SCHUBERT DECOMPOSITIONS FOR IND-VARIETIES OF GENERALIZED FLAGS

LUCAS FRESSE AND IVAN PENKOV

Abstract. Let $\mathbf{G}$ be one of the ind-groups $GL(\infty)$, $O(\infty)$, $Sp(\infty)$ and $\mathbf{P} \subset \mathbf{G}$ be a splitting parabolic ind-subgroup. The ind-variety $\mathbf{G}/\mathbf{P}$ has been identified with an ind-variety of generalized flags in [4]. In the present paper we define a Schubert cell on $\mathbf{G}/\mathbf{P}$ as a $\mathbf{B}$-orbit on $\mathbf{G}/\mathbf{P}$, where $\mathbf{B}$ is any Borel ind-subgroup of $\mathbf{G}$ which intersects $\mathbf{P}$ in a maximal ind-torus. A significant difference with the finite-dimensional case is that in general $\mathbf{B}$ is not conjugate to an ind-subgroup of $\mathbf{P}$, whence $\mathbf{G}/\mathbf{P}$ admits many non-conjugate Schubert decompositions. We study the basic properties of the Schubert cells, proving in particular that they are usual finite-dimensional cells or are isomorphic to affine ind-spaces.

We then define Schubert ind-varieties as closures of Schubert cells and study the smoothness of Schubert ind-varieties. Our approach to Schubert ind-varieties differs from an earlier approach by H. Salmasian [12].

1. Introduction

If G is a reductive algebraic group, the flag variety G/B is the most important geometric object attached to G. If $\mathbf{G}$ is a classical ind-group, $\mathbf{G} = GL(\infty), O(\infty), Sp(\infty)$, then there are infinitely many conjugacy classes of splitting Borel subgroups $\mathbf{B}$ (the notion of splitting subgroup is defined in Section 2.2), and hence there are infinitely many flag ind-varieties $\mathbf{G}/\mathbf{B}$. These smooth ind-varieties have been studied in [3, 4, 5], and in [4] each such ind-variety has been described explicitly as the ind-variety of certain generalized flags in the natural representation V of $\mathbf{G}$. A generalized flag is a chain of subspaces of V satisfying two conditions (see Definition 1), but notably such a chain is rarely ordered by an ordered subset of $\mathbb{Z}$.

In this paper we undertake a next step in the study of the generalized flag ind-varieties $\mathbf{G}/\mathbf{B}$, and more generally any ind-variety of the form $\mathbf{G}/\mathbf{P}$ where $\mathbf{P}$ is a splitting parabolic subgroup of $\mathbf{G}$. Namely, we define and study the Schubert decompositions of the ind-varieties $\mathbf{G}/\mathbf{P}$. The classical Schubert decomposition of a finite-dimensional flag variety is a key to important purely geometric theories such as Schubert calculus, as well as to geometric representation theory. In particular, Schubert varieties, i.e. closures of Schubert cells play a crucial role in the geometric theory of category $\mathcal{O}$. The challenge to find ind-analogues of those theories motivates our study of Schubert decompositions of ind-varieties of generalized flags.

We define the Schubert cells on $\mathbf{G}/\mathbf{P}$ as the $\mathbf{B}$-orbits on $\mathbf{G}/\mathbf{P}$ for any Borel ind-subgroup $\mathbf{B}$ which contains a common splitting maximal ind-torus with $\mathbf{P}$. The essential difference with the finite-dimensional case is that $\mathbf{B}$ is not necessarily conjugate to a Borel subgroup of $\mathbf{P}$. This leads to the existence of many non-conjugate Schubert decompositions of a given ind-variety of generalized flags $\mathbf{G}/\mathbf{P}$. We compute the dimensions of the cells of all Schubert decompositions of $\mathbf{G}/\mathbf{P}$ for any splitting Borel subgroup $\mathbf{B} \subset \mathbf{G}$. We also point out the Bruhat decomposition into double cosets of the ind-group $\mathbf{G}$ which results from a Schubert decomposition of $\mathbf{G}/\mathbf{P}$.

In the last part of the paper we study the smoothness of Schubert ind-varieties which we define as closures of Schubert cells. We establish a criterion for smoothness which allows us to conclude that certain known criteria for smoothness of finite-dimensional Schubert varieties pass to the limit at infinity.

In his work [12], H. Salmasian introduced Schubert ind-subvarieties of $\mathbf{G}/\mathbf{B}$ as arbitrary direct limits of Schubert varieties on finite-dimensional flag subvarieties of $\mathbf{G}/\mathbf{B}$. He showed that such an ind-variety may be singular at all of its points. With our definition, which takes into account the natural action of $\mathbf{G}$ on $\mathbf{G}/\mathbf{B}$, a Schubert ind-variety has always a smooth big cell.

2010 *Mathematics Subject Classification.* 14M15, 14M17, 20G99.

Key words and phrases. Classical ind-group, Bruhat decomposition, Schubert decomposition, generalized flag, homogeneous ind-variety.

2. Preliminaries

In what follows $\mathbb{K}$ is an algebraically closed field of characteristic zero. All varieties and algebraic groups are defined over $\mathbb{K}$. If A is a finite or infinite set, then $|A|$ denotes its cardinality. If A is a subset of the linear space V, then $\langle A\rangle$ denotes the linear subspace spanned by A.

2.1. **Ind-varieties.** An *ind-variety* is the direct limit $\mathbf{X} = \varinjlim X_n$ of a chain of morphisms of algebraic varieties

(1) $$X_1 \stackrel{\varphi_1}{\to} X_2 \stackrel{\varphi_2}{\to} \cdots \stackrel{\varphi_{n-1}}{\to} X_n \stackrel{\varphi_n}{\to} X_{n+1} \stackrel{\varphi_{n+1}}{\to} \cdots .$$

Note that the direct limit of the chain (1) does not change if we replace the sequence $\{X_n\}_{n\geq 1}$ by a subsequence $\{X_{i_n}\}_{n\geq 1}$ and the morphisms φ_n by the compositions $\tilde{\varphi}_{i_n} := \varphi_{i_{n+1}-1}\circ\cdots\circ\varphi_{i_n+1}\circ\varphi_{i_n}$. Let $\mathbf{X}'$ be a second ind-variety obtained as the direct limit of a chain

$$X'_1 \stackrel{\varphi'_1}{\to} X'_2 \stackrel{\varphi'_2}{\to} \cdots \stackrel{\varphi'_{n-1}}{\to} X'_n \stackrel{\varphi'_n}{\to} X'_{n+1} \stackrel{\varphi'_{n+1}}{\to} \cdots .$$

A *morphism of ind-varieties* $\mathbf{f} : \mathbf{X}' \to \mathbf{X}$ is a map from $\varinjlim X'_n$ to $\varinjlim X_n$ induced by a collection of morphisms of algebraic varieties $\{f_n : X'_n \to X_{i_n}\}_{n\geq 1}$ such that $\tilde{\varphi}_{i_n}\circ f_n = f_{n+1}\circ\varphi'_n$ for all $n\geq 1$. The *identity morphism* $\mathrm{id}_{\mathbf{X}}$ is a morphism that induces the identity as a set-theoretic map $\mathbf{X}\to\mathbf{X}$. A morphism $\mathbf{f} : \mathbf{X}'\to\mathbf{X}$ is an *isomorphism* if there exists a morphism $\mathbf{g} : \mathbf{X}\to\mathbf{X}'$ such that $\mathbf{g}\circ\mathbf{f} = \mathrm{id}_{\mathbf{X}'}$ and $\mathbf{f}\circ\mathbf{g} = \mathrm{id}_{\mathbf{X}}$.

Any ind-variety $\mathbf{X}$ is endowed with a topology by declaring a subset $\mathbf{U}\subset\mathbf{X}$ *open* if its inverse image by the natural map $X_m \to \varinjlim X_n$ is open for all m. Clearly, any open (resp., closed) (resp., locally closed) subset $\mathbf{Z}$ of $\mathbf{X}$ has a structure of ind-variety induced by the ind-variety structure on $\mathbf{X}$. We call $\mathbf{Z}$ an *ind-subvariety* of $\mathbf{X}$.

In what follows we only consider chains (1) where the morphisms φ_n are inclusions, so that we can write $\mathbf{X} = \bigcup_{n\geq 1} X_n$. Then the sequence $\{X_n\}_{n\geq 1}$ is called *exhaustion* of $\mathbf{X}$.

Let $x\in\mathbf{X}$, so that $x\in X_n$ for n large enough. Let $\mathfrak{m}_{n,x}\subset\mathcal{O}_{X_n,x}$ be the maximal ideal of the localization at x of $\mathcal{O}_{X_n}$. For each $k\geq 1$ we have an epimorphism

(2) $$\alpha_{n,k} : S^k(\mathfrak{m}_{n,x}/\mathfrak{m}_{n,x}^2) \to \mathfrak{m}_{n,x}^k/\mathfrak{m}_{n,x}^{k+1}.$$

Note that the point x is smooth in X_n if and only if $\alpha_{n,k}$ is an isomorphism for all k. By taking the inverse limit, we obtain a map

$$\hat{\alpha}_k := \varprojlim \alpha_{n,k} : \varprojlim S^k(\mathfrak{m}_{n,x}/\mathfrak{m}_{n,x}^2) \to \varprojlim \mathfrak{m}_{n,x}^k/\mathfrak{m}_{n,x}^{k+1}$$

which is an epimorphism for all k. We say that x is a *smooth point* of $\mathbf{X}$ if and only if $\hat{\alpha}_k$ is an isomorphism for all k. We say that x is a *singular point* otherwise. The notion of smoothness of a point is independent of the choice of exhaustion $\{X_n\}_{n\geq 1}$ of $\mathbf{X}$. We say that $\mathbf{X}$ is *smooth* if every point $x\in\mathbf{X}$ is smooth. As general references on smooth ind-varieties see [8, Chapter 4] and [11].

Example 1. (a) Assume that every variety X_n in the chain (1) is an affine space, every image $\varphi_n(X_n)$ is an affine subspace of X_{n+1}, and $\lim_{n\to\infty}\dim X_n = \infty$. Then, up to isomorphism, $\mathbf{X} = \varinjlim X_n$ is independent of the choice of $\{X_n,\varphi_n\}_{n\geq 1}$ with these properties. We write $\mathbf{X} = \mathbb{A}^\infty$ and call it the *infinite-dimensional affine space*. For instance, $\mathbb{A}^\infty$ admits the exhaustion $\mathbb{A}^\infty = \bigcup_{n\geq 1}\mathbb{A}^n$ where $\mathbb{A}^n$ stands for the n-dimensional affine space. The infinite-dimensional affine space $\mathbb{A}^\infty$ is a smooth ind-variety.
(b) If every variety X_n in the chain (1) is a projective space, every image $\varphi_n(X_n)$ is a projective subspace of X_{n+1}, and $\lim_{n\to\infty}\dim X_n = \infty$, then $\mathbf{X} = \varinjlim X_n$ is independent of the choice of $\{X_n,\varphi_n\}_{n\geq 1}$ with these properties. We write $\mathbf{X} = \mathbb{P}^\infty = \bigcup_{n\geq 1}\mathbb{P}^n$ and call $\mathbb{P}^\infty$ the *infinite-dimensional projective space*. The infinite-dimensional projective space $\mathbb{P}^\infty$ is also a smooth ind-variety.

A *cell decomposition* of an ind-variety $\mathbf{X}$ is a decomposition $\mathbf{X} = \bigsqcup_{i\in I}\mathbf{X}_i$ into locally closed ind-subvarieties $\mathbf{X}_i$, each being a finite-dimensional or infinite-dimensional affine space, and such that the closure of each $\mathbf{X}_i$ in $\mathbf{X}$ is a union of some subsets $\mathbf{X}_j$ $(j\in I)$.

