On the structure of injective tensor $\mathfrak{sp}_\infty$-, $\mathfrak{so}_\infty$- modules and of injective resolutions of simple tensor $\mathfrak{sp}_\infty$-, $\mathfrak{so}_\infty$-modules

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Abstract

The main purpose of this Bachelor's thesis is to understand and simplify certain combinatorial results in the study of tensor representations of the locally finite Lie algebras $\mathfrak{sp}_\infty$ and $\mathfrak{so}_\infty$. We focus on the socle filtrations of injective tensor $\mathfrak{sp}_\infty-, \mathfrak{so}_\infty-$modules and on minimal injective resolutions of simple tensor $\mathfrak{sp}_\infty-, \mathfrak{so}_\infty-$modules. In particular, we give direct formulas for the lengths of such filtrations and resolutions. We also describe the top socle layer and the last injective resolution term, with statements that do not depend on the algorithmically-computed Littlewood-Richardson coefficients.

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1 Introduction

For the finite-dimensional complex Lie algebras $\mathfrak{sp}(2n)$ and $\mathfrak{so}(n)$, all finite-dimensional representations form a semisimple category according to Hermann Weyl's semisimplicity theorem. However, in the case of the classical infinite-dimensional locally finite Lie algebras $\mathfrak{sp}_\infty$ and $\mathfrak{so}_\infty$, there are no non-trivial finite-dimensional representations. Natural analogues of finite-dimensional representations are the tensor representations and, as shown in [PS11], these do not form semisimple categories.

Indecomposable injective objects, as well as injective resolutions of simple objects in the respective categories of tensor $\mathfrak{sp}_\infty-$ and $\mathfrak{so}_\infty-$modules have been studied in detail ([PS11], [DCPS], [SS15]). The published explicit formulas for the layers of the socle filtrations of injective objects and for terms of injective resolutions involve Littlewood-Richardson coefficients. These coefficients are computed algorithmically and require several steps for checking whether or not they are nonzero. Hence the socle layers of injective tensor modules, as well as the terms of injective resolutions, are difficult to study without the usage of a computer program with an implemented Littlewood-Richardson coefficient calculator.

In this Bachelor's thesis, we are simplifying some aspects of the already known theory, by producing direct descriptions of modules appearing in the top layers of the socle filtrations of injective $\mathfrak{sp}_\infty-, \mathfrak{so}_\infty-$modules, and in the last terms of minimal injective resolutions of $\mathfrak{sp}_\infty-, \mathfrak{so}_\infty-$modules. Additionally, we provide explicit formulas for the length of the socle filtrations and the length of minimal injective resolutions of such modules. These formulas do not depend on Littlewood-Richardson coefficients and are written directly in terms of the partition λ that determines the respective $\mathfrak{sp}_\infty-$, $\mathfrak{so}_\infty-$module Γ_λ (as defined in section 2).

On an organizational note, this thesis starts with introducing the necessary background in combinatorics and algebra, using [FH91] and [PS11] as main references for the theory. Section 3 includes a discussion of socle filtrations, using the theorem describing the socle layers of an injective tensor $\mathfrak{sp}_\infty-$module ([PS11]) as a key starting point. We provide a more explicit version of this theorem by finding the length of the socle filtration, and then describe some direct summands of the last socle layer, that are guaranteed to appear. Section 4 presents an algorithm for computing injective resolutions, a combinatorial result involving column-quasi-symmetric partitions, and some consequences of the description of the terms of minimal injective resolutions of simple tensor $\mathfrak{sp}_\infty-$modules given by [SS15]. In Section 5, we showcase the analogous results for the Lie algebra $\mathfrak{so}_\infty$. We complete the thesis by a brief conclusion and a possible future research question.

As an appendix we present computer-generated examples of minimal injective resolutions of simple tensor $\mathfrak{sp}_\infty-$modules.

2 Preliminaries

2.1 Partitions, Young diagrams, and Littlewood-Richardson coefficients

This thesis studies combinatorial aspects of injective tensor $\mathfrak{sp}_\infty-, \mathfrak{so}_\infty-$modules, and we start with some necessary background for understanding the Littlewood-Richardson rule for semistandard skew tableaux. In this section, we make use of [FH91] and [And98] for the theory, and present it in a similar manner to the Bachelor's theses [Zok] and [Wei].

Definition 2.1 (Partition). *A partition λ of n is a finite non-increasing sequence of positive integers $\lambda_1, \lambda_2, ..., \lambda_r$ such that $\sum_{i=1}^{r} \lambda_r = n$. We use the notation $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$, and say that the λ_i, for $i = \overline{1, r}$, are the parts of the partition λ.*

A useful way to compare partitions is by the **lexicographic order** (more details in [wika]).

Definition 2.2 (Lexicographic order on partitions). *Let $\lambda = (\lambda_1, \lambda_2, ...)$ and $\mu = (\mu_1, \mu_2, ...)$ be two partitions. We say that λ precedes μ or μ succeeds λ in the lexicographic order, and write $\lambda <_{lex} \mu$ or $\mu >_{lex} \lambda$, if $\lambda \neq \mu$ and for the smallest k where $\lambda_k \neq \mu_k$ we have $\lambda_k < \mu_k$.*

Each partition λ can be represented graphically via a **Young diagram**. Such diagram consists of finitely many boxes, organized in succeeding rows. For $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$, the i^{th} row has λ_i boxes.

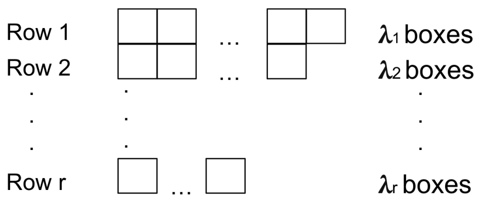

Figure 1: Young diagram of $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$.

Example 2.3. *Let us look at $\lambda = (4, 4, 2, 1)$ which is a partition of 11. The Young diagram of λ has two rows with 4 boxes, a row with 2 boxes, and a final row with 1 box.*

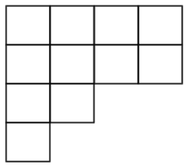

Figure 2: Young diagram of the partition (4,4,2,1).

A **Young tableau** is a Young diagram with boxes filled with numbers from 1 to **the degree of the partition** $|\lambda| := \sum_{i=1}^{r} \lambda_i$.

Definition 2.4 (**Subpartition**). *Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ be a partition. We say that a partition $\mu = (\mu_1, \mu_2, ..., \mu_s)$ is a **subpartition** of λ if $s \leq r$ and $\mu_i \leq \lambda_i$, for $1 \leq i \leq s$.*

Remark. If μ is a subpartition of λ, then the Young diagram of μ is graphically included in the Young diagram of λ so that the top left boxes of both diagrams coincide.

Example 2.5. *Let us draw the Young diagrams of $\lambda = (4, 4, 2, 1)$ and $\mu = (4, 2, 1, 1)$ (one of the subpartitions of λ) according to our graphical convention. The highlighted boxes are part of both Young diagrams.*

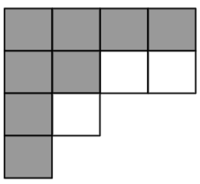

Figure 3: The Young diagram of $(4, 4, 2, 1)$ and the Young diagram of $(4, 2, 1, 1)$ with highlighted boxes for emphasising the subpartition property.

Given a Young diagram corresponding to partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$, we can transpose it (reflect it by the main diagonal) to obtain the **transposed Young diagram** - this corresponds to the **conjugate partition** λ^T. Moreover, $|\lambda| = |\lambda^T|$ and

$$\lambda^T = (\underbrace{r, r, ..., r,}_{\lambda_r \text{ times}} \underbrace{r-1, r-1, ..., r-1}_{\lambda_{r-1} - \lambda_r \text{ times}}, ..., \underbrace{1, 1, ..., 1}_{\lambda_1 - \lambda_2 \text{ times}}).$$

Example 2.6. *Given $\lambda = (4,3,1,1,1)$, we draw its Young diagram and then transpose it as in the figure 4.*

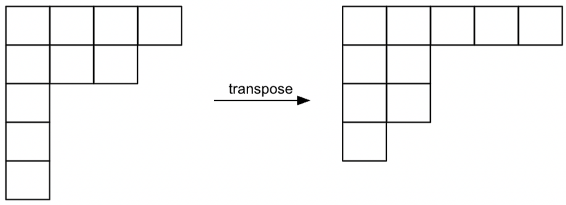

Figure 4: The Young diagrams of the partition (4,3,1,1,1) and its conjugate

The transposed diagram is the Young diagram of the partition

$$\lambda^T = (5,2,2,1) = (\underbrace{5}_{\lambda_5 = 1 \ time}, \underbrace{2,2}_{\lambda_2 - \lambda_3 = 2 \ times}, \underbrace{1}_{\lambda_1 - \lambda_2 = 1 \ time}),$$

and $|\lambda| = |\lambda^T| = 10$.

Another operation we can do on partitions is **multiplication by a positive integer**. For $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ and some natural number k, we define $k\lambda$ as the partition $(k\lambda_1, k\lambda_2, ..., k\lambda_n)$.

Lemma 2.7. *Given a partition λ and its conjugate λ^T, the Young diagram of $(k\lambda)^T$ is obtained from the Young diagram of λ^T by repeating each row of λ^T k times: if $\lambda^T = ((\lambda^T)_1, (\lambda^T)_2, ..., (\lambda^T)_m)$, then*

$$(k\lambda)^T = (\underbrace{(\lambda^T)_1, (\lambda^T)_1, ..., (\lambda^T)_1}_{k \ times}, \underbrace{(\lambda^T)_2, (\lambda^T)_2, ..., (\lambda^T)_2}_{k \ times}, ..., \underbrace{(\lambda^T)_m,)(\lambda^T)_m, ..., (\lambda^T)_m}_{k \ times}).$$

Proof. Straightforward. QED

Example 2.8. *Given $\lambda = (4,3,1,1,1)$, we have $2\lambda = (8,6,2,2,2)$. Hence, by computing the conjugate partitions, we obtain $\lambda^T = (5,2,2,1)$ and $2\lambda^T = (5,5,2,2,2,2,1,1)$. See figure 5.*

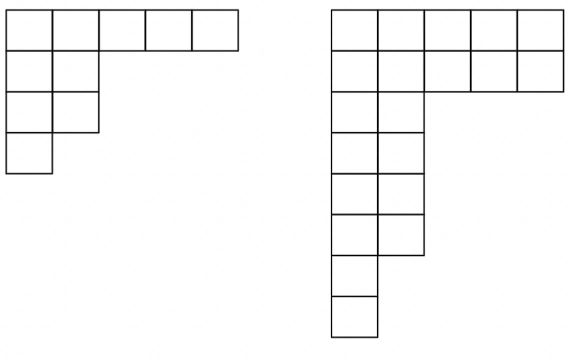

Figure 5: Young diagrams for $\lambda^T = (5,2,2,1)$ and $2\lambda^T = (5,5,2,2,2,2,1,1)$.

Next, we introduce some special types of partitions, which will be particularly useful in section 4 of this thesis when discussing injective resolutions of $\mathfrak{sp}_\infty-$modules. We define them according to the description in [SS15], and give examples in figure 6.

Definition 2.9 (**Column-quasi-symmetric partitions. Row-quasi-symmetric partitions**). *Let λ be a partition. We say that λ is a* ***column-quasi-symmetric (CQS)*** *partition if for each box b along the main diagonal of the Young diagram of λ, the number of boxes below b in the same column minus 1 equals the number of boxes to the right of b in the same row.*
In a similar manner, λ is a ***row-quasi-symmetric (RQS)*** *partition if for each main diagonal box b, the number of boxes to the right of b on the same row minus 1 equals the number of boxes below b in the same column.*

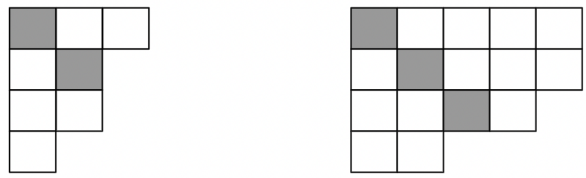

Figure 6: Young diagrams for a CQS partition (left) and a RQS partition (right). The boxes on the main diagonal are highlighted.

Before introducing the Littlewood-Richardson rule, we introduce **Schur polynomials**. These form a basis of the vector space of homogeneous symmetric polynomials of degree n in k variables $x_1, x_2, ..., x_k$. The Schur polynomials are used in the theory of symmetric functions and in representation theory - more details can be found in [FH91] (Appendix A, and chapter 1.4). However, we will focus on the fact that they are indexed by partitions, and will provide explanations and relevant examples to showcase the connection with Young diagrams.

Definition 2.10 (**Schur polynomial**). *Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ be a partition of n. Fix a positive integer $k \geq n$ and set $\lambda_{r+1} = \lambda_{r+2} = ... = \lambda_k = 0$. The* ***Schur polynomial*** *corresponding to λ and in variables $x_1, x_2, ..., x_k$ is*

$$S_\lambda^k = \frac{|((x_j^{\lambda_i+k-i})_{1\leq i,j\leq k})|}{\Delta},$$

where $|((x_j^{\lambda_i+k-i})_{1\leq i,j\leq k})| = det \begin{pmatrix} x_1^{\lambda_1+k-1} & x_2^{\lambda_1+k-1} & ... & x_k^{\lambda_1+k-1} \\ x_1^{\lambda_2+k-2} & x_2^{\lambda_2+k-2} & ... & x_k^{\lambda_2+k-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_k} & x_2^{\lambda_k} & ... & x_k^{\lambda_k} \end{pmatrix}$ *and* $\Delta = \prod_{1\leq i<j\leq k} (x_i - x_j)$.

Remark. Note that deg $S_\lambda^k = n$.

Example 2.11. *Let us look at the Schur polynomials of degree 2 in 4 variables x_1, x_2, x_3, x_4. The partition set we are interested in is $\{(2), (1, 1)\}$. Our Schur polynomials are:*

$$S_{(2)}^4 := x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4,$$

$$S_{(1,1)}^4 := x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4.$$

Next, we explain how the product of two Schur polynomials is expressed as a linear combination of Schur polynomials. First, we take a basic example of multiplying the polynomials above by $S_{(1)}^4$, and then we present the Pieri formula for multiplying a Schur polynomial with another Schur polynomial corresponding to a partition of length one (row-partition).

Example 2.12. *We multiply the Schur polynomials of degree 2 in 4 variables (from example 2.11) by the Schur polynomial of degree 1 in 4 variables $S_{(1)}^4 := x_1 + x_2 + x_3 + x_4$. The result is:*

$$S^4_{(2)} \cdot S^4_{(1)} = (x_1^3+x_2^3+x_3^3+x_4^3)+2(x_1^2x_2+x_1^2x_3+x_1^2x_4+x_2^2x_1+x_2^2x_3+x_2^2x_4+x_3^2x_1+x_3^2x_2+x_3^2x_4+x_4^2x_1+x_4^2x_2+x_4^2x_3)$$

$$+3(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) = S^4_{(3)} + S^4_{(2,1)},$$

$$S^4_{(1,1)} \cdot S^4_{(1)} = (x_1^2x_2 + x_1^2x_3 + x_1^2x_4 + x_2^2x_1 + x_2^2x_3 + x_2^2x_4 + x_3^2x_1 + x_3^2x_2 + x_3^2x_4 + x_4^2x_1 + x_4^2x_2 + x_4^2x_3)$$

$$+3(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) = S^4_{(2,1)} + S^4_{(1,1,1)}.$$

Theorem 2.13 (Pieri's formula). *Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ be a partition, let (m) be a row-partition, and let $k \geq |\lambda|$. Then, by multiplying the Schur polynomials in k variables S^k_λ and $S^k_{(m)}$ we obtain*

$$S^k_\lambda \cdot S^k_{(m)} = \sum_\mu S^k_\mu,$$

where the sum is over all partitions μ of length r and of length $r+1$ such that, respectively, $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq ... \geq \mu_r \geq \lambda_r$ or $\mu_1 \geq \lambda_1 \geq \mu_2 \geq \lambda_2 \geq ... \geq \mu_r \geq \lambda_r \geq \mu_{r+1}$ and $|\mu| = |\lambda| + m$.

Equivalently, the theorem shows how the Young diagrams of the resulting partitions μ are obtained by attaching the m boxes from the Young diagram of (m) to the Young diagram of λ so that no two attached boxes are on the same column of the resulting Young diagrams. We will refer to this as **Pieri's rule on Young diagrams**.

Example 2.14. *Let $\lambda = (3, 2, 2, 1)$. We want to attach the three boxes from the Young diagram of (3) the Young diagram of λ. According to the Pieri formula,*

$$S^k_{(3,2,2,1)} \cdot S^k_{(3)} = S^k_{(6,2,2,1)} + S^k_{(5,3,2,1)} + S^k_{(5,2,2,2)} + S^k_{(5,2,2,1,1)} + S^k_{(4,3,2,2)} + S^k_{(4,3,2,1,1)} + S^k_{(4,2,2,2,1)} + S^k_{(3,3,2,2,1)},$$

which leads to the resulting set of partitions $\{(6,2,2,1), (5,3,2,1), (5,2,2,2), (5,2,2,1,1), (4,3,2,2),$ $(4,3,2,1,1), (4,2,2,2,1), (3,3,2,2,1)\}$. The resulting Young diagrams are the ones in figure 7, where the attached boxes are highlighted.

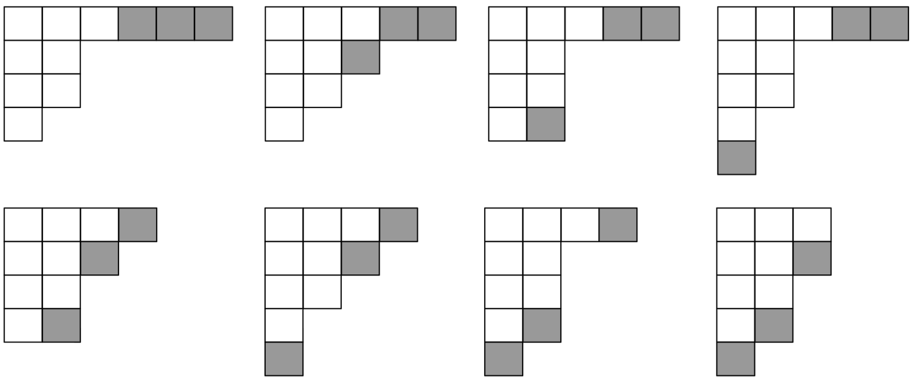

Figure 7

While Pieri's formula is a useful tool to multiply any Schur polynomial by a special Schur polynomial indexed by a row-partition, we need a generalized version for multiplying arbitrary Schur polynomials. We can express the product as a linear combination of Schur polynomials with coefficients $\mathrm{N}^\mu_{\lambda,\gamma}$, as follows:

$$S^k_\lambda \cdot S^k_\gamma = \sum_\mu \mathrm{N}^\mu_{\lambda,\gamma} S^k_\mu. \tag{1}$$

Remark. When γ is a row-partition, then $\mathrm{N}^\mu_{\lambda,\gamma} = \mathrm{N}^\mu_{\gamma,\lambda}$ equal either 0 or 1.

The coefficients $N^{\mu}_{\lambda,\gamma}$ are called **Littlewood-Richardson coefficients** and they can be computed either from the expansion of the product of Schur polynomials, or with a combinatorial method involving Young diagrams ([FH91] Appendix A). The combinatorial definition of the partitions μ which appear in the right-hand-side of the expression (1), is as follows:

Definition 2.15 ($\lambda-$**expansion of a Young diagram**). *Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ be a partition considered as a Young diagram. A $\lambda-$**expansion** is a Young diagram (or a partition) μ partially filled with numbers, obtained by following the steps:*

- *add λ_1 boxes to the Young diagram of μ according to Pieri's rule, and write number 1 in these boxes;*
- *add λ_2 boxes to the Young diagram obtained in the previous step, following Pieri's rule. Then write 2 in those boxes;*
- *continue until the final λ_r boxes filled with number r are added to the Young diagram, according to Pieri's rule.*

*Let t be any number in $\{1, 2, ..., |\lambda|\}$. We say that the expansion is **strict** if, among the first t entries from right to left, and from the top row to the last row, each integer k between 1 and $r-1$ appears at least as many times as the number $k+1$.*

Example 2.16. *In this example, we will look at all strict $(2,1)-$expansions of the partition $(4,3)$. They appear in the right-hand-side of the following product of Schur polynomials:*

$$S^k_{(4,3)} \cdot S^k_{(2,1)} = S^k_{(6,4)} + S^k_{(6,3,1)} + S^k_{(5,5)} + 2S^k_{(5,4,1)} + S^k_{(5,3,2)} + S^k_{(5,3,1,1)} + S^k_{(4,4,2)} + S^k_{(4,4,1,1)} + S^k_{(4,3,2,1)}.$$

In terms of expansions of Young diagrams, we obtain the same results, as all possible strict $(2,1)-$expansions of $(4,3)$ are in figure 8.

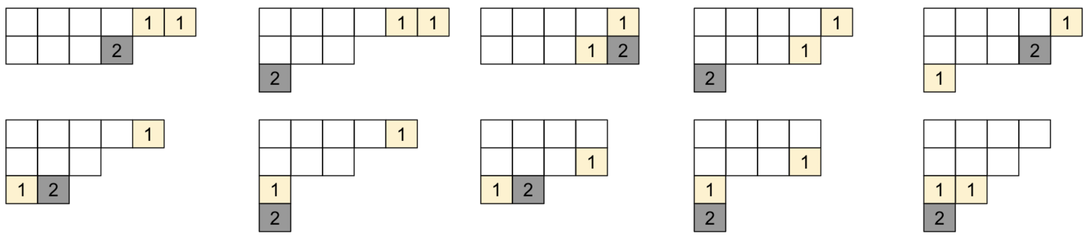

Figure 8: The strict $(2,1)-$expansions of $(4,3)$. The 2 boxes with number 1 highlighted in yellow, and the box with number 2 in grey, for emphasis of the Pieri rule.

Theorem 2.17 (**Littlewood-Richardson combinatorial rule**, [FH91]). *Given three partitions μ, λ, γ, the Littlewood-Richardson coefficients $N^{\mu}_{\lambda,\gamma}$ represent the number of ways the Young diagram for μ can be obtained by a strict $\gamma-$expansion of the Young diagram of λ.*

We notice that the theorem helps us find all partitions μ that can be obtained by attaching γ to λ - these are all partitions with nonzero Littlewood-Richardson coefficient, i.e. with $N^{\mu}_{\lambda,\gamma} \neq 0$. However, we can state the Littlewood-Richardson combinatorial rule in another manner, too. For this, we follow the outline presented in the Bachelor's theses [Zok] and [Wei].

Let λ be a partition and μ be an expansion. We can look at the set-theoretic difference of these Young diagrams. The remaining skeleton of boxes is called a **skew diagram** of shape μ/λ.

Definition 2.18 (**Semistandard skew tableau**). *Let λ, γ, μ be partitions such that λ is a subpartition of μ and $|\gamma| = |\mu| - |\lambda|$. A* ***semistandard skew tableau*** *of shape μ/λ and weight γ is a skew diagram filled with numbers such that:*

- *each positive integer $k \leq |\gamma|$ occurs exactly γ_k times,*
- *each column's entries are strictly increasing (left to right),*
- *each row's entries are weakly increasing (top to bottom row).*

Moreover, if after removal of one or more leftmost column we still have a semistandard skew tableau, then our semistandard skew tableau of shape μ/λ and weight γ is called a ***Littlewood-Richardson tableau.***

Then, all partitions γ for which $\mathrm{N}_{\lambda,\gamma}^{\mu} \neq 0$ can be found through Littlewood-Richardson tableaux. Additionally, we can rephrase the combinatorial rule as follows:

Theorem 2.19 (**Littlewood-Richardson combinatorial rule**, [wikc]). *Given three partitions μ, λ, γ, the Littlewood-Richardson coefficient $N_{\lambda,\gamma}^{\mu}$ is equal to the number of Littlewood-Richardson tableaux of shape μ/λ and weight γ.*

Corollary 2.20. *The conditions for a strict $\gamma-$expansion of a Young diagram of λ to the Young diagram of μ are equivalent to the conditions in the definition of a Littlewood-Richardson tableau of shape μ/λ and weight γ.*

2.2 Modules and socle filtrations of modules

In order to introduce the notions of socle filtrations, we need to discuss some algebraic background. We will follow the structure of the early preliminaries sections in [Zok] and [Wei].

Definition 2.21 (**Module over a Lie algebra**). *Let $\mathfrak{g}$ be a Lie algebra. A $\mathfrak{g}-$module M is a vector space equipped with a bilinear map $\mathfrak{g} \times M \to M, (x, m) \mapsto x \cdot m$ satisfying*

$$[x, y] \cdot m = x \cdot (y \cdot m) - y \cdot (x \cdot m), \quad \forall x, y \in \mathfrak{g}, \forall m \in M.$$

Let us take an R-module M, where R is a ring or a Lie algebra.

Definition 2.22 (**Simple module. Semisimple module**). *Let M be a nonzero $R-$module. We say that M is a* ***simple module*** *if it does not have any proper submodules.*

We call M a ***semisimple*** *module if one of the following equivalent conditions is satisfied:*

- *M is a direct sum of simple submodules, or*
- *M is the sum of all its simple submodules, or*
- *Any submodule $M_0 \subset M$ admits a direct complement $M_1 \subset M$ (i.e. $M = M_0 \oplus M_1$).*

The equivalence of the conditions in the definition above is proven in [Lan02] (chapter XVII).

Definition 2.23 (**Indecomposable modules. Decomposable modules**). *Let M be a nonzero $R-$module. If M cannot be written as a direct sum of two nonzero submodules, then it is called* ***indecomposable.*** *Otherwise, it is* ***decomposable.***

Definition 2.24 (**Filtration of a module**). *Let M be an $R-$module. A* ***filtration*** *of module M is a sequence of nested submodules of M*

$$... \subset M_{i-1} \subset M_i \subset M_{i+1} \subset ....$$

If the filtration is ***strict****, i.e. if $M_{i-1} \neq M_i$, for all i, then the number of indices is the* ***length*** *of the filtration.*

For the rest of this section, we will be concerned about a particular type of filtration: the **socle** filtration. We will use the theory and notation in [PS11] for these concepts.

