}^{r_{im}}$ passing through the point $\psi_{im}(x)$;

(A.ii) the variety $\Pi_{im}(x) := \{\mathbb{P}^{n_m} \in \Pi_{im} \mid \mathbb{P}^{n_m} \subset B_{im}(x)\}$ is connected for any $m \geqslant 1$;

(A.iii) the projective ind-subspaces $\mathbf{P}^\infty \in \mathbf{\Pi}_i(x) := \underset{\to}{\lim}\Pi_{im}(x)$ fill $\mathbf{B}_i(x)$;

(A.iv) for any $d \in \mathbb{Z}_{\geqslant 1}$ there exists a $m_0(d) \in \mathbb{Z}_{\geqslant 1}$ such that, for any d-dimensional variety

Y and any $m \geqslant m_0(d)$, any morphism $\Pi_{im}(x) \to Y$ is a constant map.

In particular, (A.ii) and (A.iii) imply that the varieties Π_{im}, B_{im}, $B_{im}(x)$ are connected.

Let $\mathbf{X}$ satisfy the properties L and A as above. A vector bundle $\mathbf{E}$ on $\mathbf{X}$ is called $\mathbf{B}_i$-*uniform,* if for any projective line $\mathbb{P}^1 \in \mathbf{B}_i$ on $\mathbf{X}$, the restricted bundle $\mathbf{E}|_{\mathbb{P}^1}$ is isomorphic to $\oplus_{j=1}^{\mathrm{rk}\mathbf{E}}\mathcal{O}_{\mathbb{P}^1}(k_j)$ for some integers k_j not depending on the choice of $\mathbb{P}^1$. If in addition all $k_j = 0$, then $\mathbf{E}$ is called $\mathbf{B}_i$-linearly trivial. We call $\mathbf{E}$ *uniform* (respectively, *linearly trivial*) if it is $\mathbf{B}_i$-uniform (respectively, $\mathbf{B}_i$-linearly trivial) for any $i \in \Theta_{\mathbf{X}}$. Moreover, we say that $\mathbf{X}$ *satisfies the property* T if any linearly trivial vector bundle on $\mathbf{X}$ is trivial.

Our general version of the BVTS Theorem is the following.

THEOREM 1. *Let* $\mathbf{E}$ *be a vector bundle on a linear ind-variety* $\mathbf{X}$.

(i) If $\mathbf{X}$ *satisfies the properties* L *and* A *for for some fixed line bundles* $\{\mathbf{L}_i\}_{i\in\Theta_{\mathbf{X}}}$*, and corresponding families* $\{\mathbf{B}_i\}_{i\in\Theta_{\mathbf{X}}}$ *of projective lines on* $\mathbf{X}$*, then* $\mathbf{E}$ *has a filtration by vector subbundles*

$$0 = \mathbf{E}_0 \subset \mathbf{E}_1 \subset ... \subset \mathbf{E}_t = \mathbf{E} \tag{2.3}$$

with uniform successive quotients $\mathbf{E}_k/\mathbf{E}_{k-1}$, $k = 1, ..., t$.

(ii) If, in addition, $\mathbf{X}$ *satisfies the property* T*, then the filtration (2.3) splits and its quotients are of the form*

$$\mathbf{E}_k/\mathbf{E}_{k-1} \cong \mathrm{rk}(\mathbf{E}_k/\mathbf{E}_{k-1})(\bigotimes_{i\in\Theta_{\mathbf{X}}} \mathbf{L}_i^{\otimes a_{ik}}), \qquad a_{ik} \in \mathbb{Z},\ i \in \Theta_{\mathbf{X}},\ 1 \leqslant k \leqslant t.$$

In particular, $\mathbf{E}$ *is isomorphic to a direct sum of line bundles.*

§ 3. Proof of the main theorem

3.1. Preliminaries on vector bundles. If $C \subset X$ is a smooth irreducible rational curve in an algebraic variety X and E is a vector bundle on X, then by a classical theorem often attributed to Grothendieck, $E|_C$ is isomorphic to $\bigoplus_i \mathcal{O}_C(\delta_i)$ for some $\delta_1 \geqslant \delta_2 \geqslant ... \geqslant \delta_{\mathrm{rk}E}$. We call the ordered $\mathrm{rk}E$-tuple $(\delta_1, ..., \delta_{\mathrm{rk}E})$ *the splitting type* of $E|_C$ and denote it $\mathrm{Split}(E|_C)$. We order splitting types lexicographically, i.e. $(\delta_1, ..., \delta_{\mathrm{rk}E}) > (\delta'_1, ..., \delta'_{\mathrm{rk}E})$ if $\delta_1 = \delta'_1, ..., \delta_{k-1} = \delta'_{k-1}$, $\delta_k > \delta'_k$ for some k, $1 \leqslant k \leqslant \mathrm{rk}E$.

Let $\mathbf{X}$ be a locally complete linear ind-variety satisfying the properties L and A, and let $x \in \mathbf{X}$ and $i \in \Theta_{\mathbf{X}}$. In the notation of (A.i), let $\mathbb{P}^{r_{im}} = \mathbb{P}(H^0(X_m, L_{im})^*)$ and $y = \psi_{im}(x) = \mathbb{C}u$, $0 \neq u \in H^0(X_m, L_{im})^*$, so that $B_{im}(x) \subset \mathbb{P}_y^{r_{im}-1} := \mathbb{P}(H^0(X_m, L_{im})^*/\mathbb{C}u)$. Fix a projective subspace $\mathbb{P}^{n_m} \subset B_{im}(x)$, where $\mathbb{P}^{n_m} \in \Pi_{im}(x)$. Then $\mathcal{O}_{\mathbb{P}(H^0(X_m,L_{im})^*/\mathbb{C}u)}(1)|_{\mathbb{P}^{n_m}} \simeq \mathcal{O}_{\mathbb{P}^{n_m}}(N(i))$ for some $N(i) > 0$. Consider the locally closed subvariety $Y_m := \{(z, l) \in \mathbb{P}^{r_{im}} \times \mathbb{P}^{n_m} | z \in l \setminus \{y\}\}$ of $\mathbb{P}^{r_{im}} \times \mathbb{P}^{n_m}$, and let $Y_m \hookrightarrow \mathbb{P}^{r_{im}}$ be the embedding induced by the projection $\mathbb{P}^{r_{im}} \times \mathbb{P}^{n_m} \to \mathbb{P}^{r_{im}}$. Then Y_m is isomorphic to the total space of the line bundle $\mathcal{O}_{\mathbb{P}^{n_m}}(N(i))$ (see for instance [9; Appendix B]) and

$$\mathcal{O}_{\mathbb{P}^{r_{im}}}(1)|_{Y_m} \simeq \pi_m^* \mathcal{O}_{\mathbb{P}^{n_m}}(N(i)), \tag{3.1}$$

where $\pi_m : Y_m \to \mathbb{P}^{n_m}$ is the natural projection. Moreover, by construction we have a commutative diagram of morphisms

$$\begin{array}{ccccc} \overline{Y}_m & \xleftarrow{\tau_m} & Y_m & \xrightarrow{\pi_m} & \mathbb{P}^{n_m} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{P}^{r_{im}} & \xleftarrow{\phi_y} & \tilde{\mathbb{P}}^{r_{im}} & \xrightarrow{\pi_y} & \mathbb{P}_y^{r_{im}-1}, \end{array} \tag{3.2}$$

where $\tau_m : Y_m \hookrightarrow \overline{Y}_m := Y_m \cup \{y\}$ is the inclusion, $\varphi_y : \tilde{\mathbb{P}}^{r_{im}} \to \mathbb{P}^{r_{im}}$ is the blow-up of $\mathbb{P}^{r_{im}}$ with centre at y, and π_y is the natural projection which is a $\mathbb{P}^1$-bundle. In addition, we have an open embedding

$$\iota_m : Y_m \hookrightarrow \tilde{Y} := \tilde{\mathbb{P}}^{r_{im}} \times_{\mathbb{P}_y^{r_{im}-1}} \mathbb{P}^{n_m}$$

and projections $\overline{Y}_m \overset{\tilde{\tau}_m}{\leftarrow} \widetilde{Y}_m \overset{\tilde{\tau}_m}{\to} \mathbb{P}^{n_m}$ such that

$$\tau_m = \tilde{\tau}_m \circ \iota_m, \quad \pi_m = \tilde{\pi}_m \circ \iota_m. \tag{3.3}$$

By (A.i) $\psi_{im} : \psi_{im}^{-1}(\overline{Y}_m) \to \overline{Y}_m$ is an isomorphism. Hence we may consider $\mathbf{E}|_{\psi_{im}^{-1}(\overline{Y}_m)}$ as a vector bundle on $\overline{Y}_m$ and denote it by $\mathbf{E}|_{\overline{Y}_m}$. We also set $\mathbf{E}|_{Y_m} := \tau_m^*(\mathbf{E}|_{\overline{Y}_m})$.

For an arbitrary projective line $\mathbb{P}^1 \subset \mathbb{P}^{n_m}$, we consider the surface $S = S(x, \mathbb{P}^1) := \pi_y^{-1}(\mathbb{P}^1)$ with natural projections $\pi_S := \pi_y|_S : S \to \mathbb{P}^1$ and $\sigma_S := \phi_y|_S : S \to \mathbf{X}$. It follows from (3.1) that S is a surface of type $F_{N(i)}$.

Let $\mathbf{E}$ be a vector bundle of rank $\mathbf{r}$ on $\mathbf{X}$. For any $i \in \Theta_{\mathbf{X}}$ and $x \in \mathbf{X}$ we set $C(i) := c_1(\mathbf{E}|_l) \in \mathbb{Z}$, where c_1 stands for first Chern class and $l \in \mathbf{B}_i(x)$. Since $\mathbf{B}_i(x)$ is connected, $C(i)$ is well defined. Furthermore, we have $\delta_1(\mathbf{E}|_l) \geqslant C(i)/\mathbf{r} \geqslant \delta_B(\mathbf{E}|_l)$. Hence there are well-defined integers

$$\delta_1^{\min} := \min_{l \in \mathbf{B}_i(x)} \delta_1(\mathbf{E}|_l), \qquad \delta_{\mathrm{rk}E}^{\max} := \max_{l \in \mathbf{B}_i(x)} \delta_{\mathrm{rk}E}(\mathbf{E}|_l),$$

and there exist lines $l_{\min}, l_{\max} \in \mathbf{B}_i(x)$ such that $\delta_1(\mathbf{E}|_{l_{\min}}) = \delta_1^{\min}$, $\delta_{\mathrm{rk}E}(\mathbf{E}|_{l_{\max}}) = \delta_{\mathrm{rk}E}^{\max}$. The inequality $C(i) \geqslant \delta_1^{\min} + (\mathbf{r}-1)\delta_{\mathrm{rk}E}(\mathbf{E}|_{l_{\min}})$ implies

$$\delta_1^{\min} - \delta_{\mathrm{rk}E}(\mathbf{E}|_{l_{\min}}) \leqslant \delta_1^{\min} - C(i)/(\mathbf{r}-1), \quad \delta_1(\mathbf{E}|_{l_{\max}}) - \delta_{\mathrm{rk}E}^{\max} \geqslant C(i)/(\mathbf{r}-1) - \delta_{\mathrm{rk}E}^{\max}. \tag{3.4}$$

Fix $\mathbf{P}^\infty \in \mathbf{B}_i(x)$ and $l_{\min} \in \mathbf{P}^\infty$. For an arbitrary point $l_0 \in \mathbf{P}^\infty \smallsetminus \{l_{\min}\}$ consider the line $\mathbb{P}^1 = \mathrm{Span}(l_0, l_{\min})$ in $\mathbf{P}^\infty$ and the corresponding surface $S = S(x, \mathbb{P}^1)$ together with the vector bundle $E_S := \sigma_S^* \mathbf{E}$ on S. For a general point $l \in \mathbb{P}^1$ (l is a line on $\mathbf{X}$), the first inequality in (3.4) implies

$$\delta_{\mathrm{gen}} := \delta_1(\mathbf{E}|_l) - \delta_{\mathrm{rk}E}(\mathbf{E}|_l) \leqslant \delta_1^{\min} - C(i)/(\mathbf{r}-1). \tag{3.5}$$

The following lemma is a straightforward consequence of a result of Tyurin.

Lemma 3.1. *There exist polynomials P_A, $P_B \in \mathbb{Q}[x_1, ..., x_6]$ such that for any $l_0 \in \mathbf{P}^\infty \smallsetminus \{l_{\min}\}$*

$$\delta_1(\mathbf{E}|_{l_0}) \leqslant P_A(\mathbf{r}, \delta_1^{\min}, C(i), N(i), c_1^2(E_S), c_2(E_S)) =: P_A(\mathbf{E}, i), \tag{3.6}$$

$$\delta_{\mathrm{rk}E}(\mathbf{E}|_{l_0}) \geqslant P_B(\mathbf{r}, \delta_{\mathrm{rk}E}^{\max}, C(i), N(i), c_1^2(E_S), c_2(E_S)) =: P_B(\mathbf{E}, i), \tag{3.7}$$

where $c_1^2(E_S)$ and $c_2(E_S)$ are considered as integers.

Proof. By construction, S is a surface of type $F_{N(i)}$. Hence, repeating for the vector bundle E_S the proof of Lemma 5 from [T, Ch. 2, §1] we obtain that there exists a polynomial $f \in \mathbb{Q}[x_1, ..., x_6]$ such that $\delta_1(\mathbf{E}|_{l_0}) \leqslant f(\mathbf{r}, \delta_1^{\min}, \delta_{\mathrm{gen}}, N(i), c_1^2(E_S), c_2(E_S))$. Thus, in view of (3.5), there exists a polynomial $P_A \in \mathbb{Q}[x_1, ..., x_6]$ satisfying (3.6). The proof of (3.7) is similar.

The next proposition employs in a crucial way results of E. Sato. Fix $i \in \Theta_{\mathbf{X}}$, $x \in \mathbf{X}$ and $\mathbb{P}^{n_m} \in \Pi_{im}(x)$ for a large enough m. In view of (3.6) there exists a maximal (with respect to lexicographic order) splitting type $S_i(\mathbf{E}, \mathbb{P}^{n_m}) := \max_{l \in \mathbb{P}^{n_m}} \mathrm{Split}(\mathbf{E}|_l)$.

PROPOSITION 1. *The maximal splitting type $S_i(\mathbf{E},\mathbb{P}^{n_m})$ depends only on the pair $(\mathbf{E},i)$, i.e. $S_i(\mathbf{E},\mathbb{P}^{n_m})$ does not depend on x and on $\mathbb{P}^{n_m}\in\Pi_{im}(x)$.*

PROOF. Set

$$M_i(\mathbb{P}^{n_m}) := \{l\in\mathbb{P}^{n_m} \mid \mathrm{Split}(\mathbf{E}|_l) = S_i(\mathbf{E},\mathbb{P}^{n_m})\}.$$

The semicontinuity of $\mathrm{Split}(\mathbf{E}|_l)$ implies that $M_i(\mathbb{P}^{n_m})$ is a closed subvariety of $\mathbb{P}^{n_m}$. Moreover, Lemma 3.1 together with [2; Ch. 2, §2, Lemmas 3 and 4] yields the inequality

$$\mathrm{codim}_{\mathbb{P}^{n_m}} M_i(\mathbb{P}^{n_m}) \leqslant \mathbf{r}(\mathbf{r}-1)(P_A(\mathbf{E},i)-P_B(\mathbf{E},i)). \tag{3.8}$$

Consider the upper row of the diagram (3.2). Since the right-hand side of (3.8) is constant with respect to m, for large enough m we have

$$\mathrm{codim}_{\mathbb{P}^{n_m}} M_i(\mathbb{P}^{n_m}) < \min(n_m-\mathbf{r},(n_m-2\mathbf{r}^2)/2). \tag{3.9}$$

Also, clearly for large enough m

$$\mathrm{codim}_{\overline{Y}_m}\{x\} = \mathrm{codim}_{\overline{Y}_m}(\overline{Y}_m\smallsetminus Y_m) > \mathbf{r}. \tag{3.10}$$

The inequality (3.10) shows that

$$c_k(\mathbf{E}|_{Y_m}) = \tau_m^* c_k(\mathbf{E}|_{\overline{Y}_m}),\quad 0\leqslant k\leqslant\mathbf{r}, \tag{3.11}$$

where $c_k(\cdot)$ stands for k-th Chern class. Moreover, since the Chow group of codimension k of the base of an arbitrary vector bundle pulls back isomorphically to the Chow group of codimension k of the total space of the bundle, we have

$$c_k(\mathbf{E}|_{Y_m}) = \pi_m^*(c_kH^k),\ \ 0\leqslant k\leqslant\mathbf{r}, \tag{3.12}$$

where H is the class of a hyperplane divisor on $\mathbb{P}^{n_m}$, $c_0 = 1$ and $c_1,...,c_{\mathbf{r}}$ are integers. It is essential to note that the obvious compatibility of the morphisms π_m for varying m and the functoriality of Chern classes imply that these integers do not depend on x and on $\mathbb{P}^{n_m}\in\Pi_{im}(x)$. Note also that (3.1), (3.3) and the equalities (3.11) and (3.12) imply

$$\iota_m^*c_k(\tilde{\tau}_m^*(\mathbf{E}\otimes\mathbf{L}_i^{-a}|_{\overline{Y}_m})) = \tau_m^*c_k(\mathbf{E}\otimes\mathbf{L}_i^{-a}|_{\overline{Y}_m}) = c_k(\mathbf{E}|_{Y_m}\otimes\pi_m^*\mathcal{O}_{\mathbb{P}^{n_m}}(-N(i)aH)),\ \ 0\leqslant k\leqslant\mathbf{r},\ a \tag{3.13}$$

Next, consider the polynomial

$$h(t) = \sum_{k=0}^{\mathbf{r}} c_k(-t)^{\mathbf{r}-k}\in\mathbb{Z}[t] \tag{3.14}$$

where the coefficients c_k for $k\geqslant 0$ are the integers introduced above. Following closely an idea of Sato, we will now argue that the roots of $h(t)$ constitute a constant multiple of the maximal splitting type $S_i(\mathbf{E},\mathbb{P}^{n_m})$. More precisely, let $a_1>...>a_\alpha$, $\alpha\leqslant\mathbf{r}$, be the distinct elements of $S_i(\mathbf{E},\mathbb{P}^{n_m})$ of respective multiplicities $r_1,...,r_\alpha$ in $S_i(\mathbf{E},\mathbb{P}^{n_m})$. Then we claim that the roots of $h(t)$ are $N(i)a_1,...,N(i)a_\alpha$ of respective multiplicities $r_1,...,r_\alpha$.