2.2. **Ind-groups.** An *ind-group* is an ind-variety $\mathbf{G}$ endowed with a group structure such that the multiplication $\mathbf{G} \times \mathbf{G} \to \mathbf{G}$, $(g, h) \mapsto gh$, and the inversion $\mathbf{G} \to \mathbf{G}$, $g \mapsto g^{-1}$ are morphisms of ind-varieties. A *morphism of ind-groups* $\mathbf{f} : \mathbf{G}' \to \mathbf{G}$ is by definition a morphism of groups which is also a morphism of ind-varieties. A *closed ind-subgroup* is a subgroup $\mathbf{H} \subset \mathbf{G}$ which is also a closed ind-subvariety.

We only consider *locally linear ind-groups*, i.e., ind-groups admitting an exhaustion $\{G_n\}_{n\geq 1}$ by linear algebraic groups. Moreover, we focus on the *classical ind-groups* $GL(\infty)$, $O(\infty)$, $Sp(\infty)$, which are obtained as subgroups of the group $Aut(V)$ of linear automorphisms of a countable-dimensional vector space V:

- Let E be a basis of V. Define $\mathbf{G}(E)$ as the subgroup of elements $g \in Aut(V)$ such that $g(e) = e$ for all but finitely many basis vectors $e \in E$. Given any filtration $E = \bigcup_{n\geq 1} E_n$ of the basis E by finite subsets, we have

$$\mathbf{G}(E) = \bigcup_{n\geq 1} G(E_n) \tag{3}$$

  where $G(E_n)$ stands for $GL(\langle E_n\rangle)$. Thus $\mathbf{G}(E)$ is a locally linear ind-group. We also write $\mathbf{G}(E) = GL(\infty)$.
- Assume that the space V is endowed with a nondegenerate symmetric or skew-symmetric bilinear form ω. A basis E of V is called ω-isotropic if it is equipped with an involution $i_E : E \to E$ with at most one fixed point, such that $\omega(e, e') = 0$ for any $e, e' \in E$ unless $e' = i_E(e)$. Given an ω-isotropic basis E of V, define $\mathbf{G}^\omega(E)$ as the subgroup of elements $g \in \mathbf{G}(E)$ which preserve the bilinear form ω. If a filtration $E = \bigcup_{n\geq 1} E_n$ of the basis E by i_E-stable finite subsets is fixed, we have

$$\mathbf{G}^\omega(E) = \bigcup_{n\geq 1} G^\omega(E_n) \tag{4}$$

  where $G^\omega(E_n)$ stands for the subgroup of elements $g \in G(E_n)$ preserving the restriction of ω. Thereby $\mathbf{G}^\omega(E)$ has a natural structure of locally linear ind-group. We also write $\mathbf{G}^\omega(E) = Sp(\infty)$ when ω is symplectic, and $\mathbf{G}^\omega(E) = O(\infty)$ when ω is symmetric.

Remark 1. (a) Note that the group $\mathbf{G}(E) = GL(\infty)$ depends on the choice of the basis E. For this reason, in what follows, we prefer the notation $\mathbf{G}(E)$ instead of $GL(\infty)$.

An alternative construction of $GL(\infty)$ is as follows. Note that the dual space V^* is uncountable dimensional. Let $V_* \subset V^*$ be a countable-dimensional subspace such that the pairing $V_* \times V \to \mathbb{K}$ is nondegenerate. Then the group

$$\mathbf{G}(V, V_*) := \{g \in Aut(V) : g(V_*) = V_* \text{ and there are finite-codimensional subspaces of } V \text{ and } V_* \text{ fixed pointwise by } g\}$$

is an ind-group isomorphic to $GL(\infty)$. Moreover, we have $\mathbf{G}(V, V_*) = \mathbf{G}(E)$ whenever V_* is spanned by the dual family of E.

(b) The form ω induces a countable-dimensional subspace $V_* := \{\omega(v, \cdot) : v \in V\} \subset V^*$ of the dual space. Then the group

$$\mathbf{G}(V, \omega) := \{g \in \mathbf{G}(V, V_*) : g \text{ preserves } \omega\}$$

is an ind-subgroup of $\mathbf{G}(V, V_*)$ isomorphic to $Sp(\infty)$ (if ω is symplectic) or $O(\infty)$ (if ω is symmetric). The equality $\mathbf{G}(V, \omega) = \mathbf{G}^\omega(E)$ holds whenever E is an ω-isotropic basis.

(c) If ω is symplectic, then the involution $i_E : E \to E$ has no fixed point; the basis E is said to be *of type C* in this case. If ω is symmetric, then the involution $i_E : E \to E$ can have one fixed point, in which case the basis E is said to be *of type B*; if i_E has no fixed point, the basis E is said to be *of type D*. Bases of both types B and D exist in V (see [4, Lemma 4.2]).

In the rest of the paper, we fix once and for all a basis E of V and a filtration $E = \bigcup_{n\geq 1} E_n$ by finite subsets. We assume that the basis E is ω-isotropic and that the subsets E_n are i_E-stable whenever the bilinear form ω is considered.

Moreover, if the form ω is symmetric, in view of Remark 1 (b)–(c) in what follows we assume that the basis E is of type B and that every subset E_n of the filtration contains the fixed point of the involution i_E. This convention ensures that the variety of isotropic flags of a given type of each finite-dimensional space

$\langle E_n \rangle$ is connected and $G^\omega(E_n)$-homogeneous. Similarly, every i_E-stable finite subset of E considered in the sequel is assumed to contain the fixed point of i_E.

By $\mathbf{G}$ we denote one of the ind-groups $\mathbf{G}(E)$, $\mathbf{G}^\omega(E)$.

Let $\mathbf{H}$ be the subgroup of elements $g \in \mathbf{G}$ which are diagonal in the basis E. Then $\mathbf{H}$ is a closed ind-subgroup of $\mathbf{G}$ called *splitting Cartan subgroup*. A closed ind-subgroup $\mathbf{B} \subset \mathbf{G}$ which contains $\mathbf{H}$ is called *splitting Borel subgroup* if it is locally solvable (i.e., every finite-dimensional ind-subgroup $B \subset \mathbf{B}$ is solvable) and is maximal with this property. A closed ind-subgroup which contains such a splitting Borel subgroup $\mathbf{B}$ is called *splitting parabolic subgroup*. Equivalently, a closed ind-subgroup $\mathbf{P}$ of $\mathbf{G}$ containing $\mathbf{H}$ is a splitting parabolic subgroup of $\mathbf{G}$ if and only if $\mathbf{P} \cap G_n$ is a parabolic subgroup of G_n for all $n \geq 1$, where $\mathbf{G} = \bigcup_{n\geq 1} G_n$ is the natural exhaustion of (3) or (4). The quotient $\mathbf{G}/\mathbf{P} = \bigcup_{n\geq 1} G_n/(\mathbf{P} \cap G_n)$ is a *locally projective* ind-variety (i.e., an ind-variety exhausted by projective varieties); note however that $\mathbf{G}/\mathbf{P}$ is in general not a *projective* ind-variety (i.e., is not isomorphic to a closed ind-subvariety of the infinite-dimensional projective space $\mathbb{P}^\infty$): see [4, Proposition 7.2] and [5, Proposition 15.1].

In [4] it is shown that the ind-variety $\mathbf{G}/\mathbf{P}$ can be interpreted as an ind-variety of certain generalized flags. This construction is reviewed in the following section.

3. Ind-varieties of generalized flags

In Section 3.1 we recall from [3, 4] the notion of generalized flag and the correspondence between splitting parabolic subgroups $\mathbf{P}$ of $\mathbf{G}(E)$ and E-compatible generalized flags $\mathcal{F}$. We also recall from [4] the construction of the ind-varieties $\mathbf{Fl}(\mathcal{F}, E)$ of generalized flags and their correspondence with homogeneous ind-spaces of the form $\mathbf{G}(E)/\mathbf{P}$.

In Section 3.2 we recall from [3, 4] the notion of ω-isotropic generalized flags and the construction of the ind-variety $\mathbf{Fl}(\mathcal{F}, \omega, E)$ of ω-isotropic generalized flags, as well as the correspondence with splitting parabolic subgroups of $\mathbf{G}^\omega(E)$ and the corresponding homogeneous ind-spaces.

For later use, some technical aspects of the construction of the ind-varieties $\mathbf{Fl}(\mathcal{F}, E)$ and $\mathbf{Fl}(\mathcal{F}, \omega, E)$ are emphasized in Section 3.3.

3.1. **Ind-variety of generalized flags.** By *chain* of subspaces of V we mean a set of vector subspaces of V which is totally ordered by inclusion.

Definition 1 ([3, 4])**.** A *generalized flag* is a chain $\mathcal{F}$ of subspaces of V satisfying the following additional conditions:

(i) every $F \in \mathcal{F}$ has an immediate predecessor F' in $\mathcal{F}$ or an immediate successor F'' in $\mathcal{F}$;

(ii) for every nonzero vector $v \in V$, there is a pair (F', F'') of consecutive elements of $\mathcal{F}$ such that $v \in F'' \setminus F'$.

Let $A_\mathcal{F}$ denote the set of pairs (F', F'') of consecutive subspaces $F', F'' \in \mathcal{F}$. The set $A_\mathcal{F}$ is totally ordered by the inclusion of pairs. Given a totally ordered set $(A, \preceq)$, we denote by $\mathbf{Fl}_A(V)$ the set of generalized flags such that $(A_\mathcal{F}, \subset)$ is isomorphic to $(A, \preceq)$. Equivalently, $\mathbf{Fl}_A(V)$ is the set of generalized flags $\mathcal{F}$ which can be written in the form

$$\mathcal{F} = \{F'_\alpha, F''_\alpha : \alpha \in A\}, \tag{5}$$

where F'_α, F''_α are subspaces of V such that

$$\begin{cases} F'_\alpha \subsetneq F''_\alpha \text{ for all } \alpha \in A; \\ F''_\alpha \subset F'_\beta \text{ whenever } \alpha \prec \beta \text{ (possibly } F''_\alpha = F'_\beta); \\ V \setminus \{0\} = \bigsqcup_{\alpha \in A} F''_\alpha \setminus F'_\alpha. \end{cases} \tag{6}$$

Definition 2. Let L be a basis of the space V. A generalized flag $\mathcal{F} = \{F'_\alpha, F''_\alpha : \alpha \in A\} \in \mathbf{Fl}_A(V)$ is said to be *compatible with* L if there is a (necessarily surjective) map $\sigma : L \to A$ such that

$$F'_\alpha = \langle e \in L : \sigma(e) \prec \alpha \rangle, \quad F''_\alpha = \langle e \in L : \sigma(e) \preceq \alpha \rangle$$

for all $\alpha \in A$.

Every generalized flag admits a compatible basis (see [4, Proposition 4.1]). The group $\mathbf{G}(E)$ acts on generalized flags in a natural way. Let $\mathbf{H}(E) \subset \mathbf{G}(E)$ be the splitting Cartan subgroup formed by elements diagonal in E. It is easy to see that a generalized flag $\mathcal{F}$ is compatible with E if and only if it is preserved by $\mathbf{H}(E)$. We denote by $\mathbf{P}_\mathcal{F} \subset \mathbf{G}(E)$ the subgroup of all elements which preserve $\mathcal{F}$.