Definition 2.25 (**Socle of a module**). *We call the maximal semisimple submodule of an R-module* ***M*** *the* ***socle*** *of M, and write* **soc M**. *By the above equivalence condition, the socle of M is equal to the sum of all simple submodules of M.*

Definition 2.26 (**Socle filtration**). *We define the* ***socle filtration*** *of module M inductively, as follows:*

- $soc^{(1)}M = soc\, M$,
- *For* $i \geq 2$, $soc^{(i)}M = p^{-1}(soc(M/soc^{(i-1)}M))$, *where p is the natural projection.*

We say that the socle filtration of M is ***exhaustive*** *if* $\bigcup_i soc^{(i)}M = M$.

Remark. Note that soc M does not need to exist. (e.g. $\mathbb{C}[x]$ as a $\mathbb{C}[x]-$module does not have any simple submodules, hence its socle does not exist.)

Definition 2.27 (**Socle layers**). *Let M be an $R-$module with finite and exhaustive socle filtration*

$$soc^{(1)}M \subset ... \subset soc^{(n)}M = M.$$

We call the quotients of consecutive terms ***socle layers****. They are denoted by*

$$\overline{soc}^i M = soc^{(i)}M/soc^{(i-1)}M.$$

Example 2.28. *Let A be an arbitrary but fixed $n \times n$ matrix such that its Jordan normal form has Jordan blocks only of size k. Let $\mathbb{C}[x]$ act on $\mathbb{C}^n$ by left multiplication with A:*

$$x \cdot \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \mapsto A \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Let us compute the socle filtration of $\mathbb{C}^n$. First, we denote the eigenvalue corresponding to the i-th Jordan block by λ_i, where $1 \leq i \leq \frac{n}{k} =: m$. We notice that each eigenspace will be a semisimple submodule, and that the eigenspace $ker(A - \lambda_i I)$ is spanned by one or more linearly independent eigenvectors of λ_i. Hence, using the inductive construction and denoting by $\mathbb{C}_\lambda$ the copy of $\mathbb{C}$ corresponding to eigenvalue λ, we obtain:

$$soc^{(1)}\mathbb{C}^n = \mathbb{C}_{\lambda_1} \oplus \mathbb{C}_{\lambda_2} \oplus ... \oplus \mathbb{C}_{\lambda_m},$$
$$soc^{(2)}\mathbb{C}^n = \mathbb{C}^2_{\lambda_1} \oplus \mathbb{C}^2_{\lambda_2} \oplus ... \oplus \mathbb{C}^2_{\lambda_m},$$
$$\vdots$$
$$soc^{(k)}\mathbb{C}^n = \mathbb{C}^k_{\lambda_1} \oplus \mathbb{C}^k_{\lambda_2} \oplus ... \oplus \mathbb{C}^k_{\lambda_m} = \mathbb{C}^n.$$

The socle layers will be

$$\overline{soc}^i\mathbb{C}^n = soc^{(i)}\mathbb{C}^n/soc^{(i-1)}\mathbb{C}^n = \mathbb{C}_{\lambda_1} \oplus \mathbb{C}_{\lambda_2} \oplus ... \oplus \mathbb{C}_{\lambda_m},$$

and we can represent all of them in the following table:

$\mathbb{C}_{\lambda_1} \oplus \mathbb{C}_{\lambda_2} \oplus ... \oplus \mathbb{C}_{\lambda_m}$
$\vdots$
$\mathbb{C}_{\lambda_1} \oplus \mathbb{C}_{\lambda_2} \oplus ... \oplus \mathbb{C}_{\lambda_m}$
$\mathbb{C}_{\lambda_1} \oplus \mathbb{C}_{\lambda_2} \oplus ... \oplus \mathbb{C}_{\lambda_m}$

.

2.3 Injective resolutions

Next, we present some concepts from homological algebra that will be used in section 4 of the thesis. As reference we use [Wei94], [Pha], and [wikb].

Let $\mathcal{C}$ be a full subcategory of the category of left $R-$modules, where R is a ring or a Lie algebra. All modules in this section are assumed to be objects of $\mathcal{C}$.

Definition 2.29 (**Injective object in a category of modules**). *A module M in $\mathcal{C}$ is injective in $\mathcal{C}$ if it satisfies one of the following equivalent conditions:*

- *if M is a submodule of the left $R-$module N, then there exists a submodule P of N such that $M \oplus P = N$*
- *any short exact sequence $0 \to M \to N \to P \to 0$ of left $R-$modules splits*
- *If X, Y are left R-modules, $f : X \to Y$ is an injective module homomorphism and $g : X \to M$ is an arbitrary module homomorphism, then there exists a module homomorphism $h : Y \to M$ such that the following diagram commutes:*

$$\begin{array}{ccccc} 0 & \longrightarrow & X & \xrightarrow{f} & Y \\ & & \downarrow{\scriptstyle g} & \swarrow{\scriptstyle h} & \\ & & M & & \end{array}$$

- *$Hom(-, M)$, is an exact contravariant functor from the category of left R-modules to the category of abelian groups.*

Definition 2.30 (**Injective hull**). *An injective hull of an $R-$module M is a minimal injective module containing M.*

Definition 2.31 (**Right resolution. Injective resolution**). *Let M be a left $R-$module. A* ***right resolution*** *of M is an exact sequence of $R-$modules*

$$0 \to M \to I^1 \to I^2 \to ... \to I^n \to ...$$

The resolution is called ***injective*** *if each I^j is an injective $R-$module.*

Note that if each I^j in the injective resolution is the injective hull of the cokernel of the previous map, then we say that the injective resolution is **minimal**.

Injective resolutions help us define the derived functor Ext.

Definition 2.32 (**Ext groups**). *Let B be a left $R-$module. For an injective resolution*

$$0 \to B \to I^0 \to I^1 \to ... \to I^n \to ...,$$

there exists a corresponding complex

$$0 \xrightarrow{h_0} Hom_R(A, I^0) \xrightarrow{h_1} Hom_R(A, I^1) \xrightarrow{h_2} ...,$$

where A is an $R-$module. For $i \geq 0$, the i^{th} ***Ext-group*** *$Ext^i_{\mathcal{C}}(A, B)$, is the i^{th} homology of this complex, i.e.*

$$Ext^i_{\mathcal{C}}(A, B) := ker(h^i)/im(h^{i-1}).$$

Proposition 2.33. *The $Ext^i_{\mathcal{C}}(A, B)$ groups do not depend (up to isomorphism) on the choice of injective resolution of B.*

2.4 Tensor representations of $\mathfrak{gl}_\infty$, $\mathfrak{sp}_\infty$, and $\mathfrak{so}_\infty$

Next, we introduce the necessary background to understand the tensor representations of the classical locally finite Lie algebra $\mathfrak{gl}_\infty, \mathfrak{sp}_\infty$, and $\mathfrak{so}_\infty$. More details can be found in [PS11] and [DCPS].

Let V, V_* be countable dimensional $\mathbb{C}-$vector spaces, and let $< \cdot, \cdot >: V \otimes V_* \to \mathbb{C}$ be a non-degenarate pairing. Then, the Lie algebra $\mathfrak{gl}_\infty$ is defined as the vector space $V \otimes V_*$ equipped with the Lie bracket

$$[u \otimes u^*, v \otimes v^*] = < u^*, v > u \otimes v^* - < v^*, u > v \otimes u^*, \qquad u, v \in V, u^*, v^* \in V_*,$$

and the Lie subalgebra $\mathfrak{sl}_\infty$ of $\mathfrak{gl}_\infty$ is defined as the kernel of the map $< \cdot, \cdot >$. We can also provide a coordinate definition of $\mathfrak{gl}_\infty$ by giving its linear basis. It is shown in [Mac] that we can find dual bases $\{\xi_i\}_{i\in\mathfrak{J}}$ and $\{\xi_i^*\}_{i\in\mathfrak{J}}$ of V and V_*, respectively, indexed by $\mathfrak{J}$ countable, and such that $< \xi_j^*, \xi_i >= \delta_{i,j}$, for $i, j \in \mathfrak{J}$. Thus, the linear basis of $\mathfrak{gl}_\infty$ is $\{E_{i,j} = \xi_i \otimes \xi_j^*\}_{i,j\in\mathfrak{J}}$ and $[E_{i,j}, E_{k,l}] = \delta_{j,k}E_{i,l} - \delta_{i,l}E_{k,j}$.

Assume in addition that V is equipped with a non-degenerate anti-symmetric bilinear form on $\Omega : V \otimes V \to \mathbb{C}$. The simple finitary Lie algebra $\mathfrak{sp}_\infty$ is the Lie subalgebra of $\mathfrak{gl}_\infty$ for which Ω is invariant. This means by definition that

$$\mathfrak{sp}_\infty = \{g \in \mathfrak{gl}_\infty \mid \Omega(gu, v) + \Omega(u, gv) = 0, \ \forall u, v \in V\}.$$

By [Mac], all non-degenerate symplectic forms on V are equivalent, hence $\mathfrak{sp}_\infty$ does not depend up to isomorphism on the form Ω.

For a coordinate definition of $\mathfrak{sp}_\infty$, we are able to choose a basis $\{\xi_i\}_{i\in\mathbb{Z}-\{0\}}$ of V such that $\Omega(\xi_i, \xi_j) = sgn(i)\delta_{i+j,0}$, and then a linear basis of $\mathfrak{sp}_\infty$ given by $\{sgn(j)E_{i,j} - sgn(i)E_{-j,-i}\}$. It follows that

$$\mathfrak{sp}_\infty = Sym^2 V.$$

Alternatively, assume that V is equipped with a non-degenerate symmetric bilinear form $Q : V \otimes V \to \mathbb{C}$. The Lie algebra $\mathfrak{sp}_\infty$ is the Lie subalgebra of $\mathfrak{so}_\infty$ for which Q is invariant, i.e.

$$\mathfrak{so}_\infty = \{g \in \mathfrak{gl}_\infty \mid Q(gu, v) + Q(u, gv) = 0, \ \forall u, v \in V\}.$$

Again by [Mac], all such bilinear forms Q on V are equivalent.

We pick a basis $\{\xi_i\}_{i\in\mathbb{Z}-\{0\}}$ of V such that $Q(\xi_i, \xi_j) = \delta_{i+j,0}$. Then $\{E_{i,j} - E_{-j,-i}\}_{i,j\in\mathbb{Z}-\{0\}}$ is a linear basis of $\mathfrak{so}_\infty$. In particular,

$$\mathfrak{so}_\infty = \bigwedge^2 V.$$

Now, we briefly discuss the concept of **Schur-Weyl duality** following [wikd] and [FH91]. Consider the vector space $V^{\otimes n} = V \otimes V \otimes ... \otimes V$ for some positive integer n. The symmetric group $\mathfrak{S}_n$ acts on $V^{\otimes n}$ by permuting the factors:

$$\sigma \cdot (v_1 \otimes v_2 \otimes ... \otimes v_n) = v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes ... \otimes v_{\sigma^{-1}(n)}.$$

The Lie algebra $\mathfrak{gl}_\infty$ also acts on $V^{\otimes n}$:

$$g \cdot (v_1 \otimes v_2 \otimes ... \otimes v_n) = gv_1 \otimes v_2 \otimes ... \otimes v_n + v_1 \otimes gv_2 \otimes ... \otimes v_n + ... + v_1 \otimes v_2 \otimes ... \otimes gv_n.$$

Denote by V_λ the irreducible representation of $\mathfrak{S}_n$ corresponding to a partition λ with $|\lambda| = n$ ([FH91]). We define m_λ to be the dimension of V_λ, and define the simple tensor $\mathfrak{gl}_\infty-$module Γ_λ as

$$\Gamma_\lambda := V^{\otimes n} \otimes_{\mathfrak{S}_n} V_\lambda,$$

see [wikd]. Additionally, it follows that the semisimple decomposition of $V^{\otimes n}$ as a $\mathfrak{gl}_\infty-$module is

$$V^{\otimes n} = \bigoplus_{\lambda} m_\lambda \Gamma_\lambda.$$

The irreducible $\mathfrak{gl}_\infty-$module Γ_λ can be considered as an $\mathfrak{sp}_\infty-$module or an $\mathfrak{so}_\infty-$module. In general, it becomes reducible, but always has simple socle. Those simple socles are denoted respectively by $\Gamma_{<\lambda>}$ for $\mathfrak{sp}_\infty$ and $\Gamma_{[\lambda]}$ for $\mathfrak{so}_\infty$. See [PS11] for more details.

Next, let T the tensor algebra of the space V. Any finite-length $\mathfrak{sp}_\infty-$module (respectively $\mathfrak{so}_\infty-$module) isomorphic to a subquotient of a finite direct sum of copies of T, is an $\mathfrak{sp}_\infty-$ **tensor module** (respectively an $\mathfrak{so}_\infty-$ tensor module). We denote the corresponding categories of tensor modules by $\mathcal{C}_{\mathfrak{sp}_\infty}$ and $\mathcal{C}_{\mathfrak{so}_\infty}$. It is shown in [DCPS] that the modules Γ_λ are injective in both categories $\mathcal{C}_{\mathfrak{sp}_\infty}$ and $\mathcal{C}_{\mathfrak{so}_\infty}$. Moreover, due to the fact that any tensor module has an exhaustive socle filtration, we know that Γ_λ is indecomposable and that it is an injective hull of $\Gamma_{<\lambda>}$ in $\mathcal{C}_{\mathfrak{sp}_\infty}$, and an injective hull of $\Gamma_{[\lambda]}$ in $\mathcal{C}_{\mathfrak{so}_\infty}$.

3 Length and top layer of socle filtrations of injective tensor $\mathfrak{sp}_\infty$-modules

We start by recalling the formula for the socle layers of the injective tensor $\mathfrak{sp}_\infty-$module Γ_λ.

Theorem 3.1. *[PS11], [DCPS] For any partition λ, the $\mathfrak{sp}_\infty-$module Γ_λ is an injective hull of $\Gamma_{<\lambda>}$ and is hence indecomposable. The socle layers of Γ_λ are given by*

$$\overline{soc}^{k+1}\Gamma_\lambda = \bigoplus_{\mu} \left(\sum_{|\gamma|=k} N^{\lambda}_{\mu,(2\gamma)^T} \right) \Gamma_{<\mu>} \qquad \textit{for } k = 1, ..., \left[\frac{n}{2}\right], n = |\lambda|.$$

Example 3.2. *The table below shows the socle layers of the $\mathfrak{sp}_\infty-$ module $\Gamma_{(3,2,1,1)}$:*

$\Gamma_{<(1)>}$
$2\Gamma_{<(2,1)>} \oplus \Gamma_{<(1,1,1)>}$
$\Gamma_{<(3,2)>} \oplus \Gamma_{<(2,2,1)>} \oplus \Gamma_{<(2,1,1,1)>} \oplus \Gamma_{<(3,1,1)>}$
$\Gamma_{<(3,2,1,1)>}$

In more detail: the degree of $(3,2,1,1)$ is 7, so according to Theorem 3.1 we can have at most $1+\left[\frac{7}{2}\right] = 4$ socle layers. We know from section 2 of the thesis that $\overline{soc}^1\Gamma_{(3,2,1,1)} = soc\,\Gamma_{(3,2,1,1)} = \Gamma_{<(3,2,1,1)>}$. In the second layer we need to use a degree 2 partition of the form $(2\gamma)^T$, and the only one is $(1,1)$. We are interested in finding the subpartitions μ of $(3,2,1,1)$ for which there exists at least one Littlewood-Richardson tableau of shape $(3,2,1,1)/(1,1)$ and weight μ. The only possible such subpartitions are $(3,2),(2,2,1),(2,1,1,1),(3,1,1)$, and the second socle layer is:

$$\overline{soc}^2\Gamma_{(3,2,1,1)} = N^{(3,2,1,1)}_{(3,2),(1,1)}\Gamma_{<(3,2)>} \oplus N^{(3,2,1,1)}_{(2,2,1),(1,1)}\Gamma_{<(2,2,1)>} \oplus N^{(3,2,1,1)}_{(2,1,1,1),(1,1)}\Gamma_{<(2,1,1,1)>} \oplus N^{(3,2,1,1)}_{(3,1,1),(1,1)}\Gamma_{<(3,1,1)>} =$$

$$= \Gamma_{<(3,2)>} \oplus \Gamma_{<(2,2,1)>} \oplus \Gamma_{<(2,1,1,1)>} \oplus \Gamma_{<(3,1,1)>}.$$

The next two socle layers are computed in a similar fashion:

$$\overline{soc}^3\Gamma_{(3,2,1,1)} = N^{(3,2,1,1)}_{(2,1),(2,2)}\Gamma_{<(2,1)>} \oplus N^{(3,2,1,1)}_{(2,2),(1,1,1)}\Gamma_{<(1,1,1)>} \oplus N^{(3,2,1,1)}_{(1,1,1,1),(3,2,1,1)}\Gamma_{<(2,1)>} = 2\Gamma_{<(2,1)>} \oplus \Gamma_{<(1,1,1)>},$$

$$\overline{soc}^4\Gamma_{(3,2,1,1)} = N^{(3,2,1,1)}_{(2,2,1,1),(1)}\Gamma_{<(1)>} = \Gamma_{<(1)>}.$$

Theorem 3.1 computes the length of the socle filtration of the injective tensor module Γ_λ only implicitly. Our first result in this thesis is the following explicit formula (proposition 3.4) for the length of this socle filtration. For this, we introduce the notion of **even degree** of a partition.

Definition 3.3 (Even degree of a partition). *Given a partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$, let $|\lambda|^{even}$ denote the sum of the entries on even positions $\lambda_2, \lambda_4, \lambda_6, ...$, and call it the **even degree** of λ.*

The following is our first result:

Proposition 3.4. *For any partition λ, the last nonzero socle layer in the socle filtration of the $\mathfrak{sp}_\infty$-module Γ_λ is:*

$$\overline{soc}^{k+1}\Gamma_\lambda = \bigoplus_\mu \left(\sum_{|\gamma|=k} N^\lambda_{\mu,(2\gamma)^T} \right) \Gamma_{<\mu>} \qquad \text{for } k = |\lambda|^{even}.$$

Moreover, the module $\Gamma_{<(s)>}$ with $s = |\lambda| - 2|\lambda|^{even}$ is always a simple constituent of the last nonzero socle layer and appears with multiplicity one.

Proof. We divide the proof into two steps: showing that $\overline{\mathrm{soc}}^{k+1}\Gamma_\lambda$ is indeed nonzero for $k = |\lambda|^{even}$, and proving that $\overline{\mathrm{soc}}^{k+1}\Gamma_\lambda = 0$ for any $k > |\lambda|^{even}$. For both steps we need to find the maximal $(2\gamma)^T$ subpartition of $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$.

Let $r = 2m$ or $r = 2m+1$ for some positive integer m. Then the even degree of λ is $|\lambda|^{even} = \lambda_2+\lambda_4+...+\lambda_{2m}$. We claim that the unique maximal subpartition of λ of type $(2\gamma)^T$ has degree $2|\lambda|^{even}$ and is

$$(2\gamma)^T_{max} := (\lambda_2, \lambda_2, \lambda_4, \lambda_4, ..., \lambda_{2m}, \lambda_{2m}).$$

We first show the maximality and uniqueness of the partition $(2\gamma)^T_{max}$. Suppose that $\eta = (2\gamma)^T$ is a subpartition of λ, for some partition γ. Then η has even length $2p$ and $\eta_{2i-1} = \eta_{2i} \leq \lambda_{2i}$ for any $i \leq min(m, p)$. Thus

$$|\eta| = 2 \sum_{i=1}^{min(m,p)} \eta_{2i} \leq 2 \sum_{i=1}^{m} \lambda_{2i} = 2|\lambda|^{even}.$$

Equality holds only when all inequalities $\eta_{2i-1} = \eta_{2i} \leq \lambda_{2i}$ are equalities and $m = p$. Therefore, $\eta_{2i-1} = \eta_{2i} = \lambda_{2i}$ for $1 \leq i \leq m$ is the only situation which leads to a subpartition of the form $(2\gamma)^T$ of λ of maximal degree, hence $(2\gamma)^T_{max} = (\lambda_2, \lambda_2, \lambda_4, \lambda_4, ..., \lambda_{2m}, \lambda_{2m})$ is indeed the only such subpartition of λ.

Since $(2\gamma)^T_{max}$ has degree $2|\lambda|^{even}$, we conclude that $\mathrm{N}^\lambda_{\mu,(2\gamma)^T} = 0$ for any partition μ of degree less than $|\lambda| - 2|\lambda|^{even}$, i.e. that any $(k+1)^{th}$ socle layer of Γ_λ is zero for $k > |\lambda|^{even}$.

Next, we need to prove that $\overline{\mathrm{soc}}^{k+1}\Gamma_\lambda = \bigoplus_\mu \left(\sum_{|\gamma|=k} \mathrm{N}^\lambda_{\mu,(2\gamma)^T} \right) \Gamma_{<\mu>}$ is nonzero for $k = |\lambda|^{even}$. For this, we only need to show the existence of a partition μ for which $\mathrm{N}^\lambda_{\mu,(2\gamma)^T_{max}} > 0$. We take μ to be the row-partition (s) for $s := |\lambda| - 2 \cdot |\lambda|^{even} = |\lambda| - |(2\gamma)^T_{max}|$. Then we can conclude from Pieri's rule on Young diagrams that $\mathrm{N}^\lambda_{(s),(2\gamma)^T_{max}} = 1$, since we are attaching the boxes from μ only on the odd-numbered rows of the Young diagram with no overlap on the columns. This means that $\Gamma_{<(s)>}$ appears in the top layer of the socle filtration with multiplicity 1.

QED

We now prove a stronger result concerning the last socle layer of Γ_λ.

Proposition 3.5. *Let $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ be a partition. If $r = 2m$ for some positive integer m, we pick the nonzero values among the set of differences $\{\lambda_1 - \lambda_2, \lambda_3 - \lambda_4, ..., \lambda_{2m-1} - \lambda_{2m}\}$. If $r = 2m+1$ for some positive integer m, we pick the nonzero values from $\{\lambda_1 - \lambda_2, \lambda_3 - \lambda_4, ..., \lambda_{2m-1} - \lambda_{2m}, \lambda_{2m+1}\}$. Rearranging these nonzero values in a descending order, we obtain a partition $(y_1, y_2, ..., y_l)$. Then for any partition μ with $\mu \geq_{lex} (y_1, y_2, ..., y_l)$, $|\mu| = |(y_1, y_2, ..., y_l)|$, and of length at most l, the simple $\mathfrak{sp}_\infty$−module $\Gamma_{<\mu>}$ appears in the last term of the socle filtration of Γ_λ.*

Proof. The case where $\mu = (\sum_{i=1}^{l} y_i)$ is described in the last proposition. For the general case, we need to show that we can find at least one Littlewood-Richardson tableau of size $\lambda/(2\gamma)^T{}_{max}$ and of weight $\mu = (\mu_1, \mu_2, ..., \mu_t) \geq_{lex} (y_1, y_2, ..., y_l)$, $t \leq l$. The skew diagram of size $\lambda/(2\gamma)^T{}_{max}$ contains only columns of length 1, so the Littlewood-Richardson rule is easier to check.

We construct such a Littlewood-Richardson tableau of size $\lambda/(2\gamma)^T{}_{\max}$ and of weight $(\mu_1, \mu_2, ..., \mu_t)$ by placing the numbers $\underbrace{1, 1, ..., 1}_{\mu_1 \text{ times}}, \underbrace{2, 2, ..., 2}_{\mu_2 \text{ times}}, ..., \underbrace{t, t, ..., t}_{\mu_t \text{ times}}$ in a step-by-step manner. In the procedure we describe below, we count the rows from the bottom row to the top row of the skew diagram, and say that a row is *incomplete* if it has at least one box with no number written inside it.

1. This is a recursive procedure. In the rightmost empty box of the first incomplete row, we write the greatest number x from the unplaced numbers of the multiset $\underbrace{1, 1, ..., 1}_{\mu_1 \text{ times}}, \underbrace{2, 2, ..., 2}_{\mu_2 \text{ times}}, ..., \underbrace{t, t, ..., t}_{\mu_t \text{ times}}$ corresponding to the weight μ. In the rightmost empty box from the next incomplete row above, we write $x - 1$, and so on until placing 1. We repeat this step until all the boxes of the tableau are filled. Note that the total number of steps we need to place all numbers is μ_1, and the number $i \in \{1, 2, ..., t\}$ is fully placed in the boxes of the tableau exactly when performing the first μ_i steps.
2. After all numbers are placed, some rows may may not be monotonous. We rearrange the boxes of those rows so that each row's entries are in a non-strict increasing order from left to right.

After applying this algorithm, one can immediately observe that the equivalent conditions from definitions 2.15 and 2.18 are satisfied, and the statement follows.

QED

We conjecture that, apart from the modules described in the proposition above, there are no other modules in the last socle layer. While we do not have a proof for this, it is confirmed by multiple examples done via programming.

Example 3.6. *Let us use the step-by-step construction described in the proof of Proposition 3.5 to fill in the boxes of the skew diagram of shape $(18, 16, 16, 12, 12, 10, 10, 4, 4)/(16, 16, 12, 12, 10, 10, 4, 4)$, where the partition of nonzero differences is $(6, 4, 4, 2, 2)$. We will look at a particular example of tableau with weight given by partition $\mu = (10, 6, 2) \geq_{lex} (6, 4, 4, 2, 2)$. The length of the partition μ is 3, so the boxes will only contain the numbers 1, 2, and 3.*

Recall that we are counting the rows from the bottom row to the top row. Applying step 1 for the first time, we place 3 in the rightmost box of the first (lowest) row, then we place 2 in the rightmost box of the row above, and 1 in the rightmost box of the third row. Then, we repeat the process with the rightmost free boxes of these incomplete rows. Note that we are done placing the number 3, after exactly $\mu_3 = 2$ algorithm steps, as in figure 9.