The argument in [3; pp. 138-139] shows that in order to prove this claim of $h(t)$ it suffices to establish the vanishing of $c_k(\mathbf{E}|_{Y_m} \otimes \pi_m^* \mathcal{O}_{\mathbb{P}^{n_m}}(-N(i)a_jH))$ for $\mathbf{r} - r_j + 1 \leqslant k \leqslant \mathbf{r}$, $1 \leqslant j \leqslant \alpha$. By (3.13) it is enough to prove the vanishing of $c_k(\tilde{\tau}_m^*(\mathbf{E}\otimes\mathbf{L}_i^{-a_j}|_{\overline{Y}_m}))$. However, the proof of this fact is practically the same as in [3]. Namely, one defines inductively vector bundles $F_1 := \tilde{\phi}_m^*\mathbf{E}|_{\tilde{\pi}_m^{-1}(M_i(\mathbb{P}^{n_m}))}$, $F_2, ..., F_\alpha$ such that $\mathrm{rk}F_j = \sum_{p=j}^{\alpha} r_p$ on $\tilde{\pi}_m^{-1}(M_i(\mathbb{P}^{n_m}))$ which fit into the exact triples

$$0 \to r_j\mathcal{O}_{\tilde{\pi}_m^{-1}(M_i(\mathbb{P}^{n_m}))} \to F_j \otimes (\mathbf{L}_i^{a_{j-1}-a_j}|_{\tilde{\pi}_m^{-1}(M_i(\mathbb{P}^{n_m}))}) \to F_{j+1} \to 0, \quad 1 \leqslant j \leqslant \alpha - 1, \tag{3.15}$$

where $a_0 := 0$. Using (3.9) and applying the argument from [3; p. 139] to the triples (3.15), we obtain $c_k(\tilde{\tau}_m^*(\mathbf{E} \otimes \mathbf{L}_i^{-a_j}|_{\overline{Y}_m})) = 0$ as desired.

Since $h(t)$ is independent of x and $\mathbb{P}^{n_m} \in \Pi_{im}(x)$, the same applies to $S_i(\mathbf{E}, \mathbb{P}^{n_m})$, i.e. the proposition is proved.

3.2. Proof of Theorem 2.1.

Proof. According to (3.9), the dimension of $M_i(\mathbb{P}^{n_m})$ is greater than half of the dimension of $\mathbb{P}^{n_m}$, hence the varieties $M_i(\mathbb{P}^{n_m})$ are connected for large enough m. Consider the variety $\Gamma_{im}(x) := \{(l, \mathbb{P}^{n_m}) \in B_{im}(x) \times \Pi_{im}(x) \mid l \in M_i(\mathbb{P}^{n_m})\}$ with projections $B_{im}(x) \overset{p_1}{\leftarrow} \Gamma_{im}(x) \overset{p_2}{\to} \Pi_{im}(x)$. Since $M_i(\mathbb{P}^{n_m}) = p_2^{-1}(\mathbb{P}^{n_m})$ is connected for any $\mathbb{P}^{n_m} \in \Pi_{im}(x)$ and $\Pi_{im}(x)$ is connected by (A.ii), it follows that $\Gamma_{im}(x)$ is connected. By definition, $\Gamma_{im}(x)$ is described as

$$\Gamma_{im}(x) = \bigsqcup_{\mathbb{P}^{n_m} \in \Pi_{im}(x)} M_i(\mathbb{P}^{n_m}), \qquad m \geqslant 1. \tag{3.16}$$

Similarly, consider the varieties $\Gamma_{im} := \{(x, l, \mathbb{P}^{n_m}) \in X_m \times B_{im} \times \Pi_{im} \mid (l, \mathbb{P}^{n_m}) \in \Gamma_{im}(x)\}$. By construction,

$$\Gamma_{im} = \bigsqcup_{x \in X_m} \Gamma_{im}(x),$$

so that each Γ_{im} is connected. Moreover, there is a well-defined ind-variety $\mathbf{\Gamma}_i := \varinjlim \Gamma_{im}$.

Let $(x, l, \mathbb{P}^{n_m}) \in \mathbf{\Gamma}_i$. If $\delta_1^{\max}$ is the maximal entry of $\mathrm{Split}(\mathbf{E}|_l) = S_i(\mathbf{E}, \mathbb{P}^{n_m})$, there is a well-defined subbundle $\mathbf{E}_1(l)$ of $\mathbf{E}|_l$:

$$\mathbf{E}_1(l) := \mathrm{im}(H^0(l, \mathbf{E}|_l(-\delta_1^{\max})) \otimes \mathcal{O}_l \overset{ev}{\to} \mathbf{E}|_l(-\delta_1^{\max})) \otimes \mathcal{O}_l(\delta_1^{\max}). \tag{3.17}$$

Set $\mathbf{r}_1 := \mathrm{rk}\mathbf{E}_1(l)$ and consider the relative grassmannian $\rho_1 : \mathbf{G}(\mathbf{r}_1, \mathbf{E}) \to \mathbf{X}$. According to Proposition 1, $\delta_1^{\max}$ and $\mathbf{r}_1$ do not depend on the point $(x, l, \mathbb{P}^{n_m}) \in \mathbf{\Gamma}_i$. Thus there is a morphism of ind-varieties

$$\mathbf{f}_{i1} : \mathbf{\Gamma}_i \to \mathbf{G}(\mathbf{r}_1, \mathbf{E}), \quad (x, l) \mapsto \mathbf{E}_1(l)|_x, \quad x \in \mathbf{X}. \tag{3.18}$$

Since $\mathbf{f}_{i1}(\Gamma_{im}(x)) \subset \rho_1^{-1}(x) = G(\mathbf{r}_1, \mathbf{E}|_x)$, by (3.16) we have

$$\mathbf{f}_{i1}(M_i(\mathbb{P}^{n_m})) \subset G(\mathbf{r}_1, \mathbf{E}|_x), \quad \mathbb{P}^{n_m} \in \Pi_{im}(x).$$

According to (3.8) $\mathrm{codim}_{\mathbb{P}^{n_m}} M_i(\mathbb{P}^{n_m})$ is bounded as $m \to \infty$. This means that, for large enough m, the morphism $\mathbf{f}_{i1} : M_i(\mathbb{P}^{n_m}) \to G(\mathbf{r}_1, \mathbf{E}|_x)$ satisfies the conditions of [3; Prop. 3.2], in which we set $n = n_m$, $X = M_i(\mathbb{P}^{n_m})$, $Y = G(\mathbf{r}_1, \mathbf{E}|_x)$ and $f = \mathbf{f}_{i1}$. By this proposition, $\mathbf{f}_{i1}|_{M_i(\mathbb{P}^{n_m})}$ is a constant map, hence it induces a morphism

$$\phi_{i1}(x):\ \Pi_{im}(x) \to G(\mathbf{r}_1, \mathbf{E}|_x), \quad \mathbb{P}^{n_m} \mapsto \mathbf{f}_{i1}(M_i(\mathbb{P}^{n_m})), \qquad x \in \mathbf{X}.$$

Now (A.iv) implies that there exists a positive integer m_1 such that the morphism $\phi_{i1}(x)$ is a constant map for any $m \geqslant m_1$. We thus obtain that (3.18) induces a constant morphism

$$\boldsymbol{\phi}_{i1}(x) : \mathbf{\Pi}_i(x) \to G(\mathbf{r}_1, \mathbf{E}|_x).$$

Consider the ind-variety $\mathbf{\Sigma}_i = \varinjlim \Sigma_{im}$, where $\Sigma_{im} := \{(x, \mathbb{P}^{n_m}) \in X_m \times \Pi_{im} \mid \mathbb{P}^{n_m} \in \Pi_{im}(x)\}$, and let $\mathbf{p}_i : \mathbf{\Sigma}_i \to \mathbf{X}$ be the natural projection with fibre $\mathbf{\Pi}_i(x)$, $x \in \mathbf{X}$. The above constant morphisms $\boldsymbol{\phi}_{i1}(x)$ extend to a morphism $\boldsymbol{\phi}_{i1} : \mathbf{\Sigma}_i \to \mathbf{G}(\mathbf{r}_1, \mathbf{E})$ which is constant on the fibres of $\mathbf{p}_i$. In addition, the morphism $\mathbf{f}_{i1}|_{\mathbf{M}_i(x)}$ is a constant map. We thus obtain a well-defined morphism

$$\mathbf{\Phi}_{i1} :\ \mathbf{X} \to \mathbf{G}(\mathbf{r}_1, \mathbf{E}), \quad x \mapsto \mathbf{f}_{i1}(\mathbf{\Gamma}_i(x)). \tag{3.19}$$

Let $\boldsymbol{\mathcal{S}}$ be the tautological bundle of rank $\mathbf{r}_1$ on $\mathbf{G}(\mathbf{r}_1, \mathbf{E})$. Set $\mathbf{E}_{1i} := \mathbf{\Phi}_{i1}^* \boldsymbol{\mathcal{S}}$. It follows now from (3.17), (3.18) and (3.19) that $\mathbf{E}_{1i}$ is a subbundle of $\mathbf{E}$ such that

$$\mathbf{E}_{1i}|_l = \mathbf{E}_1(l) \simeq \mathbf{r}_1 \mathcal{O}_l(\delta_1^{\max}), \qquad l \in \mathbf{M}_i.$$

Using the semicontinuity of $\dim H^0(l, \mathbf{E}_{1i}(-\delta_1^{\max})|_l)$, one checks immediately that the last equality is true for any $l \in \mathbf{B}_i$.

Applying the above argument to the quotient $\mathbf{E}' = \mathbf{E}/\mathbf{E}_{1i}$ etc., we obtain a filtration of the bundle $\mathbf{E}$

$$0 \subset \mathbf{E}_{1i} \subset \mathbf{E}_{2i} \subset ... \subset \mathbf{E}_{\alpha i} = \subset \mathbf{E}_{1i}$$

with $\mathbf{B}_i$-uniform successive quotients $\mathbf{F}_{ki} = \mathbf{E}_{ki}/\mathbf{E}_{k-1,i}$.

Fix now $j \in \Theta_{\mathbf{X}}$, $i \neq j$. By applying the same procedure to all bundles $\mathbf{F}_{ki}$, we obtain a bundle filtration of $\mathbf{E}$ whose quotients are $\mathbf{B}_i$-uniform and $\mathbf{B}_j$-uniform. After finitely many iterations we finally obtain a filtration

$$0 = \mathbf{E}_0 \subset \mathbf{E}_1 \subset ... \subset \mathbf{E}_s = \mathbf{E} \tag{3.20}$$

of $\mathbf{E}$ with uniform successive quotients. This yields (i).

Note that any uniform vector bundle on $\mathbf{X}$ becomes linearly trivial after twisting by an appropriate line bundle. This means that each successive quotient $\mathbf{E}_k/\mathbf{E}_{k-1}$ is isomorphic to $\mathbf{M}_k \otimes \mathbf{F}_k$ where $\mathbf{M}_k$ is a line bundle and $\mathbf{F}_k$ is linearly trivial. In addition, assume that the property T is satisfied. Then the bundles $\mathbf{F}_k$ are trivial, i.e.

$$\mathbf{E}_k/\mathbf{E}_{k-1} \simeq \mathrm{rk}(\mathbf{E}_k/\mathbf{E}_{k-1})\mathbf{M}_k, \quad 1 \leqslant k \leqslant s.$$

Furthermore, for $p < k$

$$\mathrm{Ext}^1(\mathbf{M}_k, \mathbf{M}_p) = H^1(\mathbf{X}, \mathbf{M}_k^* \otimes \mathbf{M}_p),$$

and, according to a well known fact [10; Theorem 4.5] (see also [11; Proposition 10.3]),

$$H^1(\mathbf{X}, \mathbf{M}_k^* \otimes \mathbf{M}_p) = \varprojlim H^1(X_m, (\mathbf{M}_k^* \otimes \mathbf{M}_p)|_{X_m}).$$

However, the above construction shows that

$$(\mathbf{M}_k^* \otimes \mathbf{M}_p)|_{X_m} \simeq \otimes_i L_{im_i}^{\otimes a_i}$$

with some a_i negative. Therefore the vanishing part of property L yields

$$H^1(X_m, (\mathbf{M}_k^* \otimes \mathbf{M}_p)|_{X_m}) = 0$$

for $m \geqslant 1$, and hence $\mathrm{Ext}^1(\mathbf{M}_k, \mathbf{M}_p) = 0$ for $1 \leqslant p \leqslant k \leqslant s$. This is sufficient to conclude that the filtration (3.20) splits, i.e. (ii) follows.

The rest of the paper (with exception of the appendix) is devoted to examples of linear ind-varieties satisfying the properties L, A and T.

§ 4. Linear ind-grassmannians satisfying the properties L, A, T

4.1. Finite-dimensional orthogonal and symplectic grassmannians. Let V be a finite-dimensional vector space. In what follows we will consider, both symmetric and symplectic, quadratic forms Φ on V. Under the assumption that Φ is fixed, for any subspace $W \subset V$ we set $W^\perp := \{v \in V \mid \Phi(v, w) = 0$ for any $w \in W\}$. Recall that W is *isotropic* (or Φ-*isotropic*) if $W \subset W^\perp$.

Let $\Phi \in S^2V^*$ be a non-degenerate symmetric form on V. For $\dim V \geqslant 3$ and $1 \leqslant k \leqslant \left[\frac{\dim V}{2}\right]$, the *orthogonal grassmannian* $GO(k, V)$ is defined as the subvariety of $G(k, V)$ consisting of all Φ-isotropic k-dimensional subspaces of V. Unless $\dim V = 2n$, $k = n$, $GO(k, V)$ is a smooth irreducible variety. For $\dim V = 2n$, $k = n$, $GO(k, V)$ is smooth and has two irreducible components, both of which are isomorphic to $GO(n-1, \tilde{V})$ for $\dim \tilde{V} = 2n - 1$.

If $\Phi \in \wedge^2V^*$ is a non-degenerate symplectic form on V, $\dim V = 2n$, we recall that the *symplectic grassmannian* $GS(k, V)$ is a smooth irreducible subvariety of $G(k, V)$ consisting of all Φ-isotropic k-dimensional subspaces of V.

4.2. Definition of linearind-grassmannians. We start by recalling the definition of standard extension of grassmannians [12].

By a *standard extension of grassmannians* we understand an embedding of grassmannians $f : G(k, V) \to G(k, V')$ for $\dim V \geqslant \dim V$, $k' \geqslant k$, given by the formula

$$f : V_k \mapsto V_k \oplus W, \tag{4.1}$$

for some fixed isomorphism $V' \simeq V \oplus \hat{W}$ and a fixed subspace $W \subset \hat{W}$ of dimension $k' - k$. Respectively, by a *standard extension of orthogonal* respectively, of *symplectic grassmannians* we understand an embedding of isotropic grassmannians $f : GO(k, V) \to GO(k, V')$ (respectively, $f : GS(k, V) \to GS(k, V')$) given by the formula (4.1) for some fixed orthogonal (respectively, symplectic) isomorphism $V' \simeq V \oplus \hat{W}$ and a fixed isotropic subspace $W \subset \hat{W}$ of dimension $k' - k$, cf. [12; Definitions 3.2. and 3.5]. Note that standard extensions are linear morphisms.

Next we recall the definition of a standard ind-grassmannian [12].