Proposition 1 ([3, 4])**.** (a) *If $\mathcal{F}$ is a generalized flag compatible with E, then $\mathbf{P}_{\mathcal{F}}$ is a splitting parabolic subgroup of $\mathbf{G}(E)$ containing $\mathbf{H}(E)$.*
(b) *The map $\mathcal{F} \mapsto \mathbf{P}_{\mathcal{F}}$ is a bijection between generalized flags compatible with E and splitting parabolic subgroups of $\mathbf{G}(E)$ containing $\mathbf{H}(E)$.*
(c) *A splitting parabolic subgroup $\mathbf{P}_{\mathcal{F}}$ is a splitting Borel subgroup if and only if the generalized flag $\mathcal{F}$ is maximal (i.e., $\dim F''/F' = 1$ for every pair (F', F'') of consecutive elements of $\mathcal{F}$).*

Remark 2. Proposition 1 (c) can be interpreted as a version of Lie's theorem for the action of any splitting Borel subgroup on the space V. A general version of Lie's theorem has been proved by J. Hennig in [6].

Definition 3 ([4])**.** (a) We say that a generalized flag $\mathcal{F}$ is *weakly compatible with* E if $\mathcal{F}$ is compatible with a basis L of V such that $E \setminus E \cap L$ is a finite set (equivalently $\mathrm{codim}_V \langle E \cap L \rangle$ is finite).
(b) Two generalized flags $\mathcal{F}, \mathcal{G}$ are said to be *E-commensurable* if both $\mathcal{F}$ and $\mathcal{G}$ are weakly compatible with E, and there are an isomorphism of ordered sets $\phi : \mathcal{F} \to \mathcal{G}$ and a finite-dimensional subspace $U \subset V$ such that

(i) $\phi(F) + U = F + U$ for all $F \in \mathcal{F}$,
(ii) $\dim \phi(F) \cap U = \dim F \cap U$ for all $F \in \mathcal{F}$.

Remark 3. (a) Clearly, if $\mathcal{F}, \mathcal{G}$ are E-commensurable with respect to a finite-dimensional subspace U, then $\mathcal{F}, \mathcal{G}$ are E-commensurable with respect to any finite-dimensional subspace $U' \subset V$ such that $U' \supset U$.
(b) E-commensurability is an equivalence relation on the set of generalized flags weakly compatible with E.

Let $\mathcal{F}$ be a generalized flag compatible with E. We denote by $\mathbf{Fl}(\mathcal{F}, E)$ the set of all generalized flags which are E-commensurable with $\mathcal{F}$.

Proposition 2 ([4])**.** *The set $\mathbf{Fl}(\mathcal{F}, E)$ is endowed with a natural structure of ind-variety. Moreover, this ind-variety is $\mathbf{G}(E)$-homogeneous and the map $g \mapsto g\mathcal{F}$ induces an isomorphism of ind-varieties $\mathbf{G}(E)/\mathbf{P}_{\mathcal{F}} \cong \mathbf{Fl}(\mathcal{F}, E)$.*

3.2. **Ind-variety of isotropic generalized flags.** In this section we assume that the space V is endowed with a nondegenerate symmetric or skew-symmetric bilinear form ω. We write $U^\perp$ for the orthogonal subspace of the subspace $U \subset V$ with respect to ω. We assume that the basis E is ω-isotropic, i.e., endowed with an involution $i_E : E \to E$ with at most one fixed point and such that, for any $e, e' \in E$, $\omega(e, e') = 0$ unless $e' = i_E(e)$.

Definition 4 ([3, 4])**.** A generalized flag $\mathcal{F}$ is said to be *ω-isotropic* if $F^\perp \in \mathcal{F}$ whenever $F \in \mathcal{F}$, and if the map $F \mapsto F^\perp$ is an involution of $\mathcal{F}$.

For $\mathcal{F}$ as in Definition 4, the involution $F \mapsto F^\perp$ is an anti-automorphism of the ordered set $(\mathcal{F}, \subset)$, i.e., it reverses the inclusion relation. Moreover, it induces an involutive anti-automorphism $(F'_\alpha, F''_\alpha) \mapsto ((F''_\alpha)^\perp, (F'_\alpha)^\perp)$ of the set $(A_{\mathcal{F}}, \subset)$ of pairs of consecutive subspaces of $\mathcal{F}$. Given a totally ordered set $(A, \preceq, i_A)$ equipped with an involutive anti-automorphism $i_A : A \to A$, we denote by $\mathbf{Fl}^\omega_A(V)$ the set of generalized flags $\mathcal{F} \in \mathbf{Fl}_A(V)$ (see (5)–(6)) which are ω-isotropic and satisfy the condition

$$((F''_\alpha)^\perp, (F'_\alpha)^\perp) = (F'_{i_A(\alpha)}, F''_{i_A(\alpha)}) \ \text{ for all } \alpha \in A. \tag{7}$$

Remark 4. Note that the set A decomposes as

$$A = A_\ell \sqcup A_0 \sqcup A_r$$

where $A_\ell = \{\alpha \in A : \alpha \prec i_A(\alpha)\}$, $A_0 = \{\alpha \in A : \alpha = i_A(\alpha)\}$ (formed by at most one element), $A_r = \{\alpha \in A : \alpha \succ i_A(\alpha)\}$, and the map i_A restricts to bijections $A_\ell \to A_r$ and $A_r \to A_\ell$.

Given any $\mathcal{F} \in \mathbf{Fl}^\omega_A(V)$, we set $\mathcal{T}' = \bigcup_{\alpha \in A_\ell} F''_\alpha$ and $\mathcal{T}'' = \bigcap_{\alpha \in A_r} F'_\alpha$. Clearly, $\mathcal{T}' \subset \mathcal{T}''$, moreover it is easy to see that $(\mathcal{T}')^\perp = \mathcal{T}''$. We have either $\mathcal{T}' = \mathcal{T}''$ or $\mathcal{T}' \subsetneq \mathcal{T}''$. In the former case, the anti-automorphism i_A has no fixed point, hence $A = A_\ell \sqcup A_r$. In the latter case, the subspaces $\mathcal{T}', \mathcal{T}''$ necessarily belong to $\mathcal{F}$, moreover we have $(\mathcal{T}', \mathcal{T}'') = (F'_{\alpha_0}, F''_{\alpha_0})$ where $\alpha_0 \in A$ is the unique fixed point of i_A; thus $A = A_\ell \sqcup \{\alpha_0\} \sqcup A_r$ in this case.

The following lemma shows that the notions of compatibility and weak-compatibility with a basis (Definitions 2–3) translate in a natural way to the context of ω-isotropic generalized flags and bases.

Lemma 1. *Let $\mathcal{F} \in \mathbf{Fl}^\omega_A(V)$, with $(A, \preceq, i_A)$ as above.*
(a) *Let L be an ω-isotropic basis with corresponding involution $i_L : L \to L$. Assume that $\mathcal{F}$ is compatible with L in the sense of Definition 2, via a surjective map $\sigma : L \to A$. Then the map σ satisfies $\sigma \circ i_L = i_A \circ \sigma$.*
(b) *Assume that $\mathcal{F}$ is weakly compatible with E. Then there is an ω-isotropic basis L such that the set $E \setminus E \cap L$ is finite and $\mathcal{F}$ is compatible with L.*

Proof. (a) For every $e \in L$, we have $e \in F''_{\sigma(e)} \setminus F'_{\sigma(e)}$. Then $i_L(e) \in (F'_{\sigma(e)})^\perp \setminus (F''_{\sigma(e)})^\perp$. The definition of i_A yields $\sigma(i_L(e)) = i_A(\sigma(e))$.
(b) Let L be a basis of V such that $E \setminus E \cap L$ is finite and $\mathcal{F}$ is compatible with L. Take a subset $E' \subset E$ stable by the involution i_E, such that i_E has no fixed point in E', $E \setminus E'$ is finite, and $E' \subset E \cap L$. Then $V'' := \langle E \setminus E' \rangle$ is a finite-dimensional space and the restriction of ω to V'' is nondegenerate. The intersections $\mathcal{F}|_{V''} := \{F \cap V'' : F \in \mathcal{F}\}$ form an isotropic flag of V''. Since V'' is finite dimensional, it is routine to find an ω-isotropic basis E'' of V'' such that $\mathcal{F}|_{V''}$ is compatible with E''. Then $E' \cup E''$ is an ω-isotropic basis of V, and $\mathcal{F}$ is compatible with $E' \cup E''$. ☐

The group $\mathbf{G}^\omega(E)$ acts in a natural way on ω-isotropic generalized flags. Let $\mathbf{H}^\omega(E) \subset \mathbf{G}^\omega(E)$ be the splitting Cartan subgroup formed by elements diagonal in E. An ω-isotropic generalized flag is compatible with the basis E if and only if it is preserved by $\mathbf{H}^\omega(E)$. Given an ω-isotropic generalized flag $\mathcal{F}$ compatible with E, we denote by $\mathbf{P}^\omega_\mathcal{F} \subset \mathbf{G}^\omega(E)$ the subgroup of all elements which preserve $\mathcal{F}$. Moreover, we denote by $\mathbf{Fl}(\mathcal{F}, \omega, E)$ the set of all ω-isotropic generalized flags which are E-commensurable with $\mathcal{F}$.

Proposition 3 ([3, 4])**.** (a) *The map $\mathcal{F} \mapsto \mathbf{P}^\omega_\mathcal{F}$ is a bijection between ω-isotropic generalized flags compatible with E and splitting parabolic subgroups of $\mathbf{G}^\omega(E)$ containing $\mathbf{H}^\omega(E)$.*
(b) *A splitting parabolic subgroup $\mathbf{P}^\omega_\mathcal{F}$ is a splitting Borel subgroup of $\mathbf{G}^\omega(E)$ if and only if the generalized flag $\mathcal{F}$ is maximal.*
(c) *The set $\mathbf{Fl}(\mathcal{F}, \omega, E)$ is endowed with a natural structure of ind-variety. This ind-variety is $\mathbf{G}^\omega(E)$-homogeneous and the map $g \mapsto g\mathcal{F}$ induces an isomorphism of ind-varieties $\mathbf{G}^\omega(E)/\mathbf{P}^\omega_\mathcal{F} \cong \mathbf{Fl}(\mathcal{F}, \omega, E)$.*

3.3. **Structure of ind-variety on $\mathbf{Fl}(\mathcal{F}, E)$ and $\mathbf{Fl}(\mathcal{F}, \omega, E)$.** In this section we present the structure of ind-variety on $\mathbf{Fl}(\mathcal{F}, E)$ and $\mathbf{Fl}(\mathcal{F}, \omega, E)$ mentioned in Propositions 2–3.

We assume that $\mathcal{F}$ is a generalized flag compatible with the basis E. Let $(A, \preceq)$ be a totally ordered set such that $\mathcal{F} \in \mathbf{Fl}_A(V)$. Hence we can write $\mathcal{F} = \{F'_\alpha, F''_\alpha : \alpha \in A\}$. Let $\sigma : E \to A$ be the surjective map corresponding to $\mathcal{F}$ in the sense of Definition 2.