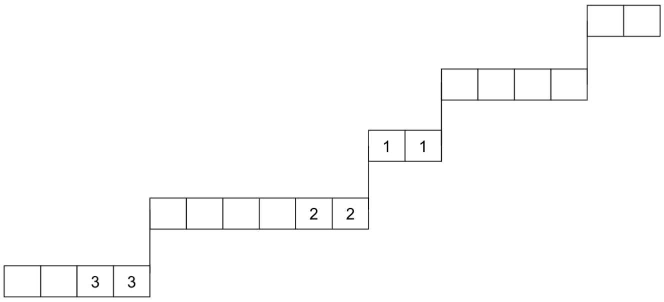

Figure 9

Next, we consider the remaining empty boxes and start writing the unplaced numbers

$$\underbrace{1,...,1}_{\mu_1-\mu_3=8\ times}\quad,\quad\underbrace{2,...,2}_{\mu_2-\mu_3=4\ times}\quad.$$

In the next two steps, we place 2 *twice in the first incomplete row, and* 1 *twice in the row above. The row where 2 was placed now has no more empty boxes. The next two steps place the last appearances of* 2 *in the boxes of the new first incomplete row (here, row 2) - its entries end up being 2,2,1,1,2,2, making the row non-increasing. The two placements of the number* 1 *which correspond to the last* 2 *entries will be in the boxes from the above incomplete row (here, row 4). These last 4 steps can be understood through the figure below.*

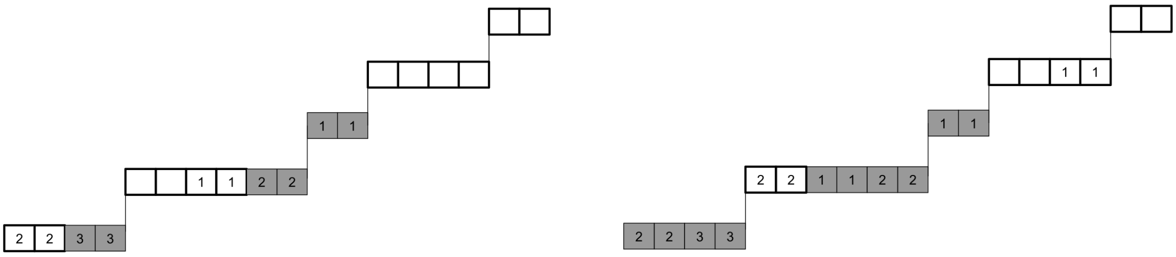

Figure 10

After this, we have to place 1 *in the remaining four boxes. We rearrange the boxes of the non-monotonous row of the tableau such that they are in an ascending order, and obtain the desired Littlewood-Richardson tableau.*

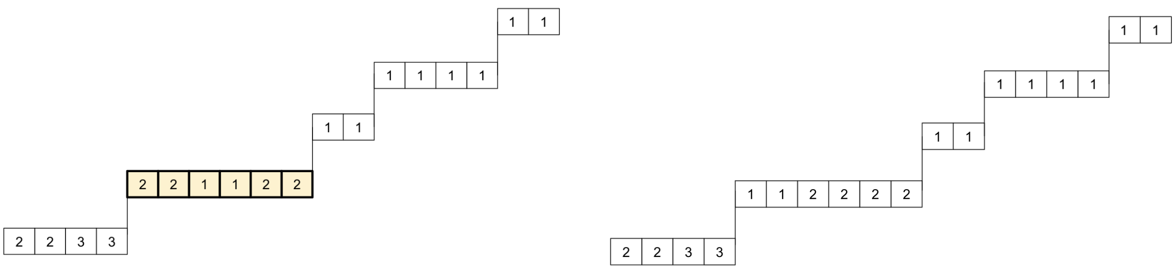

Figure 11

One can see that, even with the rearrangement of the entries of a row, the strict $\mu-$*expansion rule is still satisfied, as each number has all its predecessors written in upper rows. This shows that counting any set* *of boxes (from top to bottom, right to left), 1 appears at least as many times as 2 and 2 appears at least as many times as 3. The property is emphasized in the colour-coordinated figure below.*

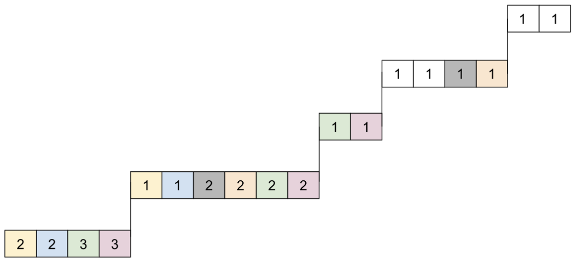

Figure 12

Corollary 3.7. *For any partition λ, the trivial representation $\Gamma_{<\emptyset>}$ is a simple constituent of the last nonzero socle layer in the socle filtration of the $\mathfrak{sp}_\infty$-module Γ_λ if and only if $\lambda = (2\gamma)^T$ for some partition γ. Furthermore, $\Gamma_{<\emptyset>}$ is the only term in the last socle layer of Γ_λ and appears with multiplicity 1.*

Proof. Straightforward. QED

Corollary 3.8. *Let λ be a partition such that the partition of nonzero differences as defined in Proposition 3.5 has length one, i.e. has the form (y_1). Then the last nonzero socle layer in the socle filtration of the $\mathfrak{sp}_\infty$-module Γ_λ equals $\Gamma_{<(y_1)>}$. Moreover, these modules Γ_λ are the only ones with simple last socle layer.*

Proof. Let $(2\gamma)^T_{max}$ be the unique maximal subpartition of λ of form $(2\gamma)^T$. Then the semistandard skew tableau of shape $\lambda/(2\gamma)^T_{max}$ will be a row with y_1 boxes. The only Littlewood-Richardson tableau of this shape is the one with weight (y_1), and $N^\lambda_{(2\gamma)^T_{max},(y_1)} = 1$ by Pieri's rule on Young diagrams. So, $\Gamma_{<(y_1)>}$ appears in the last socle layer of the socle filtration of Γ_λ with multiplicity 1.

Suppose there exists a partition η such that the partition of nonzero differences $(y'_1, y'_2, ..., y'_l)$, defined as in Proposition 3.5, has length at least 2. Then from the same proposition, we know that $\Gamma_{<(y'_1,y'_2,...,y'_l)>}$ appears in the last socle layer. By Proposition 3.4, $\Gamma_{<(y'_1+y'_2+...+y'_l)>}$ is a simple constituent of the last socle layer too, and $\Gamma_{<(y'_1,y'_2,...,y'_l)>}$ and $\Gamma_{<(y'_1+y'_2+...+y'_l)>}$ are nonisomorphic. Therefore, if the partition of nonzero differences has length strictly greater than 1, then the last layer of the socle filtration of Γ_η has at least two direct summands. QED

4 Length and last term of the minimal injective resolutions of simple tensor $\mathfrak{sp}_\infty$−modules

In this section, we are concerned with minimal injective resolutions of $\mathfrak{sp}_\infty$− modules. In particular, we want to find the length and explicitly describe the last term of such resolutions.

We start by describing an algorithm for computing minimal injective resolutions of simple tensor $\mathfrak{sp}_\infty$−modules. Let us denote by Γ^k_λ the k-th term in the injective resolution

$$0 \to \Gamma_{<\lambda>} \xrightarrow{\alpha_0} \Gamma_\lambda = \Gamma^0_\lambda \xrightarrow{\alpha_1} \Gamma^1_\lambda \xrightarrow{\alpha_2} ... \xrightarrow{\alpha_s} \Gamma^s_\lambda \to 0.$$

To calculate Γ^k_λ, we follow the steps below:

1. Γ^1_λ equals the injective hull $\bigoplus_\mu \Gamma_\mu$ of all constituents $\Gamma_{<\mu>}$ of $\overline{\mathrm{soc}}^2\Gamma_\lambda$, and

$$\text{im } \alpha_1 = \begin{array}{|c|}\hline \vdots \\ \hline \overline{\mathrm{soc}}^4\Gamma_\lambda \\ \hline \overline{\mathrm{soc}}^3\Gamma_\lambda \\ \hline \overline{\mathrm{soc}}^2\Gamma_\lambda \\ \hline \end{array}$$

2. The next resolution term Γ^2_λ equals the injective hull $\bigoplus\limits_{\eta}\Gamma_\eta$ of all simple constituents $\Gamma_{<\eta>}$(with possible repetitions) of $(\overline{\text{soc}}^2\Gamma^1_\lambda)/\alpha_1(\overline{\text{soc}}^3\Gamma_\lambda)$. The image of α_2 is the composition of the surjection $\Gamma^1_\lambda \to \Gamma^1_\lambda/\ im\alpha_1$ with the injection of $\Gamma^1_\lambda/\ im\alpha_1$ into its injective hull. This injective hull coincides with the injective hull of $(\overline{\text{soc}}^2\Gamma^1_\lambda)/(\overline{\text{soc}}^3\Gamma_\lambda)$.

3. By repeating the method described above, one can compute all the next resolution terms Γ^{k+1}_λ as the injective hull of all the modules of $(\overline{\text{soc}}^2\Gamma^k_\lambda)/\alpha_k(\overline{\text{soc}}^3\Gamma^{k-1}_\lambda)$.

The fact that the so described algorithm is valid follows from the results of [DCPS] and [SS15].

Given this algorithm, let us take an actual example, showcasing the steps.

Example 4.1. *In this example, we show that the injective resolution of* $\Gamma_{<(3,1,1,1)>}$ *is*

$$0 \to \Gamma_{<(3,1,1,1)>} \xrightarrow{\alpha_0} \Gamma_{(3,1,1,1)} \xrightarrow{\alpha_1} \Gamma_{(3,1)} \oplus \Gamma_{(2,1,1)} \xrightarrow{\alpha_2} \Gamma_{(2)} \oplus \Gamma_{(1,1)} \xrightarrow{\alpha_3} \Gamma_\emptyset \xrightarrow{\alpha_4} 0.$$

We start by writing the 0-term of the injective resolution of $\Gamma_{<(3,1,1,1)>}$:

$$\Gamma^0_{(3,1,1,1)} = \Gamma_{(3,1,1,1)},$$

with socle filtration

$$\begin{array}{|c|}\hline \Gamma_{<(2)>} \\ \hline \Gamma_{<(3,1)>} \oplus \Gamma_{<(2,1,1)>} \\ \hline \Gamma_{<(3,1,1,1)>} \\ \hline \end{array}.$$

In step 1, we find the injective hull of $\Gamma_{<(3,1)>} \oplus \Gamma_{<(2,1,1)>}$*, and we obtain* $\Gamma^1_{(3,1,1,1)} = \Gamma_{(3,1)} \oplus \Gamma_{(2,1,1)}$ *and*

$$im\ \alpha_1 = \begin{array}{|c|}\hline \Gamma_{<(2)>} \\ \hline \Gamma_{<(3,1)>} \oplus \Gamma_{<(2,1,1)>} \\ \hline \end{array}.$$

Step 2 claims that, since

$$\Gamma^1_{(3,1,1,1)} = \Gamma_{(3,1)} \oplus \Gamma_{(2,1,1)} = \begin{array}{|c|}\hline \Gamma_{<(2)>} \\ \hline \Gamma_{<(3,1)>} \\ \hline \end{array} \oplus \begin{array}{|c|}\hline \Gamma_{<(2)>} \oplus \Gamma_{<(1,1)>} \\ \hline \Gamma_{<(2,1,1)>} \\ \hline \end{array}$$

we obtain $\Gamma^2_{(3,1,1,1)} = \Gamma_{(2)} \oplus \Gamma_{(1,1)}$ *and*

$$im\ \alpha_2 = \begin{array}{|c|}\hline \Gamma_{<(2)>} \oplus \Gamma_{<(1,1)>} \\ \hline \end{array}.$$

Repeating the instructions from the algorithm we get that $\Gamma^3_{(3,1,1,1)}$ *is the injective hull of all simple constituents of* $\overline{soc}^2\Gamma^2_{(3,1,1,1)}/\alpha_2(\overline{soc}^3\Gamma^1_{(3,1,1,1)})$*. Hence,* $\Gamma^3_{(3,1,1,1)} = \Gamma_\emptyset$*, which is the last term in the minimal injective resolution.*

If we continue to compute the resolution terms for a general case, we can notice certain patterns in the partitions parametrizing the terms of Γ^k_λ. The key result is that the resolution terms contain only modules Γ_μ with partitions μ such that $\mathrm{N}^\lambda_{\mu,\delta} \neq 0$ for column-quasi-symmetric (CQS) partitions δ. This is implied by [SS15] (page 45) In other words, we have the following:

Theorem 4.2. *For any partition* λ *and the corresponding* $\mathfrak{sp}_\infty$*−module* Γ_λ*, the terms of a minimal injective resolution of* $\Gamma_{<\lambda>}$ *are*

$$\Gamma^k_\lambda = \bigoplus_\mu \left(\sum_{|\delta|=2k, \delta\, CQS} N^\lambda_{\mu,\delta} \right) \Gamma_\mu, \qquad for\ k = 1, ..., \left[\frac{n}{2}\right], n = |\lambda|.$$

Example 4.3. *Let us compare the claim in theorem 4.2 with the result in the above example 4.1.*

The verification for the 0^{th} *term is straightforward.*

Next, we look for CQS-subpartitions δ *of* $(3,1,1,1)$ *of degree* 2. *There is only one such partition:* $\delta_1 = (1,1)$. *We have* $\Gamma^1_{(3,1,1,1)} = \Gamma_{(3,1)} \oplus \Gamma_{(2,1,1)}$, *and both Young diagrams of* $(3,1)$ *and* $(2,1,1)$ *lead to the Young diagram of* $(3,1,1,1)$ *through a strict* $(1,1)-$*expansion, hence the needed Littlewood-Richardson coefficients are positive.*

Similarly, for the second resolution term, the only $CQS-$*subpartition of degree* 4 *of* $(3,1,1,1)$ *is* $\delta_2 = (2,1,1)$. *We want to check if* $\Gamma^2_{(3,1,1,1)} = \Gamma_{(2)} \oplus \Gamma_{(1,1)}$. *By the Littlewood-Richardson rule, we can obtain one tableau of shape* $(3,1,1,1)/(2,1,1)$ *and weight* (2) *(Pieri rule on Young diagrams), and one tableau of shape* $(3,1,1,1)/(2,1,1)$ *and weight* $(1,1)$.

For the last term, $\Gamma^3_{(3,1,1,1)} = \Gamma_\emptyset$, *we have that* $N^{(3,1,1,1)}_{\emptyset,(3,1,1,1)} = 1$ *since* $(3,1,1,1)$ *is a CQS-partition. Moreover, it is the maximal CQS-subpartition of* $(3,1,1,1)$.

In a fashion similar to the previous section, we now want to have an explicit formula for the length of the minimal injective resolution, as well as for the modules in the last term of the resolution. In section 2, we looked for the maximal subpartition of the form $(2\gamma)^T$; now we have to search for the unique maximal CQS-subpartition of λ and for a formula for its degree.

In the proof below, we will refer to a CQS partition with only one row and one column as a **corner column-quasi-symmetric (CCQS)** partition.

Proposition 4.4. *Any partition* λ *contains a unique maximal column-quasi-symmetric subpartition* δ_{max}. *Moreover,* $|\delta_{max}| = 2\sum_{i=1}^{m} min(\lambda_i - i + 1, (\lambda^T)_i - i)$, *where* m *is the number of boxes on the main diagonal of the Young diagram of* λ.

Proof. Let us start by recalling that the number of boxes on column i of the Young diagram of any partition δ is equal to the number of boxes on the i^{th} row of the Young diagram of δ^T, i.e. $(\delta^T)_i$. Then, by definition 2.9, δ is a $CQS-$partition if for any index i of main diagonal boxes,

$$(\delta^T)_i - i = \delta_i - i + 1.$$

Next, given a partition $\lambda = (\lambda_1, \lambda_2, ..., \lambda_r)$ with m boxes on the main diagonal (hence, $\lambda_i \geq i$ for $1 \leq i \leq m$, and $\lambda_i < i$ for $m < i \leq r$), we will construct the maximal $CQS-$subpartition δ_{max} of λ. The construction algorithm will lead to the uniqueness of δ_{max}.

Denote by b_i the i^{th} box on the main diagonal with $1 \leq i \leq m$. For each b_i we want to find a maximal $CCQS-$partition $\delta(i)$ whose Young diagram has b_i as its top left box, its row boxes fit in the i^{th} row of the Young diagram of λ (to the right of the main diagonal), and its column boxes fit in the i^{th} column of the Young diagram of λ (below the main diagonal). This way, we find the maximal CQS subpartition by describing its Young diagram as the union of the Young diagrams of corner column-quasi-symmetric partitions with top left boxes on the main diagonal $\delta(i)$ for $1 \leq i \leq m$:

- We start by considering the Young diagram of $\delta(1)$ (the "outer corner" Young diagram), which has the maximal possible number of boxes from row 1 and column 1.
- Then, the first "inner corner" Young diagram of the $CCQS-$partition $\delta(2)$ consists of boxes from row 2 and column 2 of the Young diagram of λ.
- We continue in this manner until $\delta(m)$ is constructed.

The Young diagram of δ_{max}, i.e. the union of the Young diagrams of $\delta(1), \delta(2), ..., \delta(m)$, is maximal because each $\delta(i)$ is maximal and m is the last main diagonal box. The partition δ_{max} is CQS since the necessary condition $(\delta^T_{max})_i - i = (\delta_{max})_i - i + 1$, for $1 \leq i \leq m$, follows directly from the $CCQS$ property of each

$\delta(i): \quad (\delta(i)^T)_1 = \delta(i)_1 + 1$, for $1 \leq i \leq m$.

Hence, it is enough to obtain an explicit formula for each $\delta(i)$ in terms of λ to show the existence.

From the Young diagram of $\delta(i)$, the $CCQS-$partition's row has

$$\delta(i)_1 \leq \lambda_i - i + 1 \text{ boxes (including the box } b_i),$$

and its column has

$$(\delta(i)^T)_1 \leq (\lambda^T)_i - i + 1 \text{ boxes (including the box } b_i).$$

The partition $\delta(i)$ is $CCQS$, hence it must satisfy $(\delta(i)^T)_1 = \delta(i)_1 + 1$. This leads to the inequalities:

$$\delta(i)_1 \leq \lambda_i - i + 1 \text{ and } \delta(i)_1 \leq (\lambda^T)_i - i \Rightarrow \delta(i)_1 \leq min(\lambda_i - i + 1, (\lambda^T)_i - i).$$

Therefore, by choosing $\delta(i)_1 = min(\lambda_i - i + 1, (\lambda^T)_i - i)$, we have found the unique maximal $CCQS-$partition $\delta(i)$ that fits in the i^{th} row and column, to the right and, respectively, below the box b_i on the main diagonal of the Young diagram of λ. Explicitly, the partition is

$$\delta(i) = (\delta(i)_1, \underbrace{1, 1, ..., 1}_{\delta(i)_1 \text{ times}}), \quad \text{with degree } |\delta(i)| = 2\delta(i)_1.$$

The above implies that δ_{max}, exists and is unique, and its degree is

$$|\delta_{max}| = \sum_{i=1}^{m} |\delta(i)| = 2\sum_{i=1}^{m} |\delta(i)_1| = 2\sum_{i=1}^{m} min(\lambda_i - i + 1, (\lambda^T)_i - i).$$

QED

Example 4.5. *In this example, we find the maximal $CQS-$subpartition of $\lambda = (6, 5, 4, 4, 2, 1, 1)$. We start by drawing its Young diagram, as seen in figure 13, and computing the conjugate partition $\lambda^T = (7, 5, 4, 4, 2, 1)$. There are four boxes on the main diagonal of the Young diagrams of both λ and λ^T, therefore we need to find four partitions $\delta(i)$, $1 \leq i \leq 4$. We compute each $\delta(i)_1$ and obtain the $CCQS-$partitions:*

$$\delta(1)_1 = min(\lambda_1, (\lambda^T)_1 - 1) = min(6, 6) = 6 \quad \Rightarrow \quad \delta(1) = (6, 1, 1, 1, 1, 1, 1),$$

$$\delta(2)_1 = min(\lambda_2 - 1, (\lambda^T)_2 - 2) = min(4, 3) = 3 \quad \Rightarrow \quad \delta(2) = (3, 1, 1, 1),$$

$$\delta(3)_1 = min(\lambda_3 - 2, (\lambda^T)_3 - 3) = min(2, 1) = 1 \quad \Rightarrow \quad \delta(3) = (1, 1),$$

$$\delta(4)_1 = min(\lambda_4 - 3, (\lambda^T)_4 - 4) = min(1, 0) = 0 \quad \Rightarrow \quad \delta(4) \textit{is the empty partition.}$$

Their Young diagrams can be seen in figure 14.

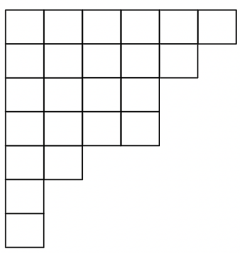

Figure 13: Young diagram of $\lambda = (6, 5, 4, 4, 2, 1, 1)$

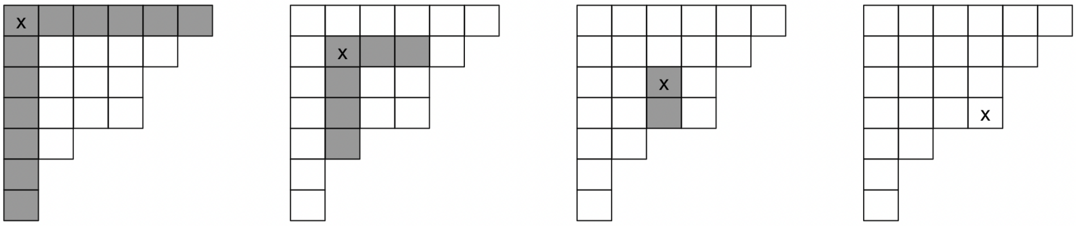

Figure 14: Highlighted, from left to right: $\delta(1), \delta(2), \delta(3), \delta(4)$ drawn on top of the Young diagram of λ, for emphasis on their maximality. Boxes on the main diagonal are marked with an $\times$.

The maximal CQS−subpartition δ_{max} is $(6,4,3,3,2,1,1)$, its Young diagram is graphical union of the Young diagrams of $\delta(1), \delta(2), \delta(3), \delta(4)$, and its degree is 20.

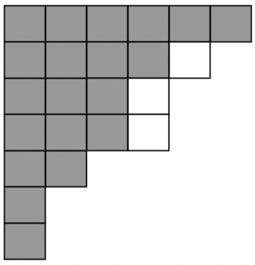

Figure 15: The Young diagram of the maximal CQS subpartition, inside the Young diagram of λ.

Proposition 4.6. *For any partition λ, let δ_{max} be its maximal CQS-subpartition. Denote by l the number of boxes in the longest column of the skew diagram λ/δ_{max}. Then the last nonzero term of a minimal injective resolution of the $\mathfrak{sp}_\infty$−module $\Gamma_{<\lambda>}$ has the form $\Gamma_\mu \oplus (\bigoplus_\eta \Gamma_\eta)$ for some partition μ of length l and possibly further partitions η of length at least l such that $|\mu| = |\eta| = |\lambda| - |\delta_{max}|$, and not necessarily distinct to μ.*

Proof. We define the partition μ as $(c_1, c_2, c_3, ..., c_l)$, where c_i is the number of columns of the skew diagram $\lambda/\delta_{\max}$ with at least i boxes, for all $i = 1, ..., l$. To show that $\mathrm{N}^{\lambda}_{\delta_{\max},\mu} \geq 1$, we construct a Littlewood-Richardson tableau of shape $\lambda/\delta_{\max}$ and weight μ. For this, we fill the boxes of the skew diagram $\lambda/\delta_{\max}$ column by column, by placing 1 in the top box of each column, then 2 in the box below, 3 in the next one, and so on, until the l−th box in the longest columns. This yields a Littlewood-Richardson tableau of $\lambda/\delta_{\max}$ and weight μ, and by attaching the Young diagram of δ_{max} to this skew tableau, we realize λ as a strict μ−expansion of the Young diagram of δ_{max}. Hence, $\mathrm{N}^{\lambda}_{\delta_{\max},\mu} \geq 1$ by Theorem 2.19.

Any other module Γ_η appearing in the last nonzero term of the resolution must satisfy $\mathrm{N}^{\lambda}_{\delta_{\max},\eta} \geq 1$, therefore there must exist at least one Littlewood-Richardson tableau of shape $\lambda/\delta_{\max}$ and weight η. Since the skew diagram $\lambda/\delta_{\max}$ contains at least a column of length l, when filling its boxes to obtain a Littlewood-Richardson tableau of weight η, we are not allowed to place the same number twice in the same column. Hence the maximal columns need to contain l distinct numbers, i.e. η has length at least l. Therefore, there is no $\mathfrak{sp}_\infty$−module Γ_η with length less than l in the last term of the injective resolution of Γ_λ.

QED

Example 4.7. *Let $\lambda = (4,3,3,2,2,2,2)$ be a partition with unique maximal CQS-subpartition $\delta_{max} = (4,3,2,2,1)$. We draw the Young diagram of δ_{max} inside the Young diagram of λ (figure 16) and see that the semistandard skew tableau of shape λ/δ_{max} contains a column of length 1, one of length 2, and one of* *length* 3.

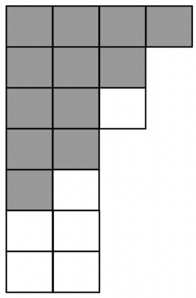

Figure 16: The Young diagram of the partition $(4,3,3,2,2,2,2)$, with the Young diagram of its maximal CQS-subpartition $(4,3,2,2,1)$ highlighted inside.

When numbering the boxes of the tableau according to the parts of a weight μ, *we must have a strictly increasing order on each column, so the length of* μ *is at least* 3. *We see in figure 17 that the partition* $\mu = (3,2,1)$ *defined as in the proof of Proposition 4.6 can serve as a weight for a Littlewood-Richardson tableau of shape* $(4,3,3,2,2,2,2)/(4,3,2,2,1)$. *Additionally, there exist other partitions* μ *of length at least 3 such that* $N^{\lambda}_{\delta_{max},\mu} \neq 0$. *Two of them are* $(2,2,2)$ *and* $(2,2,1,1)$. *These all satisfy the claim that the last nonzero term of a minimal injective resolution of* $\Gamma_{<(4,3,3,2,2,2,2))>}$ *has the form* $\Gamma_{(3,2,1)} \oplus (\bigoplus_{\eta} \Gamma_{\eta})$, *with* $(3,2,1)$ *of length* 3, *and where the partitions* η *are of length at least* 3 *and* $|\eta| = |(3,2,1)| = 6$.