Definition 1. Fix an infinite chain of vector spaces

$$V_{n_1} \subset V_{n_2} \subset ... \subset V_{n_m} \subset V_{n_{m+1}} \subset ... \tag{4.2}$$

of dimensions $n_m,\ n_m < n_{m+1}$.

a) For an integer k, $1 \leqslant k < n_1$, set $\mathbf{G}(k) := \underset{\to}{\lim} G(k, V_{n_m})$ where

$$G(k, V_{n_1}) \hookrightarrow G(k, V_{n_2}) \hookrightarrow ... \hookrightarrow G(k, V_{n_m}) \hookrightarrow G(k, V_{n_{m+1}}) \hookrightarrow ...$$

is the chain of canonical inclusions of grassmannians induced by (4.2).

b) For a sequence of integers $1 \leqslant k_1 < k_2 < ...$ such that $k_m < n_m$, $\lim\limits_{m\to\infty}(n_m - k_m) = \infty$, set $\mathbf{G}(\infty) := \underset{\to}{\lim} G(k_m, V_{n_m})$ where

$$G(k_1, V_{n_1}) \hookrightarrow G(k_2, V_{n_2}) \hookrightarrow ... \hookrightarrow G(k_m, V_{n_m}) \hookrightarrow G(k_{m+1}, V_{n_{m+1}}) \hookrightarrow ...$$

is an arbitrary chain of standard extensions of grassmannians.

c) Assume that V_{n_m} are endowed with compatible non-degenerate symmetric (respectively, symplectic) forms Φ_m. In the symplectic case $\frac{n_m}{2} \in \mathbb{Z}_+$. For an integer k, $1 \leqslant k \leqslant [\frac{n_1}{2}]$, set $\mathbf{G}\mathrm{O}(k, \infty) := \underset{\to}{\lim} GO(k, V_{n_m})$ (respectively, $\mathbf{G}\mathrm{S}(k, \infty) := \underset{\to}{\lim} GS(k, V_{n_m})$) where

$$GO(k, V_{n_1}) \hookrightarrow GO(k, V_{n_2}) \hookrightarrow ... \hookrightarrow GO(k, V_{n_m}) \hookrightarrow GO(k, V_{n_{m+1}}) \hookrightarrow ...$$

(respectively,

$$GS(k, V_{n_1}) \hookrightarrow GS(k, V_{n_2}) \hookrightarrow ... \hookrightarrow GS(k, V_{n_m}) \hookrightarrow GS(k, V_{n_{m+1}}) \hookrightarrow ...)$$

is the chain of inclusions of isotropic grassmannians induced by (4.2).

d) For a sequence of integers $1 \leqslant k_1 < k_2 < ...$ such that $k_m < [\frac{n_m}{2}]$, $\lim\limits_{m\to\infty}([\frac{n_m}{2}] - k_m) = \infty$, set $\mathbf{G}\mathrm{O}(\infty, \infty) = \underset{\to}{\lim} G\mathrm{O}(k_m, V_{n_m})$ (respectively, $\mathbf{G}\mathrm{S}(\infty, \infty) := \underset{\to}{\lim} GS(k_m, V_{n_m})$) where

$$GO(k_1, V_{n_1}) \hookrightarrow GO(k_2, V_{n_2}) \hookrightarrow ... \hookrightarrow GO(k_m, V_{n_m}) \hookrightarrow GO(k_{m+1}, V_{n_{m+1}}) \hookrightarrow ... \tag{4.3}$$

(respectively,

$$GS(k_1, V_{n_1}) \hookrightarrow GS(k_2, V_{n_2}) \hookrightarrow ... \hookrightarrow GS(k_m, V_{n_m}) \hookrightarrow GS(k_{m+1}, V_{n_{m+1}}) \hookrightarrow ...) \tag{4.4}$$

is an arbitrary chain of standard extensions of isotropic grassmannians.

e) In the symplectic case, consider a sequence of integers $1 \leqslant k_1 < k_2 < ...$ such that $k_m \leqslant \frac{n_m}{2}$, $\lim\limits_{m\to\infty}(\frac{n_m}{2} - k_m) = k \in \mathbb{N}$, and set $\mathbf{G}\mathrm{S}(\infty, k) := \underset{\to}{\lim} GS(k_m, V_{n_m})$ for any chain of standard extensions (4.4). In the orthogonal case, assume first that $\dim V_{n_m}$ are even. Then set $\mathbf{G}\mathrm{O}^0(\infty, k) := \underset{\to}{\lim} GO(k_m, V_{n_m})$ for a chain (4.3) where

$k_m < \frac{n_m}{2}$, $\lim\limits_{m\to\infty}(\frac{n_m}{2} - k_m) = k \in \mathbb{N}$, $k \geqslant 2$. Finally, consider the orthogonal case under the assumption that $\dim V_{n_m}$ are odd. Then set $\mathbf{G}\mathrm{O}^1(\infty, k) := \varinjlim GO(k_m, V_{n_m})$ for a chain (4.3) where $k_m \leqslant [\frac{n_m}{2}]$, $\lim\limits_{m\to\infty}([\frac{n_m}{2}] - k_m) = k \in \mathbb{N}$.

In particular, $\mathbf{P}^\infty = \mathbf{G}(1) \simeq \mathbf{G}\mathrm{S}(1)$. Note that the above standard ind-grassmannians are well-defined, i.e. a standard ind-grassmannian does not depend, up to an isomorphism of ind-varieties, on the specific chain of standard embeddings used in its definition. Furthermore, the main result of [12] claims that, with the exception of the isomorphism $\mathbf{P}^\infty \simeq \mathbf{G}\mathrm{S}(1)$, the standard ind-grassmannians are pairwise non-isomorphic as ind-varieties.

In all cases the maximal exterior power of a tautological bundle generated by its global sections yields an ample line bundle $\mathcal{O}_{X_m}(1)$ on X_m, where $X_m = G(k_m, V_{n_m})$, $GO(k_m, V_{n_m})$, $GS(k_m, V_{n_m})$. It is well known that $\mathcal{O}_{X_m}(1)$ generates $\mathrm{Pic}X_m$. Moreover, if $i_m : X_m \hookrightarrow X_{m+1}$ is one of the embeddings in Definition 1, there is an isomorphism $i_m^*\mathcal{O}_{X_{m+1}}(1) \simeq \mathcal{O}_{X_m}(1)$. This allows us to conclude that $\mathbf{X} = \varinjlim X_m$ is a linear ind-variety and $\mathrm{Pic}\mathbf{X}$ is generated by $\mathcal{O}_{\mathbf{X}}(1) := \varprojlim \mathcal{O}_{X_m}(1)$.

4.3. BVTS Theorem for $\mathbf{G}(\infty)$, $\mathbf{G}\mathrm{O}(\infty,\infty)$, $\mathbf{G}\mathrm{S}(\infty,\infty)$, $\mathbf{G}\mathrm{O}^1(\infty,0)$ and $\mathbf{G}\mathrm{S}(\infty,0)$. We first note that, if $\mathbf{X} = \mathbf{G}(k)$, $\mathbf{G}\mathrm{O}(k,\infty)$, $\mathbf{G}\mathrm{S}(k,\infty)$ there is a tautological rank-k bundle $\mathbf{S}$ on $\mathbf{X}$. If $k \geqslant 2$, this bundle is not isomorphic to a direct sum of line bundles, hence the BVTS theorem does not hold for these ind-grassmannians. Moreover, it is known [7] that, for $\mathbf{X} = \mathbf{G}(k)$, $\mathbf{G}\mathrm{O}(k,\infty)$, $\mathbf{X} = \mathbf{G}\mathrm{S}(k,\infty)$, any simple vector bundle of finite rank on $\mathbf{X}$, i.e. a vector bundle which does not have a non-trivial proper subbundle, is a direct summand in a tensor power of $\mathbf{S}$.

THEOREM 2. *Any vector bundle $\mathbf{E}$ on $\mathbf{X} \simeq \mathbf{G}(\infty)$, $\mathbf{G}\mathrm{O}(\infty,\infty)$, $\mathbf{G}\mathrm{S}(\infty,\infty)$, $\mathbf{G}\mathrm{O}^1(\infty,0)$, $\mathbf{G}\mathrm{S}(\infty,0)$ is isomorphic to $\bigoplus\limits_i \mathcal{O}_{\mathbf{X}}(k_i)$ for some $k_i \in \mathbb{Z}$.*

For $\mathbf{X} = \mathbf{G}(\infty)$ this is proved in [4] (see also [5; Section 4]). For the remaining standard ind-grassmannians the claim of Theorem 2 follows from Theorem 1 and the following theorem.

THEOREM 3. *Let $\mathbf{X} \simeq \mathbf{G}\mathrm{O}(\infty,\infty)$, $\mathbf{G}\mathrm{S}(\infty,\infty)$, $\mathbf{G}\mathrm{O}^1(\infty,0)$ or $\mathbf{G}\mathrm{S}(\infty,0)$. Then $\mathbf{X}$ satisfies the properties* L, A *and* T.[1]

PROOF. $\mathbf{X}$ satisfies the property L as $\mathcal{O}_{\mathbf{X}}(1)$ generates $\mathrm{Pic}\mathbf{X}$, and $H^1(X_m, \mathcal{O}_{X_m}(a))$ vanishes for all a and sufficiently large m by Borel-Weil-Bott's Theorem. Furthermore, the property T follows from Proposition 9 below.

It remains to establish the property A. Part (A.i) holds here simply because $\mathcal{O}_{\mathbf{X}}(1)$ is very ample. We therefore discuss parts (A.ii)-(A.iv).

Let $\mathbf{X} = \mathbf{G}\mathrm{O}(\infty,\infty) = \varinjlim GO(k_m, V_{n_m})$. For $m \geqslant 1$, the base B_m of the family of projective lines on $GO(k_m, V_{n_m})$ coincides with the variety of isotropic flags of type $(k_m - 1, k_m + 1)$ in V_{n_m}:

$$B_m = \{(V_{k_m-1}, V_{k_m+1}) \in GO(k_m - 1, V_{n_m}) \times GO(k_m + 1, V_{n_m}) \mid V_{k_m-1} \subset V_{k_m+1}\} \tag{4.5}$$

[1] The reader can check that $\mathbf{G}(\infty)$ also satisfies the properties L, A, T.

[12; Lemma 2.2(i)]. Furthermore, set

$$\Pi_m := \{(V_{k_m}, V_{k_m+1}) \in GO(k_m, V_{n_m}) \times GO(k_m+1, V_{n_m}) \mid V_{k_m} \subset V_{k_m+1}\}. \quad (4.6)$$

A point $y = (V_{k_m}, V_{k_m+1}) \in \Pi_m$ corresponds to the projective subspace $G(k_m - 1, V_{k_m}) \times \{V_{k_m+1}\} \subset B_m$.

It is easy to see that $\mathbf{\Pi} := \varinjlim \Pi_m$ is a well-defined ind-variety and that a point of $\mathbf{\Pi}$ represents a projective ind-subspace of $\mathbf{B} := \varinjlim B_m$.

Next, (4.5) together with [12; Lemma 2.2(iv)], implies that for any point $x = \{V_{k_m}\} \in GO(k_m, V_{n_m})$,

$$B_{\tilde m}(x) := \{\mathbb{P}^1 \in B_{\tilde m} \mid \mathbb{P}^1 \ni x\} \simeq \quad (4.7)$$

$$\mathbb{P}((\phi_{\tilde m-1}\circ\ldots\circ\phi_m)(V_{k_m})^*)\times GO(1, (\phi_{\tilde m-1}\circ\ldots\circ\phi_m)(V_{k_m})^{\perp}/(\phi_{\tilde m-1}\circ\ldots\circ\phi_m)(V_{k_m})), \ \ \tilde m \geqslant m,$$

and

$$\Pi_{\tilde m}(x) := \{((\phi_{\tilde m-1}\circ\ldots\circ\phi_m)(V_{k_m}), V_{k_{\tilde m+1}}) \in \Pi_{\tilde m}\} \simeq GO(1, V_{k_{\tilde m}}^{\perp}/V_{k_{\tilde m}}), \quad \tilde m \geqslant m. \quad (4.8)$$

Since the quadrics $GO(1, (\phi_{\tilde m-1}\circ\ldots\circ\phi_m)(V_{k_m})^{\perp}/(\phi_{\tilde m-1}\circ\ldots\circ\phi_m)(V_{k_m}))$ are connected, (A.ii) follows from (4.5). Furthermore, as for each $\tilde m \geqslant 1$ the variety $GO(1, V_{k_{\tilde m}}^{\perp}/V_{k_{\tilde m}})$ is a smooth quadric hypersurface in the projective space $\mathbb{P}^{n_{\tilde m}-2k_{\tilde m}-1}$, (4.7) and (4.8) directly imply (A.iii) and (A.iv).

In the remaining cases the same argument goes through if one makes the following modifications.

If $\mathbf{X} = \mathbf{G}\mathrm{S}(\infty,\infty)$, the formulas for B_m, Π_m and $\Pi_{\tilde m}(x)$ are the same as (4.5), (4.6), (4.7) and (4.8) respectively, with GO substituted by GS (use [12; Lemma 2.5]). Note also that $GS(1, V_{k_{\tilde m}}^{\perp}/V_{k_{\tilde m}})$ is isomorphic to the projective space $\mathbb{P}(V_{k_{\tilde m}}^{\perp}/V_{k_{\tilde m}})$.

For $\mathbf{X} = \mathbf{G}\mathrm{O}^1(\infty,0) = \varinjlim GO(k_m, V_{2k_m+1})$ we first identify $GO(k_m, V_{2k_m+1})$ with an irreducible component $GO(k_m+1, V_{2k_m+2})^*$ of $GO(k_m+1, V_{2k_m+2})$ - see [12; Section 2.3]. Consequently, $\mathbf{X} \simeq \varinjlim GO(k_m, V_{2k_m})^*$. Next, instead of (4.5)-(4.6) one has $B_m \simeq GO(k_m-2, V_{2k_m})$, $\Pi_m \simeq GO(k_m-1, V_{2k_m})$, $m \geqslant 1$. Respectively, instead of (4.7)-(4.8) one has $B_{\tilde m}(x) \simeq G(k_{\tilde m}-2, (\phi_{\tilde m-1}\circ\ldots\circ\phi_m)(V_{k_m}))$ for $x = \{V_{k_m}\}$. The latter fact can be proved by an argument similar to that of [12; Lemma 2.2]. In addition, $\Pi_{\tilde m}(x) \simeq \mathbb{P}((\phi_{\tilde m-1}\circ\ldots\circ\phi_m)(V_{k_m})^*)$, $\tilde m \geqslant m$.

For $\mathbf{X} = \mathbf{G}\mathrm{S}(\infty,0) = \varinjlim GS(k_m, V_{2k_m})$ one can show that (4.5)-(4.6) can be replaced by $B_m \simeq GS(k_m-1, V_{2k_m})$, $\Pi_m \simeq GS(k_m, V_{2k_m})$. Respectively, (4.7)-(4.8) for $x = \{V_{k_m}\} \in GS(k_m, V_{2k_m})$ can be replaced by $B_{\tilde m}(x) \simeq G(k_{\tilde m}-1, (\phi_{\tilde m-1}\circ\ldots\circ\phi_m)(V_{k_m}))$, and $\Pi_{\tilde m}(x) \simeq \{\mathbb{P}((\phi_{\tilde m-1}\circ\ldots\circ\phi_m)(V_{k_m})^*)\}$ is a point for $\tilde m \geqslant m$.

§ 5. Linear sections of $\mathbf{G}(\infty)$, $\mathbf{G}\mathrm{O}(\infty,\infty)$, $\mathbf{G}\mathrm{S}(\infty,\infty)$

5.1. Linear sections of finite-dimensional grassmannians. Let $G = G(k,V)$, $GO(k,V)$, $GS(k,V)$. Assume $1 \leqslant k < \dim V - 1$ for $G = G(k,V)$, and $1 \leqslant$

$k < [\frac{\dim V}{2}]$ for $G = GO(k,V)$, $GS(k,V)$. Put $N := \dim H^0(\mathcal{O}_G(1))$ and $V_N := H^0(\mathcal{O}_G(1))^*$. We consider G as a subvariety of $\mathbb{P}(V_N)$ via the Plücker embedding. For a given integer c, $1 \leqslant c \leqslant k-1$, set

$$X := G \cap \mathbb{P}(U),$$

where $U \subset V_N$ is a subspace of codimension c. We call X a *linear section of G of codimension c.*

Note that there is a single family of maximal projective spaces of dimension k on G with base $\tilde{G}$, where $\tilde{G} = G(k+1,V)$ if $G = G(k,V)$, respectively, $\tilde{G} = GO(k+1,V)$ if $G = GO(k,V)$, and $\tilde{G} = GS(k+1,V)$ if $G = GS(k,V)$ (see [12; Lemmas 2.2(i) and 2.5(i)]). Consider the graph of incidence $\Sigma := \{(V_k, V_{k+1}) \in G \times \tilde{G} \mid V_k \subset V_{k+1}\}$ with projections $\tilde{G} \overset{p}{\leftarrow} \Sigma \overset{q}{\rightarrow} G$ and set $\pi := p|_{q^{-1}(X)} : q^{-1}(X) \to \tilde{G}$. The condition $1 \leqslant c \leqslant k-1$ implies that π is a surjective projective morphism.

Proposition 2. *For a subspace $U \subset V_N$ of codimension c in general position the following statements hold.*

(i) The varieties X and $q^{-1}(X)$ are smooth and

$$\pi_* \mathcal{O}_{q^{-1}(X)} = \mathcal{O}_{\tilde{G}}. \tag{5.1}$$

(ii) $Z(U) := \{x \in \tilde{G} \mid \dim \pi^{-1}(x) > k-c\}$ is a proper closed subset of $\tilde{G}$ and

$$\operatorname{codim}_{\tilde{G}} Z(U) \geqslant 3, \quad \operatorname{codim}_{q^{-1}(X)} \pi^{-1}(Z(U)) \geqslant 2. \tag{5.2}$$

(iii) The projection $\pi : q^{-1}(X) \backslash \pi^{-1}(Z(U)) \to \tilde{G} \backslash Z(U)$ is a projective $\mathbb{P}^{k-c}$-bundle.

Proof. We give the proof for the case $G = GO(k,V)$. The other cases are very similar and we leave them to the reader.

(i) Since the projective subspace $\mathbb{P}(U)$ is in general position in $\mathbb{P}(V_N)$, we have $\operatorname{codim}_G X = c$ and hence there is a Koszul resolution of the $\mathcal{O}_G$-sheaf $\mathcal{O}_X$

$$0 \to \mathcal{O}_G(-c) \to ... \to \binom{c}{i} \mathcal{O}_G(-i) \to ... \to c\mathcal{O}_G(-i) \to \mathcal{O}_G \to \mathcal{O}_X \to 0. \tag{5.3}$$

The pullback of (5.3) under the projection q is a $\mathcal{O}_\Sigma$-resolution of the sheaf $\mathcal{O}_{q^{-1}(X)} = \pi^* \mathcal{O}_{\tilde{G}}$ of the form

$$0 \to \mathcal{L}_c \to ... \to \mathcal{L}_1 \to \mathcal{O}_\Sigma \to \pi^* \mathcal{O}_{\tilde{G}} \to 0 \tag{5.4}$$

where $\mathcal{L}_i := q^*(\binom{c}{i} \mathcal{O}_G(-i))$, $i = 1, ..., c$.