Let $I \subset E$ be a finite subset. The generalized flag $\mathcal{F}$ gives rise to a (finite) flag $\mathcal{F}|_I$ of the finite-dimensional vector space $\langle I \rangle$ by letting

$$\mathcal{F}|_I := \{F \cap \langle I \rangle : F \in \mathcal{F}\} = \{F'_\alpha \cap \langle I \rangle, F''_\alpha \cap \langle I \rangle : \alpha \in A\}.$$

Let

$$d'_\alpha := \dim F'_\alpha \cap \langle I \rangle = |\{e \in I : \sigma(e) \prec \alpha\}| \quad \text{and} \quad d''_\alpha := \dim F''_\alpha \cap \langle I \rangle = |\{e \in I : \sigma(e) \preceq \alpha\}|.$$

We denote by $\mathrm{Fl}(\mathcal{F}, I)$ the projective variety of flags in the space $\langle I \rangle$ of the form $\{M'_\alpha, M''_\alpha : \alpha \in A\}$ where $M'_\alpha, M''_\alpha \subset \langle I \rangle$ are linear subspaces such that

$$\dim M'_\alpha = d'_\alpha,\ \dim M''_\alpha = d''_\alpha,\ M'_\alpha \subset M''_\alpha \text{ for all } \alpha \in A, \text{ and } M''_\alpha \subset M'_\beta \text{ whenever } \alpha \prec \beta.$$

If $J \subset E$ is another finite subset such that $I \subset J$, we define an embedding $\iota_{I,J} : \mathrm{Fl}(\mathcal{F}, I) \hookrightarrow \mathrm{Fl}(\mathcal{F}, J)$, $\{M'_\alpha, M''_\alpha : \alpha \in A\} \mapsto \{N'_\alpha, N''_\alpha : \alpha \in A\}$ by letting

$$N'_\alpha = M'_\alpha \oplus \langle e \in J \setminus I : \sigma(e) \prec \alpha \rangle \quad \text{and} \quad N''_\alpha = M''_\alpha \oplus \langle e \in J \setminus I : \sigma(e) \preceq \alpha \rangle \quad \text{for all } \alpha \in A.$$

If we consider a filtration $E = \bigcup_{n \geq 1} E_n$ of the basis E by finite subsets, then we obtain a chain of morphisms of projective varieties

$$\mathrm{Fl}(\mathcal{F}, E_1) \overset{\iota_1}{\hookrightarrow} \mathrm{Fl}(\mathcal{F}, E_2) \overset{\iota_2}{\hookrightarrow} \cdots \overset{\iota_{n-1}}{\hookrightarrow} \mathrm{Fl}(\mathcal{F}, E_n) \overset{\iota_n}{\hookrightarrow} \mathrm{Fl}(\mathcal{F}, E_{n+1}) \overset{\iota_{n+1}}{\hookrightarrow} \cdots \tag{8}$$

where $\iota_n := \iota_{E_n, E_{n+1}}$.

Proposition 4 ([4])**.** *The set* $\mathbf{Fl}(\mathcal{F},E)$ *is the direct limit of the chain of morphisms (8). Hence* $\mathbf{Fl}(\mathcal{F},E)$ *is endowed with a structure of ind-variety. Moreover, this structure is independent of the filtration* $\{E_n\}_{n\geq 1}$ *of the basis* E*.*

We assume next that the space V is endowed with a nondegenerate symmetric or skew-symmetric bilinear form ω, that the basis E is ω-isotropic with corresponding involution $i_E : E \to E$, that the ordered set $(A,\preceq)$ is equipped with an anti-automorphism $i_A : A \to A$, and that the surjection $\sigma : E \to A$ satisfies $\sigma \circ i_E = i_A \circ \sigma$ so that $\mathcal{F}$ is an ω-isotropic generalized flag.

Consider an i_E-stable finite subset $I \subset E$. Then the restriction of ω to the space $\langle I\rangle$ is nondegenerate. Let $\mathrm{Fl}(\mathcal{F},\omega,I) \subset \mathrm{Fl}(\mathcal{F},I)$ be the (closed) subvariety formed by flags $\{M'_\alpha, M''_\alpha : \alpha \in A\}$ such that

$$((M''_\alpha)^{\perp_I}, (M'_\alpha)^{\perp_I}) = (M'_{i_A(\alpha)}, M''_{i_A(\alpha)}) \ \text{ for all } \alpha \in A,$$

where the notation $\perp_I$ stands for orthogonal subspace in the space $(\langle I\rangle,\omega)$. If $J \subset E$ is another i_E-stable finite subset, then the embedding $\iota_{I,J}$ restricts to an embedding $\iota^\omega_{I,J} : \mathrm{Fl}(\mathcal{F},\omega,I) \hookrightarrow \mathrm{Fl}(\mathcal{F},\omega,J)$. Consequently, for a filtration $E = \bigcup_{n\geq 1} E_n$ by i_E-stable finite subsets, we obtain a chain of morphisms of projective varieties

$$\mathrm{Fl}(\mathcal{F},\omega,E_1) \overset{\iota^\omega_1}{\hookrightarrow} \mathrm{Fl}(\mathcal{F},\omega,E_2) \overset{\iota^\omega_2}{\hookrightarrow} \cdots \overset{\iota^\omega_{n-1}}{\hookrightarrow} \mathrm{Fl}(\mathcal{F},\omega,E_n) \overset{\iota^\omega_n}{\hookrightarrow} \mathrm{Fl}(\mathcal{F},\omega,E_{n+1}) \overset{\iota^\omega_{n+1}}{\hookrightarrow} \cdots \tag{9}$$

where $\iota^\omega_n := \iota^\omega_{E_n,E_{n+1}}$.

Proposition 5 ([4])**.** *The set* $\mathbf{Fl}(\mathcal{F},\omega,E)$ *is the direct limit of the chain of morphisms (9). Hence* $\mathbf{Fl}(\mathcal{F},\omega,E)$ *is endowed with a structure of ind-variety, independent of the filtration* $\{E_n\}_{n\geq 1}$*. Moreover,* $\mathbf{Fl}(\mathcal{F},\omega,E)$ *is a closed ind-subvariety of* $\mathbf{Fl}(\mathcal{F},E)$*.*

4. Schubert decomposition of $\mathbf{Fl}(\mathcal{F},E)$ and $\mathbf{Fl}(\mathcal{F},\omega,E)$

Let $\mathbf{G}$ be one of the groups $\mathbf{G}(E)$ or $\mathbf{G}^\omega(E)$. Let $\mathbf{P}$ and $\mathbf{B}$ be respectively a splitting parabolic subgroup and a splitting Borel subgroup of $\mathbf{G}$, both containing the splitting Cartan subgroup $\mathbf{H} = \mathbf{H}(E)$ or $\mathbf{H}^\omega(E)$. From the previous section we know that the homogeneous space $\mathbf{G}/\mathbf{P}$ can be viewed as an ind-variety of generalized flags of the form $\mathbf{Fl}(\mathcal{F},E)$ or $\mathbf{Fl}(\mathcal{F},\omega,E)$. In this section we describe the decomposition of $\mathbf{G}/\mathbf{P}$ into $\mathbf{B}$-orbits. The main results are stated in Theorem 1 in the case of $\mathbf{G} = \mathbf{G}(E)$ and in Theorem 2 in the case of $\mathbf{G} = \mathbf{G}^\omega(E)$. In both cases it is shown that the $\mathbf{B}$-orbits form a cell decomposition of $\mathbf{G}/\mathbf{P}$, and their dimensions and closures are expressed in combinatorial terms. In Section 4.3 we derive the decomposition of the ind-group $\mathbf{G}$ into double cosets. Unlike the case of Kac–Moody groups, the $\mathbf{B}$-orbits of $\mathbf{G}/\mathbf{P}$ can be infinite dimensional. The cases where all orbits are finite dimensional (resp., infinite dimensional) are characterized in Section 4.4. In Section 4.5 we focus on the situation where $\mathbf{G}/\mathbf{P}$ is an ind-grassmannian.

In this section the results are stated without proofs. The proofs are given in Section 5.

4.1. Decomposition of $\mathbf{Fl}(\mathcal{F},E)$.

Let $\mathbf{G} = \mathbf{G}(E)$, $\mathbf{H} = \mathbf{H}(E)$, and $\mathbf{P}$, $\mathbf{B}$ be as above. By Propositions 1–2 there is a generalized flag $\mathcal{F}$ compatible with E such that $\mathbf{P} = \mathbf{P}_\mathcal{F}$ is the subgroup of elements $g \in \mathbf{G}(E)$ preserving $\mathcal{F}$, and the homogenenous space $\mathbf{G}(E)/\mathbf{P}$ is isomorphic to the ind-variety of generalized flags $\mathbf{Fl}(\mathcal{F},E)$. The precise description of the decomposition of $\mathbf{Fl}(\mathcal{F},E)$ into $\mathbf{B}$-orbits is the object of this section. It requires some preliminaries and notation.

We denote by $\mathbf{W}(E)$ the group of permutations $w : E \to E$ such that $w(e) = e$ for all but finitely many $e \in E$. In particular, $\mathbf{W}(E)$ is isomorphic to the infinite symmetric group $\mathfrak{S}_\infty$. Note that we have

$$\mathbf{W}(E) = \bigcup_{n\geq 1} W(E_n)$$

where $W(E_n)$ is the Weyl group of $G(E_n)$.

Let $(A,\preceq_A) := (A_\mathcal{F},\subset)$ be the set of pairs of consecutive elements of $\mathcal{F}$, so that $\mathcal{F} \in \mathbf{Fl}_A(V)$ and in fact $\mathbf{Fl}(\mathcal{F},E) \subset \mathbf{Fl}_A(V)$. Let $\mathfrak{S}(E,A)$ be the set of surjective maps $\sigma : E \to A$. For $\sigma \in \mathfrak{S}(E,A)$, we denote by $\mathcal{F}_\sigma$ the generalized flag $\mathcal{F}_\sigma = \{F'_{\sigma,\alpha}, F''_{\sigma,\alpha} : \alpha \in A\}$ given by

$$F'_{\sigma,\alpha} = \langle e \in E : \sigma(e) \prec_A \alpha\rangle \quad \text{and} \quad F''_{\sigma,\alpha} = \langle e \in E : \sigma(e) \preceq_A \alpha\rangle. \tag{10}$$

Thus $\{\mathcal{F}_\sigma : \sigma \in \mathfrak{S}(E,A)\}$ are exactly the generalized flags of $\mathbf{Fl}_A(V)$ compatible with the basis E (see Definition 2). Let $\sigma_0 : E \to A$ be the surjective map such that $\mathcal{F} = \mathcal{F}_{\sigma_0}$.

Remark 5. The totally ordered set $(A, \preceq_A)$ and the surjective map $\sigma_0 : E \to A$ give rise to a partial order $\preceq_{\mathbf{P}}$ on E, defined by letting $e \prec_{\mathbf{P}} e'$ if $\sigma_0(e) \prec_A \sigma_0(e')$. Note that the partial order $\preceq_{\mathbf{P}}$ has the property

(11) the relation "e is not comparable with e'" (i.e., neither $e \prec_{\mathbf{P}} e'$ nor $e' \prec_{\mathbf{P}} e$) is an equivalence relation.

In fact, fixing a splitting parabolic subgroup $\mathbf{P} \subset \mathbf{G}(E)$ containing $\mathbf{H}(E)$ is equivalent to fixing a partial order $\preceq_{\mathbf{P}}$ on E satisfying property (11). Moreover, $\mathbf{P}$ is a splitting Borel subgroup if and only if the order $\preceq_{\mathbf{P}}$ is total.

The group $\mathbf{W}(E)$ acts on the set $\mathfrak{S}(E, A)$, hence on E-compatible generalized flags of $\mathbf{Fl}_A(V)$, by the procedure $(w, \sigma) \mapsto \sigma \circ w^{-1}$. Let $\mathbf{W}_{\mathbf{P}}(E) \subset \mathbf{W}(E)$ be the subgroup of permutations such that $\sigma_0 \circ w^{-1} = \sigma_0$. Equivalently, $\mathbf{W}_{\mathbf{P}}(E)$ is the subgroup of permutations $w \in \mathbf{W}(E)$ which preserve the fibers $\sigma_0^{-1}(\alpha)$ $(\alpha \in A)$ of the map σ_0.