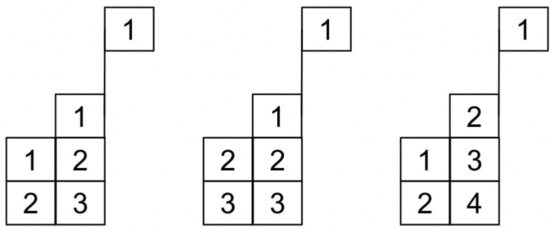

Figure 17: Littlewood-Richardson tableaux of shape $(4,3,3,2,2,2,2)/(4,3,2,2,1)$ and weight $(3,2,1)$ (left), $(2,2,2)$ (middle), and $(2,2,1,1)$ (right).

Corollary 4.8. *For any partition* λ, *let* δ_{max} *be its maximal column-quasi-symmetric subpartition. Then the length of a minimal injective resolution of the corresponding* $\mathfrak{sp}_\infty$*−module* $\Gamma_{<\lambda>}$ *is equal to* $\frac{|\delta_{max}|}{2} + 1$.

Corollary 4.9. *The* $\mathfrak{sp}_\infty$*−module* $\Gamma_\emptyset$ *appears as the last term of the injective resolution of some* $\mathfrak{sp}_\infty$*−module* $\Gamma_{<\lambda>}$ *if and only if the partition* λ *is column-quasi-symmetric.*

5 The case of tensor $\mathfrak{so}_\infty$−modules

In the papers [DCPS] and [Ser17], it has been proven that the category of tensor $\mathfrak{sp}_\infty$−modules is equivalent to the category of tensor $\mathfrak{so}_\infty$−modules. Under this equivalence, the $\mathfrak{sp}_\infty$−modules $\Gamma_{<\lambda>}$ and Γ_λ are mapped to the $\mathfrak{so}_\infty$−modules $\Gamma_{[\lambda^T]}$ and Γ_{λ^T}, respectively. Hence all results transfer to the $\mathbf{so}_\infty$−case by simply transposing Young diagrams.

Using the theory in [PS11] and [SS15] and our results from section 3 and 4, we give the following analogues of propositions 3.4, 3.5, 4.6, and of corollary 4.8:

Proposition 5.1. *For any partition λ, the last nonzero socle layer in the socle filtration of the $\mathfrak{so}_\infty$-module Γ_λ is:*

$$\overline{soc}^{k+1}\Gamma_\lambda = \bigoplus_\mu \left(\sum_{|\gamma|=k} N^\lambda_{\mu,2\gamma} \right) \Gamma_{[\mu]} \qquad \text{for } k = |\lambda^T|^{even}.$$

In particular, the module $\Gamma_{[\eta]}$ with $\eta = \underbrace{(1,1,...,1)}_{|\lambda|-2|\lambda^T|^{even} \ times}$ is always a simple constituent of the last nonzero socle layer, and the module appears with multiplicity one.

Proposition 5.2. *Let λ be a partition and $\lambda^T = ((\lambda^T)_1, (\lambda^T)_2, ..., (\lambda^T)_r)$ be its conjugate partition. If $r = 2m$ for some positive integer m, we pick the nonzero values among the set of differences $\{(\lambda^T)_1 - (\lambda^T)_2, (\lambda^T)_3 - (\lambda^T)_4, ..., (\lambda^T)_{2m-1} - (\lambda^T)_{2m}\}$. If $r = 2m+1$ for some positive integer m, we pick the nonzero values from $\{(\lambda^T)_1 - (\lambda^T)_2, (\lambda^T)_3 - (\lambda^T)_4, ..., (\lambda^T)_{2m-1} - (\lambda^T)_{2m}, (\lambda^T)_{2m+1}\}$. Rearranging these values in descending order, we obtain a partition $(y_1, y_2, ..., y_l)$. Then for any partition μ with $\mu \geq_{lex} (y_1, y_2, ..., y_l)$, $|\mu| = |(y_1, y_2, ..., y_l)|$, and of length at most l, the $\mathfrak{so}_\infty-$module $\Gamma_{[\mu^T]}$ appears in the last term of the socle filtration of the $\mathfrak{so}_\infty$-module Γ_λ.*

Proposition 5.3. *For any partition λ, let δ_{max} be its maximal RQS-subpartition. Let l be the number of boxes in the longest column of the semistandard skew tableau of shape λ/δ_{max}. Then the last nonzero term of a minimal injective resolution of the $\mathfrak{so}_\infty-$module $\Gamma_{[\lambda]}$ has the form*
$\Gamma_\mu \oplus (\bigoplus_\eta \Gamma_\eta)$ for some partition μ of length l and possibly further partitions η of length at least l such that $|\mu| = |\eta| = |\lambda| - |\delta_{max}|$.

Corollary 5.4. *For any partition λ, let δ_{max} be its maximal row-quasi-symmetric subpartition. The length of a minimal injective resolution of the corresponding $\mathfrak{so}_\infty-$module $\Gamma_{[\lambda]}$ is equal to $\frac{|\delta_{max}|}{2} + 1$.*

6 Conclusion and outlook

In the previous sections we proposed explicit combinatorial formulas for the last layer of the socle filtration of an indecomposable injective tensor $\mathfrak{sp}_\infty-$module, as well as for the last term of a minimal injective resolution of a simple tensor $\mathfrak{sp}_\infty-$module. Using the equivalence of the categories of tensor $\mathfrak{sp}_\infty-$ and $\mathfrak{so}_\infty-$modules, we applied our results also to tensor $\mathfrak{so}_\infty-$modules.

It makes sense to try to find explicit information about injective tensor $\mathfrak{sp}_\infty-$ and $\mathfrak{so}_\infty-$modules and injective resolutions. In particular, I. Penkov and V. Tsanov made the following conjecture:

Consider the socle layers of an injective tensor $\mathfrak{sp}_\infty-$module Γ_λ, and let $s(k+1) := \sum_{|\gamma|=k} \mathrm{N}^\lambda_{\mu,(2\gamma)^T}$, i.e., let $s(k+1)$ be the sum of the multiplicities of all simple constituents $\Gamma_{<\mu>}$ of $\overline{\mathrm{soc}}^{k+1}\Gamma_\lambda$. The conjecture claims that the socle filtration layers have a diamond shape structure, meaning that the inequalities

$$1 = s(1) \leq s(2) \leq s(3) \leq ... \leq s(m) \geq s(m-1) \geq s(m-2) \geq ... \geq s(|\lambda|^{even}+1),$$

hold for $m = \lfloor \frac{|\lambda|^{even}}{2} \rfloor + 1$.

7 Appendix

This appendix displays the minimal injective resolutions of the $\mathfrak{sp}_\infty$−modules $\Gamma_{<\lambda>}$, with $|\lambda| \leq 12$. Any resolution

$$0 \to \Gamma_{<\lambda>} \xrightarrow{\alpha_0} \Gamma_\lambda = \Gamma^0_\lambda \xrightarrow{\alpha_1} \Gamma^1_\lambda \xrightarrow{\alpha_2} ... \xrightarrow{\alpha_s} \Gamma^s_\lambda \to 0$$

is formatted in a vertical setting, with the bottom row being $\Gamma^0_\lambda = \Gamma_\lambda$, the next row being Γ^1_λ, and so on up to the top row, which is the last term of a minimal injective resolution of $\Gamma_{<\lambda>}$:

$$\begin{array}{c} \Gamma^s_\lambda \\ \hline \Gamma^{s-1}_\lambda \\ \hline \vdots \\ \hline \Gamma^1_\lambda \\ \hline \Gamma^0_\lambda = \Gamma_\lambda \end{array}.$$

Minimal injective resolutions of $\Gamma_{<\lambda>}$ with $|\lambda| = 1$:

$$\Gamma_{(1)}$$

Minimal injective resolutions of $\Gamma_{<\lambda>}$ with $|\lambda| = 2$:

$$\Gamma_{(2)} \qquad \begin{array}{c} \Gamma_\emptyset \\ \hline \Gamma_{(1,1)} \end{array}$$

Minimal injective resolutions of $\Gamma_{<\lambda>}$ with $|\lambda| = 3$:

$$\Gamma_{(3)} \qquad \begin{array}{c} \Gamma_{(1)} \\ \hline \Gamma_{(2,1)} \end{array} \qquad \begin{array}{c} \Gamma_{(1)} \\ \hline \Gamma_{(1,1,1)} \end{array}$$

Minimal injective resolutions of $\Gamma_{<\lambda>}$ with $|\lambda| = 4$:

$$\Gamma_{(4)} \qquad \begin{array}{c} \Gamma_{(2)} \\ \hline \Gamma_{(3,1)} \end{array} \qquad \begin{array}{c} \Gamma_{(1,1)} \\ \hline \Gamma_{(2,2)} \end{array} \qquad \begin{array}{c} \Gamma_\emptyset \\ \hline \Gamma_{(2)} \oplus \Gamma_{(1,1)} \\ \hline \Gamma_{(2,1,1)} \end{array} \qquad \begin{array}{c} \Gamma_{(1,1)} \\ \hline \Gamma_{(1,1,1,1)} \end{array}$$

Minimal injective resolutions of $\Gamma_{<\lambda>}$ with $|\lambda| = 5$:

$$\Gamma_{(5)} \qquad \begin{array}{c} \Gamma_{(3)} \\ \hline \Gamma_{(4,1)} \end{array} \qquad \begin{array}{c} \Gamma_{(2,1)} \\ \hline \Gamma_{(3,2)} \end{array} \qquad \begin{array}{c} \Gamma_{(1)} \\ \hline \Gamma_{(3)} \oplus \Gamma_{(2,1)} \\ \hline \Gamma_{(3,1,1)} \end{array} \qquad \begin{array}{c} \Gamma_{(1)} \\ \hline \Gamma_{(2,1)} \oplus \Gamma_{(1,1,1)} \\ \hline \Gamma_{(2,2,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(1)} \\ \hline \Gamma_{(2,1)} \oplus \Gamma_{(1,1,1)} \\ \hline \Gamma_{(2,1,1,1)} \end{array} \qquad \begin{array}{c} \Gamma_{(1,1,1)} \\ \hline \Gamma_{(1,1,1,1,1)} \end{array}$$

Minimal injective resolutions of $\Gamma_{<\lambda>}$ with $|\lambda| = 6$:

$$\Gamma_{(6)} \qquad \begin{array}{c} \Gamma_{(4)} \\ \hline \Gamma_{(5,1)} \end{array} \qquad \begin{array}{c} \Gamma_{(3,1)} \\ \hline \Gamma_{(4,2)} \end{array} \qquad \begin{array}{c} \Gamma_{(2)} \\ \hline \Gamma_{(4)} \oplus \Gamma_{(3,1)} \\ \hline \Gamma_{(4,1,1)} \end{array} \qquad \begin{array}{c} \Gamma_{(2,2)} \\ \hline \Gamma_{(3,3)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2)} \oplus \Gamma_{(1,1)} \\ \hline \Gamma_{(3,1)} \oplus \Gamma_{(2,2)} \oplus \Gamma_{(2,1,1)} \\ \hline \Gamma_{(3,2,1)} \end{array} \qquad \begin{array}{c} \Gamma_\emptyset \\ \hline \Gamma_{(2)} \oplus \Gamma_{(1,1)} \\ \hline \Gamma_{(3,1)} \oplus \Gamma_{(2,1,1)} \\ \hline \Gamma_{(3,1,1,1)} \end{array} \qquad \begin{array}{c} \Gamma_\emptyset \\ \hline \Gamma_{(1,1)} \\ \hline \Gamma_{(2,1,1)} \\ \hline \Gamma_{(2,2,2)} \end{array} \qquad \begin{array}{c} \Gamma_{(2)} \oplus \Gamma_{(1,1)} \\ \hline \Gamma_{(2,2)} \oplus \Gamma_{(2,1,1)} \oplus \Gamma_{(1,1,1,1)} \\ \hline \Gamma_{(2,2,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(1,1)} \\ \hline \Gamma_{(2,1,1)} \oplus \Gamma_{(1,1,1,1)} \\ \hline \Gamma_{(2,1,1,1,1)} \end{array} \qquad \begin{array}{c} \Gamma_{(1,1,1,1)} \\ \hline \Gamma_{(1,1,1,1,1,1)} \end{array}$$

Minimal injective resolutions of $\Gamma_{<\lambda>}$ with $|\lambda| = 7$:

$$\Gamma_{(7)} \qquad \begin{array}{c} \Gamma_{(5)} \\ \hline \Gamma_{(6,1)} \end{array} \qquad \begin{array}{c} \Gamma_{(4,1)} \\ \hline \Gamma_{(5,2)} \end{array} \qquad \begin{array}{c} \Gamma_{(3)} \\ \hline \Gamma_{(5)} \oplus \Gamma_{(4,1)} \\ \hline \Gamma_{(5,1,1)} \end{array} \qquad \begin{array}{c} \Gamma_{(3,2)} \\ \hline \Gamma_{(4,3)} \end{array}$$

$$\begin{array}{c}\Gamma_{(3)}\oplus\Gamma_{(2,1)}\\ \hline \Gamma_{(4,1)}\oplus\Gamma_{(3,2)}\oplus\Gamma_{(3,1,1)}\\ \hline \Gamma_{(4,2,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(1)}\\ \hline \Gamma_{(3)}\oplus\Gamma_{(2,1)}\\ \hline \Gamma_{(4,1)}\oplus\Gamma_{(3,1,1)}\\ \hline \Gamma_{(4,1,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(2,1)}\\ \hline \Gamma_{(3,2)}\oplus\Gamma_{(2,2,1)}\\ \hline \Gamma_{(3,3,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(1)}\\ \hline \Gamma_{(2,1)}\oplus\Gamma_{(1,1,1)}\\ \hline \Gamma_{(3,1,1)}\oplus\Gamma_{(2,2,1)}\\ \hline \Gamma_{(3,2,2)}\end{array}$$

$$\begin{array}{c}\Gamma_{(1)}\\ \hline \Gamma_{(3)}\oplus 2\Gamma_{(2,1)}\oplus\Gamma_{(1,1,1)}\\ \hline \Gamma_{(3,2)}\oplus\Gamma_{(3,1,1)}\oplus\Gamma_{(2,2,1)}\oplus\Gamma_{(2,1,1,1)}\\ \hline \Gamma_{(3,2,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(1)}\\ \hline \Gamma_{(2,1)}\oplus\Gamma_{(1,1,1)}\\ \hline \Gamma_{(3,1,1)}\oplus\Gamma_{(2,1,1,1)}\\ \hline \Gamma_{(3,1,1,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(1)}\\ \hline \Gamma_{(2,1)}\oplus\Gamma_{(1,1,1)}\\ \hline \Gamma_{(2,2,1)}\oplus\Gamma_{(2,1,1,1)}\\ \hline \Gamma_{(2,2,2,1)}\end{array}$$

$$\begin{array}{c}\Gamma_{(2,1)}\oplus\Gamma_{(1,1,1)}\\ \hline \Gamma_{(2,2,1)}\oplus\Gamma_{(2,1,1,1)}\oplus\Gamma_{(1,1,1,1,1)}\\ \hline \Gamma_{(2,2,1,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(1,1,1)}\\ \hline \Gamma_{(2,1,1,1)}\oplus\Gamma_{(1,1,1,1,1)}\\ \hline \Gamma_{(2,1,1,1,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(1,1,1,1,1)}\\ \hline \Gamma_{(1,1,1,1,1,1,1)}\end{array}$$

Minimal injective resolutions of $\Gamma_{<\lambda>}$ with $|\lambda| = 8$:

$$\Gamma_{(8)}\qquad \begin{array}{c}\Gamma_{(6)}\\ \hline \Gamma_{(7,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(5,1)}\\ \hline \Gamma_{(6,2)}\end{array}\qquad \begin{array}{c}\Gamma_{(4)}\\ \hline \Gamma_{(6)}\oplus\Gamma_{(5,1)}\\ \hline \Gamma_{(6,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(4,2)}\\ \hline \Gamma_{(5,3)}\end{array}$$

$$\begin{array}{c}\Gamma_{(4)}\oplus\Gamma_{(3,1)}\\ \hline \Gamma_{(5,1)}\oplus\Gamma_{(4,2)}\oplus\Gamma_{(4,1,1)}\\ \hline \Gamma_{(5,2,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(2)}\\ \hline \Gamma_{(4)}\oplus\Gamma_{(3,1)}\\ \hline \Gamma_{(5,1)}\oplus\Gamma_{(4,1,1)}\\ \hline \Gamma_{(5,1,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(3,3)}\\ \hline \Gamma_{(4,4)}\end{array}\qquad \begin{array}{c}\Gamma_{(3,1)}\oplus\Gamma_{(2,2)}\\ \hline \Gamma_{(4,2)}\oplus\Gamma_{(3,3)}\oplus\Gamma_{(3,2,1)}\\ \hline \Gamma_{(4,3,1)}\end{array}$$

$$\begin{array}{c}\Gamma_{(2)}\\ \hline \Gamma_{(3,1)}\oplus\Gamma_{(2,1,1)}\\ \hline \Gamma_{(4,1,1)}\oplus\Gamma_{(3,2,1)}\\ \hline \Gamma_{(4,2,2)}\end{array}\qquad \begin{array}{c}\Gamma_{(2)}\oplus\Gamma_{(1,1)}\\ \hline \Gamma_{(4)}\oplus 2\Gamma_{(3,1)}\oplus\Gamma_{(2,2)}\oplus\Gamma_{(2,1,1)}\\ \hline \Gamma_{(4,2)}\oplus\Gamma_{(4,1,1)}\oplus\Gamma_{(3,2,1)}\oplus\Gamma_{(3,1,1,1)}\\ \hline \Gamma_{(4,2,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{\emptyset}\\ \hline \Gamma_{(2)}\oplus\Gamma_{(1,1)}\\ \hline \Gamma_{(3,1)}\oplus\Gamma_{(2,1,1)}\\ \hline \Gamma_{(4,1,1)}\oplus\Gamma_{(3,1,1,1)}\\ \hline \Gamma_{(4,1,1,1,1)}\end{array}$$

$$\begin{array}{c}\Gamma_{(1,1)}\\ \hline \Gamma_{(2,2)}\oplus\Gamma_{(2,1,1)}\\ \hline \Gamma_{(3,2,1)}\oplus\Gamma_{(2,2,2)}\\ \hline \Gamma_{(3,3,2)}\end{array}\qquad \begin{array}{c}\Gamma_{(2)}\\ \hline \Gamma_{(3,1)}\oplus\Gamma_{(2,2)}\oplus\Gamma_{(2,1,1)}\\ \hline \Gamma_{(3,3)}\oplus\Gamma_{(3,2,1)}\oplus\Gamma_{(2,2,1,1)}\\ \hline \Gamma_{(3,3,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{\emptyset}\\ \hline \Gamma_{(1,1)}\oplus\Gamma_{(2)}\oplus\Gamma_{(1,1)}\\ \hline \Gamma_{(3,1)}\oplus\Gamma_{(2,2)}\oplus 2\Gamma_{(2,1,1)}\oplus\Gamma_{(1,1,1,1)}\\ \hline \Gamma_{(3,2,1)}\oplus\Gamma_{(3,1,1,1)}\oplus\Gamma_{(2,2,2)}\oplus\Gamma_{(2,2,1,1)}\\ \hline \Gamma_{(3,2,2,1)}\end{array}$$

$$\begin{array}{c}\Gamma_{(2)}\oplus\Gamma_{(1,1)}\\ \hline \Gamma_{(3,1)}\oplus\Gamma_{(2,2)}\oplus 2\Gamma_{(2,1,1)}\oplus\Gamma_{(1,1,1,1)}\\ \hline \Gamma_{(3,2,1)}\oplus\Gamma_{(3,1,1,1)}\oplus\Gamma_{(2,2,1,1)}\oplus\Gamma_{(2,1,1,1,1)}\\ \hline \Gamma_{(3,2,1,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(1,1)}\\ \hline \Gamma_{(2,1,1)}\oplus\Gamma_{(1,1,1,1)}\\ \hline \Gamma_{(3,1,1,1)}\oplus\Gamma_{(2,1,1,1,1)}\\ \hline \Gamma_{(3,1,1,1,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(2)}\\ \hline \Gamma_{(2,1,1)}\\ \hline \Gamma_{(2,2,1,1)}\\ \hline \Gamma_{(2,2,2,2)}\end{array}$$

$$\begin{array}{c}\Gamma_{(1,1)}\\ \hline \Gamma_{(2,2)}\oplus\Gamma_{(2,1,1)}\oplus\Gamma_{(1,1,1,1)}\\ \hline \Gamma_{(2,2,2)}\oplus\Gamma_{(2,2,1,1)}\oplus\Gamma_{(2,1,1,1,1)}\\ \hline \Gamma_{(2,2,2,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(2,1,1)}\oplus\Gamma_{(1,1,1,1)}\\ \hline \Gamma_{(2,2,1,1)}\oplus\Gamma_{(2,1,1,1,1)}\oplus\Gamma_{(1,1,1,1,1,1)}\\ \hline \Gamma_{(2,2,1,1,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(1,1,1,1)}\\ \hline \Gamma_{(2,1,1,1,1)}\oplus\Gamma_{(1,1,1,1,1,1)}\\ \hline \Gamma_{(2,1,1,1,1,1,1)}\end{array}$$

$$\begin{array}{c}\Gamma_{(1,1,1,1,1,1)}\\ \hline \Gamma_{(1,1,1,1,1,1,1,1)}\end{array}$$

Minimal injective resolutions of $\Gamma_{<\lambda>}$ with $|\lambda| = 9$:

$$\Gamma_{(9)}\qquad \begin{array}{c}\Gamma_{(7)}\\ \hline \Gamma_{(8,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(6,1)}\\ \hline \Gamma_{(7,2)}\end{array}\qquad \begin{array}{c}\Gamma_{(5)}\\ \hline \Gamma_{(7)}\oplus\Gamma_{(6,1)}\\ \hline \Gamma_{(7,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(5,2)}\\ \hline \Gamma_{(6,3)}\end{array}$$

$$\begin{array}{c}\Gamma_{(5)}\oplus\Gamma_{(4,1)}\\ \hline \Gamma_{(6,1)}\oplus\Gamma_{(5,2)}\oplus\Gamma_{(5,1,1)}\\ \hline \Gamma_{(6,2,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(3)}\\ \hline \Gamma_{(5)}\oplus\Gamma_{(4,1)}\\ \hline \Gamma_{(6,1)}\oplus\Gamma_{(5,1,1)}\\ \hline \Gamma_{(6,1,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(4,3)}\\ \hline \Gamma_{(5,4)}\end{array}\qquad \begin{array}{c}\Gamma_{(4,1)}\oplus\Gamma_{(3,2)}\\ \hline \Gamma_{(5,2)}\oplus\Gamma_{(4,3)}\oplus\Gamma_{(4,2,1)}\\ \hline \Gamma_{(5,3,1)}\end{array}$$

$$\begin{array}{c}\Gamma_{(3)}\\ \hline \Gamma_{(4,1)}\oplus\Gamma_{(3,1,1)}\\ \hline \Gamma_{(5,1,1)}\oplus\Gamma_{(4,2,1)}\\ \hline \Gamma_{(5,2,2)}\end{array}\qquad \begin{array}{c}\Gamma_{(3)}\oplus\Gamma_{(2,1)}\\ \hline \Gamma_{(5)}\oplus 2\Gamma_{(4,1)}\oplus\Gamma_{(3,2)}\oplus\Gamma_{(3,1,1)}\\ \hline \Gamma_{(5,2)}\oplus\Gamma_{(5,1,1)}\oplus\Gamma_{(4,2,1)}\oplus\Gamma_{(4,1,1,1)}\\ \hline \Gamma_{(5,2,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(1)}\\ \hline \Gamma_{(3)}\oplus\Gamma_{(2,1)}\\ \hline \Gamma_{(4,1)}\oplus\Gamma_{(3,1,1)}\\ \hline \Gamma_{(5,1,1)}\oplus\Gamma_{(4,1,1,1)}\\ \hline \Gamma_{(5,1,1,1,1)}\end{array}$$

$$\begin{array}{c}\Gamma_{(3,2)}\\ \hline \Gamma_{(4,3)}\oplus\Gamma_{(3,3,1)}\\ \hline \Gamma_{(4,4,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(2,1)}\\ \hline \Gamma_{(3,2)}\oplus\Gamma_{(3,1,1)}\oplus\Gamma_{(2,2,1)}\\ \hline \Gamma_{(4,2,1)}\oplus\Gamma_{(3,3,1)}\oplus\Gamma_{(3,2,2)}\\ \hline \Gamma_{(4,3,2)}\end{array}\qquad \begin{array}{c}\Gamma_{(3)}\oplus\Gamma_{(2,1)}\\ \hline \Gamma_{(4,1)}\oplus 2\Gamma_{(3,2)}\oplus\Gamma_{(3,1,1)}\oplus\Gamma_{(2,2,1)}\\ \hline \Gamma_{(4,3)}\oplus\Gamma_{(4,2,1)}\oplus\Gamma_{(3,3,1)}\oplus\Gamma_{(3,2,1,1)}\\ \hline \Gamma_{(4,3,1,1)}\end{array}$$

$$\begin{array}{c}\Gamma_{(1)}\\ \hline 2\Gamma_{(2,1)}\oplus\Gamma_{(1,1,1)}\oplus\Gamma_{(3)}\\ \hline \Gamma_{(4,1)}\oplus\Gamma_{(3,2)}\oplus 2\Gamma_{(3,1,1)}\oplus\Gamma_{(2,2,1)}\oplus\Gamma_{(2,1,1,1)}\\ \hline \Gamma_{(4,2,1)}\oplus\Gamma_{(4,1,1,1)}\oplus\Gamma_{(3,2,2)}\oplus\Gamma_{(3,2,1,1)}\\ \hline \Gamma_{(4,2,2,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(1)}\\ \hline \Gamma_{(3)}\oplus 2\Gamma_{(2,1)}\oplus\Gamma_{(1,1,1)}\\ \hline \Gamma_{(4,1)}\oplus\Gamma_{(3,2)}\oplus 2\Gamma_{(3,1,1)}\oplus\Gamma_{(2,2,1)}\oplus\Gamma_{(2,1,1,1)}\\ \hline \Gamma_{(4,2,1)}\oplus\Gamma_{(4,1,1,1)}\oplus\Gamma_{(3,2,1,1)}\oplus\Gamma_{(3,1,1,1,1)}\\ \hline \Gamma_{(4,2,1,1,1)}\end{array}$$