For any $x \in \tilde{G}$ we have $p^{-1}(x) \simeq \mathbb{P}^k$, so the condition $c \leqslant k-1$ implies $H^j(p^{-1}(x), \mathcal{L}_i|_{p^{-1}(x)}) \simeq H^j(\mathbb{P}^k, \mathcal{O}_{\mathbb{P}^k}(-i)) = 0$ for $j \geqslant 0$, $i = 1, ..., c$. Hence the Base-change Theorem [13; Ch. III, Theorem 12.11] for the flat projective morphism p shows that $R^j p_* \mathcal{L}_i = 0$. In addition, by the same reason $R^j p_* \mathcal{O}_\Sigma = 0$, $j > 0$, and clearly $p_* \mathcal{O}_\Sigma = \mathcal{O}_{\tilde{G}}$. Therefore, applying the functor $R^{\cdot} p_*$ to (5.4) we obtain (5.1).

(ii) We now prove (5.2). Fix an arbitrary point $V_{k+1} \in \tilde{G}$. Since $p^{-1}(x) = \mathbb{P}(V_{k+1}^*)$ (see [12; Lemma 2.2(i)]), there is an induced monomorphism

$$0 \to V_{k+1}^* \to V_N. \tag{5.5}$$

Consider the varieties

$\Gamma_i := \{(W, V_{k+1}) \in G(N-c, V_N) \times \tilde{G} \mid \dim(W \cap V_{k+1}^*) \geqslant k-c+i+2\},\ \ 0 \leqslant i \leqslant c-1,$

together with the natural projections

$$G(N-c, V_N) \xleftarrow{p_i} \Gamma_i \xrightarrow{q_i} \tilde{G}.$$

For an arbitrary $U \in G(N-c, V_N)$ denote

$$Z_i(U) := q_i(p_i^{-1}(U)), \quad 0 \leqslant i \leqslant c-1.$$

By construction, $Z_0(U) = Z(U)$, $Z_i(U)$ are closed subvarieties of $\tilde{G}$, and we have a filtration

$$\emptyset =: Z_c(U) \subset Z_{c-1}(U) \subset ... \subset Z_0(U) = Z(U) \tag{5.6}$$

such that $Z_i(U)' := Z_i(U) \backslash Z_{i+1}(U))$ are locally closed subvarieties of $\tilde{G}$. Consequently, $B_i(U)' := \pi^{-1}(Z_i(U)')$ are locally closed subvarieties of $q^{-1}(X)$. Moreover, $\pi|_{B_i(U)'} : B_i(U)' \to Z_i(U)'$ is a $\mathbb{P}^{k+1-c+i}$-bundle, so that $\dim B_i(U)' = \dim Z_i(U)' + k + 1 - c + i$. Equivalently,

$$\mathrm{codim}_{q^{-1}(X)} B_i(U)' = \mathrm{codim}_{\tilde{G}} Z_i(U)' - (i+1). \tag{5.7}$$

Note also that $Z(U) = \cup_{i=0}^{c-1} Z_i(U)'$, hence

$$\pi^{-1}(Z(U)) = \pi^{-1}\left(\cup_{i=0}^{c-1} Z_i(U)'\right) = \cup_{i=0}^{c-1} B_i(U)'. \tag{5.8}$$

We now calculate the dimensions of $Z_i(U)$ under the assumption that U is in general position. For this, let $Y := q_i^{-1}(x)$ be the fibre of the projection q_i over a point $x = V_{k+1} \in Z_i(U)$. Consider the variety $\tilde{Y} = \{(W, V_{k-c+i+2}) \in G(N-c, V_N) \times G(k-c+i+2, V_{k+1}^*) \mid W \supset V_{k-c+i+2} \subset V_{k+1}^*\}$. The natural projection $\tilde{Y} \to G(k-c+i+2, V_{k+1}^*)$ is a fibration with the grassmannian $G(N-k-i-2, \mathbb{C}^{N-k-i-2+c})$ as a fibre. On the other hand, one has a birational surjective morphism $\tilde{Y} \to Y$, $(W, V_{k-c+i+2}) \mapsto W$. Therefore, in view of (5.5), $\dim Y = \dim \tilde{Y} = \dim G(k-c+i+2, V_{k+1}^*) + \dim G(N-k-i-2, \mathbb{C}^{N-k-i-2+c}) = cN - c^2 + (i+1)(c-k-i-2)$. As q_i is surjective, this yields

$$\dim \Gamma_i = \dim \tilde{G} + \dim Y = \dim \tilde{G} + cN - c^2 + (i+1)(c-k-i-2).$$

Since p_i is also surjective, for a point $U \in G(N-c, V_N)$ in general position we have $\dim Z_i(U) = \dim \Gamma_i - \dim G(N-c, V_N) = \dim \tilde{G} - (i+1)(k+i+2-c)$, i.e.

$$\mathrm{codim}_{\tilde{G}} Z_i(U) = (i+1)(k+i+2-c), \quad 0 \leqslant i \leqslant c-1. \tag{5.9}$$

This together with (5.7) implies $\mathrm{codim}_{q^{-1}(X)} B_i(U)' = (i+1)(k+i+1-c)$, $0 \leqslant i \leqslant c-1$. Therefore, in view of (5.8) and the assumption $c \leqslant k-1$, we obtain

$$\mathrm{codim}_{q^{-1}(X)} \pi^{-1}(Z(U)) = \min_{0 \leqslant i \leqslant c-1} \mathrm{codim}_{q^{-1}(X)} B_i(U)' = k+1-c \geqslant 2. \tag{5.10}$$

The inequality $\mathrm{codim}_{\tilde{G}} Z(U) \geqslant 3$ follows now from (5.7), and the proposition is proved.

COROLLARY 1. *Under the assumptions of Proposition 2, let $\mathcal{E}$ be a vector bundle on $q^{-1}(X)$ trivial along the fibres of the morphism $\pi : q^{-1}(X) \to \tilde{G}$. Then there is an isomorphism $ev : \pi^*\pi_*\mathcal{E} \stackrel{\simeq}{\to} \mathcal{E}$.*

PROOF. Apply Proposition 7 from the appendix to the morphism $\pi : q^{-1}(X) \to \tilde{G}$, the subvariety $Z(U)$ in $\tilde{G}$, and the vector bundle $\mathcal{E}$ on $q^{-1}(X)$.

LEMMA 5.1. *Let $X = G \cap \mathbb{P}(U)$ be a linear section of G of codimension c for $1 \leqslant c \leqslant (k-1)/2$, and let $\mathbb{P}^1$ be a projective line on $\tilde{G}$. Then there exists a rational curve $C \subset q^{-1}(X)$ such that $\pi|_C$ is an isomorphism of C with $\mathbb{P}^1$, and $q|_C$ is either an isomorphism or a constant map.*

PROOF. We only consider the case $G = GO(k, V)$. It is clear that

$$\mathbb{P}^1 = \{V_{k+1} \in \tilde{G} \mid V_k \subset V_{k+1} \subset V_{k+2}\}$$

for a unique isotropic flag $V_k \subset V_{k+2}$ in V. If $V_k \in X$, we set $C := \{(V_k, V_{k+1}) \in \Sigma \mid V_k \subset V_{k+1} \subset V_{k+2}\}$. Then $\pi|_C : C \to \mathbb{P}^1$ is an isomorphism and $q(C)$ equals the point $\{V_k\} \in G$.

Assume that $V_k \notin X$. It is straightforward to check that the intersection $q(p^{-1}(\mathbb{P}^1)) \cap \mathbb{P}^{N-2}$ for a hyperplane $\mathbb{P}^{N-2} \subset \mathbb{P}(V_N)$ such that $V_k \notin \mathbb{P}^{N-2}$, is isomorphic to the direct product $\mathbb{P}(V_k^*) \times \mathbb{P}(V_{k+2}/V_k)$ imbedded by Segre in $\mathbb{P}^{N-2}$. Let $\mathbb{P}_a^{k-1}$, $\mathbb{P}_b^{k-1}$ be the fibres in $\mathbb{P}(V_k^*) \times \mathbb{P}(V_{k+2}/V_k)$ over two points $a, b \in \mathbb{P}(V_{k+2}/V_k)$. The projection $pr_1 : \mathbb{P}(V_k^*) \times \mathbb{P}(V_{k+2}/V_k) \to \mathbb{P}(V_k^*)$ induces an isomorphism $f : \mathbb{P}_a^{k-1} \stackrel{\simeq}{\to} \mathbb{P}_b^{k-1}$.

Set $\mathbb{P}_a^{k-c-1} := \mathbb{P}_a^{k-1} \cap \mathbb{P}(U)$, $\mathbb{P}_b^{k-c-1} := \mathbb{P}_b^{k-1} \cap \mathbb{P}(U)$. Since $1 \leqslant c \leqslant (k-1)/2$, the intersection $\mathbb{P}_b^{k-c-1} \cap f(\mathbb{P}_a^{k-c-1}) \subset \mathbb{P}_b^{k-1}$ is nonempty. Consider a point x in this latter intersection. By construction, the fibre $\mathbb{P}_x^1 := pr_1^{-1}(x)$ lies in $q(p^{-1}(\mathbb{P}^1)) \cap X$. Finally, the preimage of $\mathbb{P}_x^1$ in $p^{-1}(\mathbb{P}^1)$ is a rational curve C as desired.

PROPOSITION 3. *Let X be a linear section of G of codimension c for $1 \leqslant c \leqslant (k-1)/2$. Then a linearly trivial vector bundle E on X is trivial.*

PROOF. Consider the vector bundle $\mathcal{E} := q^*E$ on $q^{-1}(X)$. Since E is linearly trivial, for any $x \in \tilde{G}$ $\mathcal{E}|_{\pi^{-1}(x)}$ is a linearly trivial bundle on the projective space $\pi^{-1}(x)$. A well-known theorem [14; Ch. I, Theorem 3.2.1] implies that $\mathcal{E}|_{\pi^{-1}(x)}$ is trivial. Therefore $ev : \pi^*\pi_*\mathcal{E} \to \mathcal{E}$ is an isomorphism by Corollary **??**.

Next, Lemma 5.1 allows us to conclude that $\pi_*\mathcal{E}$ is linearly trivial. Indeed, if $\mathbb{P}^1 \subset \tilde{G}$ is a projective line and $C \subset q^{-1}(X)$ is a rational curve as in Lemma 5.1, then $\pi_*\mathcal{E}|_{\mathbb{P}^1} \simeq \mathcal{E}|_C$, and hence $\mathcal{E}|_C$ is trivial because of the linear triviality of E.

Consequently, $\pi_*\mathcal{E}$ is trivial by Proposition 7.4. from the appendix. Then $\mathcal{E} \simeq \pi^*\pi_*\mathcal{E}$ is also trivial. Finally, since $q : q^{-1}(X) \to X$ is a flat projective morphism with irreducible fibres, $E = q_*\mathcal{E}$ is trivial by Proposition 6.

5.2. Linear sections of $\mathbf{G}(\infty)$, $\mathbf{G}\mathrm{O}(\infty,\infty)$, $\mathbf{G}\mathrm{S}(\infty,\infty)$ of small codimension. Let $\mathbf{G} = \mathbf{G}(\infty)$, $\mathbf{G}\mathrm{O}(\infty,\infty)$, $\mathbf{G}\mathrm{S}(\infty,\infty)$, in particular $\mathbf{G} = \varinjlim G(k_m, V_{n_m})$, $\varinjlim GO(k_m, V_{n_m})$, $\varinjlim GS(k_m, V_{n_m})$, see Definition 1. Fix a nondecreasing sequence $\{c_m\}_{m \geqslant 1}$ of integers satisfying the condition

$$1 \leqslant c_m \leqslant (k_m - 1)/2. \tag{5.11}$$

Consider an ind-variety $\mathbf{X} = \varinjlim X_m$ such that, for each $m \geqslant 1$, X_m is a smooth linear section of codimension c_m of $G_m = G(k_m, V_{n_m})$, $GO(k_m, V_{n_m})$, $GS(k_m, V_{n_m})$

and the embedding $\phi_m : X_m \hookrightarrow X_{m+1}$ is induced by the embedding $G_m \hookrightarrow G_{m+1}$. In what follows we call such ind-varieties *linear sections of* $\mathbf{G}$ *of small codimension.*

The existence of linear sections $\mathbf{X}$ of $\mathbf{G}$ of small codimension is a consequence of the Bertini Theorem. Moreover, such a linear section $\mathbf{X}$ is a linear ind-variety and Pic$\mathbf{X}$ is generated by $\mathcal{O}_{\mathbf{X}}(1) := \underleftarrow{\lim}\mathcal{O}_{X_m}(1)$. This follows from the observation that $\mathcal{O}_{X_m}(1)$ generates PicX_m by the Lefschetz Theorem, and from the linearity of the embeddings $G_m \hookrightarrow G_{m+1}$.

Proposition 4. *Let* $\mathbf{G} = \mathbf{G}(\infty)$, $\mathbf{G}\mathrm{O}(\infty,\infty)$, $\mathbf{G}\mathrm{S}(\infty,\infty)$. *There exists a linear section* $\mathbf{X}$ *of* $\mathbf{G}$ *of small codimension which is not isomorphic to either of the ind-varieties* $\mathbf{G}(\infty)$, $\mathbf{G}\mathrm{O}(\infty,\infty)$ *or* $\mathbf{G}\mathrm{S}(\infty,\infty)$.

Proof. Let $\mathbf{G} = \underrightarrow{\lim}G_m$ where $G_m = G(k_m, V_{n_m}), GO(k_m, V_{2n_m+1}), GS(k_m, V_{2n_m})$. Fix $m \geqslant 1$ and let $\mathbb{P}^1$ be a projective line on G_m. Then there exist unique maximal projective subspaces $\mathbb{P}^{k_m}$ and $\mathbb{P}^{s_m}$ on G_m which intersect in $\mathbb{P}^1$. For $G_m = G(k_m, V_{n_m})$ one has $s_m = n_m - k_m$, and for $G_m = GO(k_m, V_{n_m})$, $GS(k_m, V_{n_m})$ one has $s_m = [\frac{n_m}{2}] - k_m$, see [12; Lemmas 2.3(iii) and 2.6(ii)].

For $\tilde{m} \geqslant m$ the embeddings $\phi_{\tilde{m}} : G_{\tilde{m}} \hookrightarrow G_{\tilde{m}+1}$ in the direct limit $\underrightarrow{\lim}G_m$ are given by formula (4.1), which makes it easy to check that $\mathbb{P}^{k_m}$ and $\mathbb{P}^{s_m}$ admit extensions $\mathbb{P}^{k_{\tilde{m}}} \subset G_{\tilde{m}}$, $\mathbb{P}^{s_{\tilde{m}}} \subset G_{\tilde{m}}$ such that $\mathbb{P}^{k_m} \subset \mathbb{P}^{k_{\tilde{m}}}$, $\mathbb{P}^{s_m} \subset \mathbb{P}^{s_{\tilde{m}}}$ and

$$\mathbb{P}^1 = \mathbb{P}^{k_{\tilde{m}}} \cap \mathbb{P}^{s_{\tilde{m}}}, \quad \tilde{m} \geqslant m.$$

Denoting $\mathbf{P}^\infty_\alpha := \underrightarrow{\lim}\mathbb{P}^{k_{\tilde{m}}}$, $\mathbf{P}^\infty_\beta := \underrightarrow{\lim}\mathbb{P}^{s_{\tilde{m}}}$, we have

$$\mathbb{P}^1 = \mathbf{P}^\infty_\alpha \cap \mathbf{P}^\infty_\beta. \tag{5.12}$$

We now choose $n_{\tilde{m}}$ and $k_{\tilde{m}}$ in a specific way. Namely, we assume that $n_{\tilde{m}} = 3t_{\tilde{m}}$ for $G_m = G(k_m, V_{n_m})$, and $[\frac{n_{\tilde{m}}}{2}] = 3t_{\tilde{m}}$ for $G_m = GO(k_m, V_{n_m})$, $GS(k_m, V_{n_m})$, $k_{\tilde{m}} = 2t_{\tilde{m}}$, $t_{\tilde{m}} \in \mathbb{Z}_{\geqslant 1}$. Set $c_{\tilde{m}} := t_{\tilde{m}} - 1$. Then

$$s_{\tilde{m}} - c_{\tilde{m}} = 1 \tag{5.13}$$

and the inequality (5.11) together with the conditions $\lim\limits_{\tilde{m}\to\infty} k_{\tilde{m}} = \lim\limits_{\tilde{m}\to\infty} s_{\tilde{m}} = \infty$ are satisfied. Next, using (5.13) and the Bertini Theorem we choose a tower of projective subspaces $\mathbb{P}^{N_{\tilde{m}}-c_{\tilde{m}}-1} \subset \mathbb{P}(V_{N_{\tilde{m}}})$ for $\tilde{m} \geqslant m$ in general positon so that $\mathbb{P}^{N_{\tilde{m}}-c_{\tilde{m}}-1} \cap \mathbb{P}^{s_{\tilde{m}}} = \mathbb{P}^1$ and $\mathbb{P}^{N_{\tilde{m}}-c_{\tilde{m}}-1} \cap \mathbb{P}^{k_{\tilde{m}}} = \mathbb{P}^{(k_{\tilde{m}}/2)+1}$ for some projective subspaces $\mathbb{P}^{(k_{\tilde{m}}/2)+1}$ of $G_{\tilde{m}}$. As a result, we obtain a linear section $\mathbf{X} := \underrightarrow{\lim}(\mathbb{P}^{N_{\tilde{m}}-c_{\tilde{m}}-1} \cap G_{\tilde{m}})$ of $\mathbf{G}$ of small codimension and a projective line $\mathbb{P}^1 \subset \mathbf{X}$ such that $\mathbb{P}^1$ is contained in a unique linear ind-projective subspace $\mathbf{P}^\infty$ of $\mathbf{X}$, namely $\mathbf{P}^\infty := \underrightarrow{\lim}\mathbb{P}^{(k_{\tilde{m}}/2)+1}$. If $\mathbf{X}$ were isomorphic to $\mathbf{G}(\infty)$, $\mathbf{G}\mathrm{O}(\infty,\infty)$, $\mathbf{G}\mathrm{S}(\infty,\infty)$, this would contradict to (5.12).