Lemma 2. *The map $w \mapsto \mathcal{F}_{\sigma_0 \circ w^{-1}}$ induces a bijection between the quotient $\mathbf{W}(E)/\mathbf{W}_{\mathbf{P}}(E)$ and the set of E-compatible generalized flags of the ind-variety $\mathbf{Fl}(\mathcal{F}, E)$.*

Let $\mathbf{W}(E) \cdot \sigma_0 = \{\sigma_0 \circ w^{-1} : w \in \mathbf{W}(E)\}$ denote the $\mathbf{W}(E)$-orbit of σ_0.

The splitting Borel subgroup $\mathbf{B}$ is the subgroup $\mathbf{B} = \mathbf{P}_{\mathcal{F}_0}$ of elements $g \in \mathbf{G}(E)$ preserving a maximal generalized flag $\mathcal{F}_0$ compatible with E (see Proposition 1). Equivalently $\mathbf{B}$ corresponds to a total order $\preceq_{\mathbf{B}}$ on the basis E (see Remark 5). Then, the generalized flag $\mathcal{F}_0 = \{F'_{0,e}, F''_{0,e} : e \in E\}$ is given by $F'_{0,e} = \langle e' \in E : e' \prec_{\mathbf{B}} e\rangle$ and $F''_{0,e} = \langle e' \in E : e' \preceq_{\mathbf{B}} e\rangle$ for all $e \in E$.

Relying on the total order $\preceq_{\mathbf{B}}$, we define a notion of inversion number and an analogue of the Bruhat order on the set $\mathfrak{S}(E, A)$.

Number of inversions $n_{\mathrm{inv}}(\sigma)$. We say that a pair $(e, e') \in E \times E$ is an *inversion* of $\sigma \in \mathfrak{S}(E, A)$ if $e \prec_{\mathbf{B}} e'$ and $\sigma(e) \succ_A \sigma(e')$. Then

$$n_{\mathrm{inv}}(\sigma) := |\{(e, e') \in E \times E : (e, e') \text{ is an inversion of } \sigma\}|$$

is the *inversion number of* σ. Note that the inversion number $n_{\mathrm{inv}}(\sigma)$ may be infinite.

Partial order $\leq$ on $\mathfrak{S}(E, A)$. We now define a partial order on the set $\mathfrak{S}(E, A)$, analogous to the Bruhat order. For $(e, e') \in E \times E$ with $e \neq e'$, we denote by $t_{e,e'}$ the element of $\mathbf{W}(E)$ which exchanges e with e' and fixes every other element $e'' \in E$. Let $\sigma, \tau \in \mathfrak{S}(E, A)$. We set $\sigma \hat{<} \tau$ if $\tau = \sigma \circ t_{e,e'}$ for a pair $(e, e') \in E \times E$ satisfying $e \prec_{\mathbf{B}} e'$ and $\sigma(e) \prec_A \sigma(e')$. We set $\sigma < \tau$ if there is a chain $\tau_0 = \sigma \hat{<} \tau_1 \hat{<} \tau_2 \hat{<} \ldots \hat{<} \tau_k = \tau$ of elements of $\mathfrak{S}(E, A)$ (with $k \geq 1$).

Element $\sigma_{\mathcal{G}} \in \mathfrak{S}(E, A)$. Given a generalized flag $\mathcal{G} = \{G'_\alpha, G''_\alpha : \alpha \in A\} \in \mathbf{Fl}_A(V)$ weakly compatible with E, we define an element $\sigma_{\mathcal{G}} \in \mathfrak{S}(E, A)$ which measures the relative position of $\mathcal{G}$ to the maximal generalized flag $\mathcal{F}_0$. Set

$$\sigma_{\mathcal{G}}(e) = \min\{\alpha \in A : G''_\alpha \cap F''_{0,e} \neq G''_\alpha \cap F'_{0,e}\} \quad \text{for all } e \in E. \tag{12}$$

[It can be checked directly that the so obtained map $\sigma_{\mathcal{G}} : E \to A$ is indeed surjective, hence an element of $\mathfrak{S}(E, A)$. This fact is also shown in Section 5.2 in the proof of Theorem 2.]

We are now in position to formulate the statement which describes the decomposition of $\mathbf{Fl}(\mathcal{F}, E)$ into $\mathbf{B}$-orbits.

Theorem 1. *Let $\mathbf{P}_{\mathcal{F}}$ be the splitting parabolic subgroup of $\mathbf{G}(E)$ containing $\mathbf{H}(E)$, and corresponding to a generalized flag $\mathcal{F} = \mathcal{F}_{\sigma_0} \in \mathbf{Fl}_A(V)$ (with $\sigma_0 \in \mathfrak{S}(E, A)$) compatible with E. Let $\mathbf{B}$ be any splitting Borel subgroup of $\mathbf{G}(E)$ containing $\mathbf{H}(E)$.*

(a) *We have the decomposition*

$$\mathbf{G}(E)/\mathbf{P}_{\mathcal{F}} = \mathbf{Fl}(\mathcal{F}, E) = \bigsqcup_{\sigma \in \mathbf{W}(E) \cdot \sigma_0} \mathbf{B}\mathcal{F}_\sigma = \bigsqcup_{w \in \mathbf{W}(E)/\mathbf{W}_{\mathbf{P}}(E)} \mathbf{B}\mathcal{F}_{\sigma_0 \circ w^{-1}}.$$

(b) *A generalized flag $\mathcal{G} \in \mathbf{Fl}(\mathcal{F}, E)$ belongs to the $\mathbf{B}$-orbit $\mathbf{B}\mathcal{F}_\sigma$ $(\sigma \in \mathbf{W}(E) \cdot \sigma_0)$ if and only if $\sigma_{\mathcal{G}} = \sigma$.*

(c) *The orbit $\mathbf{B}\mathcal{F}_\sigma$ $(\sigma \in \mathbf{W}(E) \cdot \sigma_0)$ is a locally closed ind-subvariety of $\mathbf{Fl}(\mathcal{F}, E)$ isomorphic to the affine space $\mathbb{A}^{n_{\mathrm{inv}}(\sigma)}$ (which is infinite dimensional if $n_{\mathrm{inv}}(\sigma)$ is infinite).*

(d) *For $\sigma, \tau \in \mathbf{W}(E) \cdot \sigma_0$, the inclusion $\mathbf{B}\mathcal{F}_\sigma \subset \overline{\mathbf{B}\mathcal{F}_\tau}$ holds if and only if $\sigma \leq \tau$.*

Remark 6. If $\sigma \in \mathbf{W}(E)\cdot\sigma_0$, say $\sigma = \sigma_0 \circ w$ with $w \in \mathbf{W}(E)$, then the inversion number of σ is also given by the formula

$$n_{\mathrm{inv}}(\sigma) = |\{(e,e') \in E \times E : e \prec_{\mathbf{B}} e' \text{ and } w(e) \succ_{\mathbf{P}} w(e')\}|$$

(see Remark 5). Note that the inversion number $n_{\mathrm{inv}}(\sigma)$ cannot be directly interpreted as a Bruhat length because we do not assume $\mathbf{B}$ to be conjugate to a subgroup of $\mathbf{P}$.

4.2. **Decomposition of $\mathbf{Fl}(\mathcal{F},\omega,E)$.** In this section the basis E is ω-isotropic with corresponding involution $i_E : E \to E$ (see Section 3.2). Let $\mathbf{P} \subset \mathbf{G}^\omega(E)$ be a splitting parabolic subgroup containing $\mathbf{H}^\omega(E)$, or equivalently let $\mathcal{F}$ be an ω-isotropic generalized flag compatible with E (see Proposition 3). Let $\mathbf{B} \subset \mathbf{G}^\omega(E)$ be a splitting Borel subgroup containing $\mathbf{H}^\omega(E)$. We study the decomposition of the ind-variety $\mathbf{G}^\omega(E)/\mathbf{P} \cong \mathbf{Fl}(\mathcal{F},\omega,E)$ into $\mathbf{B}$-orbits.

Let $(A,\preceq_A,i_A)$ be a totally ordered set with involutive anti-automorphism i_A, such that $\mathcal{F} \in \mathbf{Fl}^\omega_A(V)$. We denote by $\mathfrak{S}^\omega(E,A)$ the set of surjective maps $\sigma : E \to A$ such that $\sigma(i_E(e)) = i_A(\sigma(e))$ for all $e \in E$. By Lemma 1, $\{\mathcal{F}_\sigma : \sigma \in \mathfrak{S}^\omega(E,A)\}$ are exactly the elements of $\mathbf{Fl}^\omega_A(V)$ compatible with E (the notation $\mathcal{F}_\sigma$ is introduced in (10)). Let $\sigma_0 \in \mathfrak{S}^\omega(E,A)$ be such that $\mathcal{F} = \mathcal{F}_{\sigma_0}$.

The group $\mathbf{W}^\omega(E)$ is defined as the group of permutations $w : E \to E$ such that $w(e) = e$ for all but finitely many $e \in E$ and $w(i_E(e)) = i_E(w(e))$ for all $e \in E$. Note that $\mathbf{W}^\omega(E)$ acts on the set $\mathfrak{S}^\omega(E,A)$ by the procedure $(w,\sigma) \mapsto \sigma \circ w^{-1}$. Let $\mathbf{W}^\omega_{\mathbf{P}}(E)$ be the subgroup of elements $w \in \mathbf{W}^\omega(E)$ such that $\sigma_0 \circ w^{-1} = \sigma_0$ and let $\mathbf{W}^\omega(E)\cdot\sigma_0 := \{\sigma_0 \circ w^{-1} : w \in \mathbf{W}^\omega(E)\}$ be the $\mathbf{W}^\omega(E)$-orbit of σ_0.

Lemma 3. *The map $w \mapsto \mathcal{F}_{\sigma_0\circ w^{-1}}$ induces a bijection between $\mathbf{W}^\omega(E)/\mathbf{W}^\omega_{\mathbf{P}}(E)$ and the set of E-compatible elements of $\mathbf{Fl}(\mathcal{F},\omega,E)$.*

The splitting Borel subgroup $\mathbf{B}$ is the subgroup $\mathbf{B} = \mathbf{P}^\omega_{\mathcal{F}_0}$ of elements preserving some maximal ω-isotropic generalized flag $\mathcal{F}_0$ compatible with E. We can write $\mathcal{F}_0 = \{F'_{0,e}, F''_{0,e} : e \in E\}$ with $F'_{0,e} = \langle e' \in E : e' \prec_{\mathbf{B}} e\rangle$ and $F''_{0,e} = \langle e' \in E : e' \preceq_{\mathbf{B}} e\rangle$, where $\preceq_{\mathbf{B}}$ is a total order on E. Moreover, the fact that $\mathcal{F}_0$ is ω-isotropic implies that the involution $i_E : E \to E$ is an anti-automorphism of the ordered set $(E,\preceq_{\mathbf{B}})$.

Number of inversions $n^\omega_{\mathrm{inv}}(\sigma)$. Let $\sigma \in \mathfrak{S}^\omega(E,A)$. We define an *$\omega$-isotropic inversion* of σ as a pair $(e,e') \in E \times E$ such that

$$e \prec_{\mathbf{B}} e', \quad e \prec_{\mathbf{B}} i_E(e), \quad e' \neq i_E(e'), \quad \text{and} \quad \sigma(e) \succ_A \sigma(e').$$

Let

$$n^\omega_{\mathrm{inv}}(\sigma) = |\{(e,e') \in E \times E : (e,e') \text{ is an } \omega\text{-isotropic inversion of } \sigma\}|.$$

Partial order $\leq_\omega$ on $\mathfrak{S}^\omega(E,A)$. Given $(e,e') \in E \times E$ with $e \neq e'$, $i_E(e) \neq e$, $i_E(e') \neq e'$, we set

$$t^\omega_{e,e'} = t_{e,e'} \circ t_{i_E(e),i_E(e')} \text{ if } e' \neq i_E(e), \quad t^\omega_{e,e'} = t_{e,e'} \text{ if } e' = i_E(e).$$

Thus $t^\omega_{e,e'} \in \mathbf{W}^\omega(E)$. Let $\sigma,\tau \in \mathfrak{S}^\omega(E,A)$. We set $\sigma \hat{<}_\omega \tau$ if $\tau = \sigma \circ t^\omega_{e,e'}$ for a pair (e,e') satisfying $e \prec_{\mathbf{B}} e'$ and $\sigma(e) \prec_A \sigma(e')$. Finally we set $\sigma <_\omega \tau$ if there is a chain $\tau_0 = \sigma \hat{<}_\omega \tau_1 \hat{<}_\omega \tau_2 \hat{<}_\omega \ldots \hat{<}_\omega \tau_k = \tau$ of elements of $\mathfrak{S}^\omega(E,A)$.