$$\begin{array}{c}\Gamma_{(1)}\\ \hline \Gamma_{(2,1)}\oplus\Gamma_{(1,1,1)}\\ \hline \Gamma_{(3,1,1)}\oplus\Gamma_{(2,1,1,1)}\\ \hline \Gamma_{(4,1,1,1)}\oplus\Gamma_{(3,1,1,1,1)}\\ \hline \Gamma_{(4,1,1,1,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(1,1,1)}\\ \hline \Gamma_{(2,2,1)}\\ \hline \Gamma_{(3,2,2)}\\ \hline \Gamma_{(3,3,3)}\end{array}\qquad \begin{array}{c}\Gamma_{(1)}\\ \hline 2\Gamma_{(2,1)}\oplus\Gamma_{(1,1,1)}\\ \hline \Gamma_{(3,2)}\oplus\Gamma_{(3,1,1)}\oplus 2\Gamma_{(2,2,1)}\oplus\Gamma_{(2,1,1,1)}\\ \hline \Gamma_{(3,3,1)}\oplus\Gamma_{(3,2,2)}\oplus\Gamma_{(3,2,1,1)}\oplus\Gamma_{(2,2,2,1)}\\ \hline \Gamma_{(3,3,2,1)}\end{array}$$

$$\begin{array}{c}\Gamma_{(3)}\oplus\Gamma_{(2,1)}\\ \hline \Gamma_{(3,2)}\oplus\Gamma_{(3,1,1)}\oplus\Gamma_{(2,2,1)}\oplus\Gamma_{(2,1,1,1)}\\ \hline \Gamma_{(3,3,1)}\oplus\Gamma_{(3,2,1,1)}\oplus\Gamma_{(2,2,1,1,1)}\\ \hline \Gamma_{(3,3,1,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(1)}\\ \hline \Gamma_{(1,1,1)}\oplus\Gamma_{(3)}\oplus\Gamma_{(2,1)}\\ \hline \Gamma_{(3,1,1)}\oplus\Gamma_{(2,2,1)}\oplus\Gamma_{(2,1,1,1)}\\ \hline \Gamma_{(3,2,1,1)}\oplus\Gamma_{(2,2,2,1)}\\ \hline \Gamma_{(3,2,2,2)}\end{array}$$

$$\begin{array}{c}\Gamma_{(1)}\\ \hline 2\Gamma_{(2,1)}\oplus 2\Gamma_{(1,1,1)}\\ \hline \Gamma_{(3,2)}\oplus\Gamma_{(3,1,1)}\oplus 2\Gamma_{(2,2,1)}\oplus 2\Gamma_{(2,1,1,1)}\oplus\Gamma_{(1,1,1,1,1)}\\ \hline \Gamma_{(3,2,2)}\oplus\Gamma_{(3,2,1,1)}\oplus\Gamma_{(3,1,1,1,1)}\oplus\Gamma_{(2,2,2,1)}\oplus\Gamma_{(2,2,1,1,1)}\\ \hline \Gamma_{(3,2,2,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(2,1)}\oplus\Gamma_{(1,1,1)}\\ \hline \Gamma_{(3,1,1)}\oplus\Gamma_{(2,2,1)}\oplus 2\Gamma_{(2,1,1,1)}\oplus\Gamma_{(1,1,1,1,1)}\\ \hline \Gamma_{(3,2,1,1)}\oplus\Gamma_{(3,1,1,1,1)}\oplus\Gamma_{(2,2,1,1,1)}\oplus\Gamma_{(2,1,1,1,1,1)}\\ \hline \Gamma_{(3,2,1,1,1,1)}\end{array}$$

$$\begin{array}{c}\Gamma_{(1,1,1)}\\ \hline \Gamma_{(2,1,1,1)}\oplus\Gamma_{(1,1,1,1,1)}\\ \hline \Gamma_{(3,1,1,1,1)}\oplus\Gamma_{(2,1,1,1,1,1)}\\ \hline \Gamma_{(3,1,1,1,1,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(2,1)}\\ \hline \Gamma_{(2,2,1)}\oplus\Gamma_{(2,1,1,1)}\\ \hline \Gamma_{(2,2,2,1)}\oplus\Gamma_{(2,2,1,1,1)}\\ \hline \Gamma_{(2,2,2,2,2)}\end{array}\qquad \begin{array}{c}\Gamma_{(1,1,1)}\\ \hline \Gamma_{(2,2,1)}\oplus\Gamma_{(2,1,1,1)}\oplus\Gamma_{(1,1,1,1,1)}\\ \hline \Gamma_{(2,2,2,1)}\oplus\Gamma_{(2,2,1,1,1)}\oplus\Gamma_{(2,1,1,1,1,1)}\\ \hline \Gamma_{(2,2,2,1,1,1)}\end{array}$$

$$\begin{array}{c}\Gamma_{(2,1,1,1)}\oplus\Gamma_{(1,1,1,1,1)}\\ \hline \Gamma_{(2,2,1,1,1)}\oplus\Gamma_{(2,1,1,1,1,1)}\oplus\Gamma_{(1,1,1,1,1,1,1)}\\ \hline \Gamma_{(2,2,1,1,1,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(1,1,1,1,1)}\\ \hline \Gamma_{(2,1,1,1,1,1)}\oplus\Gamma_{(1,1,1,1,1,1,1)}\\ \hline \Gamma_{(2,1,1,1,1,1,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(1,1,1,1,1,1,1)}\\ \hline \Gamma_{(1,1,1,1,1,1,1,1,1)}\end{array}$$

Minimal injective resolutions of $\Gamma_{<\lambda>}$ with $|\lambda| = 10$:

$$\Gamma_{(10)}\qquad \begin{array}{c}\Gamma_{(8)}\\ \hline \Gamma_{(9,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(7,1)}\\ \hline \Gamma_{(8,2)}\end{array}\qquad \begin{array}{c}\Gamma_{(6)}\\ \hline \Gamma_{(8)}\oplus\Gamma_{(7,1)}\\ \hline \Gamma_{(8,1,1)}\end{array}\qquad \begin{array}{c}\Gamma_{(6,2)}\\ \hline \Gamma_{(7,3)}\end{array}$$

$$\begin{array}{c} \Gamma_{(6)} \oplus \Gamma_{(5,1)} \\ \hline \Gamma_{(7,1)} \oplus \Gamma_{(6,2)} \oplus \Gamma_{(6,1,1)} \\ \hline \Gamma_{(7,2,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(4)} \\ \hline \Gamma_{(6)} \oplus \Gamma_{(5,1)} \\ \hline \Gamma_{(7,1)} \oplus \Gamma_{(6,1,1)} \\ \hline \Gamma_{(7,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(5,3)} \\ \hline \Gamma_{(6,4)} \end{array}$$

$$\begin{array}{c} \Gamma_{(5,1)} \oplus \Gamma_{(4,2)} \\ \hline \Gamma_{(6,2)} \oplus \Gamma_{(5,3)} \oplus \Gamma_{(5,2,1)} \\ \hline \Gamma_{(6,3,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(4)} \\ \hline \Gamma_{(5,1)} \oplus \Gamma_{(4,1,1)} \\ \hline \Gamma_{(6,1,1)} \oplus \Gamma_{(5,2,1)} \\ \hline \Gamma_{(6,2,2)} \end{array}$$

$$\begin{array}{c} \Gamma_{(4)} \oplus \Gamma_{(3,1)} \\ \hline \Gamma_{(6)} \oplus 2\Gamma_{(5,1)} \oplus \Gamma_{(4,2)} \oplus \Gamma_{(4,1,1)} \\ \hline \Gamma_{(6,2)} \oplus \Gamma_{(6,1,1)} \oplus \Gamma_{(5,2,1)} \oplus \Gamma_{(5,1,1,1)} \\ \hline \Gamma_{(6,2,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2)} \\ \hline \Gamma_{(4)} \oplus \Gamma_{(3,1)} \\ \hline \Gamma_{(5,1)} \oplus \Gamma_{(4,1,1)} \\ \hline \Gamma_{(6,1,1)} \oplus \Gamma_{(5,1,1,1)} \\ \hline \Gamma_{(6,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(4,4)} \\ \hline \Gamma_{(5,5)} \end{array}$$

$$\begin{array}{c} \Gamma_{(4,2)} \oplus \Gamma_{(3,3)} \\ \hline \Gamma_{(5,3)} \oplus \Gamma_{(4,4)} \oplus \Gamma_{(4,3,1)} \\ \hline \Gamma_{(5,4,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(3,1)} \\ \hline \Gamma_{(4,2)} \oplus \Gamma_{(4,1,1)} \oplus \Gamma_{(3,2,1)} \\ \hline \Gamma_{(5,2,1)} \oplus \Gamma_{(4,3,1)} \oplus \Gamma_{(4,2,2)} \\ \hline \Gamma_{(5,3,2)} \end{array}$$

$$\begin{array}{c} \Gamma_{(4)} \oplus \Gamma_{(3,1)} \oplus \Gamma_{(2,2)} \\ \hline \Gamma_{(5,1)} \oplus 2\Gamma_{(4,2)} \oplus \Gamma_{(4,1,1)} \oplus \Gamma_{(3,3)} \oplus \Gamma_{(3,2,1)} \\ \hline \Gamma_{(5,3)} \oplus \Gamma_{(5,2,1)} \oplus \Gamma_{(4,3,1)} \oplus \Gamma_{(4,2,1,1)} \\ \hline \Gamma_{(5,3,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2)} \\ \hline 2\Gamma_{(3,1)} \oplus \Gamma_{(2,1,1)} \oplus \Gamma_{(4)} \\ \hline \Gamma_{(5,1)} \oplus \Gamma_{(4,2)} \oplus 2\Gamma_{(4,1,1)} \oplus \Gamma_{(3,2,1)} \oplus \Gamma_{(3,1,1,1)} \\ \hline \Gamma_{(5,2,1)} \oplus \Gamma_{(5,1,1,1)} \oplus \Gamma_{(4,2,2)} \oplus \Gamma_{(4,2,1,1)} \\ \hline \Gamma_{(5,2,2,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2)} \oplus \Gamma_{(1,1)} \\ \hline \Gamma_{(4)} \oplus 2\Gamma_{(3,1)} \oplus \Gamma_{(2,2)} \oplus \Gamma_{(2,1,1)} \\ \hline \Gamma_{(5,1)} \oplus \Gamma_{(4,2)} \oplus 2\Gamma_{(4,1,1)} \oplus \Gamma_{(3,2,1)} \oplus \Gamma_{(3,1,1,1)} \\ \hline \Gamma_{(5,2,1)} \oplus \Gamma_{(5,1,1,1)} \oplus \Gamma_{(4,2,1,1)} \oplus \Gamma_{(4,1,1,1,1)} \\ \hline \Gamma_{(5,2,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{\emptyset} \\ \hline \Gamma_{(2)} \oplus \Gamma_{(1,1)} \\ \hline \Gamma_{(3,1)} \oplus \Gamma_{(2,1,1)} \\ \hline \Gamma_{(4,1,1)} \oplus \Gamma_{(3,1,1,1)} \\ \hline \Gamma_{(5,1,1,1)} \oplus \Gamma_{(4,1,1,1,1)} \\ \hline \Gamma_{(5,1,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2,2)} \\ \hline \Gamma_{(3,3)} \oplus \Gamma_{(3,2,1)} \\ \hline \Gamma_{(4,3,1)} \oplus \Gamma_{(3,3,2)} \\ \hline \Gamma_{(4,4,2)} \end{array}$$

$$\begin{array}{c} \Gamma_{(3,1)} \\ \hline \Gamma_{(4,2)} \oplus \Gamma_{(3,3)} \oplus \Gamma_{(3,2,1)} \\ \hline \Gamma_{(4,4)} \oplus \Gamma_{(4,3,1)} \oplus \Gamma_{(3,3,1,1)} \\ \hline \Gamma_{(4,4,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2,1,1)} \\ \hline \Gamma_{(3,2,1)} \oplus \Gamma_{(2,2,2)} \\ \hline \Gamma_{(4,2,2)} \oplus \Gamma_{(3,3,2)} \\ \hline \Gamma_{(4,3,3)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2)} \oplus \Gamma_{(1,1)} \\ \hline 2\Gamma_{(3,1)} \oplus 2\Gamma_{(2,2)} \oplus 2\Gamma_{(2,1,1)} \\ \hline \Gamma_{(4,2)} \oplus \Gamma_{(4,1,1)} \oplus \Gamma_{(3,3)} \oplus 3\Gamma_{(3,2,1)} \oplus \Gamma_{(3,1,1,1)} \oplus \Gamma_{(2,2,2)} \oplus \Gamma_{(2,2,1,1)} \\ \hline \Gamma_{(4,3,1)} \oplus \Gamma_{(4,2,2)} \oplus \Gamma_{(4,2,1,1)} \oplus \Gamma_{(3,3,2)} \oplus \Gamma_{(3,3,1,1)} \oplus \Gamma_{(3,2,2,1)} \\ \hline \Gamma_{(4,3,2,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2)} \\ \hline \Gamma_{(4)} \oplus 2\Gamma_{(3,1)} \oplus \Gamma_{(2,2)} \oplus \Gamma_{(2,1,1)} \\ \hline \Gamma_{(4,2)} \oplus \Gamma_{(4,1,1)} \oplus \Gamma_{(3,3)} \oplus 2\Gamma_{(3,2,1)} \oplus \Gamma_{(3,1,1,1)} \oplus \Gamma_{(2,2,1,1)} \\ \hline \Gamma_{(4,3,1)} \oplus \Gamma_{(4,2,1,1)} \oplus \Gamma_{(3,3,1,1)} \oplus \Gamma_{(3,2,1,1,1)} \\ \hline \Gamma_{(4,3,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2)} \oplus \Gamma_{(1,1)} \\ \hline \Gamma_{(2,1,1)} \oplus \Gamma_{(1,1,1,1)} \oplus \Gamma_{(4)} \oplus \Gamma_{(3,1)} \oplus \Gamma_{(2,2)} \\ \hline \Gamma_{(4,1,1)} \oplus \Gamma_{(3,2,1)} \oplus \Gamma_{(3,1,1,1)} \oplus \Gamma_{(2,2,1,1)} \\ \hline \Gamma_{(4,2,1,1)} \oplus \Gamma_{(3,2,2,1)} \\ \hline \Gamma_{(4,2,2,2)} \end{array}$$

$$\begin{array}{c} \Gamma_{\emptyset} \\ \hline \Gamma_{(1,1)} \oplus \Gamma_{(2)} \oplus \Gamma_{(1,1)} \\ \hline 2\Gamma_{(3,1)} \oplus \Gamma_{(2,2)} \oplus 3\Gamma_{(2,1,1)} \oplus \Gamma_{(1,1,1,1)} \\ \hline \Gamma_{(4,2)} \oplus \Gamma_{(4,1,1)} \oplus \Gamma_{(3,2,1)} \oplus \Gamma_{(3,2,1)} \oplus 2\Gamma_{(3,1,1,1)} \oplus \Gamma_{(2,2,2)} \oplus \Gamma_{(2,2,1,1)} \oplus \Gamma_{(2,1,1,1,1)} \\ \hline \Gamma_{(4,2,2)} \oplus \Gamma_{(4,2,1,1)} \oplus \Gamma_{(4,1,1,1,1)} \oplus \Gamma_{(3,2,2,1)} \oplus \Gamma_{(3,2,1,1,1)} \\ \hline \Gamma_{(4,2,2,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2)} \oplus \Gamma_{(1,1)} \\ \hline \Gamma_{(3,1)} \oplus \Gamma_{(2,2)} \oplus 2\Gamma_{(2,1,1)} \oplus \Gamma_{(1,1,1,1)} \\ \hline \Gamma_{(4,1,1)} \oplus \Gamma_{(3,2,1)} \oplus 2\Gamma_{(3,1,1,1)} \oplus \Gamma_{(2,2,1,1)} \oplus \Gamma_{(2,1,1,1,1)} \\ \hline \Gamma_{(4,2,1,1)} \oplus \Gamma_{(4,1,1,1,1)} \oplus \Gamma_{(3,2,1,1,1)} \oplus \Gamma_{(3,1,1,1,1,1)} \\ \hline \Gamma_{(4,2,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(1,1)} \\ \hline \Gamma_{(2,1,1)} \oplus \Gamma_{(1,1,1,1)} \\ \hline \Gamma_{(3,1,1,1)} \oplus \Gamma_{(2,1,1,1,1)} \\ \hline \Gamma_{(4,1,1,1,1)} \oplus \Gamma_{(3,1,1,1,1,1)} \\ \hline \Gamma_{(4,1,1,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(1,1)} \\ \hline \Gamma_{(2,2)} \oplus \Gamma_{(2,1,1)} \oplus \Gamma_{(1,1,1,1)} \\ \hline \Gamma_{(3,2,1)} \oplus \Gamma_{(2,2,2)} \oplus \Gamma_{(2,2,1,1)} \\ \hline \Gamma_{(3,3,2)} \oplus \Gamma_{(3,2,2,1)} \\ \hline \Gamma_{(3,3,3,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{\emptyset} \\ \hline \Gamma_{(2)} \oplus \Gamma_{(1,1)} \\ \hline \Gamma_{(2,1,1)} \oplus \Gamma_{(3,1)} \oplus \Gamma_{(2,1,1)} \\ \hline \Gamma_{(3,2,1)} \oplus \Gamma_{(3,1,1,1)} \oplus \Gamma_{(2,2,2)} \oplus \Gamma_{(2,2,1,1)} \\ \hline \Gamma_{(3,3,1,1)} \oplus \Gamma_{(3,2,2,1)} \oplus \Gamma_{(2,2,2,2)} \\ \hline \Gamma_{(3,3,2,2)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2)} \oplus \Gamma_{(1,1)} \\ \hline \Gamma_{(3,1)} \oplus 2\Gamma_{(2,2)} \oplus 2\Gamma_{(2,1,1)} \oplus \Gamma_{(1,1,1,1)} \\ \hline \Gamma_{(3,3)} \oplus 2\Gamma_{(3,2,1)} \oplus \Gamma_{(3,1,1,1)} \oplus \Gamma_{(2,2,2)} \oplus 2\Gamma_{(2,2,1,1)} \oplus \Gamma_{(2,1,1,1,1)} \\ \hline \Gamma_{(3,3,2)} \oplus \Gamma_{(3,3,1,1)} \oplus \Gamma_{(3,2,2,1)} \oplus \Gamma_{(3,2,1,1,1)} \oplus \Gamma_{(2,2,2,1,1)} \\ \hline \Gamma_{(3,3,2,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(3,1)} \oplus \Gamma_{(2,1,1)} \\ \hline \Gamma_{(3,2,1)} \oplus \Gamma_{(3,1,1,1)} \oplus \Gamma_{(2,2,1,1)} \oplus \Gamma_{(2,1,1,1,1)} \\ \hline \Gamma_{(3,3,1,1)} \oplus \Gamma_{(3,2,1,1,1)} \oplus \Gamma_{(2,2,1,1,1,1)} \\ \hline \Gamma_{(3,3,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2)} \oplus \Gamma_{(1,1)} \\ \hline 2\Gamma_{(2,1,1)} \oplus \Gamma_{(1,1,1,1)} \oplus \Gamma_{(3,1)} \oplus \Gamma_{(2,2)} \\ \hline \Gamma_{(3,2,1)} \oplus \Gamma_{(3,1,1,1)} \oplus \Gamma_{(2,2,2)} \oplus 2\Gamma_{(2,2,1,1)} \oplus \Gamma_{(2,1,1,1,1)} \\ \hline \Gamma_{(3,2,2,1)} \oplus \Gamma_{(3,2,1,1,1)} \oplus \Gamma_{(2,2,2,2)} \oplus \Gamma_{(2,2,2,1,1)} \\ \hline \Gamma_{(3,2,2,2,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(1,1)} \\ \hline \Gamma_{(2,2)} \oplus 2\Gamma_{(2,1,1)} \oplus 2\Gamma_{(1,1,1,1)} \\ \hline \Gamma_{(3,2,1)} \oplus \Gamma_{(3,1,1,1)} \oplus \Gamma_{(2,2,2)} \oplus 2\Gamma_{(2,2,1,1)} \oplus 2\Gamma_{(2,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1)} \\ \hline \Gamma_{(3,2,2,1)} \oplus \Gamma_{(3,2,1,1,1)} \oplus \Gamma_{(3,1,1,1,1,1)} \oplus \Gamma_{(2,2,2,1,1)} \oplus \Gamma_{(2,2,1,1,1,1)} \\ \hline \Gamma_{(3,2,2,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2,1,1)} \oplus \Gamma_{(1,1,1,1)} \\ \hline \Gamma_{(3,1,1,1)} \oplus \Gamma_{(2,2,1,1)} \oplus 2\Gamma_{(2,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1)} \\ \hline \Gamma_{(3,2,1,1,1)} \oplus \Gamma_{(3,1,1,1,1,1)} \oplus \Gamma_{(2,2,1,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1,1)} \\ \hline \Gamma_{(3,2,1,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(1,1,1,1)} \\ \hline \Gamma_{(2,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1)} \\ \hline \Gamma_{(3,1,1,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1,1)} \\ \hline \Gamma_{(3,1,1,1,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2,2)} \\ \hline \Gamma_{(2,2,1,1)} \\ \hline \Gamma_{(2,2,2,1,1)} \\ \hline \Gamma_{(2,2,2,2,2)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2,1,1)} \\ \hline \Gamma_{(2,2,2)} \oplus \Gamma_{(2,2,1,1)} \oplus \Gamma_{(2,1,1,1,1)} \\ \hline \Gamma_{(2,2,2,2)} \oplus \Gamma_{(2,2,2,1,1)} \oplus \Gamma_{(2,2,1,1,1,1)} \\ \hline \Gamma_{(2,2,2,2,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(1,1,1,1)} \\ \hline \Gamma_{(2,2,1,1)} \oplus \Gamma_{(2,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1)} \\ \hline \Gamma_{(2,2,2,1,1)} \oplus \Gamma_{(2,2,1,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1,1)} \\ \hline \Gamma_{(2,2,2,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1)} \\ \hline \Gamma_{(2,2,1,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1,1,1)} \\ \hline \Gamma_{(2,2,1,1,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(1,1,1,1,1,1)} \\ \hline \Gamma_{(2,1,1,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1,1,1)} \\ \hline \Gamma_{(2,1,1,1,1,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(1,1,1,1,1,1,1,1)} \\ \hline \Gamma_{(1,1,1,1,1,1,1,1,1,1)} \end{array}$$