Now we show that a linear section $\mathbf{X}$ of $\mathbf{G}$ of small codimension satisfies the properties L, A and T. The property L is clear as Pic$\mathbf{X}$ is generated by the class of $\mathcal{O}_{\mathbf{X}}(1)$, and $H^1(X_m, \mathcal{O}_{X_m}(a))$ vanishes for $\dim X_m > 1$ and $a < 0$ by Kodaira's Theorem. The property T is established in Proposition 3. It remains to establish the property A. Part (A.i) is clear as $\mathcal{O}_{X_m}(1)$ is very ample. For parts (A.ii)-(A.iv) we consider in detail only the case when $\mathbf{X}$ is a linear section of $\mathbf{G}\mathrm{O}(\infty,\infty)$.

Let $B(G_m)$ be the family of all projective lines in $G_m = GO(k_m, V_{n_m})$, and B_m be its subfamily consisting of those projective lines which lie in X_m. By definition, X_m is the intersection of G_m with a subspace $\mathbb{P}(U_m)$ of $\mathbb{P}(V_{N_m})$ for a fixed $U_m \in G(N_m - c_m, V_{N_m})$ in general position. The grassmannians $G(2, V_{N_m})$ and $G(2, U_m)$ can be be thought of as the grassmannians of projective lines in $\mathbb{P}(V_{N_m})$ and $\mathbb{P}(U_m)$ respectively. Then $B_m = B(G_m) \cap G(2, U_m)$ where the intersection is taken in $G(2, V_{N_m})$. We show next that B_m is irreducible.

Let B be an irreducible component of B_m. Since $G(2, V_{N_m})$ is smooth, the subadditivity of codimensions [15; Thm. 17.24]) yields

$$\operatorname{codim}_{B(G_m)} B \leqslant \operatorname{codim}_{G(2,V_{N_m})} G(2, U_m) = 2c_m. \tag{5.14}$$

Consider the graph of incidence $\Sigma_m := \{(x, \mathbb{P}^{k_m}) \in G_m \times \tilde{G}_m \mid x \in \mathbb{P}^{k_m}\}$ with its projections $\tilde{G}_m \overset{p_m}{\leftarrow} \Sigma_m \overset{q_m}{\rightarrow} G_m$, where $\tilde{G}_m := GO(k_m + 1, V_{n_m})$. Let $B(\Sigma_m)$ be the family of all projective lines in Σ_m lying in the fibres of the projection p_m. Denote by $B(q_m^{-1}(X_m))$ the subfamily of $B(\Sigma_m)$ consisting of those projective lines which lie in $q_m^{-1}(X_m)$. The projection $q_m : \Sigma_m \to G_m$ induces a morphism $r_{G_m} : B(\Sigma_m) \to B(G_m)$ which is bijective since any projective line on G_m lies in a unique maximal projective space $\mathbb{P}^{k_m}$ (see [12; Section 2]). The space $\mathbb{P}^{k_m}$ is an isomorphic image via q_m of some fibre of p_m. Respectively, the restricted morphism $r_{X_m} := r_{G_m}|_{B(q_m^{-1}(X_m))} : B(q_m^{-1}(X_m)) \to B_m$ is a bijection. Hence, for any irreducible component B' of $B(q_m^{-1}(X_m))$, (5.14) yields the inequality

$$\operatorname{codim}_{B(\Sigma_m)} B' \leqslant \operatorname{codim}_{G(2,V_{N_m})} G(2, U_m) = 2c_m. \tag{5.15}$$

The projection p_m induces a projection $\rho_m : B(\Sigma_m) \to \tilde{G}_m$. Let

$$\emptyset = Z_{c_m}(U_m) \subset Z_{c_m - 1}(U_m) \subset \dots \subset Z_0(U_m) \subset \tilde{G}_m$$

be the filtration (5.6) of $\tilde{G}_m$ by closed subvarieties $Z_i(U_m)$ of codimensions in $\tilde{G}_m$ given by (5.9) where we put $c = c_m$, $k = k_m$. This filtration yields a decomposition $B(q_m^{-1}(X_m)) = \bigsqcup_{0 \leqslant i \leqslant c_m} B_i$, where $B_0 := B(q_m^{-1}(X_m)) \cap \rho_m^{-1}(\tilde{G}_m \setminus Z_0(U_m))$, $B_i := B(q_m^{-1}(X_m)) \cap \rho_m^{-1}(Z_{i-1}(U_m) \setminus Z_i(U_m))$, $i = 1, \dots, c_m$. Formula (5.9) implies that $\operatorname{codim}_{B(\Sigma_m)} B_0 = 2c_m$, $\operatorname{codim}_{B(\Sigma_m)} B_i > 2c_m$ for $i = 1, \dots, c_m$. This together with (5.15) yields the irreducibility of $B(q_m^{-1}(X_m))$, hence of B_m as well.

Now let $\Pi(G_m)$ be the family of projective spaces $\mathbb{P}^{k_m - 1}$ lying in $B(G_m)$ and defined by the right-hand side of (4.6). Set

$$\mathbf{\Pi} := \lim_{\to} \Pi_m, \tag{5.16}$$

where $\Pi_m := \{\mathbb{P}^{k_m - c_m - 1} \subset B_m \mid \mathbb{P}^{k_m - c_m - 1}$ is a linear subspace of some $\mathbb{P}^{k_m - 1} \in \Pi(G_m)\}$.

Fix m and let $\pi := p_m|_{q_m^{-1}(X_m)} : q_m^{-1}(X_m) \to \tilde{G}_m$ be the projection. Consider the relative grassmannian $G_\pi := \{\mathbb{P}^1 \subset q_m^{-1}(X_m) \mid \mathbb{P}^1$ lies linearly in a fibre of the projection $\pi\}$ with induced projections $\rho : G_\pi \to \tilde{G}_m$ and $q_\pi : G_\pi \to B_m$. By definition, the fibre $\rho^{-1}(x)$ over an arbitrary point $x \in \tilde{G}_m$ is the grassmannian of projective lines in the projective space $\pi^{-1}(x)$. (Note that for a point $x \in \tilde{G}_m$

in general position the fibre $\pi^{-1}(x)$ is a projective space $\mathbb{P}^{k_m-c_m}$, hence $\rho^{-1}(x)\simeq G(2,\mathbb{C}^{k_m-c_m+1})$.) Furthermore, the projection q_π is birational.

By construction, the projective spaces $\mathbb{P}^{k_m-c_m-1}\in\Pi_m$ are isomorphic images under q_π of projective spaces $\mathbb{P}^{k_m-c_m-1}$ lying linearly in the fibres of $\rho: G_\pi\to\tilde{G}_m$. Considering the set $G_\rho := \{\mathbb{P}^{k_m-c_m}\subset q_m^{-1}(X_m)\mid \mathbb{P}^{k_m-c_m}$ lies linearly in a fibre of $\pi: q_m^{-1}(X_m)\to\tilde{G}_m\}$, we obtain a $\mathbb{P}^{k_m-c_m}$-fibration $q_\rho:\Pi_m\to G_\rho$ with fibre $q_\rho^{-1}(y)=\mathbb{P}^{k_m-c_m}$ over a given point $y=\{\mathbb{P}^{k_m-c_m}\}\in G_\rho$. Therefore, to check the irreducibility of Π_m it suffices to check the irreducibility of G_ρ. Note that the projection π induces a projection $\tau: G_\rho\to\tilde{G}_m$ such that the fibre of τ over a point $V_{k_m+1}\in\tilde{G}_m$ coincides with the grassmannian $G(k_m-c_m+1,W)$, where $W\subset V_{k_m+1}$ is the subspace defined by the condition $\mathbb{P}(W)=\pi^{-1}(x)$. As above, (5.9) implies that, for $i\geqslant 1$, the locally closed subsets $\tau^{-1}(Z_i(W))$ of G_ρ have dimensions strictly less than that of the open subset $\tau^{-1}(Z(W)\setminus Z_i(W))$. This proves the irreducibility of G_ρ, hence of Π_m.

Next, (4.7) implies that, for any point $x=V_{k_m}\in X_m$ and $\tilde{m}\geqslant m$, the base $B_{\tilde{m}}(x)$ of the family of projective lines on X_m passing through x is a linear section of the variety $\mathbb{P}((\phi_{\tilde{m}-1}\circ\ldots\circ\phi_m)(V_{k_m})^*)\times GO(1,(\phi_{\tilde{m}-1}\circ\ldots\circ\phi_m)(V_{k_m})^\perp/(\phi_{\tilde{m}-1}\circ\ldots\circ\phi_m)(V_{k_m}))\subset\mathbb{P}(V^*_{k_{\tilde{m}}}\otimes V^\perp_{k_{\tilde{m}}}/V_{k_{\tilde{m}}})$ by a projective subspace of codimension $c_{\tilde{m}}$ in $\mathbb{P}(V^*_{k_{\tilde{m}}}\otimes V^\perp_{k_{\tilde{m}}}/V_{k_{\tilde{m}}})$. Let $b_{\tilde{m}}(x): B_{\tilde{m}}(x)\to Q_{(\tilde{m})}(x):=GO(1,V^\perp_{k_{\tilde{m}}}/V_{k_{\tilde{m}}})$ be the natural projection. Note that the fibres of $b_{\tilde{m}}(x)$ are projective spaces of dimension at least $k_{\tilde{m}}-c_{\tilde{m}}-1$, and, for points x of X_m and $z\in Q_{\tilde{m}}(x)$ in general position the fibre $b_{\tilde{m}}(x)^{-1}(z)$ is a projective space $\mathbb{P}^{k_{\tilde{m}}-c_{\tilde{m}}-1}$ by the Bertini Theorem. Moreover, we have an ind-variety $\mathbf{B}(x)=\varinjlim B_{\tilde{m}}(x)$, $\tilde{m}\geqslant m$.

In a similar way we obtain that $\mathbf{\Pi}(x):=\{\mathbf{P}^\infty\in\mathbf{\Pi}\mid\mathbf{P}^\infty\ni x\}$ is the ind-variety $\varinjlim\Pi_{\tilde{m}}(x)$, $\tilde{m}\geqslant m$, where $\Pi_{\tilde{m}}(x):=\{\mathbb{P}^{k_{\tilde{m}}-c_{\tilde{m}}-1}\subset B_m|\ \mathbb{P}^{k_{\tilde{m}}-c_{\tilde{m}}-1}$ lies as a linear projective subspace in a fibre of the projection $b_{\tilde{m}}(x)\}$. Let $p(x):\Pi_{\tilde{m}}(x)\to Q_{(\tilde{m})}(x)$ be the induced projection. By construction, for any point $z\in Q_{(\tilde{m})}(x)$ the fibre $p(x)^{-1}(z)$ is the grassmannian $G(k_{\tilde{m}}-c_{\tilde{m}},\mathbb{C}^{\dim(b_{\tilde{m}}(x)^{-1}(z))+1})$. (In particular, this grassmannian is just a point for $x\in X_{\tilde{m}}$ and $z\in Q_{(\tilde{m})}(x)$ in general position.) This implies the property (A.ii) since $Q_{(\tilde{m})}(x)$ is an irreducible quadric hypersurface.

The property (A.iii) is evident. As for the property (A.iv), let Y be a fixed variety and $f:\Pi_{\tilde{m}}(x)\to Y$ be a morphism. For $\tilde{m}\to\infty$ the fibres of $p(x)$ are either points or are grassmannians whose dimensions tend to infinity. Therefore f maps each fibre of $p(x)$ to a point, i.e. f factors through the induced morphism $g: Q_{(\tilde{m})}(x)\to Y$. As $Q_{(\tilde{m})}(x)$ is a smooth quadric hypersurface whose dimension tends to infinity as $\tilde{m}\to\infty$, it follows that for large enough $\tilde{m}$ the morphism g is a constant map. Hence, f is constant too, and (A.iv) is proved.

Theorem 1 yields now the following.

Theorem 4. *A vector bundle on a linear section $\mathbf{X}$ of small codimension of $\mathbf{G}(\infty)$, $\mathbf{GO}(\infty,\infty)$, $\mathbf{GS}(\infty,\infty)$ is isomorphic to a direct sum of line bundles $\mathcal{O}_{\mathbf{X}}(a_i)$ for $a_i\in\mathbb{Z}$.*

§ 6. Ind-products and their subvarieties satisfying the properties L, A, T

6.1. Finite or countable ind-products satisfying the properties L, A, T. Let $\mathbf{X}^\xi = \varinjlim X_m^\xi$, $\xi \in \Xi$, be a countable collection of ind-varieties. We assume that for each $\xi \in \Xi$ and each $m \geqslant 1$ we have a fixed inclusion $X_m^\xi \subset X_{m+1}^\xi$. On every $\mathbf{X}^\xi$ we fix a point x_0^ξ. Without loss of generality we assume that $x_0^\xi \in X_1^\xi$. Fix a bijection $\nu : \mathbb{N} \xrightarrow{\sim} \Xi$ and denote $\underline{m} := \{1, 2, ..., m\}$. Set

$$_\nu X_m := \underset{\xi \in \nu(\underline{m})}{\times} X_m^\xi$$

and consider the embeddings

$$_\nu X_m \hookrightarrow {_\nu X_{m+1}} = {_\nu X_m} \times X_m^{\nu(m+1)}, \quad x \mapsto \big(x, x_0^{\nu(m+1)}\big), \ m \geqslant 1.$$

We call the ind-variety $\mathbf{X} := \varinjlim {_\nu X_m}$ an *ind-product* of the ind-varieties $\{\mathbf{X}^\xi\}_{\xi \in \Xi}$ and denote it as

$$\mathbf{X} = \underset{\xi \in \Xi}{\times} \mathbf{X}^\xi. \tag{6.1}$$

Note that $\mathbf{X}$ does not depend, up to an isomorphism of ind-varieties, on the choice of the bijection $\nu : \mathbb{N} \xrightarrow{\sim} \Xi$, and thus the notation (6.1) is consistent. Indeed, let $\nu' : \mathbb{N} \xrightarrow{\sim} \Xi$ be another bijection, and let $\psi := \nu'^{-1} \circ \nu : \mathbb{N} \xrightarrow{\sim} \mathbb{N}$ be the induced bijection. An isomorphism $\mathbf{f} : \lim {_\nu X_m} \xrightarrow{\sim} \lim {_{\nu'} X_m}$, $\mathbf{f} = \{f_m : {_\nu X_m} \to {_{\nu'} X_{\tilde{m}(m)}}\}$, and its inverse $\mathbf{g} = \mathbf{f}^{-1} : \lim {_{\nu'} X_m} \xrightarrow{\sim} \lim {_\nu X_m}$, $\mathbf{g} = \{g_m : {_{\nu'} X_m} \to {_\nu X_{\tilde{m}(m)}}\}$, for $\tilde{m}(m) := \min\limits_{m' > m} \{m' \mid \psi(\underline{m}) \subset \underline{m'}\}$, are given by the formulas

$$f_m : {_\nu X_m} = \underset{\xi \in \nu(\underline{m})}{\times} X_m^\xi \to {_{\nu'} X_{\tilde{m}(m)}} = \big(\underset{\xi \in \nu(\underline{m})}{\times} X_{\tilde{m}(m)}^\xi\big) \times \big(\underset{\xi \in \nu'(\underline{\tilde{m}(m)}) \setminus \nu(\underline{m})}{\times} X_{\tilde{m}(m)}^\xi\big),$$

$$x \mapsto \big(x, \{x_0^\xi\}_{\xi \in \nu'(\underline{\tilde{m}(m)}) \setminus \nu(\underline{m})}\big),$$

$$g_m : {_{\nu'} X_m} = \underset{\xi \in \nu'(\underline{m})}{\times} X_m^\xi \to {_\nu X_{\tilde{m}(m)}} = \big(\underset{\xi \in \nu'(\underline{m})}{\times} X_{\tilde{m}(m)}^\xi\big) \times \big(\underset{\xi \in \nu(\underline{\tilde{m}(m)}) \setminus \nu'(\underline{m})}{\times} X_{\tilde{m}(m)}^\xi\big),$$

$$x \mapsto \big(x, \{x_0^\xi\}_{\xi \in \nu(\underline{\tilde{m}(m)}) \setminus \nu'(\underline{m})}\big).$$

Note in addition that in principle $\mathbf{X}$ depends on the choice of points x_0^ξ, however we suppress this dependence in the notation (6.1).

The reason we call $\mathbf{X}$ an ind-product rather than a product is that $\mathbf{X}$ is not a direct product in the category of ind-varieties. Of course, there are well-defined projections of ind-varieties $\mathbf{p}_\xi : \mathbf{X} \to \mathbf{X}^\xi$, $\mathbf{p}_\xi = \varinjlim p_{\xi m}$, where $p_{\xi m} : {_\nu X_m} \to X_m^\xi$ is the projection onto the ξ-factor for $\xi \in \nu(\underline{m})$, and the constant map $p_{\xi m} : {_\nu X_m} \to x_0^{\nu(m)}$ for $\xi \notin \nu(\underline{m})$. However, $\mathbf{X}$ fails to satisfy the universality property of a product.

If Ξ is finite, we define the ind-product $\underset{\xi \in \Xi}{\times} \mathbf{X}^\xi$ as the set-theoretic direct product of the $\mathbf{X}^\xi$'s. Then $\underset{\xi \in \Xi}{\times} \mathbf{X}^\xi = \varinjlim \big(\underset{\xi \in \Xi}{\times} X_m^\xi\big)$ is an ind-variety, and $\underset{\xi \in \Xi}{\times} \mathbf{X}^\xi$ clearly satisfies the universality property of a direct product in the category of ind-varieties.