Theorem 2. *Let $\mathbf{P}^\omega_{\mathcal{F}}$ be the splitting parabolic subgroup of $\mathbf{G}^\omega(E)$ containing $\mathbf{H}^\omega(E)$, and corresponding to an E-compatible generalized flag $\mathcal{F} = \mathcal{F}_{\sigma_0} \in \mathbf{Fl}^\omega_A(V)$ (with $\sigma_0 \in \mathfrak{S}^\omega(E,A)$). Let $\mathbf{B}$ be any splitting Borel subgroup of $\mathbf{G}^\omega(E)$ containing $\mathbf{H}^\omega(E)$.*
(a) *We have the decomposition*

$$\mathbf{G}^\omega(E)/\mathbf{P}^\omega_{\mathcal{F}} = \mathbf{Fl}(\mathcal{F},\omega,E) = \bigsqcup_{\sigma\in\mathbf{W}^\omega(E)\cdot\sigma_0} \mathbf{B}\mathcal{F}_\sigma = \bigsqcup_{w\in\mathbf{W}^\omega(E)/\mathbf{W}^\omega_{\mathbf{P}}(E)} \mathbf{B}\mathcal{F}_{\sigma_0\circ w^{-1}}.$$

(b) *For $\mathcal{G} \in \mathbf{Fl}(\mathcal{F},\omega,E)$ the map $\sigma_{\mathcal{G}} : E \to A$ (see (12)) belongs to $\mathbf{W}^\omega(E)\cdot\sigma_0$. Moreover, $\mathcal{G}$ belongs to $\mathbf{B}\mathcal{F}_\sigma$ ($\sigma \in \mathbf{W}^\omega(E)\cdot\sigma_0$) if and only if $\sigma_{\mathcal{G}} = \sigma$.*
(c) *The orbit $\mathbf{B}\mathcal{F}_\sigma$ ($\sigma \in \mathbf{W}^\omega(E)\cdot\sigma_0$) is a locally closed ind-subvariety of $\mathbf{Fl}(\mathcal{F},\omega,E)$ isomorphic to the affine space $\mathbb{A}^{n^\omega_{\mathrm{inv}}(\sigma)}$ (possibly infinite-dimensional).*
(d) *For $\sigma,\tau \in \mathbf{W}^\omega(E)\cdot\sigma_0$, the inclusion $\mathbf{B}\mathcal{F}_\sigma \subset \overline{\mathbf{B}\mathcal{F}_\tau}$ holds if and only if $\sigma \leq_\omega \tau$.*

4.3. **Bruhat decomposition of the ind-group $\mathbf{G} = \mathbf{G}(E)$ or $\mathbf{G}^\omega(E)$.** Let $\mathbf{H} = \mathbf{H}(E)$ or $\mathbf{H}^\omega(E)$, and let $\mathbf{W} = \mathbf{W}(E)$ or $\mathbf{W}^\omega(E)$. If $\mathbf{W} = \mathbf{W}(E)$, the linear extension of $w \in \mathbf{W}$ is an element $\hat{w} \in \mathbf{G}(E)$. If $\mathbf{W} = \mathbf{W}^\omega(E)$, we can find scalars $\lambda_e \in \mathbb{K}^*$ ($e \in E$) such that the map $e \mapsto \lambda_e w(e)$ linearly extends to an element $\hat{w} \in \mathbf{G}^\omega(E)$. In both situations it is easy to deduce that $\mathbf{W}$ is isomorphic to the quotient $N_{\mathbf{G}}(\mathbf{H})/\mathbf{H}$.

Given a splitting parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ containing $\mathbf{H}$, we denote by $\mathbf{W}_{\mathbf{P}}$ the corresponding subgroup of $\mathbf{W}$. The following statement describes the decomposition of the ind-group $\mathbf{G}$ into double cosets. It is a consequence of Theorems 1–2.

Corollary 1. *Let $\mathbf{G}$ be one of the ind-groups $\mathbf{G}(E)$ or $\mathbf{G}^\omega(E)$, and let $\mathbf{P}$ and $\mathbf{B}$ be respectively a splitting parabolic and a splitting subgroup of $\mathbf{G}$ containing $\mathbf{H}$. Then we have a decomposition*

$$\mathbf{G} = \bigsqcup_{w \in \mathbf{W}/\mathbf{W}_{\mathbf{P}}} \mathbf{B}\hat{w}\mathbf{P}.$$

Remark 7. (a) Note that the unique assumption on the splitting parabolic and Borel subgroups $\mathbf{P}$ and $\mathbf{B}$ in Corollary 1 is that they contain a common splitting Cartan subgroup, in particular it is not required that $\mathbf{B}$ be conjugate to a subgroup of $\mathbf{P}$.
(b) The ind-group $\mathbf{G}$ admits a natural exhaustion $\mathbf{G} = \bigcup_{n \geq 1} G_n$ by finite-dimensional subgroups of the form $G_n = G(E_n)$ or $G_n = G^\omega(E_n)$ (see Section 2.2). Moreover, the intersections $P_n := \mathbf{P} \cap G_n$ and $B_n := \mathbf{B} \cap G_n$ are respectively a parabolic subgroup and a Borel subgroup of G_n, containing a common Cartan subgroup. Then the decomposition of Corollary 1 can be retrieved by considering usual Bruhat decompositions of the groups G_n into double cosets for P_n and B_n.

4.4. **On the existence of cells of finite or infinite dimension.** In Theorems 1–2 it appears that the decomposition of an ind-variety of generalized flags into $\mathbf{B}$-orbits may comprise orbits of infinite dimension. The following result determines precisely the situations in which infinite-dimensional orbits arise.

Theorem 3. *Let $\mathbf{G}$ be one of the groups $\mathbf{G}(E)$ or $\mathbf{G}^\omega(E)$. Let $\mathbf{P}, \mathbf{B} \subset \mathbf{G}$ be splitting parabolic and Borel subgroups containing the splitting Cartan subgroup $\mathbf{H}$ of $\mathbf{G}$.*
(a) *The following conditions are equivalent:*
- (i) *$\mathbf{B}$ is conjugate (under $\mathbf{G}$) to a subgroup of $\mathbf{P}$;*
- (ii) *At least one $\mathbf{B}$-orbit of $\mathbf{G}/\mathbf{P}$ is finite dimensional;*
- (iii) *One $\mathbf{B}$-orbit of $\mathbf{G}/\mathbf{P}$ is a single point (and this orbit is necessarily unique).*

(b) *Let $\preceq_{\mathbf{B}}$ be the total order on the basis E induced by $\mathbf{B}$. Assume that $\mathbf{P} \neq \mathbf{G}$. The following conditions are equivalent:*
- (i) *$\mathbf{B}$ is conjugate (under $\mathbf{G}$) to a subgroup of $\mathbf{P}$, and the ordered set $(E, \preceq_{\mathbf{B}})$ is isomorphic (as ordered set) to a subset of $(\mathbb{Z}, \leq)$;*
- (ii) *Every $\mathbf{B}$-orbit of $\mathbf{G}/\mathbf{P}$ is finite dimensional.*

Remark 8. (a) Theorem 3 provides in particular a criterion for a given splitting Borel subgroup to be conjugate to a subgroup of a given splitting parabolic subgroup. This criterion is applied in the next section.
(b) Following [4], we call a generalized flag $\mathcal{G}$ a *flag* if the chain $(\mathcal{G}, \subset)$ is isomorphic as ordered set to a subset of $(\mathbb{Z}, \leq)$. Then the second part of condition (b) (i) in Theorem 3 can be rephrased by saying that the maximal generalized flag $\mathcal{F}_0$ is a flag. Another characterization of flags is provided by [4, Proposition 7.2] which says that the ind-variety of generalized flags $\mathbf{Fl}(\mathcal{G}, E)$ (resp., $\mathbf{Fl}(\mathcal{G}, \omega, E)$) is *projective* (i.e., isomorphic as ind-variety to a closed ind-subvariety of the infinite-dimensional projective space $\mathbb{P}^\infty$) if and only if $\mathcal{G}$ is a flag.

4.5. **Decomposition of ind-grassmannians.** A minimal (nontrivial) generalized flag $\mathcal{F} = \{0, F, V\}$ of the space V is determined by the proper nonzero subspace $F \subset V$. If $\mathcal{F}$ is compatible with the basis E, then the surjective map $\sigma_0 : E \to \{1, 2\}$ such that $F = \langle e \in E : \sigma_0(e) = 1 \rangle$ can be simply viewed as the subset $\sigma_0 \subset E$ such that $F = \langle \sigma_0 \rangle$.

In this case the ind-variety $\mathbf{Fl}(\mathcal{F}, E)$ is an *ind-grassmannian* and we denote it by $\mathbf{Gr}(F, E)$.

- If $k := \dim F$ is finite, a subspace $F_1 \subset V$ is E-commensurable with F if and only if $\dim F_1 = k$. Thus the ind-variety $\mathbf{Gr}(F, E)$ only depends on k, and we write $\mathbf{Gr}(k) = \mathbf{Gr}(F, E)$ in this case.

- If $k := \mathrm{codim}_V F$ is finite, the ind-variety $\mathbf{Gr}(F,E)$ depends on E and k (but not on F). It is also isomorphic to $\mathbf{Gr}(k)$. Indeed, the basis $E \subset V$ gives rise to a dual family $E^* \subset V^*$. The linear space $V_* := \langle E^* \rangle$ is then countable dimensional. Let $U^\# := \{\phi \in V_* : \phi(u) = 0\ \forall u \in U\}$ be the orthogonal subspace in V_* of a subspace $U \subset V$. The map $U \mapsto U^\#$ realizes an isomorphism of ind-varieties between $\mathbf{Gr}(F,E)$ and $\{F' \subset V_* : \dim F' = k\} \cong \mathbf{Gr}(k)$.
- If F is both infinite dimensional and infinite codimensional, the ind-variety $\mathbf{Gr}(F,E)$ depends on (F,E), although all ind-varieties of this type are isomorphic; their isomorphism class is denoted $\mathbf{Gr}(\infty)$. Moreover, $\mathbf{Gr}(\infty)$ and $\mathbf{Gr}(k)$ are not isomorphic as ind-varieties (see [10]).

Let $\mathfrak{S}(E)$ be the set of subsets $\sigma \subset E$. The group $\mathbf{W}(E)$ acts on $\mathfrak{S}(E)$ in a natural way. The $\mathbf{W}(E)$-orbit of σ_0 is the subset $\mathbf{W}(E) \cdot \sigma_0 = \{\sigma \in \mathfrak{S}(E) : |\sigma_0 \setminus \sigma| = |\sigma \setminus \sigma_0| < +\infty\}$. We write $F_\sigma = \langle \sigma \rangle$ (for $\sigma \in \mathfrak{S}(E)$).