Minimal injective resolutions of $\Gamma_{<\lambda>}$ with $|\lambda| = 11$:

$$\Gamma_{(11)}$$

$$\begin{array}{c} \Gamma_{(9)} \\ \hline \Gamma_{(10,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(8,1)} \\ \hline \Gamma_{(9,2)} \end{array}$$

$$\begin{array}{c} \Gamma_{(7)} \\ \hline \Gamma_{(9)} \oplus \Gamma_{(8,1)} \\ \hline \Gamma_{(9,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(7,2)} \\ \hline \Gamma_{(8,3)} \end{array}$$

$$\begin{array}{c} \Gamma_{(7)} \oplus \Gamma_{(6,1)} \\ \hline \Gamma_{(8,1)} \oplus \Gamma_{(7,2)} \oplus \Gamma_{(7,1,1)} \\ \hline \Gamma_{(8,2,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(5)} \\ \hline \Gamma_{(7)} \oplus \Gamma_{(6,1)} \\ \hline \Gamma_{(8,1)} \oplus \Gamma_{(7,1,1)} \\ \hline \Gamma_{(8,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(6,3)} \\ \hline \Gamma_{(7,4)} \end{array}$$

$$\begin{array}{c} \Gamma_{(6,1)} \oplus \Gamma_{(5,2)} \\ \hline \Gamma_{(7,2)} \oplus \Gamma_{(6,3)} \oplus \Gamma_{(6,2,1)} \\ \hline \Gamma_{(7,3,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(5)} \\ \hline \Gamma_{(6,1)} \oplus \Gamma_{(5,1,1)} \\ \hline \Gamma_{(7,1,1)} \oplus \Gamma_{(6,2,1)} \\ \hline \Gamma_{(7,2,2)} \end{array}$$

$$\begin{array}{c} \Gamma_{(5)} \oplus \Gamma_{(4,1)} \\ \hline \Gamma_{(7)} \oplus 2\Gamma_{(6,1)} \oplus \Gamma_{(5,2)} \oplus \Gamma_{(5,1,1)} \\ \hline \Gamma_{(7,2)} \oplus \Gamma_{(7,1,1)} \oplus \Gamma_{(6,2,1)} \oplus \Gamma_{(6,1,1,1)} \\ \hline \Gamma_{(7,2,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(3)} \\ \hline \Gamma_{(5)} \oplus \Gamma_{(4,1)} \\ \hline \Gamma_{(6,1)} \oplus \Gamma_{(5,1,1)} \\ \hline \Gamma_{(7,1,1)} \oplus \Gamma_{(6,1,1,1)} \\ \hline \Gamma_{(7,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(5,4)} \\ \hline \Gamma_{(6,5)} \end{array}$$

$$\begin{array}{c} \Gamma_{(5,2)}\oplus \Gamma_{(4,3)} \\ \hline \Gamma_{(6,3)}\oplus \Gamma_{(5,4)}\oplus \Gamma_{(5,3,1)} \\ \hline \Gamma_{(6,4,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(4,1)} \\ \hline \Gamma_{(5,2)}\oplus \Gamma_{(5,1,1)}\oplus \Gamma_{(4,2,1)} \\ \hline \Gamma_{(6,2,1)}\oplus \Gamma_{(5,3,1)}\oplus \Gamma_{(5,2,2)} \\ \hline \Gamma_{(6,3,2)} \end{array}$$

$$\begin{array}{c} \Gamma_{(5)}\oplus \Gamma_{(4,1)}\oplus \Gamma_{(3,2)} \\ \hline \Gamma_{(6,1)}\oplus 2\Gamma_{(5,2)}\oplus \Gamma_{(5,1,1)}\oplus \Gamma_{(4,3)}\oplus \Gamma_{(4,2,1)} \\ \hline \Gamma_{(6,3)}\oplus \Gamma_{(6,2,1)}\oplus \Gamma_{(5,3,1)}\oplus \Gamma_{(5,2,1,1)} \\ \hline \Gamma_{(6,3,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(3)} \\ \hline 2\Gamma_{(4,1)}\oplus \Gamma_{(3,1,1)}\oplus \Gamma_{(5)} \\ \hline \Gamma_{(6,1)}\oplus \Gamma_{(5,2)}\oplus 2\Gamma_{(5,1,1)}\oplus \Gamma_{(4,2,1)}\oplus \Gamma_{(4,1,1,1)} \\ \hline \Gamma_{(6,2,1)}\oplus \Gamma_{(6,1,1,1)}\oplus \Gamma_{(5,2,2)}\oplus \Gamma_{(5,2,1,1)} \\ \hline \Gamma_{(6,2,2,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(3)}\oplus \Gamma_{(2,1)} \\ \hline \Gamma_{(5)}\oplus 2\Gamma_{(4,1)}\oplus \Gamma_{(3,2)}\oplus \Gamma_{(3,1,1)} \\ \hline \Gamma_{(6,1)}\oplus \Gamma_{(5,2)}\oplus 2\Gamma_{(5,1,1)}\oplus \Gamma_{(4,2,1)}\oplus \Gamma_{(4,1,1,1)} \\ \hline \Gamma_{(6,2,1)}\oplus \Gamma_{(6,1,1,1)}\oplus \Gamma_{(5,2,1,1)}\oplus \Gamma_{(5,1,1,1,1)} \\ \hline \Gamma_{(6,2,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(1)} \\ \hline \Gamma_{(3)}\oplus \Gamma_{(2,1)} \\ \hline \Gamma_{(4,1)}\oplus \Gamma_{(3,1,1)} \\ \hline \Gamma_{(5,1,1)}\oplus \Gamma_{(4,1,1,1)} \\ \hline \Gamma_{(6,1,1,1)}\oplus \Gamma_{(5,1,1,1,1)} \\ \hline \Gamma_{(6,1,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(4,3)} \\ \hline \Gamma_{(5,4)}\oplus \Gamma_{(4,4,1)} \\ \hline \Gamma_{(5,5,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(3,2)} \\ \hline \Gamma_{(4,3)}\oplus \Gamma_{(4,2,1)}\oplus \Gamma_{(3,3,1)} \\ \hline \Gamma_{(5,3,1)}\oplus \Gamma_{(4,4,1)}\oplus \Gamma_{(4,3,2)} \\ \hline \Gamma_{(5,4,2)} \end{array}$$

$$\begin{array}{c} \Gamma_{(4,1)}\oplus \Gamma_{(3,2)} \\ \hline \Gamma_{(5,2)}\oplus 2\Gamma_{(4,3)}\oplus \Gamma_{(4,2,1)}\oplus \Gamma_{(3,3,1)} \\ \hline \Gamma_{(5,4)}\oplus \Gamma_{(5,3,1)}\oplus \Gamma_{(4,4,1)}\oplus \Gamma_{(4,3,1,1)} \\ \hline \Gamma_{(5,4,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(3,1,1)} \\ \hline \Gamma_{(4,2,1)}\oplus \Gamma_{(3,2,2)} \\ \hline \Gamma_{(5,2,2)}\oplus \Gamma_{(4,3,2)} \\ \hline \Gamma_{(5,3,3)} \end{array}$$

$$\begin{array}{c} \Gamma_{(3)}\oplus \Gamma_{(2,1)} \\ \hline 2\Gamma_{(4,1)}\oplus 2\Gamma_{(3,2)}\oplus 2\Gamma_{(3,1,1)} \\ \hline \Gamma_{(5,2)}\oplus \Gamma_{(5,1,1)}\oplus \Gamma_{(4,3)}\oplus 3\Gamma_{(4,2,1)}\oplus \Gamma_{(4,1,1,1)}\oplus \Gamma_{(3,3,1)}\oplus \Gamma_{(3,2,2)}\oplus \Gamma_{(3,2,1,1)} \\ \hline \Gamma_{(5,3,1)}\oplus \Gamma_{(5,2,2)}\oplus \Gamma_{(5,2,1,1)}\oplus \Gamma_{(4,3,2)}\oplus \Gamma_{(4,3,1,1)}\oplus \Gamma_{(4,2,2,1)} \\ \hline \Gamma_{(5,3,2,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(3)}\oplus \Gamma_{(2,1)} \\ \hline \Gamma_{(5)}\oplus 2\Gamma_{(4,1)}\oplus 2\Gamma_{(3,2)}\oplus \Gamma_{(3,1,1)}\oplus \Gamma_{(2,2,1)} \\ \hline \Gamma_{(5,2)}\oplus \Gamma_{(5,1,1)}\oplus \Gamma_{(4,3)}\oplus 2\Gamma_{(4,2,1)}\oplus \Gamma_{(4,1,1,1)}\oplus \Gamma_{(3,3,1)}\oplus \Gamma_{(3,2,1,1)} \\ \hline \Gamma_{(5,3,1)}\oplus \Gamma_{(5,2,1,1)}\oplus \Gamma_{(4,3,1,1)}\oplus \Gamma_{(4,2,1,1,1)} \\ \hline \Gamma_{(5,3,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(3)}\oplus \Gamma_{(2,1)} \\ \hline \Gamma_{(3,1,1)}\oplus \Gamma_{(2,1,1,1)}\oplus \Gamma_{(5)}\oplus \Gamma_{(4,1)}\oplus \Gamma_{(3,2)} \\ \hline \Gamma_{(5,1,1)}\oplus \Gamma_{(4,2,1)}\oplus \Gamma_{(4,1,1,1)}\oplus \Gamma_{(3,2,1,1)} \\ \hline \Gamma_{(5,2,1,1)}\oplus \Gamma_{(4,2,2,1)} \\ \hline \Gamma_{(5,2,2,2)} \end{array}$$

$$\begin{array}{c} \Gamma_{(1)} \\ \hline 2\Gamma_{(2,1)}\oplus \Gamma_{(1,1,1)}\oplus \Gamma_{(3)} \\ \hline 2\Gamma_{(4,1)}\oplus \Gamma_{(3,2)}\oplus 3\Gamma_{(3,1,1)}\oplus \Gamma_{(2,2,1)}\oplus \Gamma_{(2,1,1,1)} \\ \hline \Gamma_{(5,2)}\oplus \Gamma_{(5,1,1)}\oplus 2\Gamma_{(4,2,1)}\oplus 2\Gamma_{(4,1,1,1)}\oplus \Gamma_{(3,2,2)}\oplus \Gamma_{(3,2,1,1)}\oplus \Gamma_{(3,1,1,1,1)} \\ \hline \Gamma_{(5,2,2)}\oplus \Gamma_{(5,2,1,1)}\oplus \Gamma_{(5,1,1,1,1)}\oplus \Gamma_{(4,2,2,1)}\oplus \Gamma_{(4,2,1,1,1)} \\ \hline \Gamma_{(5,2,2,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(1)} \\ \hline \Gamma_{(3)}\oplus 2\Gamma_{(2,1)}\oplus \Gamma_{(1,1,1)} \\ \hline \Gamma_{(4,1)}\oplus \Gamma_{(3,2)}\oplus 2\Gamma_{(3,1,1)}\oplus \Gamma_{(2,2,1)}\oplus \Gamma_{(2,1,1,1)} \\ \hline \Gamma_{(5,1,1)}\oplus \Gamma_{(4,2,1)}\oplus 2\Gamma_{(4,1,1,1)}\oplus \Gamma_{(3,2,1,1)}\oplus \Gamma_{(3,1,1,1,1)} \\ \hline \Gamma_{(5,2,1,1)}\oplus \Gamma_{(5,1,1,1,1)}\oplus \Gamma_{(4,2,1,1,1)}\oplus \Gamma_{(4,1,1,1,1,1)} \\ \hline \Gamma_{(5,2,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(1)} \\ \hline \Gamma_{(2,1)}\oplus \Gamma_{(1,1,1)} \\ \hline \Gamma_{(3,1,1)}\oplus \Gamma_{(2,1,1,1)} \\ \hline \Gamma_{(4,1,1,1)}\oplus \Gamma_{(3,1,1,1,1)} \\ \hline \Gamma_{(5,1,1,1,1)}\oplus \Gamma_{(4,1,1,1,1,1)} \\ \hline \Gamma_{(5,1,1,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2,2,1)} \\ \hline \Gamma_{(3,3,1)}\oplus \Gamma_{(3,2,2)} \\ \hline \Gamma_{(4,3,2)}\oplus \Gamma_{(3,3,3)} \\ \hline \Gamma_{(4,4,3)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2,1)} \\ \hline 2\Gamma_{(3,2)}\oplus \Gamma_{(3,1,1)}\oplus \Gamma_{(2,2,1)} \\ \hline \Gamma_{(4,3)}\oplus \Gamma_{(4,2,1)}\oplus 2\Gamma_{(3,3,1)}\oplus \Gamma_{(3,2,2)}\oplus \Gamma_{(3,2,1,1)} \\ \hline \Gamma_{(4,4,1)}\oplus \Gamma_{(4,3,2)}\oplus \Gamma_{(4,3,1,1)}\oplus \Gamma_{(3,3,2,1)} \\ \hline \Gamma_{(4,4,2,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(3)} \\ \hline \Gamma_{(4,1)}\oplus \Gamma_{(3,2)}\oplus \Gamma_{(3,1,1)} \\ \hline \Gamma_{(4,3)}\oplus \Gamma_{(4,2,1)}\oplus \Gamma_{(3,3,1)}\oplus \Gamma_{(3,2,1,1)} \\ \hline \Gamma_{(4,4,1)}\oplus \Gamma_{(4,3,1,1)}\oplus \Gamma_{(3,3,1,1,1)} \\ \hline \Gamma_{(4,4,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2,1)}\oplus \Gamma_{(1,1,1)} \\ \hline \Gamma_{(3,2)}\oplus \Gamma_{(3,1,1)}\oplus 2\Gamma_{(2,2,1)}\oplus \Gamma_{(2,1,1,1)} \\ \hline \Gamma_{(4,2,1)}\oplus \Gamma_{(3,3,1)}\oplus 2\Gamma_{(3,2,2)}\oplus \Gamma_{(3,2,1,1)}\oplus \Gamma_{(2,2,2,1)} \\ \hline \Gamma_{(4,3,2)}\oplus \Gamma_{(4,2,2,1)}\oplus \Gamma_{(3,3,3)}\oplus \Gamma_{(3,3,2,1)} \\ \hline \Gamma_{(4,3,3,1)} \end{array}$$

$$\begin{array}{c}
\Gamma_{(1)} \\ \hline
\Gamma_{(3)} \oplus 2\Gamma_{(2,1)} \oplus \Gamma_{(1,1,1)} \\ \hline
2\Gamma_{(3,1,1)} \oplus 2\Gamma_{(2,2,1)} \oplus \Gamma_{(2,1,1,1)} \oplus \Gamma_{(4,1)} \oplus \Gamma_{(3,2)} \\ \hline
\Gamma_{(4,2,1)} \oplus \Gamma_{(4,1,1,1)} \oplus \Gamma_{(3,3,1)} \oplus \Gamma_{(3,2,2)} \oplus \Gamma_{(3,2,1,1)} \oplus \Gamma_{(3,2,1,1)} \oplus \Gamma_{(2,2,2,1)} \\ \hline
\Gamma_{(4,3,1,1)} \oplus \Gamma_{(4,2,2,1)} \oplus \Gamma_{(3,3,2,1)} \oplus \Gamma_{(3,2,2,2)} \\ \hline
\Gamma_{(4,3,2,2)}
\end{array}$$

$$\begin{array}{c}
\Gamma_{(1)} \\ \hline
3\Gamma_{(2,1)} \oplus \Gamma_{(3)} \oplus \Gamma_{(1,1,1)} \\ \hline
\Gamma_{(4,1)} \oplus 3\Gamma_{(3,2)} \oplus 3\Gamma_{(3,1,1)} \oplus 3\Gamma_{(2,2,1)} \oplus 2\Gamma_{(2,1,1,1)} \\ \hline
\Gamma_{(4,3)} \oplus 2\Gamma_{(4,2,1)} \oplus \Gamma_{(4,1,1,1)} \oplus 2\Gamma_{(3,3,1)} \oplus 2\Gamma_{(3,2,2)} \oplus 3\Gamma_{(3,2,1,1)} \oplus \Gamma_{(3,1,1,1,1)} \oplus \Gamma_{(2,2,2,1)} \oplus \Gamma_{(2,2,1,1,1)} \\ \hline
\Gamma_{(4,3,2)} \oplus \Gamma_{(4,3,1,1)} \oplus \Gamma_{(4,2,2,1)} \oplus \Gamma_{(4,2,1,1,1)} \oplus \Gamma_{(3,3,2,1)} \oplus \Gamma_{(3,3,1,1,1)} \oplus \Gamma_{(3,2,2,1,1)} \\ \hline
\Gamma_{(4,3,2,1,1)}
\end{array}$$

$$\begin{array}{c}
\Gamma_{(3)} \oplus \Gamma_{(2,1)} \\ \hline
\Gamma_{(4,1)} \oplus \Gamma_{(3,2)} \oplus 2\Gamma_{(3,1,1)} \oplus \Gamma_{(2,2,1)} \oplus \Gamma_{(2,1,1,1)} \\ \hline
\Gamma_{(4,2,1)} \oplus \Gamma_{(4,1,1,1)} \oplus \Gamma_{(3,3,1)} \oplus 2\Gamma_{(3,2,1,1)} \oplus \Gamma_{(3,1,1,1,1)} \oplus \Gamma_{(2,2,1,1,1)} \\ \hline
\Gamma_{(4,3,1,1)} \oplus \Gamma_{(4,2,1,1,1)} \oplus \Gamma_{(3,3,1,1,1)} \oplus \Gamma_{(3,2,1,1,1,1)} \\ \hline
\Gamma_{(4,3,1,1,1,1)}
\end{array}$$

$$\begin{array}{c}
\Gamma_{(1)} \\ \hline
\Gamma_{(3)} \oplus 2\Gamma_{(2,1)} \oplus 2\Gamma_{(1,1,1)} \\ \hline
2\Gamma_{(3,1,1)} \oplus 2\Gamma_{(2,2,1)} \oplus 2\Gamma_{(2,1,1,1)} \oplus \Gamma_{(1,1,1,1,1)} \oplus \Gamma_{(4,1)} \oplus \Gamma_{(3,2)} \\ \hline
\Gamma_{(4,2,1)} \oplus \Gamma_{(4,1,1,1)} \oplus \Gamma_{(3,2,2)} \oplus 2\Gamma_{(3,2,1,1)} \oplus \Gamma_{(3,1,1,1,1)} \oplus \Gamma_{(2,2,2,1)} \oplus \Gamma_{(2,2,1,1,1)} \\ \hline
\Gamma_{(4,2,2,1)} \oplus \Gamma_{(4,2,1,1,1)} \oplus \Gamma_{(3,2,2,2)} \oplus \Gamma_{(3,2,2,1,1)} \\ \hline
\Gamma_{(4,2,2,2,1)}
\end{array}$$

$$\begin{array}{c}
\Gamma_{(1)} \\ \hline
2\Gamma_{(2,1)} \oplus 2\Gamma_{(1,1,1)} \\ \hline
\Gamma_{(3,2)} \oplus 2\Gamma_{(3,1,1)} \oplus 2\Gamma_{(2,2,1)} \oplus 3\Gamma_{(2,1,1,1)} \oplus \Gamma_{(1,1,1,1,1)} \\ \hline
\Gamma_{(4,2,1)} \oplus \Gamma_{(4,1,1,1)} \oplus \Gamma_{(3,2,2)} \oplus 2\Gamma_{(3,2,1,1)} \oplus 2\Gamma_{(3,1,1,1,1)} \oplus \Gamma_{(2,2,2,1)} \oplus \Gamma_{(2,2,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1)} \\ \hline
\Gamma_{(4,2,2,1)} \oplus \Gamma_{(4,2,1,1,1)} \oplus \Gamma_{(4,1,1,1,1,1)} \oplus \Gamma_{(3,2,2,1,1)} \oplus \Gamma_{(3,2,1,1,1,1)} \\ \hline
\Gamma_{(4,2,2,1,1,1)}
\end{array}$$

$$\begin{array}{c}
\Gamma_{(2,1)} \oplus \Gamma_{(1,1,1)} \\ \hline
\Gamma_{(3,1,1)} \oplus \Gamma_{(2,2,1)} \oplus 2\Gamma_{(2,1,1,1)} \oplus \Gamma_{(1,1,1,1,1)} \\ \hline
\Gamma_{(4,1,1,1)} \oplus \Gamma_{(3,2,1,1)} \oplus 2\Gamma_{(3,1,1,1,1)} \oplus \Gamma_{(2,2,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1)} \\ \hline
\Gamma_{(4,2,1,1,1)} \oplus \Gamma_{(4,1,1,1,1,1)} \oplus \Gamma_{(3,2,1,1,1,1)} \oplus \Gamma_{(3,1,1,1,1,1,1)} \\ \hline
\Gamma_{(4,2,1,1,1,1,1)}
\end{array}$$

$$\begin{array}{c}
\Gamma_{(1,1,1)} \\ \hline
\Gamma_{(2,1,1,1)} \oplus \Gamma_{(1,1,1,1,1)} \\ \hline
\Gamma_{(3,1,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1)} \\ \hline
\Gamma_{(4,1,1,1,1,1)} \oplus \Gamma_{(3,1,1,1,1,1,1)} \\ \hline
\Gamma_{(4,1,1,1,1,1,1,1)}
\end{array}$$

$$\begin{array}{c}
\Gamma_{(1)} \\ \hline
\Gamma_{(2,1)} \oplus \Gamma_{(1,1,1)} \\ \hline
\Gamma_{(2,2,1)} \oplus \Gamma_{(3,1,1)} \oplus \Gamma_{(2,1,1,1)} \\ \hline
\Gamma_{(3,2,2)} \oplus \Gamma_{(3,2,1,1)} \oplus \Gamma_{(2,2,2,1)} \\ \hline
\Gamma_{(3,3,2,1)} \oplus \Gamma_{(3,2,2,2)} \\ \hline
\Gamma_{(3,3,3,2)}
\end{array}$$

$$\begin{array}{c}
\Gamma_{(2,1)} \oplus \Gamma_{(1,1,1)} \\ \hline
\Gamma_{(3,2)} \oplus 2\Gamma_{(2,2,1)} \oplus \Gamma_{(2,1,1,1)} \oplus \Gamma_{(1,1,1,1,1)} \\ \hline
\Gamma_{(3,3,1)} \oplus \Gamma_{(3,2,2)} \oplus \Gamma_{(3,2,1,1)} \oplus \Gamma_{(2,2,2,1)} \oplus \Gamma_{(2,2,1,1,1)} \\ \hline
\Gamma_{(3,3,3)} \oplus \Gamma_{(3,3,2,1)} \oplus \Gamma_{(3,2,2,1,1)} \\ \hline
\Gamma_{(3,3,3,1,1)}
\end{array}$$

$$\begin{array}{c}
\Gamma_{(1)} \\ \hline
\Gamma_{(3)} \oplus 2\Gamma_{(2,1)} \oplus \Gamma_{(1,1,1)} \\ \hline
2\Gamma_{(3,1,1)} \oplus 2\Gamma_{(2,2,1)} \oplus 2\Gamma_{(2,1,1,1)} \oplus \Gamma_{(3,2)} \\ \hline
\Gamma_{(3,3,1)} \oplus \Gamma_{(3,2,2)} \oplus 2\Gamma_{(3,2,1,1)} \oplus \Gamma_{(3,1,1,1,1)} \oplus 2\Gamma_{(2,2,2,1)} \oplus \Gamma_{(2,2,1,1,1)} \\ \hline
\Gamma_{(3,3,2,1)} \oplus \Gamma_{(3,3,1,1,1)} \oplus \Gamma_{(3,2,2,2)} \oplus \Gamma_{(3,2,2,1,1)} \oplus \Gamma_{(2,2,2,2,1)} \\ \hline
\Gamma_{(3,3,2,2,1)}
\end{array}$$

$$\begin{array}{c} \Gamma_{(2,1)} \oplus \Gamma_{(1,1,1)} \\ \hline \Gamma_{(3,2)} \oplus \Gamma_{(3,1,1)} \oplus 2\Gamma_{(2,2,1)} \oplus 2\Gamma_{(2,1,1,1)} \oplus \Gamma_{(1,1,1,1,1)} \\ \hline \Gamma_{(3,3,1)} \oplus \Gamma_{(3,2,2)} \oplus 2\Gamma_{(3,2,1,1)} \oplus \Gamma_{(3,1,1,1,1)} \oplus \Gamma_{(2,2,2,1)} \oplus 2\Gamma_{(2,2,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1)} \\ \hline \Gamma_{(3,3,2,1)} \oplus \Gamma_{(3,3,1,1,1)} \oplus \Gamma_{(3,2,2,1,1)} \oplus \Gamma_{(3,2,1,1,1,1)} \oplus \Gamma_{(2,2,2,1,1,1)} \\ \hline \Gamma_{(3,3,2,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(3,1,1)} \oplus \Gamma_{(2,1,1,1)} \\ \hline \Gamma_{(3,2,1,1)} \oplus \Gamma_{(3,1,1,1,1)} \oplus \Gamma_{(2,2,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1)} \\ \hline \Gamma_{(3,3,1,1,1)} \oplus \Gamma_{(3,2,1,1,1,1)} \oplus \Gamma_{(2,2,1,1,1,1,1)} \\ \hline \Gamma_{(3,3,1,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2,1)} \\ \hline \Gamma_{(2,1,1,1)} \oplus \Gamma_{(3,2)} \oplus \Gamma_{(2,2,1)} \\ \hline \Gamma_{(3,2,1,1)} \oplus \Gamma_{(2,2,2,1)} \oplus \Gamma_{(2,2,1,1,1)} \\ \hline \Gamma_{(3,2,2,1,1)} \oplus \Gamma_{(2,2,2,2,1)} \\ \hline \Gamma_{(3,2,2,2,2)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2,1)} \oplus \Gamma_{(1,1,1)} \\ \hline 2\Gamma_{(2,2,1)} \oplus 2\Gamma_{(2,1,1,1)} \oplus \Gamma_{(1,1,1,1,1)} \oplus \Gamma_{(3,1,1)} \\ \hline \Gamma_{(3,2,2)} \oplus \Gamma_{(3,2,1,1)} \oplus \Gamma_{(3,1,1,1,1)} \oplus 2\Gamma_{(2,2,2,1)} \oplus 2\Gamma_{(2,2,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1)} \\ \hline \Gamma_{(3,2,2,2)} \oplus \Gamma_{(3,2,2,1,1)} \oplus \Gamma_{(3,2,1,1,1,1)} \oplus \Gamma_{(2,2,2,2,1)} \oplus \Gamma_{(2,2,2,1,1,1)} \\ \hline \Gamma_{(3,2,2,2,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(1,1,1)} \\ \hline \Gamma_{(2,2,1)} \oplus 2\Gamma_{(2,1,1,1)} \oplus 2\Gamma_{(1,1,1,1,1)} \\ \hline \Gamma_{(3,2,1,1)} \oplus \Gamma_{(3,1,1,1,1)} \oplus \Gamma_{(2,2,2,1)} \oplus 2\Gamma_{(2,2,1,1,1)} \oplus 2\Gamma_{(2,1,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1,1)} \\ \hline \Gamma_{(3,2,2,1,1)} \oplus \Gamma_{(3,2,1,1,1,1)} \oplus \Gamma_{(3,1,1,1,1,1,1)} \oplus \Gamma_{(2,2,2,1,1,1)} \oplus \Gamma_{(2,2,1,1,1,1,1)} \\ \hline \Gamma_{(3,2,2,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2,1,1,1)} \oplus \Gamma_{(1,1,1,1,1)} \\ \hline \Gamma_{(3,1,1,1,1)} \oplus \Gamma_{(2,2,1,1,1)} \oplus 2\Gamma_{(2,1,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1,1)} \\ \hline \Gamma_{(3,2,1,1,1,1)} \oplus \Gamma_{(3,1,1,1,1,1,1)} \oplus \Gamma_{(2,2,1,1,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1,1,1)} \\ \hline \Gamma_{(3,2,1,1,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(1,1,1,1,1)} \\ \hline \Gamma_{(2,1,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1,1)} \\ \hline \Gamma_{(3,1,1,1,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1,1,1)} \\ \hline \Gamma_{(3,1,1,1,1,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2,2,1)} \\ \hline \Gamma_{(2,2,2,1)} \oplus \Gamma_{(2,2,1,1,1)} \\ \hline \Gamma_{(2,2,2,2,1)} \oplus \Gamma_{(2,2,2,1,1,1)} \\ \hline \Gamma_{(2,2,2,2,2,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2,1,1,1)} \\ \hline \Gamma_{(2,2,2,1)} \oplus \Gamma_{(2,2,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1)} \\ \hline \Gamma_{(2,2,2,2,1)} \oplus \Gamma_{(2,2,2,1,1,1)} \oplus \Gamma_{(2,2,1,1,1,1,1)} \\ \hline \Gamma_{(2,2,2,2,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(1,1,1,1,1)} \\ \hline \Gamma_{(2,2,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1,1)} \\ \hline \Gamma_{(2,2,2,1,1,1)} \oplus \Gamma_{(2,2,1,1,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1,1,1)} \\ \hline \Gamma_{(2,2,2,1,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2,1,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1,1)} \\ \hline \Gamma_{(2,2,1,1,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1,1,1,1)} \\ \hline \Gamma_{(2,2,1,1,1,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(1,1,1,1,1,1,1)} \\ \hline \Gamma_{(2,1,1,1,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1,1,1,1)} \\ \hline \Gamma_{(2,1,1,1,1,1,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(1,1,1,1,1,1,1,1,1)} \\ \hline \Gamma_{(1,1,1,1,1,1,1,1,1,1,1)} \end{array}$$