Now let Ξ be finite or countable and let the ind-varieties $\mathbf{X}^\xi = \lim\limits_{\to} X^\xi_m$ satisfy the properties L, A and T. This means that on each $\mathbf{X}^\xi$ there exists a collection $\mathbf{L}^\xi := \{\mathbf{L}^\xi_\theta \mid \theta \in \Theta_{\mathbf{X}^\xi}\}$ of line bundles and a collection $\mathbf{B}^\xi := \{\mathbf{B}^\xi_\theta = \lim\limits_{\to} B^\xi_{\theta m} \mid \theta \in \Theta_{\mathbf{X}^\xi}\}$ of ind-varieties of projective lines such that $\mathbf{X}^\xi$ satisfies the properties L, A and T. The collections $\{\mathbf{L}^\xi\}_{\xi\in\Xi}$ yield the following countable collection of line bundles on $\mathbf{X}$

$$\mathbf{L} = \{\mathbf{p}^*_\xi \mathbf{L}^\xi_\theta \mid \theta \in \Theta_{\mathbf{X}^\xi},\ \xi \in \Xi\}. \tag{6.2}$$

Moreover, any projective line $\mathbb{P}^1$ on $\mathbf{X}^\xi$ determines a projective line $\mathbb{P}^1_{\mathbf{X}}$ on $\mathbf{X}$ such that $\mathbf{p}_\xi(\mathbb{P}^1_{\mathbf{X}}) = \mathbb{P}^1$ and $\mathbf{p}_{\xi'}(\mathbb{P}^1_{\mathbf{X}}) = \{x^{\xi'}_0\}$ for $\xi' \neq \xi$. Therefore each ind-variety $\mathbf{B}^\xi_\theta$ of projective lines on $\mathbf{X}^\xi$ "lifts"to an ind-variety of projective lines on $\mathbf{X}$. In this way we obtain a collection $\mathbf{B}$ of ind-varieties of projective lines on $\mathbf{X}$. Since each $\mathbf{X}^\xi$ satisfies the properties L, A and T, it is easy to check that $\mathbf{X}$ satisfies the same properties with respect to the collections $\mathbf{L}$ and $\mathbf{B}$.

This, together with Theorem 1, leads to the following theorem.

THEOREM 5. *A vector bundle on* $\mathbf{X} = \underset{\xi\in\Xi}{\times} \mathbf{X}^\xi$, *where each ind-variety* $\mathbf{X}^\xi$ *satisfies the properties L, A and T, is isomorphic to a direct sum of line bundles.*

6.2. Linear sections of ind-products. In this subsection we assume that Ξ is finite, $\Xi = \{1, 2, ..., l\}$, and that the ind-varieties $\mathbf{X}^i = \lim\limits_{\to} X^i_m$, $i \in \Xi$, are copies of the standard ind-grassmannians $\mathbf{G}(\infty)$, $\mathbf{G}\mathrm{O}(\infty,\infty)$, $\mathbf{G}\mathrm{S}(\infty,\infty)$. By the above, $\underset{i\in\Xi}{\times} \mathbf{X}^i$ is a direct limit $\lim\limits_{\to}(\underset{i\in\Xi}{\times} X^i_m)$. Each X^i_m is a (possibly isotropic) grassmannian which we consider as lying via the Plücker embedding in $\mathbb{P}^{N_{im}-1} = \mathbb{P}(V_{N_{im}})$, and the embeddings $\underset{i\in\Xi}{\times} X^i_m \hookrightarrow \underset{i\in\Xi}{\times} X^i_{m+1}$ are induced by standard extensions $X^i_m \hookrightarrow X^i_{m+1}$. We also assume that $\underset{i\in\Xi}{\times} X^i_m$ lies in $\mathbb{P}^{N_m-1}$ via the Segre embedding $\underset{i\in\Xi}{\times} \mathbb{P}^{N_{im}-1} \hookrightarrow \mathbb{P}^{N_m-1}$.

For each $m \geqslant 1$ and $i \in \Xi$ set

$$\widehat{X}^i_m := X^1_m \times ... \times X^{i-1}_m \times X^{i+1}_m \times ... \times X^l_m,$$

and for $X^i_m = G(k_{im}, V_{n_{im}})$ (respectively, $X^i_m = GO(k_{im}, V_{n_{im}})$ or $X^i_m = GS(k_{im}, V_{n_{im}})$), set ${X^i_m}^+ := G(k_{im}+1, V_{n_{im}})$ (respectively, ${X^i_m}^+ = GO(k_{im}+1, V_{n_{im}})$ or ${X^i_m}^+ = GS(k_{im}+1, V_{n_{im}})$). Consider the flag variety $\Sigma_{im} := \{(V_{k_{im}}, V_{k_{im}+1}) \in X^i_m \times {X^i_m}^+ \mid V_{k_{im}} \subset V_{k_{im}+1}\}$ with natural projections ${X^i_m}^+ \leftarrow \Sigma_{im} \to X^i_m$. There are induced projections

$${X^i_m}^+ \times \widehat{X}^i_m \overset{p_{im}}{\leftarrow} \Sigma_{im} \times \widehat{X}^i_m \overset{q_{im}}{\to} X^i_m \times \widehat{X}^i_m \simeq \widehat{X}_m,\quad i \in \Xi, \tag{6.3}$$

and we put

$$\pi_{im} := p_{im}|_{q^{-1}_{im}(X_m)} : q^{-1}_{im}(X_m) \to {X^i_m}^+ \times \widehat{X}^i_m,\quad i \in \Xi. \tag{6.4}$$

Now let $\{c_m\}_{m\geqslant 1}$ be a nondecreasing sequence of integers satisfying the conditions

$$1 \leqslant c_m \leqslant \min\{\frac{k_{im}-1}{2} \mid i \in \Xi\}, \tag{6.5}$$

where $X_m^i = G(k_{im}, V_{n_{im}}), GO(k_{im}, V_{n_{im}}), GS(k_{im}, V_{n_{im}})$. For each $m \geqslant 1$ consider a linear section X_m of $\underset{i\in\Xi}{\times} X_m^i$,

$$X_m := (\underset{i\in\Xi}{\times} X_m^i) \cap \mathbb{P}^{N_m - c_m - 1},$$

where $\mathbb{P}^{N_m-c_m-1} = \mathbb{P}(U_m)$ for $U_m \in G(N_m - c_m, V_{N_m})$. We call X_m a *linear section of* $\underset{i\in\Xi}{\times} X_m^i$ *of codimension* c_m.

Proposition 5. *For a given $m \geqslant 1$ such that $k_{im} \geqslant 2$ for all $i \in \Xi$, fix an integer c_m satisfying (6.5). Then for a projective subspace $\mathbb{P}^{N_m-c_m-1} = \mathbb{P}(U_m)$ of general position in $\mathbb{P}^{N_m-1}$, $U_m \in G(N_m - c_m, V_{N_m})$, and any $i \in \Xi$ the following statements hold:*

(i) the varieties X_m and $q_{im}^{-1}(X_m)$ are smooth;

(ii) $\operatorname{codim}_{\underset{i\in\Xi}{\times} X_m^i} X_m = \operatorname{codim}_{\Sigma_{im}\times \widehat{X}_m^i} q_{im}^{-1}(X_m) = c_m$, $\pi_{im*}\mathcal{O}_{q_{im}^{-1}(X_m)} = \mathcal{O}_{X_m^{i}{}^+ \times \widehat{X}_m^i}$, *and, for a point $x = (x_1, x_2) \in X_m^i{}^+ \times \widehat{X}_m^i$ in general position, the projective subspace $\mathbb{P}^{k_{im}} = q_{im}p_{im}^{-1}(x)$ of $X_m^i \times \{x_2\}$ satisfies the condition*

$$\operatorname{codim}_{\mathbb{P}^{k_{im}}}(\mathbb{P}^{k_{im}} \cap \mathbb{P}^{N_m-c_m-1}) = c_m,$$

so that $Z_m^i(U_m) := \{x \in X_m^i{}^+ \times \widehat{X}_m^i \mid \dim \pi_{im}^{-1}(x) > k_{im} - c_m\}$ is a proper closed subset of $X_m^i{}^+ \times \widehat{X}_m^i$;

(iii) let $B_m^i(U_m) := \pi_{im}^{-1}(Z_m^i(U_m))$; then $\operatorname{codim}_{q_{im}^{-1}(X_m)} B_m^i(U_m) \geqslant 2$, $\operatorname{codim}_{X_m^i{}^+\times\widehat{X}_m^i} Z_m^i(U_m$
3*;*

(iv) the projection $\pi_{im} : q_{im}^{-1}(X_m) \setminus B_m^i(U_m) \to X_m^i{}^+ \times \widehat{X}_m^i \setminus Z_m^i(U_m)$ is a projective $\mathbb{P}^{k_{im}-c_m}$-bundle.

Proof. Similar to the proof of Proposition 2.

Corollary 2. *Under the assumptions of Proposition 5, let $i \in \Xi$ and let $\mathcal{E}$ be a vector bundle on $q_{im}^{-1}(X_m)$ trivial along the fibres of the morphism $\pi_{im} : q_{im}^{-1}(X_m) \to X_m^i{}^+ \times \widehat{X}_m^i$. Then the sheaf $\pi_{im*}\mathcal{E}$ is locally free and the canonical morphism $ev : \pi_{im}^*\pi_{im*}\mathcal{E} \xrightarrow{\simeq} \mathcal{E}$ is an isomorphism.*

Proof. Proposition 5 implies that the data $X = X_m^i{}^+ \times \widehat{X}_m^i$, $Y = q_{im}^{-1}(X_m)$, $B = B_m^i(U_m)$, $Z = Z_m^i(U_m)$, $E = \mathcal{E}$ satisfy the conditions of Proposition 6 from the appendix. Therefore this latter proposition yields the corollary.

Below we will need the following lemma, the proof of which is similar to that of Lemma 5.1.

Lemma 6.1. *For $i = 1, 2$, let $X_i = G(k_i, V_i)$, $GO(k_i, V_i)$, $GS(k_i, V_i)$ and let X be a linear section of codimension c of $X_1 \times X_2$, where $1 \leqslant c \leqslant \min\{\frac{k_1+1}{2}, \frac{k_2+1}{2}\}$. Then for any projective line $\mathbb{P}_i^1 \subset X_i$ there exists a projective line $\mathbb{P}^1$ such that $pr_i|_{\mathbb{P}^1}$ is an isomorphism of $\mathbb{P}^1$ and $\mathbb{P}_i^1$. Here pr_i stands for the natural projection $X_1 \times X_2 \to X_i$.*

Consider an ind-variety $\mathbf{X} = \varinjlim X_m$ such that, for each $m \geqslant 1$, X_m is a smooth linear section of codimension c_m of $\underset{i\in\Xi}{\times} X_{im}$ where c_m satisfies (6.5), and the embeddings $\phi_m : X_m \hookrightarrow X_{m+1}$ are induced by the corresponding embeddings $\underset{i\in\Xi}{\times} X^i_m \hookrightarrow \underset{i\in\Xi}{\times} X^i_{m+1}$. In what follows we call $\mathbf{X}$ a *linear section of* $\underset{i\in\Xi}{\times}\mathbf{X}^i$ *of small codimension.* The existence of linear sections $\mathbf{X}$ of $\underset{i\in\Xi}{\times}\mathbf{X}^i$ of small codimension follows immediately from the Bertini Theorem.

Theorem 6. *Let* $\mathbf{X}$ *be a linear section of* $\underset{i\in\Xi}{\times}\mathbf{X}^i$ *of small codimension. Then a vector bundle on* $\mathbf{X}$ *is a direct sum of line bundles.*

Proof. We give a proof for the case when all $\mathbf{X}^i$ are isomorphic to $\mathbf{G}\mathrm{O}(\infty,\infty)$. The case when some $\mathbf{X}^i$ are isomorphic to $\mathbf{G}(\infty)$ or $\mathbf{G}\mathrm{S}(\infty,\infty)$ is treated similarly.

First we construct families $\mathbf{B}_i$, $i \in \Xi$, on $\mathbf{X}$. For each $m \geqslant 1$ and each i, $1 \leqslant i \leqslant l$, consider the natural projection $p_{im} : X_m \to \hat{X}^i_m$, and for a varying point $y \in \hat{X}^i_m$ set $X^i_m(y) := p^{-1}_{im}(y)$. By definition, $X^i_m(y)$ is a linear section of the grassmannian $GO(k_{im}, V_{n_{im}})$. Let $B_i(m)(y)$ be the family of projective lines in $GO(k_{im}, V_{n_{im}})$ lying on $X^i_m(y)$, and set $B_i(m) := \underset{y\in\hat{X}^i_m}{\cup} B_i(m)(y)$. Then $\mathbf{B}_i := \underset{\to}{\lim} B_i(m)$, $i \in \Xi$. Furthermore, the ind-varieties $\mathbf{\Pi}_i$, $i \in \Xi$, parametrizing certain families of ind-projective spaces $\mathbb{P}^\infty$ in $\mathbf{B}_i$, are defined in the same way as the ind-variety $\mathbf{\Pi}$ in subsection 5.2 – see (5.16).

Next, recall the collection of line bundles (6.2) on $\underset{i\in\Xi}{\times}\mathbf{X}^i$. In our case $\Theta_{\mathbf{X}^\xi}$ consists of a single point for each i, hence we can write simply $\mathbf{L} = \{\mathbf{L}_i\}_{i\in\Xi}$. We now define a family of line bundles $\mathbf{L}_\mathbf{X}$ by putting $\mathbf{L}_\mathbf{X} := \{\mathbf{L}_i|_\mathbf{X}\}_{i\in\Xi}$. Then by the Lefschetz Theorem $\mathbf{L}_\mathbf{X}$ freely generates Pic$\mathbf{X}$; in addition, the relation (2.2) is clearly satisfied. To see that $\mathbf{X}$ satisfies the property L, it remains to notice that $H^1(X_m, \underset{i\in\Xi}{\otimes}\mathbf{L}_i^{\otimes a_i}|_{X_m}) = 0$ for all a_i. Indeed, the vanishing of $H^1(\underset{i\in\Xi}{\times}X^i_m, \underset{i\in\Xi}{\otimes}\mathbf{L}_i^{\otimes a_i}|_{\underset{i\in\Xi}{\times}X^i_m})$ follows from Kunneth's and Bott's formulas. Since $\underset{i\in\Xi}{\otimes}\mathbf{L}_i^{\otimes a_i}|_{X_m}$ admits a Koszul resolution similar to (5.3), this is sufficient to conclude that $H^1(X_m, \underset{i\in\Xi}{\otimes}\mathbf{L}_i^{\otimes a_i}|_{X_m}) = 0$.

Using Proposition 5 and repeating the argument from subsection 5.2, it is easy to check that $\mathbf{X}$ satisfies the property A. Let us show that $\mathbf{X}$ satisfies the property T. The case $|\Xi| = 1$ was treated in the proof of Theorem 4. It is enough to give the proof for the case $|\Xi| = 2$; the proof for $|\Xi| \geqslant 3$ goes along the same lines. We thus assume that X_m is a linear section of $X^1_m \times X^2_m$. According to (6.3) and (6.4) we have a commutative diagram

$$\begin{tikzcd} q^{-1}_{im}(X_m) \arrow[r, "q_{1m}"] \arrow[d, "\pi_{1m}"'] & X_m \arrow[d, "\rho"] \\ X^{1\,+}_m \times X^2_m \arrow[r, "\lambda"'] & X^2_m, \end{tikzcd} \tag{6.6}$$

where $l = 2$, $i = 1$, so that $\hat{X}^1_m = X^2_m$, and λ and ρ are the natural projections.

Let $\underset{\leftarrow}{\lim} E_m$ be a $\mathbf{B}_i$-trivial vector bundle on $\mathbf{X} = \underset{\to}{\lim} X_m$ for $i = 1, 2$. This means that each vector bundle E_m is a $B_i(m)$-trivial bundle on X_m, i.e. $E_m|_{\mathbb{P}^1}$ is trivial for any $\mathbb{P}^1 \in B_i(m)$, $i = 1, 2$. Consider the vector bundle

$$\tilde{E}_m := q^*_{1m}E_m.$$

Since E_m is $B_1(m)$-trivial, i.e. linearly trivial on the fibres of ρ, E_m is trivial by Proposition 3. It follows that $\tilde{E}_m$ is trivial along the fibres of π_{1m}. Therefore, by Corollary 2 there is an isomorphism

$$ev : \pi_{1m}^* \pi_{1m*} \tilde{E}_m \xrightarrow{\simeq} \tilde{E}_m.$$

Moreover, as in the proof of Proposition 3, we obtain that the bundle $\pi_{1m*}\tilde{E}_m|_{\lambda^{-1}(y)}$ is trivial for any $y \in X_m^2$.

Thus Proposition 6 below and diagram (6.6) imply that $E_m^\lambda := \lambda_* \pi_{1m*} \tilde{E}_m$ is a vector bundle on X_m^2 such that $\tilde{E}_m \simeq \pi_{1m}^* \lambda^* E_m^\lambda \simeq q_{1m}^* \rho^* E_m^\lambda$. Applying again Proposition 6 we obtain

$$E_m \simeq q_{1m*} \tilde{E}_m \simeq q_{1m*} q_{1m}^* \rho^* E_m^\lambda \simeq \rho^* E_m^\lambda. \tag{6.7}$$

Since E_m is $B_2(m)$-trivial, it follows from (6.7) and Lemma 6.1 that E_m^λ is a linearly trivial bundle on X_m^2. Hence, E_m^λ is trivial by Proposition 9 below, and E_m is trivial as well. In this we showed that $\mathbf{X}$ satisfies the property T.