The following statement describes the decomposition of the ind-grassmannian $\mathbf{Gr}(F,E)$ into $\mathbf{B}$-orbits. It is a direct consequence of Theorem 1.

Proposition 6. *Let $\mathbf{B} \subset \mathbf{G}(E)$ be a splitting Borel subgroup containing $\mathbf{H}(E)$.*
(a) *We have the decomposition*

$$\mathbf{Gr}(F,E) = \bigsqcup_{\sigma \in \mathbf{W}(E)\cdot\sigma_0} \mathbf{B}F_\sigma.$$

(b) *For $F' \in \mathbf{Gr}(F,E)$, we have $F' \in \mathbf{B}F_\sigma$ if and only if*

$$\sigma = \sigma_{F'} := \{e \in E : F' \cap \langle e' \in E : e' \prec_{\mathbf{B}} e \rangle \neq F' \cap \langle e' \in E : e' \preceq_{\mathbf{B}} e \rangle\}.$$

(c) *For $\sigma \in \mathbf{W}(E) \cdot \sigma_0$, the orbit $\mathbf{B}F_\sigma$ is a locally closed ind-subvariety of $\mathbf{Gr}(F,E)$ isomorphic to an affine space $\mathbb{A}^{d_\sigma}$ of (possibly infinite) dimension*

$$d_\sigma = n_{\mathrm{inv}}(\sigma) := |\{(e,e') \in E \times E : e \prec_{\mathbf{B}} e',\ e \notin \sigma,\ e' \in \sigma\}|.$$

(d) *For $\sigma, \tau \in \mathbf{W}(E) \cdot \sigma_0$, the inclusion $\mathbf{B}F_\sigma \subset \overline{\mathbf{B}F_\tau}$ holds if and only if $\sigma \leq \tau$, where the relation $\sigma \leq \tau$ means that, if $e_1 \prec_{\mathbf{B}} e_2 \prec_{\mathbf{B}} \ldots \prec_{\mathbf{B}} e_\ell$ are the elements of $\sigma \setminus \tau$ and $f_1 \prec_{\mathbf{B}} f_2 \prec_{\mathbf{B}} \ldots \prec_{\mathbf{B}} f_\ell$ are the elements of $\tau \setminus \sigma$, then $e_i \prec_{\mathbf{B}} f_i$ for all $i \in \{1, \ldots, \ell\}$.*

Example 2 (Case of the ind-grassmannian $\mathbf{Gr}(k)$)**.** Let $\mathfrak{S}_k(E)$ be the set of subsets $\sigma \subset E$ of cardinality k. Given $\sigma_0 \in \mathfrak{S}_k(E)$, set $F = \langle \sigma_0 \rangle$, and consider the splitting parabolic subgroup $\mathbf{P}_F = \{g \in \mathbf{G}(E) : g(F) = F\}$ and the ind-grassmannian $\mathbf{Gr}(k) = \mathbf{Gr}(F,E) = \mathbf{G}(E)/\mathbf{P}_F$. By Proposition 6 (a), we have the decomposition

$$\mathbf{Gr}(k) = \bigsqcup_{\sigma \in \mathfrak{S}_k(E)} \mathbf{B}F_\sigma.$$

By Proposition 6 (c), the cell $\mathbf{B}F_\sigma$ is finite dimensional if and only if σ is contained in a finite ideal of the ordered set $(E, \preceq_{\mathbf{B}})$, i.e., there is a finite subset $\overline{\sigma} \subset E$ satisfying ($e \in \overline{\sigma}$ and $e' \preceq_{\mathbf{B}} e \Rightarrow e' \in \overline{\sigma}$) and containing σ. It easily follows that there are finite-dimensional $\mathbf{B}$-orbits in $\mathbf{Gr}(k)$ if and only if the maximal generalized flag $\mathcal{F}_0$ corresponding to $\mathbf{B}$ contains a subspace M of dimension k. By Theorem 3, $\mathbf{B}$ is conjugate to a subgroup of the splitting parabolic subgroup $\mathbf{P}_F$ exactly in this case. By Theorem 3 (or directly), we note that all cells $\mathbf{B}F_\sigma \subset \mathbf{Gr}(k)$ are finite dimensional if and only if $(E, \preceq_{\mathbf{B}})$ is isomorphic to $(\mathbb{N}, \leq)$ as an ordered set, in other words $\mathcal{F}_0$ is a flag of the form

$$\mathcal{F}_0 = (F_{0,0} \subset F_{0,1} \subset F_{0,2} \subset \ldots) \quad \text{with} \quad \dim F_{0,i} = i \ \text{ for all } i \geq 0. \tag{13}$$

By Proposition 6 (d), given $\sigma = \{e_1 \prec_{\mathbf{B}} e_2 \prec_{\mathbf{B}} \ldots \prec_{\mathbf{B}} e_k\}$ and $\tau = \{f_1 \prec_{\mathbf{B}} f_2 \prec_{\mathbf{B}} \ldots \prec_{\mathbf{B}} f_k\}$, we have $\mathbf{B}F_\sigma \subset \overline{\mathbf{B}F_\tau}$ if and only if $e_i \preceq_{\mathbf{B}} f_i$ for all $i \in \{1, \ldots, k\}$.

Now let $\tau_0 \subset E$ be an infinite subset whose complement $E \setminus \tau_0$ is finite of cardinality k. Let $M = \langle \tau_0 \rangle$ be the corresponding subspace of V of codimension k and let $\mathbf{P}_M \subset \mathbf{G}(E)$ be the corresponding splitting parabolic subgroup. We consider the ind-grassmannian $\mathbf{Gr}(M,E) = \mathbf{G}(E)/\mathbf{P}_M$ which is isomorphic to $\mathbf{Gr}(k) = \mathbf{G}(E)/\mathbf{P}_F$ as mentioned at the beginning of Section 4.5. If $\mathcal{F}_0$ is as in (13) and $\mathbf{B}$ is the corresponding splitting Borel subgroup, then it follows from Proposition 6 (c) that every $\mathbf{B}$-orbit of $\mathbf{Gr}(M,E)$ is infinite dimensional. By Theorem 3, this shows in particular that the splitting parabolic subgroups $\mathbf{P}_F$ and $\mathbf{P}_M$ are not conjugate under $\mathbf{G}(E)$.

Example 3 (Case of the infinite-dimensional projective space)**.** Assume that $k = \dim F = 1$. In this case $\mathbf{Gr}(k)$ is the infinite-dimensional projective space $\mathbb{P}^\infty$ (see Example 1). The decomposition becomes

$$\mathbb{P}^\infty = \bigsqcup_{e \in E} \mathbf{C}_e$$

where $\mathbf{C}_e = \mathbf{B}\langle e\rangle = \{L \text{ line} : L \subset \langle e' \in E : e' \preceq_{\mathbf{B}} e\rangle,\ L \not\subset \langle e' \in E : e' \prec_{\mathbf{B}} e\rangle\}$ for all $e \in E$. The cell $\mathbf{C}_e$ is isomorphic to an affine space of dimension $\dim \mathbf{C}_e = |\{e' \in E : e' \prec_{\mathbf{B}} e\}|$. Moreover, $\mathbf{C}_e \subset \overline{\mathbf{C}_f}$ if and only if $e \preceq_{\mathbf{B}} f$.

In this case the maximal generalized flag $\mathcal{F}_0 = \{F'_{0,e}, F''_{0,e} : e \in E\}$ corresponding to $\mathbf{B}$ can be retrieved from the cell decomposition:

$$F''_{0,e} = \sum_{L \in \overline{\mathbf{C}}_e} L \quad \text{and} \quad F'_{0,e} = \sum_{L \in \overline{\mathbf{C}}_e \setminus \mathbf{C}_e} L \ \text{ for all } e \in E.$$

More generally, let $(A, \preceq)$ be a totally ordered set and let $\mathbb{P}^\infty = \bigsqcup_{\alpha \in A} \mathbf{C}_\alpha$ be a *linear* cell decomposition such that $\mathbf{C}_\alpha \subset \overline{\mathbf{C}_\beta}$ whenever $\alpha \preceq \beta$. By "linear" we mean that each $\overline{\mathbf{C}_\alpha}$ is a projective subspace of $\mathbb{P}^\infty$, i.e., we can find a subspace $F''_\alpha \subset V$ such that $\overline{\mathbf{C}_\alpha} = \mathbb{P}(F''_\alpha)$. Setting $F'_\alpha = \sum_{\beta < \alpha} F''_\beta$, we get a generalized flag $\mathcal{F}_0 := \{F'_\alpha, F''_\alpha : \alpha \in A\}$ such that $\mathbb{P}(F''_\alpha) \setminus \mathbb{P}(F'_\alpha)$ is a (possibly infinite-dimensional) affine space for all α. The last property ensures that $\dim F''_\alpha / F'_\alpha = 1$, i.e., $\mathcal{F}_0$ is a maximal generalized flag. In this way we obtain a correspondence between maximal generalized flags (not necessarily compatible with a given basis) and linear cell decompositions of the infinite-dimensional projective space $\mathbb{P}^\infty$.

Example 4 (Case of the ind-grassmannian $\mathbf{Gr}(\infty)$)**.** Assume that the basis E is parametrized by $\mathbb{Z}$, in other words let $E = \{e_i\}_{i \in \mathbb{Z}}$. We consider the splitting Borel subgroup $\mathbf{B}$ corresponding to the natural order $\leq$ on $\mathbb{Z}$.

Let $F = \langle e_i : i \leq 0\rangle$. Then the ind-variety $\mathbf{Gr}(F, E)$ is isomorphic to $\mathbf{Gr}(\infty)$. We have $\mathbf{B} \subset \mathbf{P}_F$. It follows from Theorem 3 that every $\mathbf{B}$-orbit of $\mathbf{Gr}(F, E)$ is finite dimensional.

Let $F' = \langle e_i : i \in 2\mathbb{Z}\rangle$. Again the ind-variety $\mathbf{Gr}(F', E)$ is isomorphic to $\mathbf{Gr}(\infty)$. However in this case we see from Proposition 6 (c) that every $\mathbf{B}$-orbit of $\mathbf{Gr}(F', E)$ is infinite dimensional.

We now suppose that the space V is endowed with a nondegenerate symmetric or skew-symmetric bilinear form ω and the basis E is ω-isotropic with corresponding involution $i_E : E \to E$. Then a minimal ω-isotropic generalized flag is of the form $\mathcal{F} = (0 \subset F \subset F^\perp \subset V)$ with $F \subset V$ proper and nontrivial, possibly $F = F^\perp$. Assuming that F is compatible with the basis E, there is a subset $\sigma_0 \subset E$ such that $F = \langle \sigma_0 \rangle$ and $i_E(\sigma_0) \cap \sigma_0 = \emptyset$ as the generalized flag is ω-isotropic. The ind-variety $\mathbf{Fl}(\mathcal{F}, \omega, E)$ is also denoted $\mathbf{Gr}(F, \omega, E)$ and called *isotropic ind-grassmannian.*

- If $\dim F = k$ is finite, the ind-variety $\mathbf{Gr}(F, \omega, E)$ is the set of all k-dimensional subspaces $M \subset V$ such that $M \subset M^\perp$. This ind-variety does not depend on (F, E) and we denote it also by $\mathbf{Gr}^\omega(k)$.
- If $\dim F$ is infinite, the isomorphism class of the ind-variety $\mathbf{Gr}(F, \omega, E)$ also depends on the dimension of the quotient $F^\perp/F$. A special situation is when $\dim F^\perp/F \in \{0, 1\}$, in which case $\mathbf{Gr}(F, \omega, E)$ is formed by maximal isotropic subspaces.