Minimal injective resolutions of $\Gamma_{<\lambda>}$ with $|\lambda| = 12$:

$$\Gamma_{(12)}$$

$$\begin{array}{c} \Gamma_{(10)} \\ \hline \Gamma_{(11,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(9,1)} \\ \hline \Gamma_{(10,2)} \end{array}$$

$$\begin{array}{c} \Gamma_{(8)} \\ \hline \Gamma_{(10)} \oplus \Gamma_{(9,1)} \\ \hline \Gamma_{(10,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(8,2)} \\ \hline \Gamma_{(9,3)} \end{array}$$

$$\begin{array}{c} \Gamma_{(8)} \oplus \Gamma_{(7,1)} \\ \hline \Gamma_{(9,1)} \oplus \Gamma_{(8,2)} \oplus \Gamma_{(8,1,1)} \\ \hline \Gamma_{(9,2,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(6)} \\ \hline \Gamma_{(8)} \oplus \Gamma_{(7,1)} \\ \hline \Gamma_{(9,1)} \oplus \Gamma_{(8,1,1)} \\ \hline \Gamma_{(9,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(7,3)} \\ \hline \Gamma_{(8,4)} \end{array}$$

$$\begin{array}{c} \Gamma_{(7,1)} \oplus \Gamma_{(6,2)} \\ \hline \Gamma_{(8,2)} \oplus \Gamma_{(7,3)} \oplus \Gamma_{(7,2,1)} \\ \hline \Gamma_{(8,3,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(6)} \\ \hline \Gamma_{(7,1)} \oplus \Gamma_{(6,1,1)} \\ \hline \Gamma_{(8,1,1)} \oplus \Gamma_{(7,2,1)} \\ \hline \Gamma_{(8,2,2)} \end{array}$$

$$\begin{array}{c} \Gamma_{(6)} \oplus \Gamma_{(5,1)} \\ \hline \Gamma_{(8)} \oplus 2\Gamma_{(7,1)} \oplus \Gamma_{(6,2)} \oplus \Gamma_{(6,1,1)} \\ \hline \Gamma_{(8,2)} \oplus \Gamma_{(8,1,1)} \oplus \Gamma_{(7,2,1)} \oplus \Gamma_{(7,1,1,1)} \\ \hline \Gamma_{(8,2,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(4)} \\ \hline \Gamma_{(6)} \oplus \Gamma_{(5,1)} \\ \hline \Gamma_{(7,1)} \oplus \Gamma_{(6,1,1)} \\ \hline \Gamma_{(8,1,1)} \oplus \Gamma_{(7,1,1,1)} \\ \hline \Gamma_{(8,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(6,4)} \\ \hline \Gamma_{(7,5)} \end{array}$$

$$\begin{array}{c} \Gamma_{(6,2)} \oplus \Gamma_{(5,3)} \\ \hline \Gamma_{(7,3)} \oplus \Gamma_{(6,4)} \oplus \Gamma_{(6,3,1)} \\ \hline \Gamma_{(7,4,1)} \end{array} \qquad \begin{array}{c} \Gamma_{(5,1)} \\ \hline \Gamma_{(6,2)} \oplus \Gamma_{(6,1,1)} \oplus \Gamma_{(5,2,1)} \\ \hline \Gamma_{(7,2,1)} \oplus \Gamma_{(6,3,1)} \oplus \Gamma_{(6,2,2)} \\ \hline \Gamma_{(7,3,2)} \end{array} \qquad \begin{array}{c} \Gamma_{(6)} \oplus \Gamma_{(5,1)} \oplus \Gamma_{(4,2)} \\ \hline \Gamma_{(7,1)} \oplus 2\Gamma_{(6,2)} \oplus \Gamma_{(6,1,1)} \oplus \Gamma_{(5,3)} \oplus \Gamma_{(5,2,1)} \\ \hline \Gamma_{(7,3)} \oplus \Gamma_{(7,2,1)} \oplus \Gamma_{(6,3,1)} \oplus \Gamma_{(6,2,1,1)} \\ \hline \Gamma_{(7,3,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(4)} \\ \hline 2\Gamma_{(5,1)} \oplus \Gamma_{(4,1,1)} \oplus \Gamma_{(6)} \\ \hline \Gamma_{(7,1)} \oplus \Gamma_{(6,2)} \oplus 2\Gamma_{(6,1,1)} \oplus \Gamma_{(5,2,1)} \oplus \Gamma_{(5,1,1,1)} \\ \hline \Gamma_{(7,2,1)} \oplus \Gamma_{(7,1,1,1)} \oplus \Gamma_{(6,2,2)} \oplus \Gamma_{(6,2,1,1)} \\ \hline \Gamma_{(7,2,2,1)} \end{array} \qquad \begin{array}{c} \Gamma_{(4)} \oplus \Gamma_{(3,1)} \\ \hline \Gamma_{(6)} \oplus 2\Gamma_{(5,1)} \oplus \Gamma_{(4,2)} \oplus \Gamma_{(4,1,1)} \\ \hline \Gamma_{(7,1)} \oplus \Gamma_{(6,2)} \oplus 2\Gamma_{(6,1,1)} \oplus \Gamma_{(5,2,1)} \oplus \Gamma_{(5,1,1,1)} \\ \hline \Gamma_{(7,2,1)} \oplus \Gamma_{(7,1,1,1)} \oplus \Gamma_{(6,2,1,1)} \oplus \Gamma_{(6,1,1,1,1)} \\ \hline \Gamma_{(7,2,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2)} \\ \hline \Gamma_{(4)} \oplus \Gamma_{(3,1)} \\ \hline \Gamma_{(5,1)} \oplus \Gamma_{(4,1,1)} \\ \hline \Gamma_{(6,1,1)} \oplus \Gamma_{(5,1,1,1)} \\ \hline \Gamma_{(7,1,1,1)} \oplus \Gamma_{(6,1,1,1,1)} \\ \hline \Gamma_{(7,1,1,1,1,1)} \end{array} \qquad \begin{array}{c} \Gamma_{(5,5)} \\ \hline \Gamma_{(6,6)} \end{array} \qquad \begin{array}{c} \Gamma_{(5,3)} \oplus \Gamma_{(4,4)} \\ \hline \Gamma_{(6,4)} \oplus \Gamma_{(5,5)} \oplus \Gamma_{(5,4,1)} \\ \hline \Gamma_{(6,5,1)} \end{array} \qquad \begin{array}{c} \Gamma_{(4,2)} \\ \hline \Gamma_{(5,3)} \oplus \Gamma_{(5,2,1)} \oplus \Gamma_{(4,3,1)} \\ \hline \Gamma_{(6,3,1)} \oplus \Gamma_{(5,4,1)} \oplus \Gamma_{(5,3,2)} \\ \hline \Gamma_{(6,4,2)} \end{array}$$

$$\begin{array}{c} \Gamma_{(5,1)} \oplus \Gamma_{(4,2)} \oplus \Gamma_{(3,3)} \\ \hline \Gamma_{(6,2)} \oplus 2\Gamma_{(5,3)} \oplus \Gamma_{(5,2,1)} \oplus \Gamma_{(4,4)} \oplus \Gamma_{(4,3,1)} \\ \hline \Gamma_{(6,4)} \oplus \Gamma_{(6,3,1)} \oplus \Gamma_{(5,4,1)} \oplus \Gamma_{(5,3,1,1)} \\ \hline \Gamma_{(6,4,1,1)} \end{array} \qquad \begin{array}{c} \Gamma_{(4,1,1)} \\ \hline \Gamma_{(5,2,1)} \oplus \Gamma_{(4,2,2)} \\ \hline \Gamma_{(6,2,2)} \oplus \Gamma_{(5,3,2)} \\ \hline \Gamma_{(6,3,3)} \end{array}$$

$$\begin{array}{c} \Gamma_{(4)} \oplus \Gamma_{(3,1)} \\ \hline 2\Gamma_{(5,1)} \oplus 2\Gamma_{(4,2)} \oplus 2\Gamma_{(4,1,1)} \oplus \Gamma_{(3,2,1)} \\ \hline \Gamma_{(6,2)} \oplus \Gamma_{(6,1,1)} \oplus \Gamma_{(5,3)} \oplus 3\Gamma_{(5,2,1)} \oplus \Gamma_{(5,1,1,1)} \oplus \Gamma_{(4,3,1)} \oplus \Gamma_{(4,2,2)} \oplus \Gamma_{(4,2,1,1)} \\ \hline \Gamma_{(6,3,1)} \oplus \Gamma_{(6,2,2)} \oplus \Gamma_{(6,2,1,1)} \oplus \Gamma_{(5,3,2)} \oplus \Gamma_{(5,3,1,1)} \oplus \Gamma_{(5,2,2,1)} \\ \hline \Gamma_{(6,3,2,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(4)} \oplus \Gamma_{(3,1)} \oplus \Gamma_{(2,2)} \\ \hline \Gamma_{(6)} \oplus 2\Gamma_{(5,1)} \oplus 2\Gamma_{(4,2)} \oplus \Gamma_{(4,1,1)} \oplus \Gamma_{(3,3)} \oplus \Gamma_{(3,2,1)} \\ \hline \Gamma_{(6,2)} \oplus \Gamma_{(6,1,1)} \oplus \Gamma_{(5,3)} \oplus 2\Gamma_{(5,2,1)} \oplus \Gamma_{(5,1,1,1)} \oplus \Gamma_{(4,3,1)} \oplus \Gamma_{(4,2,1,1)} \\ \hline \Gamma_{(6,3,1)} \oplus \Gamma_{(6,2,1,1)} \oplus \Gamma_{(5,3,1,1)} \oplus \Gamma_{(5,2,1,1,1)} \\ \hline \Gamma_{(6,3,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(4)} \oplus \Gamma_{(3,1)} \\ \hline \Gamma_{(4,1,1)} \oplus \Gamma_{(3,1,1,1)} \oplus \Gamma_{(6)} \oplus \Gamma_{(5,1)} \oplus \Gamma_{(4,2)} \\ \hline \Gamma_{(6,1,1)} \oplus \Gamma_{(5,2,1)} \oplus \Gamma_{(5,1,1,1)} \oplus \Gamma_{(4,2,1,1)} \\ \hline \Gamma_{(6,2,1,1)} \oplus \Gamma_{(5,2,2,1)} \\ \hline \Gamma_{(6,2,2,2)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2)} \\ \hline 2\Gamma_{(3,1)} \oplus \Gamma_{(2,1,1)} \oplus \Gamma_{(4)} \\ \hline 2\Gamma_{(5,1)} \oplus \Gamma_{(4,2)} \oplus 3\Gamma_{(4,1,1)} \oplus \Gamma_{(3,2,1)} \oplus \Gamma_{(3,1,1,1)} \\ \hline \Gamma_{(6,2)} \oplus \Gamma_{(6,1,1)} \oplus 2\Gamma_{(5,2,1)} \oplus 2\Gamma_{(5,1,1,1)} \oplus \Gamma_{(4,2,2)} \oplus \Gamma_{(4,2,1,1)} \oplus \Gamma_{(4,1,1,1,1)} \\ \hline \Gamma_{(6,2,2)} \oplus \Gamma_{(6,2,1,1)} \oplus \Gamma_{(6,1,1,1,1)} \oplus \Gamma_{(5,2,2,1)} \oplus \Gamma_{(5,2,1,1,1)} \\ \hline \Gamma_{(6,2,2,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2)} \oplus \Gamma_{(1,1)} \\ \hline \Gamma_{(4)} \oplus 2\Gamma_{(3,1)} \oplus \Gamma_{(2,2)} \oplus \Gamma_{(2,1,1)} \\ \hline \Gamma_{(5,1)} \oplus \Gamma_{(4,2)} \oplus 2\Gamma_{(4,1,1)} \oplus \Gamma_{(3,2,1)} \oplus \Gamma_{(3,1,1,1)} \\ \hline \Gamma_{(6,1,1)} \oplus \Gamma_{(5,2,1)} \oplus 2\Gamma_{(5,1,1,1)} \oplus \Gamma_{(4,2,1,1)} \oplus \Gamma_{(4,1,1,1,1)} \\ \hline \Gamma_{(6,2,1,1)} \oplus \Gamma_{(6,1,1,1,1)} \oplus \Gamma_{(5,2,1,1,1)} \oplus \Gamma_{(5,1,1,1,1,1)} \\ \hline \Gamma_{(6,2,1,1,1,1)} \end{array} \qquad \begin{array}{c} \Gamma_{\emptyset} \\ \hline \Gamma_{(2)} \oplus \Gamma_{(1,1)} \\ \hline \Gamma_{(3,1)} \oplus \Gamma_{(2,1,1)} \\ \hline \Gamma_{(4,1,1)} \oplus \Gamma_{(3,1,1,1)} \\ \hline \Gamma_{(5,1,1,1)} \oplus \Gamma_{(4,1,1,1,1)} \\ \hline \Gamma_{(6,1,1,1,1)} \oplus \Gamma_{(5,1,1,1,1,1)} \\ \hline \Gamma_{(6,1,1,1,1,1,1)} \end{array} \qquad \begin{array}{c} \Gamma_{(3,3)} \\ \hline \Gamma_{(4,4)} \oplus \Gamma_{(4,3,1)} \\ \hline \Gamma_{(5,4,1)} \oplus \Gamma_{(4,4,2)} \\ \hline \Gamma_{(5,5,2)} \end{array}$$

$$\begin{array}{c} \Gamma_{(4,2)} \\ \hline \Gamma_{(5,3)}\oplus \Gamma_{(4,4)}\oplus \Gamma_{(4,3,1)} \\ \hline \Gamma_{(5,5)}\oplus \Gamma_{(5,4,1)}\oplus \Gamma_{(4,4,1,1)} \\ \hline \Gamma_{(5,5,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(3,2,1)} \\ \hline \Gamma_{(4,3,1)}\oplus \Gamma_{(4,2,2)}\oplus \Gamma_{(3,3,2)} \\ \hline \Gamma_{(5,3,2)}\oplus \Gamma_{(4,4,2)}\oplus \Gamma_{(4,3,3)} \\ \hline \Gamma_{(5,4,3)} \end{array}$$

$$\begin{array}{c} \Gamma_{(3,1)}\oplus \Gamma_{(2,2)} \\ \hline 2\Gamma_{(4,2)}\oplus \Gamma_{(4,1,1)}\oplus 2\Gamma_{(3,3)}\oplus 2\Gamma_{(3,2,1)} \\ \hline \Gamma_{(5,3)}\oplus \Gamma_{(5,2,1)}\oplus \Gamma_{(4,4)}\oplus 3\Gamma_{(4,3,1)}\oplus \Gamma_{(4,2,2)}\oplus \Gamma_{(4,2,1,1)}\oplus \Gamma_{(3,3,2)}\oplus \Gamma_{(3,3,1,1)} \\ \hline \Gamma_{(5,4,1)}\oplus \Gamma_{(5,3,2)}\oplus \Gamma_{(5,3,1,1)}\oplus \Gamma_{(4,4,2)}\oplus \Gamma_{(4,4,1,1)}\oplus \Gamma_{(4,3,2,1)} \\ \hline \Gamma_{(5,4,2,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(4)}\oplus \Gamma_{(3,1)} \\ \hline \Gamma_{(5,1)}\oplus 2\Gamma_{(4,2)}\oplus \Gamma_{(4,1,1)}\oplus \Gamma_{(3,3)}\oplus \Gamma_{(3,2,1)} \\ \hline \Gamma_{(5,3)}\oplus \Gamma_{(5,2,1)}\oplus \Gamma_{(4,4)}\oplus 2\Gamma_{(4,3,1)}\oplus \Gamma_{(4,2,1,1)}\oplus \Gamma_{(3,3,1,1)} \\ \hline \Gamma_{(5,4,1)}\oplus \Gamma_{(5,3,1,1)}\oplus \Gamma_{(4,4,1,1)}\oplus \Gamma_{(4,3,1,1,1)} \\ \hline \Gamma_{(5,4,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(3,1)}\oplus \Gamma_{(2,1,1)} \\ \hline \Gamma_{(4,2)}\oplus 2\Gamma_{(3,2,1)}\oplus \Gamma_{(2,2,2)}\oplus \Gamma_{(4,1,1)}\oplus \Gamma_{(3,1,1,1)} \\ \hline \Gamma_{(5,2,1)}\oplus \Gamma_{(4,3,1)}\oplus 2\Gamma_{(4,2,2)}\oplus \Gamma_{(4,2,1,1)}\oplus \Gamma_{(3,3,2)}\oplus \Gamma_{(3,2,2,1)} \\ \hline \Gamma_{(5,3,2)}\oplus \Gamma_{(5,2,2,1)}\oplus \Gamma_{(4,3,3)}\oplus \Gamma_{(4,3,2,1)} \\ \hline \Gamma_{(5,3,3,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2)} \\ \hline \Gamma_{(4)}\oplus 2\Gamma_{(3,1)}\oplus \Gamma_{(2,2)}\oplus \Gamma_{(2,1,1)} \\ \hline 2\Gamma_{(4,1,1)}\oplus 2\Gamma_{(3,2,1)}\oplus \Gamma_{(3,1,1,1)}\oplus \Gamma_{(2,2,1,1)}\oplus \Gamma_{(5,1)}\oplus \Gamma_{(4,2)}\oplus \Gamma_{(3,3)} \\ \hline \Gamma_{(5,2,1)}\oplus \Gamma_{(5,1,1,1)}\oplus \Gamma_{(4,3,1)}\oplus \Gamma_{(4,2,2)}\oplus 2\Gamma_{(4,2,1,1)}\oplus \Gamma_{(3,3,1,1)}\oplus \Gamma_{(3,2,2,1)} \\ \hline \Gamma_{(5,3,1,1)}\oplus \Gamma_{(5,2,2,1)}\oplus \Gamma_{(4,3,2,1)}\oplus \Gamma_{(4,2,2,2)} \\ \hline \Gamma_{(5,3,2,2)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2)}\oplus \Gamma_{(1,1)} \\ \hline 3\Gamma_{(3,1)}\oplus 2\Gamma_{(2,2)}\oplus 2\Gamma_{(2,1,1)}\oplus \Gamma_{(4)} \\ \hline \Gamma_{(5,1)}\oplus 3\Gamma_{(4,2)}\oplus 3\Gamma_{(4,1,1)}\oplus \Gamma_{(3,3)}\oplus 4\Gamma_{(3,2,1)}\oplus 2\Gamma_{(3,1,1,1)}\oplus \Gamma_{(2,2,2)}\oplus \Gamma_{(2,2,1,1)} \\ \hline \Gamma_{(5,3)}\oplus 2\Gamma_{(5,2,1)}\oplus \Gamma_{(5,1,1,1)}\oplus 2\Gamma_{(4,3,1)}\oplus 2\Gamma_{(4,2,2)}\oplus 3\Gamma_{(4,2,1,1)}\oplus \Gamma_{(4,1,1,1,1)}\oplus \Gamma_{(3,3,2)}\oplus \Gamma_{(3,3,1,1)}\oplus \Gamma_{(3,2,2,1)}\oplus \Gamma_{(3,2,1,1,1)} \\ \hline \Gamma_{(5,3,2)}\oplus \Gamma_{(5,3,1,1)}\oplus \Gamma_{(5,2,2,1)}\oplus \Gamma_{(5,2,1,1,1)}\oplus \Gamma_{(4,3,2,1)}\oplus \Gamma_{(4,3,1,1,1)}\oplus \Gamma_{(4,2,2,1,1)} \\ \hline \Gamma_{(5,3,2,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2)} \\ \hline \Gamma_{(4)}\oplus 2\Gamma_{(3,1)}\oplus \Gamma_{(2,2)}\oplus \Gamma_{(2,1,1)} \\ \hline \Gamma_{(5,1)}\oplus \Gamma_{(4,2)}\oplus 2\Gamma_{(4,1,1)}\oplus \Gamma_{(3,3)}\oplus 2\Gamma_{(3,2,1)}\oplus \Gamma_{(3,1,1,1)}\oplus \Gamma_{(2,2,1,1)} \\ \hline \Gamma_{(5,2,1)}\oplus \Gamma_{(5,1,1,1)}\oplus \Gamma_{(4,3,1)}\oplus 2\Gamma_{(4,2,1,1)}\oplus \Gamma_{(4,1,1,1,1)}\oplus \Gamma_{(3,3,1,1)}\oplus \Gamma_{(3,2,1,1,1)} \\ \hline \Gamma_{(5,3,1,1)}\oplus \Gamma_{(5,2,1,1,1)}\oplus \Gamma_{(4,3,1,1,1)}\oplus \Gamma_{(4,2,1,1,1,1)} \\ \hline \Gamma_{(5,3,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2)}\oplus \Gamma_{(1,1)} \\ \hline 2\Gamma_{(2,1,1)}\oplus \Gamma_{(1,1,1,1)}\oplus \Gamma_{(4)}\oplus 2\Gamma_{(3,1)}\oplus \Gamma_{(2,2)} \\ \hline 2\Gamma_{(4,1,1)}\oplus 2\Gamma_{(3,2,1)}\oplus 2\Gamma_{(3,1,1,1)}\oplus \Gamma_{(2,2,1,1)}\oplus \Gamma_{(2,1,1,1,1)}\oplus \Gamma_{(5,1)}\oplus \Gamma_{(4,2)} \\ \hline \Gamma_{(5,2,1)}\oplus \Gamma_{(5,1,1,1)}\oplus \Gamma_{(4,2,2)}\oplus 2\Gamma_{(4,2,1,1)}\oplus \Gamma_{(4,1,1,1,1)}\oplus \Gamma_{(3,2,2,1)}\oplus \Gamma_{(3,2,1,1,1)} \\ \hline \Gamma_{(5,2,2,1)}\oplus \Gamma_{(5,2,1,1,1)}\oplus \Gamma_{(4,2,2,2)}\oplus \Gamma_{(4,2,2,1,1)} \\ \hline \Gamma_{(5,2,2,2,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{\emptyset} \\ \hline 2\Gamma_{(1,1)}\oplus \Gamma_{(2)} \\ \hline 2\Gamma_{(3,1)}\oplus \Gamma_{(2,2)}\oplus 3\Gamma_{(2,1,1)}\oplus \Gamma_{(1,1,1,1)} \\ \hline \Gamma_{(4,2)}\oplus 2\Gamma_{(4,1,1)}\oplus 2\Gamma_{(3,2,1)}\oplus 3\Gamma_{(3,1,1,1)}\oplus \Gamma_{(2,2,2)}\oplus \Gamma_{(2,2,1,1)}\oplus \Gamma_{(2,1,1,1,1)} \\ \hline \Gamma_{(5,2,1)}\oplus \Gamma_{(5,1,1,1)}\oplus \Gamma_{(4,2,2)}\oplus 2\Gamma_{(4,2,1,1)}\oplus 2\Gamma_{(4,1,1,1,1)}\oplus \Gamma_{(3,2,2,1)}\oplus \Gamma_{(3,2,1,1,1)}\oplus \Gamma_{(3,1,1,1,1,1)} \\ \hline \Gamma_{(5,2,2,1)}\oplus \Gamma_{(5,2,1,1,1)}\oplus \Gamma_{(5,1,1,1,1,1)}\oplus \Gamma_{(4,2,2,1,1)}\oplus \Gamma_{(4,2,1,1,1,1)} \\ \hline \Gamma_{(5,2,2,1,1,1)} \end{array}$$

$$\begin{array}{c}
\Gamma_{(2)}\oplus\Gamma_{(1,1)}\\ \hline
\Gamma_{(3,1)}\oplus\Gamma_{(2,2)}\oplus 2\Gamma_{(2,1,1)}\oplus\Gamma_{(1,1,1,1)}\\ \hline
\Gamma_{(4,1,1)}\oplus\Gamma_{(3,2,1)}\oplus 2\Gamma_{(3,1,1,1)}\oplus\Gamma_{(2,2,1,1)}\oplus\Gamma_{(2,1,1,1,1)}\\ \hline
\Gamma_{(5,1,1,1)}\oplus\Gamma_{(4,2,1,1)}\oplus 2\Gamma_{(4,1,1,1,1)}\oplus\Gamma_{(3,2,1,1,1)}\oplus\Gamma_{(3,1,1,1,1,1)}\\ \hline
\Gamma_{(5,2,1,1,1)}\oplus\Gamma_{(5,1,1,1,1,1)}\oplus\Gamma_{(4,2,1,1,1,1)}\oplus\Gamma_{(4,1,1,1,1,1,1)}\\ \hline
\Gamma_{(5,2,1,1,1,1,1)}
\end{array}$$

$$\begin{array}{c}
\Gamma_{(1,1)}\\ \hline
\Gamma_{(2,1,1)}\oplus\Gamma_{(1,1,1,1)}\\ \hline
\Gamma_{(3,1,1,1)}\oplus\Gamma_{(2,1,1,1,1)}\\ \hline
\Gamma_{(4,1,1,1,1)}\oplus\Gamma_{(3,1,1,1,1,1)}\\ \hline
\Gamma_{(5,1,1,1,1,1)}\oplus\Gamma_{(4,1,1,1,1,1,1)}\\ \hline
\Gamma_{(5,1,1,1,1,1,1,1)}
\end{array}$$

$$\begin{array}{c}
\Gamma_{(2,2,2)}\\ \hline
\Gamma_{(3,3,2)}\\ \hline
\Gamma_{(4,3,3)}\\ \hline
\Gamma_{(4,4,4)}
\end{array}$$

$$\begin{array}{c}
\Gamma_{(2,2)}\oplus\Gamma_{(2,1,1)}\\ \hline
\Gamma_{(3,3)}\oplus 2\Gamma_{(3,2,1)}\oplus\Gamma_{(2,2,2)}\oplus\Gamma_{(2,2,1,1)}\\ \hline
\Gamma_{(4,3,1)}\oplus\Gamma_{(4,2,2)}\oplus 2\Gamma_{(3,3,2)}\oplus\Gamma_{(3,3,1,1)}\oplus\Gamma_{(3,2,2,1)}\\ \hline
\Gamma_{(4,4,2)}\oplus\Gamma_{(4,3,3)}\oplus\Gamma_{(4,3,2,1)}\oplus\Gamma_{(3,3,3,1)}\\ \hline
\Gamma_{(4,4,3,1)}
\end{array}$$