6.3. Ind-varieties of generalized flags. We first recall some basic definitions concerning generalized flags in a vector space, see [16; sections 3-5]. Let V be a countable-dimensional vector space. A set $\mathcal{C}$ of pairwise distinct subspaces of V is called a *chain* if it is linearly ordered by inclusion. A chain $\mathcal{F}$ of subspaces of V is a *generalized flag* in V if it satisfies the following conditions:

(i) each $F \in \mathcal{F}$ has an immediate successor or an immediate predecessor, i.e. $\mathcal{F} = \mathcal{F}' \cup \mathcal{F}''$, where $\mathcal{F}' \subset \mathcal{F}$ (respectively, $\mathcal{F}'' \subset \mathcal{F}$) is the set of subspaces in $\mathcal{F}$ having an immediate successor (respectively, predecessor);

(ii) $V \setminus \{0\} = \sqcup_{F' \in \mathcal{F}'} (F'' \setminus F')$, where $F'' \in \mathcal{F}''$ is the immediate successor of $F' \in \mathcal{F}'$.

We define a *flag* in V to be a generalized flag in V which is isomorphic as an ordered set to a subset of $\mathbb{Z}$. A flag can be equivalently defined as a chain of subspaces of V such that $\cap_{F \in \mathcal{F}} F = 0$, $\cup_{F \in \mathcal{F}} F = V$ and there exists a strictly monotonic map of ordered sets $\phi : \mathcal{F} \to \mathbb{Z}$.

If $\mathcal{F}$ is a generalized flag in V and $\{e_\alpha\}_{\alpha \in A}$ is a basis of V (A being a countable set), we say that $\mathcal{F}$ and $\{e_\alpha\}_{\alpha \in A}$ are *compatible* if there exists a strict partial order $\prec$ on A such that, for any $F' \in \mathcal{F}'$, $F' = \mathrm{Span}\{e_\beta \mid \beta \prec \alpha\}$ for a certain $\alpha \in A$, and $F'' = F' \oplus \mathrm{Span}\{e_\gamma \mid \gamma \text{ is not } \prec\text{-comparable to } \alpha\}$.

For the rest of this section we fix a basis $E = \{e_n\}$ of V. We call a generalized flag $\mathcal{F}$ *weakly compatible with* E if $\mathcal{F}$ is compatible with a basis L of V such that $E \setminus (E \cap L)$ is a finite set. Furthermore, we define two generalized flags $\mathcal{F}$ and $\mathcal{G}$ in V to be E-*commensurable* if both $\mathcal{F}$ and $\mathcal{G}$ are weakly compatible with E and there exists an inclusion preserving bijection $\phi : \mathcal{F} \to \mathcal{G}$ and a finite-dimensional subspace U in V such that, for every $F \in \mathcal{F}$,

(i) $F \subset \phi(F) + U$, $\phi(F) \subset F + U$, and

(ii) $\dim(F \cap U) = \dim(\phi(F) \cap U)$.

Let $\mathcal{F}l(\mathcal{F}, E)$ be the set of all generalized flags in V that are E-commensurable with $\mathcal{F}$. Following [16; Prop. 5.2]) we endow $\mathcal{F}l(\mathcal{F}, E)$ with a structure of an ind-variety in the following way. For any $m \geqslant 1$ denote $E_m := \{e_\alpha\}_{\alpha \leqslant m}$, $V_m := \mathrm{Span}(E_m)$, $E_m^c := \{e_\alpha\}_{\alpha > m}$, $V_m^c := \mathrm{Span}(E_m^c)$. Next, for any $\mathcal{G} \in \mathcal{F}l(\mathcal{F}, E)$ choose a positive integer $m_{\mathcal{G}}$ such that $\mathcal{F}$ and $\mathcal{G}$ are compatible with bases containing $E^c_{m_{\mathcal{G}}}$, and $V_{m_{\mathcal{G}}}$ contains a finite-dimensional subspace U which together with the corresponding inclusion preserving bijection $\phi_{\mathcal{G}} : \mathcal{F} \overset{\sim}{\to} \mathcal{G}$ makes $\mathcal{F}$ and $\mathcal{G}$ E-commensurable. We can pick $n_{\mathcal{F}}$ so that $n_{\mathcal{F}} \leqslant m_{\mathcal{G}}$ for every $\mathcal{G} \in \mathcal{F}l(\mathcal{F}, E)$. Set

$$\mathcal{G}_m := \{G \cap V_m \mid G \in \mathcal{G}\}, \quad m \geqslant m_{\mathcal{G}}.$$

The type of the flag $\mathcal{F}_n$ yields a sequence of integers

$$0 < d_{m,1} < ... < d_{m,s_{m-1}} < d^m_{m,s_m} = m,$$

and let $\mathcal{F}l(d_m, V_m)$ be the usual flag variety of type $d_m = (d_{m,1}, ..., d_{m,s_{m-1}})$ in V_m. Notice that $s_{m+1} = s_m$ or $s_{m+1} = s_m + 1$. Furthermore, in both cases an integer j_m is determined as follows: in the former case $d_{m+1,i} = d_{m,i}$ for $0 \leqslant i < j_m$ and $d_{m+1,i} = d_{m,i} + 1$ for $j_m \leqslant i < s_m$, and in the latter case $d_{m+1,i} = d_{m,i}$ for $0 \leqslant i < j_m$ and $d_{m+1,i} = d_{m,i-1} + 1$ for $j_m \leqslant i < s_m$.

Now we define a map $\iota_m : \mathcal{F}l(d_m, V_m) \to \mathcal{F}l(d_{m+1}, V_{m+1})$ for every $m \geqslant m_{\mathcal{F}}$. Given a flag $\mathcal{G}_m = \{0 = G^m_0 \subset G^m_1 \subset ... \subset G^m_{s_m} = V_m\} \in \mathcal{F}l(d_m, V_m)$, put $\iota_m(\mathcal{G}_m) = \mathcal{G}_{m+1} := \{0 = G^{m+1}_0 \subset G^{m+1}_1 \subset ... \subset G^{m+1}_{s_m+1} = V_{m+1}\}$, where

$$G^{m+1}_i = \begin{cases} G^m_i & \text{if } \ 0 \leqslant i < j_m, \\ G^m_i \oplus ke_{m+1} & \text{if } j_m \leqslant i \leqslant s_{m+1} \ \text{ and } \ s_{m+1} = s_m, \\ G^m_{i-1} \oplus ke_{m+1} & \text{if } j_m \leqslant i \leqslant s_{m+1} \ \text{ and } \ s_{m+1} = s_m + 1. \end{cases}$$

The maps ι_m are closed embeddings of algebraic varieties, and hence $\lim\limits_{\to} \mathcal{F}l(d_m, V_m)$ is an ind-variety. A bijection between $\mathcal{F}l(\mathcal{F}, E)$ and $\lim\limits_{\to} \mathcal{F}l(d_m, V_m mm)$ is given by

$$\tau : \mathcal{F}l(\mathcal{F}, E) \overset{\sim}{\to} \lim_{\to} \mathcal{F}l(d_m, V_m), \quad \mathcal{G} \mapsto \lim_{\to} \mathcal{G}_m$$

– see [16; Prop. 5.2].

Assume now that $\mathcal{F}$ is a flag of subspaces in V. Then $\mathcal{F} = \{... \subset F_i \subset F_{i+1} \subset ...\}_{i \in \Theta}$, where Θ is one of the four linearly ordered sets $\{1, ..., n\}$, $\mathbb{Z}$, $\mathbb{Z}_{>0}$, $\mathbb{Z}_{<0}$. Assume in addition that

$$\dim(F''/F') = \infty \tag{6.8}$$

for all $i \in \Theta$ for which $i + 1 \in \Theta$. Denote by $\hat{\mathcal{F}}(i)$ the flag $\hat{\mathcal{F}} \setminus \{F_i\} = \{... \subset F_{i-1} \subset F_{i+1} \subset ...\}$. There is natural projection $\pi_i : \mathcal{F}l(\mathcal{F}, E) \to \mathcal{F}l(\hat{\mathcal{F}}(i), E)$. Let $\hat{\mathcal{G}} = \{... \subset G_{i-1} \subset G_{i+1} \subset ...\} \in \mathcal{F}l(\hat{\mathcal{F}}(i), E)$ and $\mathcal{G} = \{... \subset G_{i-1} \subset G_i \subset G_{i+1} \subset ...\} \in \pi_i^{-1}(\hat{\mathcal{G}})$. Then the fibre $\pi_i^{-1}(\hat{\mathcal{G}})$ equals $\mathcal{F}l(G_i/G_{i-1}, E(i))$ where $E(i) = (E \cap G_{i+1}) \setminus (E \cap G_{i-1})$. Note that the ind-variety $\mathcal{F}l(G_i/G_{i-1}, E(i))$ is isomorphic to the ind-grassmannian $\mathbf{G}(\infty)$.

Moreover, there is a well-defined line bundle $\mathbf{L}_i := \mathcal{O}_{\pi_i}(1) := \lim\limits_{\leftarrow} \mathcal{O}_{\pi_{im}}(1)$ on $\mathcal{F}l(\mathcal{F}, E)$, where $\pi_{im} : \mathcal{F}l(d_m, V_m) \to \mathcal{F}l(\hat{d}_m(i), V_m)$ is the natural projection and $\hat{d}_m(i)$ is defined in the same way as d_m using the flag $\hat{\mathcal{F}}(i)$ instead of $\mathcal{F}$. The fact that the line bundles $\mathcal{O}_{\pi_{im}}(1)$ yield a well-defined bundle $\mathcal{O}_{\pi_i}(1)$ is established by a straightforward checking using the explicit form of the embeddings ι_m.

By $\mathbf{B}_i(\mathcal{G})$ we denote the ind-variety of projective lines on $\mathcal{F}l(\mathcal{F}, E)$ passing through a point $\mathcal{G} \in \mathcal{F}l(\mathcal{F}, E)$ and lying in the fibre of π_i which contains $\mathcal{G}$. Finally, we define the ind-variety $\mathbf{\Pi}_i(\mathcal{G})$ as the ind-variety $\mathbf{\Pi}(\mathcal{G})$ for the ind-grassmannian $\mathcal{F}l(G_i/G_{i-1}, E(i)) \simeq \mathbf{G}(\infty)$ as defined in subsection 4.3.

It is easy to check that $\mathcal{F}l(\mathcal{F}, E)$ satisfies the properties L, A and T with respect to the data $\Theta_{\mathcal{F}l(\mathcal{F},E)} := \Theta$, $\mathbf{L}_i$, $\mathbf{B}_i := \underset{\mathcal{G}\in\mathcal{F}l(\mathcal{F},E)}{\cup}\mathbf{B}_i(\mathcal{G})$, $\mathbf{\Pi}_i := \underset{\mathcal{G}\in\mathcal{F}l(\mathcal{F},E)}{\cup}\mathbf{\Pi}_i(\mathcal{G})$. As a result, Theorem 1 implies the following theorem.

Theorem 7. *Let V be a countable-dimensional vector space with basis E. Let $\mathcal{F}$ be a flag in V satisfying (6.8) weakly compatible with E. Then any vector bundle on $\mathcal{F}l(\mathcal{F}, E)$ is isomorphic to a direct sum of line bundles.*

It is an interesting question whether the BVTS Theorem holds on any ind-variety of generalized flags $\mathcal{F}l(\mathcal{F}, E)$ under the assumption that the generalized flag $\mathcal{F}$ satisfies (6.8) for all $F' \in \mathcal{F}'$ and their respective successors F''.

§ 7. Appendix

In this appendix we collect some general facts about coherent sheaves on projective varieties and their behaviour under flat projective morphisms, which are used throughout the paper.

Proposition 6. *Let $p : Y \to X$ be a smooth flat projective morphism of projective varieties with irreducible fibres.*

1) If E is a vector bundle on Y, trivial on the fibres of p, then the evaluation morphism $ev : p^*p_*E \to E$ is an isomorphism.

2) If F be a vector bundle on X, then the canonical morphism $F \xrightarrow{\sim} p_*p^*F$ is an isomorphism.

Proof. 1) This follows easily from the Base-change Theorem [H, Ch. III, Cor. 12.9].

2) Consider the Stein factorization $f : Y \xrightarrow{f'} X' \xrightarrow{g} X$ of f, where $X' = \mathbf{Spec}(f_*\mathcal{O}_Y)$ and $f'_*\mathcal{O}_Y = \mathcal{O}_{X'}$ (see [13; Ch. III, Cor.12.9]). Since $f_*\mathcal{O}_Y$ is an invertible sheaf by 1), it follows that g is an isomorphism. Therefore $f_*\mathcal{O}_Y = \mathcal{O}_X$. This, together with the projection formula [13; Ch. III, Exc. 8.3], gives the desired assertion.

Proposition 7. *Let $\pi : Y \to X$ be a surjective morphism of smooth irreducible projective varieties such that:*

(i) the fibres of π are projective spaces;

(ii) the variety $Z := \{x \in X \mid \dim \pi^{-1}(x) > \dim Y - \dim X\}$ has codimension at least 3 in X, and the variety $B := \pi^{-1}(Z)$ has codimension at least 2 in Y;

(iii) there exists a vector bundle F on $X \setminus Z$ such that $\pi : Y \setminus B \simeq \mathbb{P}(F) \to X \setminus Z$ is the structure map of the projectivized vector bundle F.

Next, let E be a vector bundle on Y, trivial along the fibres of π. Then the $\mathcal{O}_X$-sheaf π_*E is locally free and the evaluation morphism $ev : \pi^*\pi_*E \to E$ is an isomorphism.

PROOF. We first show that $ev : \pi^*\pi_*E \to E$ is an isomorphism. For this, consider an arbitrary open subvariety $U \subset X$ and its closed subvariety $A \subset U$ such that

$$\mathrm{codim}_U A \geqslant 2. \tag{7.1}$$

Since X is smooth and Z has codimension $\geqslant 3$ in X, it follows that $\mathrm{codim}_U(Z \cap A) \geqslant 3$ and $\mathrm{codim}_{\pi^{-1}(U)}\pi^{-1}(Z \cap A) \geqslant 2$. Next, (iii) and (7.1) imply $\mathrm{codim}_{\pi^{-1}(U)}\pi^{-1}(Z \setminus Z \cap A) \geqslant 2$. Hence

$$\mathrm{codim}_{\pi^{-1}(U)}\pi^{-1}(A) \geqslant 2. \tag{7.2}$$

Let $s \in H^0(U \setminus A, \pi_*E|_{U\setminus A})$ and let $\tilde{s} := \phi(s)$, where $\phi : H^0(U \setminus A, \pi_*E|_{U\setminus A}) \overset{\simeq}{\to} H^0(\pi^{-1}(U \setminus A), E|_{\pi^{-1}(U\setminus A)})$ is the canonical isomorphism. Since E is a locally free sheaf on a smooth variety Y, E is normal by [17; Prop. 1.6(ii)], i.e. (7.2) implies that $\tilde{s}$ extends uniquely to a section $\tilde{s}' \in H^0(\pi^{-1}(U), E|_{\pi^{-1}(U)})$. Then s extends to the section $s' := \psi(\tilde{s}') \in H^0(U, \pi_*E|_U)$, where $\psi : H^0(\pi^{-1}(U), E|_{\pi^{-1}(U)}) \overset{\simeq}{\to} H^0(U, \pi_*E|_U)$ is the canonical isomorphism. In view of (7.1) this means that the sheaf π_*E is normal.

Note that π_*E is torsion-free. Indeed, if the torsion subsheaf $Tors(\pi_*E)$ were nonzero, then since E is locally free, by (iii) any section $0 \neq s \in H^0(Y, Tors(\pi_*E))$ would be supported in Z. Then the section $0 \neq \tilde{s} := \psi^{-1}(s)$ would be supported in B, i.e. $Tors(E) \neq 0$. This contradicts the assumptions that E is locally free and Y is smooth and irreducible.

Hence, π_*E is reflexive by [17; Prop. 1.6]. Set

$$\tilde{E} := \pi^*\pi_*E/Tors(\pi^*\pi_*E).$$

Proposition 6, together with (iii), implies the existence of an isomorphism

$$\alpha : \tilde{E}|_{Y\setminus B} \overset{\simeq}{\to} E|_{Y\setminus B}. \tag{7.3}$$

Now by [17; Prop. 1.1] the sheaf π_*E can, locally on X, be included in an exact sequence

$$0 \to \pi_*E \to L_1 \to L_2, \tag{7.4}$$

with locally free sheaves L_1 and L_2. Applying to (7.4) the functor π^* we obtain the sequence

$$0 \to \tilde{E} \to \pi^*L_1 \to \pi^*L_2$$

which is exact when restricted onto $Y \setminus B$. Hence, this sequence is itself exact as $\tilde{E}$ is torsion free and the sheaves π^*L_1 and π^*L_2 are locally free. By [17; Prop. 1.1] this implies that $\tilde{E}$ is reflexive. Therefore, denoting by i the inclusion $Y \setminus B \hookrightarrow Y$ and using the isomorphism (7.3) and [17; Prop. 1.6(iii)] we obtain an isomorphism

$$\pi^*\pi_*E = \tilde{E} \cong i_*(\tilde{E}|_{Y\setminus B}) \underset{\simeq}{\overset{i_*\alpha}{\to}} i_*(E|_{Y\setminus B}) \cong E.$$

This isomorphism is nothing but the evaluation morphism ev.

It remains to show that π_*E is locally free. The isomorphism ev implies that $\pi^*\pi_*E$ is locally free. Therefore, for any $y \in Y$, $r := \dim_{\mathbb{C}(y)}(\pi^*\pi_*E \otimes \mathbb{C}(y))$ does not depend on y, and consequently, $\dim_{\mathbb{C}(x)}(\pi_*E \otimes \mathbb{C}(x)) = r$. According to [18; §5, Lemma 1], since X is smooth, the fact that $\dim_{\mathbb{C}(x)}(\pi_*E \otimes \mathbb{C}(x))$ does not depend on $x \in X$ implies that π_*E is locally free.