$$\begin{array}{c}
\Gamma_{(1,1)}\\ \hline
\Gamma_{(3,1)}\oplus\Gamma_{(2,2)}\oplus\Gamma_{(2,1,1)}\\ \hline
2\Gamma_{(3,2,1)}\oplus\Gamma_{(3,1,1,1)}\oplus\Gamma_{(4,2)}\oplus\Gamma_{(2,2,2)}\\ \hline
\Gamma_{(4,3,1)}\oplus\Gamma_{(4,2,1,1)}\oplus\Gamma_{(3,3,2)}\oplus\Gamma_{(3,3,1,1)}\oplus\Gamma_{(3,2,2,1)}\\ \hline
\Gamma_{(4,4,1,1)}\oplus\Gamma_{(4,3,2,1)}\oplus\Gamma_{(3,3,2,2)}\\ \hline
\Gamma_{(4,4,2,2)}
\end{array}$$

$$\begin{array}{c}
\Gamma_{(2)}\\ \hline
2\Gamma_{(3,1)}\oplus\Gamma_{(2,2)}\oplus\Gamma_{(2,1,1)}\\ \hline
\Gamma_{(4,2)}\oplus\Gamma_{(4,1,1)}\oplus 2\Gamma_{(3,3)}\oplus 3\Gamma_{(3,2,1)}\oplus\Gamma_{(3,1,1,1)}\oplus\Gamma_{(2,2,1,1)}\\ \hline
\Gamma_{(4,4)}\oplus 2\Gamma_{(4,3,1)}\oplus\Gamma_{(4,2,2)}\oplus\Gamma_{(4,2,1,1)}\oplus\Gamma_{(3,3,2)}\oplus 2\Gamma_{(3,3,1,1)}\oplus\Gamma_{(3,2,2,1)}\oplus\Gamma_{(3,2,1,1,1)}\\ \hline
\Gamma_{(4,4,2)}\oplus\Gamma_{(4,4,1,1)}\oplus\Gamma_{(4,3,2,1)}\oplus\Gamma_{(4,3,1,1,1)}\oplus\Gamma_{(3,3,2,1,1)}\\ \hline
\Gamma_{(4,4,2,1,1)}
\end{array}$$

$$\begin{array}{c}
\Gamma_{(4)}\oplus\Gamma_{(3,1)}\\ \hline
\Gamma_{(4,2)}\oplus\Gamma_{(4,1,1)}\oplus\Gamma_{(3,2,1)}\oplus\Gamma_{(3,1,1,1)}\\ \hline
\Gamma_{(4,3,1)}\oplus\Gamma_{(4,2,1,1)}\oplus\Gamma_{(3,3,1,1)}\oplus\Gamma_{(3,2,1,1,1)}\\ \hline
\Gamma_{(4,4,1,1)}\oplus\Gamma_{(4,3,1,1,1)}\oplus\Gamma_{(3,3,1,1,1,1)}\\ \hline
\Gamma_{(4,4,1,1,1,1)}
\end{array}$$

$$\begin{array}{c}
\Gamma_{(2)}\oplus\Gamma_{(1,1)}\\ \hline
\Gamma_{(3,1)}\oplus\Gamma_{(2,2)}\oplus 2\Gamma_{(2,1,1)}\oplus\Gamma_{(1,1,1,1)}\\ \hline
2\Gamma_{(3,2,1)}\oplus\Gamma_{(2,2,2)}\oplus 2\Gamma_{(2,2,1,1)}\oplus\Gamma_{(4,1,1)}\oplus\Gamma_{(3,1,1,1)}\\ \hline
\Gamma_{(4,2,2)}\oplus\Gamma_{(4,2,1,1)}\oplus\Gamma_{(3,3,2)}\oplus\Gamma_{(3,3,1,1)}\oplus 2\Gamma_{(3,2,2,1)}\oplus\Gamma_{(2,2,2,2)}\\ \hline
\Gamma_{(4,3,2,1)}\oplus\Gamma_{(4,2,2,2)}\oplus\Gamma_{(3,3,3,1)}\oplus\Gamma_{(3,3,2,2)}\\ \hline
\Gamma_{(4,3,3,2)}
\end{array}$$

$$\begin{array}{c}
\Gamma_{(1,1)}\\ \hline
2\Gamma_{(2,2)}\oplus\Gamma_{(3,1)}\oplus 2\Gamma_{(2,1,1)}\oplus\Gamma_{(1,1,1,1)}\\ \hline
\Gamma_{(4,2)}\oplus\Gamma_{(3,3)}\oplus 3\Gamma_{(3,2,1)}\oplus 2\Gamma_{(2,2,2)}\oplus 2\Gamma_{(2,2,1,1)}\oplus\Gamma_{(3,1,1,1)}\oplus\Gamma_{(2,1,1,1,1)}\\ \hline
\Gamma_{(4,3,1)}\oplus\Gamma_{(4,2,2)}\oplus\Gamma_{(4,2,1,1)}\oplus 2\Gamma_{(3,3,2)}\oplus\Gamma_{(3,3,1,1)}\oplus 2\Gamma_{(3,2,2,1)}\oplus\Gamma_{(3,2,1,1,1)}\oplus\Gamma_{(2,2,2,1,1)}\\ \hline
\Gamma_{(4,3,3)}\oplus\Gamma_{(4,3,2,1)}\oplus\Gamma_{(4,2,2,1,1)}\oplus\Gamma_{(3,3,3,1)}\oplus\Gamma_{(3,3,2,1,1)}\\ \hline
\Gamma_{(4,3,3,1,1)}
\end{array}$$

$$\begin{array}{c}
\Gamma_{\emptyset}\\ \hline
2\Gamma_{(2)}\oplus 2\Gamma_{(1,1)}\\ \hline
4\Gamma_{(2,1,1)}\oplus\Gamma_{(4)}\oplus 3\Gamma_{(3,1)}\oplus 2\Gamma_{(2,2)}\oplus\Gamma_{(1,1,1,1)}\\ \hline
2\Gamma_{(4,1,1)}\oplus 4\Gamma_{(3,2,1)}\oplus 3\Gamma_{(3,1,1,1)}\oplus 2\Gamma_{(2,2,2)}\oplus 3\Gamma_{(2,2,1,1)}\oplus\Gamma_{(2,1,1,1,1)}\oplus\Gamma_{(4,2)}\oplus\Gamma_{(3,3)}\\ \hline
\Gamma_{(4,3,1)}\oplus\Gamma_{(4,2,2)}\oplus 2\Gamma_{(4,2,1,1)}\oplus\Gamma_{(4,1,1,1,1)}\oplus\Gamma_{(3,3,2)}\oplus 2\Gamma_{(3,3,1,1)}\oplus 3\Gamma_{(3,2,2,1)}\oplus 2\Gamma_{(3,2,1,1,1)}\oplus\Gamma_{(2,2,2,2)}\oplus\Gamma_{(2,2,2,1,1)}\\ \hline
\Gamma_{(4,3,2,1)}\oplus\Gamma_{(4,3,1,1,1)}\oplus\Gamma_{(4,2,2,2)}\oplus\Gamma_{(4,2,2,1,1)}\oplus\Gamma_{(3,3,2,2)}\oplus\Gamma_{(3,3,2,1,1)}\oplus\Gamma_{(3,2,2,2,1)}\\ \hline
\Gamma_{(4,3,2,2,1)}
\end{array}$$

$$\begin{array}{c}
\Gamma_{(2)}\oplus\Gamma_{(1,1)}\\ \hline
2\Gamma_{(3,1)}\oplus 2\Gamma_{(2,2)}\oplus 3\Gamma_{(2,1,1)}\oplus\Gamma_{(1,1,1,1)}\\ \hline
\Gamma_{(4,2)}\oplus\Gamma_{(4,1,1)}\oplus\Gamma_{(3,3)}\oplus 4\Gamma_{(3,2,1)}\oplus 3\Gamma_{(3,1,1,1)}\oplus\Gamma_{(2,2,2)}\oplus 3\Gamma_{(2,2,1,1)}\oplus 2\Gamma_{(2,1,1,1,1)}\\ \hline
\Gamma_{(4,3,1)}\oplus\Gamma_{(4,2,2)}\oplus 2\Gamma_{(4,2,1,1)}\oplus\Gamma_{(4,1,1,1,1)}\oplus\Gamma_{(3,3,2)}\oplus 2\Gamma_{(3,3,1,1)}\oplus 2\Gamma_{(3,2,2,1)}\oplus 3\Gamma_{(3,2,1,1,1)}\oplus\Gamma_{(3,1,1,1,1,1)}\oplus\Gamma_{(2,2,2,1,1)}\oplus\Gamma_{(2,2,1,1,1,1)}\\ \hline
\Gamma_{(4,3,2,1)}\oplus\Gamma_{(4,3,1,1,1)}\oplus\Gamma_{(4,2,2,1,1)}\oplus\Gamma_{(4,2,1,1,1,1)}\oplus\Gamma_{(3,3,2,1,1)}\oplus\Gamma_{(3,3,1,1,1,1)}\oplus\Gamma_{(3,2,2,1,1,1)}\\ \hline
\Gamma_{(4,3,2,1,1,1)}
\end{array}$$

$\Gamma_{(3,1)} \oplus \Gamma_{(2,1,1)}$
$\Gamma_{(4,1,1)} \oplus \Gamma_{(3,2,1)} \oplus 2\Gamma_{(3,1,1,1)} \oplus \Gamma_{(2,2,1,1)} \oplus \Gamma_{(2,1,1,1,1)}$
$\Gamma_{(4,2,1,1)} \oplus \Gamma_{(4,1,1,1,1)} \oplus \Gamma_{(3,3,1,1)} \oplus 2\Gamma_{(3,2,1,1,1)} \oplus \Gamma_{(3,1,1,1,1,1)} \oplus \Gamma_{(2,2,1,1,1,1)}$
$\Gamma_{(4,3,1,1,1)} \oplus \Gamma_{(4,2,1,1,1,1)} \oplus \Gamma_{(3,3,1,1,1,1)} \oplus \Gamma_{(3,2,1,1,1,1,1)}$
$\Gamma_{(4,3,1,1,1,1,1)}$

$\Gamma_{(1,1)}$
$\Gamma_{(1,1,1,1)} \oplus \Gamma_{(3,1)} \oplus \Gamma_{(2,2)} \oplus \Gamma_{(2,1,1)}$
$\Gamma_{(3,1,1,1)} \oplus \Gamma_{(2,2,1,1)} \oplus \Gamma_{(2,1,1,1,1)} \oplus \Gamma_{(4,2)} \oplus \Gamma_{(3,2,1)} \oplus \Gamma_{(2,2,2)}$
$\Gamma_{(4,2,1,1)} \oplus \Gamma_{(3,2,2,1)} \oplus \Gamma_{(3,2,1,1,1)} \oplus \Gamma_{(2,2,2,1,1)}$
$\Gamma_{(4,2,2,1,1)} \oplus \Gamma_{(3,2,2,2,1)}$
$\Gamma_{(4,2,2,2,2)}$

$\Gamma_{(2)} \oplus \Gamma_{(1,1)}$
$3\Gamma_{(2,1,1)} \oplus 2\Gamma_{(1,1,1,1)} \oplus \Gamma_{(3,1)} \oplus \Gamma_{(2,2)}$
$2\Gamma_{(3,2,1)} \oplus 2\Gamma_{(3,1,1,1)} \oplus \Gamma_{(2,2,2)} \oplus 3\Gamma_{(2,2,1,1)} \oplus 2\Gamma_{(2,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1)} \oplus \Gamma_{(4,1,1)}$
$\Gamma_{(4,2,2)} \oplus \Gamma_{(4,2,1,1)} \oplus \Gamma_{(4,1,1,1,1)} \oplus 2\Gamma_{(3,2,2,1)} \oplus 2\Gamma_{(3,2,1,1,1)} \oplus \Gamma_{(3,1,1,1,1,1)} \oplus \Gamma_{(2,2,2,2)} \oplus \Gamma_{(2,2,2,1,1)} \oplus \Gamma_{(2,2,1,1,1,1)}$
$\Gamma_{(4,2,2,2)} \oplus \Gamma_{(4,2,2,1,1)} \oplus \Gamma_{(4,2,1,1,1,1)} \oplus \Gamma_{(3,2,2,2,1)} \oplus \Gamma_{(3,2,2,1,1,1)}$
$\Gamma_{(4,2,2,2,1,1)}$

$\Gamma_{(1,1)}$
$\Gamma_{(2,2)} \oplus 2\Gamma_{(2,1,1)} \oplus 2\Gamma_{(1,1,1,1)}$
$\Gamma_{(3,2,1)} \oplus 2\Gamma_{(3,1,1,1)} \oplus \Gamma_{(2,2,2)} \oplus 2\Gamma_{(2,2,1,1)} \oplus 3\Gamma_{(2,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1)}$
$\Gamma_{(4,2,1,1)} \oplus \Gamma_{(4,1,1,1,1)} \oplus \Gamma_{(3,2,2,1)} \oplus 2\Gamma_{(3,2,1,1,1)} \oplus 2\Gamma_{(3,1,1,1,1,1)} \oplus \Gamma_{(2,2,2,1,1)} \oplus \Gamma_{(2,2,1,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1,1)}$
$\Gamma_{(4,2,2,1,1)} \oplus \Gamma_{(4,2,1,1,1,1)} \oplus \Gamma_{(4,1,1,1,1,1,1)} \oplus \Gamma_{(3,2,2,1,1,1)} \oplus \Gamma_{(3,2,1,1,1,1,1)}$
$\Gamma_{(4,2,2,1,1,1,1)}$

$\Gamma_{(2,1,1)} \oplus \Gamma_{(1,1,1,1)}$
$\Gamma_{(3,1,1,1)} \oplus \Gamma_{(2,2,1,1)} \oplus 2\Gamma_{(2,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1)}$
$\Gamma_{(4,1,1,1,1)} \oplus \Gamma_{(3,2,1,1,1)} \oplus 2\Gamma_{(3,1,1,1,1,1)} \oplus \Gamma_{(2,2,1,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1,1)}$
$\Gamma_{(4,2,1,1,1,1)} \oplus \Gamma_{(4,1,1,1,1,1,1)} \oplus \Gamma_{(3,2,1,1,1,1,1)} \oplus \Gamma_{(3,1,1,1,1,1,1,1)}$
$\Gamma_{(4,2,1,1,1,1,1,1)}$

$\Gamma_{(1,1,1,1)}$
$\Gamma_{(2,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1)}$
$\Gamma_{(3,1,1,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1,1)}$
$\Gamma_{(4,1,1,1,1,1,1)} \oplus \Gamma_{(3,1,1,1,1,1,1,1)}$
$\Gamma_{(4,1,1,1,1,1,1,1,1)}$

$\Gamma_{\emptyset}$
$\Gamma_{(1,1)}$
$\Gamma_{(2,1,1)}$
$\Gamma_{(2,2,2)} \oplus \Gamma_{(3,1,1,1)}$
$\Gamma_{(3,2,2,1)}$
$\Gamma_{(3,3,2,2)}$
$\Gamma_{(3,3,3,3)}$

$\Gamma_{(2)} \oplus \Gamma_{(1,1)}$
$\Gamma_{(3,1)} \oplus \Gamma_{(2,2)} \oplus 2\Gamma_{(2,1,1)} \oplus \Gamma_{(1,1,1,1)}$
$2\Gamma_{(3,2,1)} \oplus \Gamma_{(2,2,2)} \oplus 2\Gamma_{(2,2,1,1)} \oplus \Gamma_{(3,1,1,1)} \oplus \Gamma_{(2,1,1,1,1)}$
$\Gamma_{(3,3,2)} \oplus \Gamma_{(3,3,1,1)} \oplus 2\Gamma_{(3,2,2,1)} \oplus \Gamma_{(3,2,1,1,1)} \oplus \Gamma_{(2,2,2,2)} \oplus \Gamma_{(2,2,2,1,1)}$
$\Gamma_{(3,3,3,1)} \oplus \Gamma_{(3,3,2,2)} \oplus \Gamma_{(3,3,2,1,1)} \oplus \Gamma_{(3,2,2,2,1)}$
$\Gamma_{(3,3,3,2,1)}$

$\Gamma_{(2,2)} \oplus \Gamma_{(2,1,1)} \oplus \Gamma_{(1,1,1,1)}$
$\Gamma_{(3,3)} \oplus \Gamma_{(3,2,1)} \oplus 2\Gamma_{(2,2,1,1)} \oplus \Gamma_{(2,2,2)} \oplus \Gamma_{(2,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1)}$
$\Gamma_{(3,3,2)} \oplus \Gamma_{(3,3,1,1)} \oplus \Gamma_{(3,2,2,1)} \oplus \Gamma_{(3,2,1,1,1)} \oplus \Gamma_{(2,2,2,1,1)} \oplus \Gamma_{(2,2,1,1,1,1)}$
$\Gamma_{(3,3,3,1)} \oplus \Gamma_{(3,3,2,1,1)} \oplus \Gamma_{(3,2,2,1,1,1)}$
$\Gamma_{(3,3,3,1,1,1)}$

$\Gamma_{(2)}$
$\Gamma_{(3,1)} \oplus \Gamma_{(2,2)} \oplus \Gamma_{(2,1,1)}$
$\Gamma_{(3,1,1,1)} \oplus 2\Gamma_{(2,2,1,1)} \oplus \Gamma_{(3,3)} \oplus \Gamma_{(3,2,1)}$
$\Gamma_{(3,3,1,1)} \oplus \Gamma_{(3,2,2,1)} \oplus \Gamma_{(3,2,1,1,1)} \oplus \Gamma_{(2,2,2,2)} \oplus \Gamma_{(2,2,2,1,1)}$
$\Gamma_{(3,3,2,1,1)} \oplus \Gamma_{(3,2,2,2,1)} \oplus \Gamma_{(2,2,2,2,2)}$
$\Gamma_{(3,3,2,2,2)}$

$$\begin{array}{c} \Gamma_{(1,1)} \\ \hline \Gamma_{(3,1)} \oplus \Gamma_{(2,2)} \oplus 2\Gamma_{(2,1,1)} \oplus \Gamma_{(1,1,1,1)} \\ \hline 2\Gamma_{(3,2,1)} \oplus 2\Gamma_{(3,1,1,1)} \oplus 2\Gamma_{(2,2,2)} \oplus 2\Gamma_{(2,2,1,1)} \oplus 2\Gamma_{(2,1,1,1,1)} \\ \hline \Gamma_{(3,3,2)} \oplus \Gamma_{(3,3,1,1)} \oplus 2\Gamma_{(3,2,2,1)} \oplus 2\Gamma_{(3,2,1,1,1)} \oplus \Gamma_{(3,1,1,1,1,1)} \oplus \Gamma_{(2,2,2,2)} \oplus 2\Gamma_{(2,2,2,1,1)} \oplus \Gamma_{(2,2,1,1,1,1)} \\ \hline \Gamma_{(3,3,2,2)} \oplus \Gamma_{(3,3,2,1,1)} \oplus \Gamma_{(3,3,1,1,1,1)} \oplus \Gamma_{(3,2,2,2,1)} \oplus \Gamma_{(3,2,2,1,1,1)} \oplus \Gamma_{(2,2,2,2,1,1)} \\ \hline \Gamma_{(3,3,2,2,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2,1,1)} \oplus \Gamma_{(1,1,1,1)} \\ \hline \Gamma_{(3,2,1)} \oplus \Gamma_{(3,1,1,1)} \oplus 2\Gamma_{(2,2,1,1)} \oplus 2\Gamma_{(2,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1)} \\ \hline \Gamma_{(3,3,1,1)} \oplus \Gamma_{(3,2,2,1)} \oplus 2\Gamma_{(3,2,1,1,1)} \oplus \Gamma_{(3,1,1,1,1,1)} \oplus \Gamma_{(2,2,2,1,1)} \oplus 2\Gamma_{(2,2,1,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1,1)} \\ \hline \Gamma_{(3,3,2,1,1)} \oplus \Gamma_{(3,3,1,1,1,1)} \oplus \Gamma_{(3,2,2,1,1,1)} \oplus \Gamma_{(3,2,1,1,1,1,1)} \oplus \Gamma_{(2,2,2,1,1,1,1)} \\ \hline \Gamma_{(3,3,2,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(3,1,1,1)} \oplus \Gamma_{(2,1,1,1,1)} \\ \hline \Gamma_{(3,2,1,1,1)} \oplus \Gamma_{(3,1,1,1,1,1)} \oplus \Gamma_{(2,2,1,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1,1)} \\ \hline \Gamma_{(3,3,1,1,1,1)} \oplus \Gamma_{(3,2,1,1,1,1,1)} \oplus \Gamma_{(2,2,1,1,1,1,1,1)} \\ \hline \Gamma_{(3,3,1,1,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2,2)} \oplus \Gamma_{(2,1,1)} \\ \hline 2\Gamma_{(2,2,1,1)} \oplus \Gamma_{(2,1,1,1,1)} \oplus \Gamma_{(3,2,1)} \oplus \Gamma_{(2,2,2)} \\ \hline \Gamma_{(3,2,2,1)} \oplus \Gamma_{(3,2,1,1,1)} \oplus \Gamma_{(2,2,2,2)} \oplus 2\Gamma_{(2,2,2,1,1)} \oplus \Gamma_{(2,2,1,1,1,1)} \\ \hline \Gamma_{(3,2,2,2,1)} \oplus \Gamma_{(3,2,2,1,1,1)} \oplus \Gamma_{(2,2,2,2,2)} \oplus \Gamma_{(2,2,2,2,1,1)} \\ \hline \Gamma_{(3,2,2,2,2,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2,1,1)} \oplus \Gamma_{(1,1,1,1)} \\ \hline \Gamma_{(2,2,2)} \oplus 2\Gamma_{(2,2,1,1)} \oplus 2\Gamma_{(2,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1)} \oplus \Gamma_{(3,1,1,1)} \\ \hline \Gamma_{(3,2,2,1)} \oplus \Gamma_{(3,2,1,1,1)} \oplus \Gamma_{(3,1,1,1,1,1)} \oplus \Gamma_{(2,2,2,2)} \oplus 2\Gamma_{(2,2,2,1,1)} \oplus 2\Gamma_{(2,2,1,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1,1)} \\ \hline \Gamma_{(3,2,2,2,1)} \oplus \Gamma_{(3,2,2,1,1,1)} \oplus \Gamma_{(3,2,1,1,1,1,1)} \oplus \Gamma_{(2,2,2,2,1,1)} \oplus \Gamma_{(2,2,2,1,1,1,1)} \\ \hline \Gamma_{(3,2,2,2,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(1,1,1,1)} \\ \hline \Gamma_{(2,2,1,1)} \oplus 2\Gamma_{(2,1,1,1,1)} \oplus 2\Gamma_{(1,1,1,1,1,1)} \\ \hline \Gamma_{(3,2,1,1,1)} \oplus \Gamma_{(3,1,1,1,1,1)} \oplus \Gamma_{(2,2,2,1,1)} \oplus 2\Gamma_{(2,2,1,1,1,1)} \oplus 2\Gamma_{(2,1,1,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1,1,1)} \\ \hline \Gamma_{(3,2,2,1,1,1)} \oplus \Gamma_{(3,2,1,1,1,1,1)} \oplus \Gamma_{(3,1,1,1,1,1,1,1)} \oplus \Gamma_{(2,2,2,1,1,1,1)} \oplus \Gamma_{(2,2,1,1,1,1,1,1)} \\ \hline \Gamma_{(3,2,2,1,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1)} \\ \hline \Gamma_{(3,1,1,1,1,1)} \oplus \Gamma_{(2,2,1,1,1,1)} \oplus 2\Gamma_{(2,1,1,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1,1,1)} \\ \hline \Gamma_{(3,2,1,1,1,1,1)} \oplus \Gamma_{(3,1,1,1,1,1,1,1)} \oplus \Gamma_{(2,2,1,1,1,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1,1,1,1)} \\ \hline \Gamma_{(3,2,1,1,1,1,1,1,1)} \end{array} \qquad \begin{array}{c} \Gamma_{(1,1,1,1,1,1)} \\ \hline \Gamma_{(2,1,1,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1,1,1)} \\ \hline \Gamma_{(3,1,1,1,1,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1,1,1,1)} \\ \hline \Gamma_{(3,1,1,1,1,1,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(2,2,2)} \\ \hline \Gamma_{(2,2,2,1,1)} \\ \hline \Gamma_{(2,2,2,2,1,1)} \\ \hline \Gamma_{(2,2,2,2,2,2)} \end{array} \qquad \begin{array}{c} \Gamma_{(2,2,1,1)} \\ \hline \Gamma_{(2,2,2,2)} \oplus \Gamma_{(2,2,2,1,1)} \oplus \Gamma_{(2,2,1,1,1,1)} \\ \hline \Gamma_{(2,2,2,2,2)} \oplus \Gamma_{(2,2,2,2,1,1)} \oplus \Gamma_{(2,2,2,1,1,1,1)} \\ \hline \Gamma_{(2,2,2,2,2,1,1)} \end{array} \qquad \begin{array}{c} \Gamma_{(2,1,1,1,1)} \\ \hline \Gamma_{(2,2,2,1,1)} \oplus \Gamma_{(2,2,1,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1,1)} \\ \hline \Gamma_{(2,2,2,2,1,1)} \oplus \Gamma_{(2,2,2,1,1,1,1)} \oplus \Gamma_{(2,2,1,1,1,1,1,1)} \\ \hline \Gamma_{(2,2,2,2,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(1,1,1,1,1,1)} \\ \hline \Gamma_{(2,2,1,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1,1,1)} \\ \hline \Gamma_{(2,2,2,1,1,1,1)} \oplus \Gamma_{(2,2,1,1,1,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1,1,1,1)} \\ \hline \Gamma_{(2,2,2,1,1,1,1,1,1)} \end{array} \qquad \begin{array}{c} \Gamma_{(2,1,1,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1,1,1)} \\ \hline \Gamma_{(2,2,1,1,1,1,1,1)} \oplus \Gamma_{(2,1,1,1,1,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1,1,1,1,1)} \\ \hline \Gamma_{(2,2,1,1,1,1,1,1,1,1)} \end{array}$$

$$\begin{array}{c} \Gamma_{(1,1,1,1,1,1,1,1)} \\ \hline \Gamma_{(2,1,1,1,1,1,1,1,1)} \oplus \Gamma_{(1,1,1,1,1,1,1,1,1,1)} \\ \hline \Gamma_{(2,1,1,1,1,1,1,1,1,1,1)} \end{array} \qquad \begin{array}{c} \Gamma_{(1,1,1,1,1,1,1,1,1,1)} \\ \hline \Gamma_{(1,1,1,1,1,1,1,1,1,1,1,1)} \end{array}$$

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