PROPOSITION 8. *Let Q_n be a nonsingular n-dimensional quadric in $\mathbb{P}^{n+1}$ for $n \geqslant 2$, and let E be a linearly trivial vector bundle on Q_n. Then E is trivial.*

PROOF. We argue by induction on n. For $n = 2$ the proof is easy and is the same as for a projective space as given in [14; Ch. I, Thm. 3.2.1]. Thus we may assume that $n \geqslant 3$. Consider a codimension-2 subspace $\mathbb{P}^{n-1}$ in $\mathbb{P}^{n+1}$ such that $Q_{n-2} := Q_n \cap \mathbb{P}^{n-1}$ is a smooth quadric of dimension $n-2$. If $n \geqslant 4$ then $E|_l$ is trivial for any projective line $l \subset Q_{n-2}$, hence $E|_{Q_{n-2}}$ is trivial by the induction assumption. For $n = 3$ the quadric Q_{n-2} is a smooth conic C. By Bertini's Theorem there exists a smooth quadric surface Q_2 on Q_3 passing through the conic C. Since $E|_{Q_2}$ is trivial (being linearly trivial), $E|_C$ is also trivial, i.e. our claim holds for $n = 3$.

We will now use the triviality of $E|_{Q_{n-2}}$ for $n \geqslant 4$ to show that E is trivial. Let $\sigma_Q : \tilde{Q}_n \to Q_n$ be the blow-up of Q_n with center at Q_{n-2}, and let $D := \sigma_Q^{-1}(Q_{n-2})$ be the exceptional divisor. Clearly, $D \simeq Q_{n-2} \times \mathbb{P}^1$ and there is a flat surjective morphism $\pi : \tilde{Q}_n \to \mathbb{P}^1$ fitting in the commutative diagram

$$\begin{tikzcd} D = Q_{n-2} \times \mathbb{P}^1 \arrow[r, hook, "i"] \arrow[dr, "pr_2"'] & \tilde{Q}_n \arrow[d, "\pi"] \\ & \mathbb{P}^1, \end{tikzcd} \tag{7.5}$$

where i is the embedding of the exceptional divisor. By construction, there exist two distinct points $t_1, t_2 \in \mathbb{P}^1$ such that the fibre $Q_{n-1}(t) = \pi^{-1}(t)$ is a smooth quadric for $t \in U := \mathbb{P}^1 \smallsetminus \{t_1 \cup t_2\}$, and $Q_{n-1}(t_j) := \pi^{-1}(t_j)$ for $j = 1, 2$ are quadratic cones whose vertices are points.

Consider the vector bundle $\tilde{E} := \sigma_Q^* E$ on $\tilde{Q}_n$. By construction, $\tilde{E}$ is trivial on any projective line $l \subset Q_{n-1}(t)$, $t \in U$. Hence, by the induction assumption, $E|_{Q_{n-1}(t)}$, $t \in U$, is trivial. Consequently,

$$H^i(Q_{n-1}(t), \tilde{E}(-D)|_{Q_{n-1}(t)}) = 0,\ i \geqslant 0, \quad \dim H^i(Q_{n-1}(t), \tilde{E}|_{Q_{n-1}(t)}) = \left\{ \begin{array}{ll} r, & \text{if } i = 0, \\ 0, & \text{if } i \geqslant 1, \end{array} \right. \quad t \in \tag{7.6}$$

Next, for $j = 1, 2$, let $\sigma : K_j \to Q_{n-1}(t_j)$ be the blow-up of the cone $Q_{n-1}(t_j)$ with center at the singular point. Let $f_j : K_j \to Q_{n-2}$ be the induced $\mathbb{P}^1$-bundle, the fibres of which map to projective lines on $Q_{n-1}(t_j)$ under the morphism σ. Since $\tilde{E}_{t_j} := \tilde{E}|_{Q_{n-1}(t_j)}$ is trivial along the projective lines on $Q_{n-1}(t_j)$, it follows that the bundle $\tilde{E}_{K_j} := \sigma^* \tilde{E}_{t_j}$ is trivial along the fibers of f_j. Therefore, for an arbitrary point $x \in Q_{n-2}$ we obtain

$$H^i(Q_{n-2}, \tilde{E}_{K_j} \otimes \mathbb{C}_x) = H^i(\mathbb{P}^1, r\mathcal{O}_{\mathbb{P}^1}) = 0, \quad i \geqslant 1,$$

$$H^i(Q_{n-2}, \tilde{E}_{K_j}(-\sigma^* D) \otimes \mathbb{C}_x) = H^i(\mathbb{P}^1, r\mathcal{O}_{\mathbb{P}^1}(-1)) = 0, \quad i \geqslant 0.$$

This, together with the Base-change Theorem for f_j, shows that $R^i f_{j*} \tilde{E}_{K_j} = 0$ for $i \geqslant 1$, and $R^i f_{j*}(\tilde{E}_{K_j}(-\sigma^* D)) = 0$ for $i \geqslant 0$. Hence the Leray spectral sequence for the projection f_j yields

$$H^i(K_j, \tilde{E}_{K_j}) = 0, \quad i \geqslant 1, \quad H^i(K_j, \tilde{E}_{K_j}(-\sigma^* D)) = 0, \quad i \geqslant 0, \quad j = 1, 2. \tag{7.7}$$

Next, one uses the blow-up of the embedded in $\mathbb{P}^n$ cone $Q_{n-1}(t_j)$ C,Ps show that $\sigma_*\mathcal{O}_{K_j} = \mathcal{O}_{Q_{n-1}(t_j)}$ and $R^i\sigma_*\mathcal{O}_{K_j} = 0,\ i \geqslant 1$. Therefore, setting $\tilde{E}_{t_j} := \tilde{E}|_{Q_{n-1}(t_j)}$, we have by the projection formula: $\sigma_*\tilde{E}_{K_j} = \tilde{E}_{t_j}$, $R^i\sigma_*\tilde{E}_{K_j} = 0,\ i \geqslant 1$, and $\sigma_*(\tilde{E}_{K_j}(-\sigma^*D)) = \tilde{E}_{t_j}(-D)$, $R^i\sigma_*(\tilde{E}_{K_j}(-\sigma^*D)) = 0,\ i \geqslant 1$. Now the Leray spectral sequence applied to σ shows in view of (7.7) that

$$\begin{aligned} H^i(Q_{n-1}(t_j), \tilde{E}_{t_j}) = H^i(K_j, \tilde{E}_{K_j}) = 0, \qquad & i \geqslant 1, \\ H^i(Q_{n-1}(t_j), \tilde{E}_{t_j}(-D)) = H^i(K_j, \tilde{E}_{K_j}(-\sigma^*D)) = 0, \qquad & i \geqslant 0, \quad j = 1, 2. \end{aligned} \tag{7.8}$$

The equalities (7.6) and (7.8) yield via base change for the flat morphism π

$$R^i\pi_*\tilde{E} = 0,\ i \geqslant 1, \quad R^i\pi_*(\tilde{E}(-D)) = 0, \ \ i \geqslant 0. \tag{7.9}$$

The same argument yields base change isomorphisms

$$b_t:\ \pi_*\tilde{E} \otimes \mathbb{C}_t \xrightarrow{\simeq} H^0(Q_{n-1}(t), \tilde{E}|_{Q_{n-1}(t)}), \quad t \in \mathbb{P}^1. \tag{7.10}$$

Consider the divisor $D = Q_{n-2}\times\mathbb{P}^1$ on $\tilde{Q}_n$ (see diagram (7.5)) and the projections $Q_{n-2} \xleftarrow{\mathrm{pr}_1} Q_{n-2} \times \mathbb{P}^1 \xrightarrow{\mathrm{pr}_2} \mathbb{P}^1$. By definition, $\tilde{E}|_D = pr_1^*(\tilde{E}|_{Q_{n-2}})$, hence, since $\tilde{E}|_{Q_{n-2}}$ is trivial, the base change for the flat morphism pr_2 gives the isomorphisms

$$b':\ \mathrm{pr}_{2*}(\tilde{E}|_D) \xrightarrow{\simeq} H^0(Q_{n-2}, E|_{Q_{n-2}}) \otimes \mathcal{O}_{\mathbb{P}^1} \simeq \mathbb{C}^r \otimes \mathcal{O}_{\mathbb{P}^1}, \tag{7.11}$$

$$b'_t:\ \mathrm{pr}_{2*}(\tilde{E}|_D) \otimes \mathbb{C}_t \xrightarrow{\simeq} H^0(Q_{n-2}, E|_{Q_{n-2}}) \simeq \mathbb{C}^r, \quad t \in \mathbb{P}^1. \tag{7.12}$$

Now consider the exact triple

$$0 \to \tilde{E}(-D) \to \tilde{E} \to E|_D \to 0 \tag{7.13}$$

and its restriction

$$0 \to \tilde{E}(-D)|_{Q_{n-1}(t)} \to \tilde{E}|_{Q_{n-1}(t)} \to (E|_{Q_{n-2}}) \otimes \mathbb{C}_t \to 0 \tag{7.14}$$

onto a fibre $Q_{n-1}(t)$ of the projection π over an arbitrary point $t \in \mathbb{P}^1$. Applying the functor $R^i\pi_*$ to (7.13) and using (7.9) and (7.11) we obtain the isomorphism of sheaves

$$r_D:\ \pi_*\tilde{E} \xrightarrow{\simeq} \mathrm{pr}_{2*}(\tilde{E}|_D) \simeq \mathbb{C}^r \otimes \mathcal{O}_{\mathbb{P}^1}. \tag{7.15}$$

In particular, $\pi_*\tilde{E}$ is a trivial bundle. Respectively, passing to cohomology of the exact sequence (7.14) and using (7.6), (7.8) and (7.12), we obtain the isomorphisms

$$res_t: H^0(Q_{n-1}(t), \tilde{E}|_{Q_{n-1}(t)}) \xrightarrow{\simeq} H^0(Q_{n-2}, E|_{Q_{n-2}}), \quad t \in \mathbb{P}^1. \tag{7.16}$$

By construction, the isomorphisms (7.10), (7.12), (7.15) and (7.16) constitute the commutative diagram

$$\begin{tikzcd} \pi_*\tilde{E} \otimes \mathbb{C}_t \arrow[r, "r_D\otimes\mathbb{C}_t", "\simeq"'] \arrow[d, "b_t"', "\simeq"] & \mathrm{pr}_{2*}(\tilde{E}|_D) \otimes \mathbb{C}_t \arrow[d, "b'_t"', "\simeq"] \\ H^0(Q_{n-1}(t), \tilde{E}|_{Q_{n-1}(t)}) \arrow[r, "res_t", "\simeq"'] & H^0(Q_{n-2}, E|_{Q_{n-2}}) = \mathbb{C}^r \end{tikzcd} \tag{7.17}$$

for $t \in \mathbb{P}^1$. Next, since $E|_{Q_{n-2}}$ is trivial, the evaluation map $H^0(Q_{n-2}, E|_{Q_{n-2}}) \otimes \mathcal{O}_{Q_{n-2}} \to E|_{Q_{n-2}}$ is an isomorphism, so that its composition e_t with the restriction $H^0(Q_{n-2}, E|_{Q_{n-2}}) \otimes \mathcal{O}_{Q_{n-1}(t)} \twoheadrightarrow H^0(Q_{n-2}, E|_{Q_{n-2}}) \otimes \mathcal{O}_{Q_{n-2}}$ is an epimorphism for any $t \in \mathbb{P}^1$ and fits in the commutative diagram

$$\begin{tikzcd} H^0(Q_{n-1}(t), \tilde{E}|_{Q_{n-1}(t)}) \otimes \mathcal{O}_{Q_{n-1}(t)} \arrow[r, "\pi^* res_t", "\simeq"'] \arrow[d, "ev_t"'] & H^0(Q_{n-2}, E|_{Q_{n-2}}) \otimes \mathcal{O}_{Q_{n-1}(t)} \arrow[d, two heads, "e_t"'] \\ \tilde{E}|_{Q_{n-1}(t)} \arrow[r, two heads, "res_{Q_{n-2}}"] & E|_{Q_{n-2}}. \end{tikzcd} \tag{7.18}$$

Here we understand Q_{n-2} as lying in $Q_{n-1}(t)$ as a divisor. In particular, through any point of $Q_{n-1}(t) \smallsetminus Q_{n-2}$ there passes a line, say, l interesecting Q_{n-2} at a point, say y. Therefore, since $\tilde{E}|_l$ is trivial, we have a commutative diagram of restriction maps

$$\begin{tikzcd} H^0(Q_{n-1}(t), \tilde{E}|_{Q_{n-1}(t)}) \arrow[r, "res_t", "\simeq"'] \arrow[d, "\rho"'] & H^0(Q_{n-2}, E|_{Q_{n-2}}) \arrow[d, "\simeq"] \\ H^0(l, \tilde{E}|_l) \arrow[r, "\simeq"] & H^0(y, \tilde{E}|_y). \end{tikzcd}$$

Hence, ρ is an isomorphism, and therefore the evaluation morphism ev_t in (7.18) is an isomorphism of sheaves. Composing it with the isomorphism $\pi^* b_t : \pi^*\pi_*\tilde{E}|_{Q_{n-1}(t)} \xrightarrow{\simeq} H^0(Q_{n-1}(t), \tilde{E}|_{Q_{n-1}(t)}) \otimes \mathcal{O}_{Q_{n-1}(t)}$ arising from the left vertical isomorphism in (7.17) we obtain the (evaluation) isomorphism $ev|_{Q_{n-1}(t)} : \pi^*\pi_*\tilde{E}|_{Q_{n-1}(t)} \xrightarrow{\simeq} \tilde{E}|_{Q_{n-1}(t)}$. Since this is true for any $t \in \mathbb{P}^1$, we obtain the isomorphism $ev : \pi^*\pi_*\tilde{E} \xrightarrow{\simeq} \tilde{E}$ which together with (7.15) leads to the triviality of $\tilde{E}$. Since clearly $\sigma_{Q*}\mathcal{O}_{\tilde{Q}} = \mathcal{O}_Q$, it follows that $E = \sigma_{Q*}\tilde{E} = \sigma_{Q*}(r\mathcal{O}_{\tilde{Q}}) = r\mathcal{O}_Q$, i. e. we obtain the statement of Proposition.

Proposition 9. *Let E be a linearly trivial vector bundle on $GO(k,V)$ or $GS(k,V)$. Then E is trivial.*

Proof. Consider the case $GO(k,V)$. We give a proof by induction under the assumption that $n := \frac{\dim V}{2} \in \mathbb{Z}_{>0}$. The case when $\dim V$ is odd can be treated similarly.

For $n = 2$ we have $GO(1,V) \simeq \mathbb{P}^1 \times \mathbb{P}^1$, $GO(2,V) \simeq \mathbb{P}^1$, and for these varieties our claim clearly holds. Therefore we assume that $n \geqslant 3$ and argue by induction on k. If $k = 1$, $GO(k,V)$ is a $(2n-2)$-dimensional quadric in $\mathbb{P}^{2n-1}$ so our statement holds by Proposition 8. Now let $1 \leqslant k \leqslant n-2$, and recall the graph of incidence Σ with natural projections

$$GO(k,V) \xleftarrow{q} \Sigma \xrightarrow{p} GO(k+1,V) \tag{7.19}$$

(see subsection 5.1).

Let E be a linearly trivial vector bundle on $GO(k+1,V)$. Then the bundle p^*E is linearly trivial on the fibres of q. Since these fibres are quadrics, Proposition 8 implies that p^*E is trivial on the fibres of q. Furthermore, Proposition 6 yields an isomorphism $q^*q_*p^*E \xrightarrow{\simeq} p^*E$. Hence, since p^*E is trivial along the fibres of p which are mapped by q isomorphically to projective spaces $\mathbb{P}^k$ on $GO(k,V)$, it follows that q_*p^*E is trivial along these projective subspaces $\mathbb{P}^k$ of $GO(k,V)$. Consequently, q_*p^*E is linearly trivial on $GO(k,V)$. Thus, by the induction assumption, q_*p^*E is trivial. Hence p^*E and $E = p_*p^*E$ are trivial.

It remains to consider $GO(n,V)$. Here we employ induction on n. For $n=3$ $GO(n,V) \simeq \mathbb{P}^3$, hence the statement holds in this case. For $n \geqslant 4$, consider the graph of incidence $\Pi_n := \{(V_1, V_n) \in Q_{2n-2} \times GO(n,V) \mid V_1 \subset V_n\}$ with natural projections

$$Q_{2n-2} \xleftarrow{p} \Pi_n \xrightarrow{q} GO(n,V). \tag{7.20}$$

Let E be a linearly trivial vector bundle on $GO(n,V)$. Then q^*E is trivial on lines lying in the fibres of p which are isomorphic to $GO(n-1,\mathbb{C}^{2n-2})$. Hence q^*E is trivial along the fibres of p by the induction assumption. Next, Proposition 6 yields an isomorphism $p^*p_*q^*E \xrightarrow{\simeq} q^*E$. Since q^*E is trivial on the fibres of p, it follows that p_*q^*E is trivial on the projective subspaces $\mathbb{P}^{n-1}$ of the quadric Q_{2n-2}. Therefore p_*q^*E is trivial on the lines in Q_{2n-2}, so it is trivial by Proposition 8. Finally, $q^*E \simeq p^*p_*q^*E$ and $E = q_*q^*E$ are trivial as well.

Proceed to the case of $GS(k,V)$. Substituting GO by GS in diagram (7.19), we obtain a diagram $GS(k,V) \xleftarrow{q} \Sigma' \xrightarrow{p} GS(k+1,V)$, where p is a $\mathbb{P}^k$-bundle and q is a $\mathbb{P}^{2n-2k-1}$-bundle. Respectively, substituting GO by GS, and Q_{2n-2} by $\mathbb{P}(V_n)$ in diagram (7.20), we obtain a diagram $\mathbb{P}(V_n) \xleftarrow{p} \Pi'_n \xrightarrow{q} GS_n$. This enables us to carry out an argument very similar to the one for $GO(k,V)$.

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