We denote by $\mathfrak{S}^\omega(E)$ the set of subsets $\sigma \subset E$ such that $i_E(\sigma) \cap \sigma = \emptyset$. The group $\mathbf{W}^\omega(E)$ acts on $\mathfrak{S}^\omega(E)$ in a natural way. The orbit $\mathbf{W}^\omega(E) \cdot \sigma_0$ is the set of subsets $\sigma \in \mathfrak{S}^\omega(E)$ such that $|\sigma \setminus \sigma_0| = |\sigma_0 \setminus \sigma| < +\infty$. From Theorem 2 we obtain the following description of the $\mathbf{B}$-orbits of $\mathbf{Gr}(F, \omega, E)$.

Proposition 7. *Let $\mathbf{B}$ be a splitting Borel subgroup of $\mathbf{G}^\omega(E)$ corresponding to a total order $\preceq_{\mathbf{B}}$ on E. Recall that i_E is a anti-automorphism of the ordered set $(E, \preceq_{\mathbf{B}})$.*
(a) *We have the decomposition*

$$\mathbf{Gr}(F, \omega, E) = \bigsqcup_{\sigma \in \mathbf{W}^\omega(E) \cdot \sigma_0} \mathbf{B}F_\sigma$$

where as before $F_\sigma = \langle \sigma \rangle$.
(b) *For $F' \in \mathbf{Gr}(F, \omega, E)$ we have $\sigma_{F'} \in \mathbf{W}^\omega(E) \cdot \sigma_0$ (see Proposition 6* (b)*), moreover $F' \in \mathbf{B}F_\sigma$ if and only if $\sigma = \sigma_{F'}$.*
(c) *For $\sigma \in \mathbf{W}^\omega(E) \cdot \sigma_0$, the orbit $\mathbf{B}F_\sigma$ is a locally closed ind-subvariety of $\mathbf{Gr}(F, \omega, E)$ isomorphic to an affine space of (possibly infinite) dimension*

$$n^\omega_{\mathrm{inv}}(\sigma) := |\{(e, e') \in E \times E : e \prec_{\mathbf{B}} e' \neq i_E(e'),\ e \prec_{\mathbf{B}} i_E(e),\ \big((e \notin \sigma,\ e' \in \sigma) \text{ or } (i_E(e) \in \sigma,\ i_E(e') \notin \sigma)\big)\}|.$$

(d) *For $\sigma, \tau \in \mathbf{W}^\omega(E) \cdot \sigma_0$, the inclusion $\mathbf{B}F_\sigma \subset \overline{\mathbf{B}F_\tau}$ holds if and only if $\sigma \leq \tau$, where the relation $\sigma \leq \tau$ is defined as in Proposition 6* (d).

Example 5 (Case of the isotropic ind-grassmannian $\mathbf{Gr}^\omega(k)$)**.** In this case the cells $\mathbf{B}F_\sigma$ are parametrized by the set $\mathfrak{S}_k^\omega(E)$ of finite subsets $\sigma \subset E$ of cardinality k such that $i_E(\sigma) \cap \sigma = \emptyset$. The cell $\mathbf{B}F_\sigma$ is finite dimensional if and only if σ is contained in a finite ideal $\overline{\sigma}$ of the ordered set $(E, \preceq_{\mathbf{B}})$. Thereby the ind-variety $\mathbf{Gr}^\omega(k)$ has finite-dimensional $\mathbf{B}$-orbits if and only if the ordered set $(E, \preceq_{\mathbf{B}})$ has a finite ideal with k elements. Equivalently, the maximal generalized flag $\mathcal{F}_0$ corresponding to $\mathbf{B}$ has a subspace $M \in \mathcal{F}_0$ of dimension k. Since $\mathcal{F}_0$ is maximal and ω-isotropic, it is of the form

$$\mathcal{F}_0 = \{0 = F_{0,0} \subset F_{0,1} \subset \ldots \subset F_{0,k} \subset (\ldots) \subset F_{0,k}^\perp \subset \ldots \subset F_{0,1}^\perp \subset F_{0,0}^\perp = V\}$$

with infinitely many terms between $F_{0,k}$ and $F_{0,k}^\perp$. Hence the ordered set $(\mathcal{F}_0, \subset)$ is not isomorphic to a subset of $(\mathbb{Z}, \leq)$. By Theorem 3, this implies that $\mathbf{Gr}^\omega(k)$ admits infinite-dimensional $\mathbf{B}$-orbits. Therefore, contrary to the case of the ind-grassmannian $\mathbf{Gr}(k)$ (see Example 2), there is no splitting Borel subgroup $\mathbf{B} \subset \mathbf{G}^\omega(E)$ for which all $\mathbf{B}$-orbits of the isotropic ind-grassmannian $\mathbf{Gr}^\omega(k)$ are finite dimensional.

Assume that ω is skew symmetric and $k = 1$. Then $\mathbf{Gr}^\omega(k)$ coincides with the entire infinite-dimensional projective space $\mathbb{P}^\infty$. The above discussion shows that, for every splitting Borel subgroup $\mathbf{B}$ of $\mathbf{G}^\omega(E)$, there are infinite-dimensional $\mathbf{B}$-orbits in the projective space $\mathbb{P}^\infty$. We know however from Examples 2–3 that, for a well-chosen splitting Borel subgroup of $\mathbf{G}(E)$, the infinite-dimensional projective space $\mathbb{P}^\infty$ admits a decomposition into finite-dimensional orbits. Therefore the realizations of $\mathbb{P}^\infty$ as $\mathbf{Gr}(1)$ and $\mathbf{Gr}^\omega(1)$ yield different sets of cell decompositions on $\mathbb{P}^\infty$.

Example 6 (An isotropic ind-grassmannian with decomposition into finite-dimensional cells)**.** Let $E = \{e_i : i \in 2\mathbb{Z}+1\}$ be an ω-isotropic basis of V such that $\omega(e_i, e_j) = 0$ unless $i + j = 0$. For $k \geq 1$, we let $F = \langle e_i : i \leq -k \rangle$ and consider the ind-grassmannian $\mathbf{Gr}(F, \omega, E)$. Let $\mathbf{B}$ be the splitting Borel subgroup of $\mathbf{G}^\omega(E)$ corresponding to the natural total order $\leq$ on $2\mathbb{Z}+1$. We then have $\mathbf{B} \subset \mathbf{P}_F^\omega := \{g \in \mathbf{G}^\omega(E) : g(F) = F\}$, hence by Theorem 3 (b) all the $\mathbf{B}$-orbits of the ind-grassmannian $\mathbf{Gr}(F, \omega, E)$ are finite dimensional.

5. Proof of the results stated in Section 4

Throughout this section let $\mathbf{G} = \mathbf{G}(E)$ or $\mathbf{G}^\omega(E)$, and $\mathbf{W}$ is the corresponding group $\mathbf{W}(E)$ or $\mathbf{W}^\omega(E)$ (see Sections 4.1–4.2). The proofs of the results stated in Section 4 are given in Sections 5.3–5.5. They rely on preliminary facts presented in Section 5.1 (which is concerned with the combinatorics of the group $\mathbf{W}$) and Section 5.2 (where we review some standard facts on Schubert decomposition of finite-dimensional flag varieties).

5.1. **Combinatorial properties of the group W.** We first recall certain features of the group $\mathbf{W}$:

- $\mathbf{W} \cong N_{\mathbf{G}}(\mathbf{H})/\mathbf{H}$ where $\mathbf{H} \subset \mathbf{G}$ is the splitting Cartan subgroup of elements diagonal in the basis E; specifically, to an element $w \in \mathbf{W}$, we can associate an explicit representative $\hat{w} \in N_{\mathbf{G}}(\mathbf{H})$ (see Section 4.3).
- We have a natural exhaustion
$$\mathbf{W} = \bigcup_{n \geq 1} W_n$$
where $W_n = W(E_n)$ (resp. $W_n = W^\omega(E_n)$) is the Weyl group of $G_n = G(E_n)$ (resp. $G_n = G^\omega(E_n)$).
- Let $E' = E$ if $\mathbf{G} = \mathbf{G}(E)$ and $E' = \{e \in E : i_E(e) \neq e\}$ if $\mathbf{G} = \mathbf{G}^\omega(E)$, and let
$$\hat{E} = \{(e, e') \in E' \times E' : e \neq e'\}.$$
For $(e, e') \in \hat{E}$, set $s_{e,e'} = t_{e,e'}$ if $\mathbf{G} = \mathbf{G}(E)$ and $s_{e,e'} = t_{e,e'}^\omega$ if $\mathbf{G} = \mathbf{G}^\omega(E)$ (see Sections 4.1–4.2). In both cases for each pair $(e, e') \in \hat{E}$, we get an element $s_{e,e'} \in \mathbf{W}$. Clearly $\{s_{e,e'} : (e, e') \in \hat{E}\}$ is a system of generators of $\mathbf{W}$.

5.1.1. *Analogue of Bruhat length.* As seen in Sections 4.1–4.2, fixing a splitting Borel subgroup $\mathbf{B}$ of $\mathbf{G}$ with $\mathbf{B} \supset \mathbf{H}$ is equivalent to fixing a total order $\preceq_{\mathbf{B}}$ on the basis E (resp., such that the involution $i_E : E \to E$ becomes an anti-automorphism of ordered set, in the case where $\mathbf{G} = \mathbf{G}^\omega(E)$). This total order allows us to define a system of simple transpositions for $\mathbf{W}$ by letting

$$S_{\mathbf{B}} = \{s_{e,e'} : e, e' \text{ are consecutive elements of } (E', \preceq_{\mathbf{B}})\}.$$

Note however that in general $S_{\mathbf{B}}$ does not generate the group $\mathbf{W}$. For instance, in the extreme case where $(E, \preceq_{\mathbf{B}}) = (\mathbb{Q}, \leq)$ we have $S_{\mathbf{B}} = \emptyset$.

For $w \in \mathbf{W}$, we define

$$\ell_{\mathbf{B}}(w) = \min\{m \geq 0 : w = s_1 \cdots s_m \text{ for some } s_1, \ldots, s_m \in S_{\mathbf{B}}\}$$

if the set on the right-hand side is nonempty, and

$$\ell_{\mathbf{B}}(w) = +\infty$$

otherwise.

For every $n \geq 1$, the order $\preceq_{\mathbf{B}}$ induces a total order on the finite subset $E_n \subset E$, and thus a system of simple reflections $S_{\mathbf{B},n} := \{s_{e,e'} : e, e' \text{ are consecutive elements of } (E_n \cap E', \preceq_{\mathbf{B}})\}$ of the Weyl group W_n. Let $\ell_{\mathbf{B},n}(w)$ be the usual Bruhat length of $w \in W_n$ with respect to $S_{\mathbf{B},n}$.

Proposition 8. *Let $w \in \mathbf{W}$. Then*

(a) $\ell_{\mathbf{B}}(w) = \lim_{n\to\infty} \ell_{\mathbf{B},n}(w)$*;*

(b) $\ell_{\mathbf{B}}(w) = \begin{cases} |\{(e,e') \in \hat{E} : e \prec_{\mathbf{B}} e' \text{ and } w(e) \succ_{\mathbf{B}} w(e')\}| & \text{if } \mathbf{G} = \mathbf{G}(E), \\ |\{(e,e') \in \hat{E} : e \prec_{\mathbf{B}} e',\ e \prec_{\mathbf{B}} i_E(e) \text{ and } w(e) \succ_{\mathbf{B}} w(e')\}| & \text{if } \mathbf{G} = \mathbf{G}^\omega(E); \end{cases}$

(c) $\ell_{\mathbf{B}}(w) = +\infty$ *if and only if there is $e \in E$ such that the set $\{e' \in E : e \prec_{\mathbf{B}} e' \prec_{\mathbf{B}} w(e)\}$ is infinite.*

Proof. Denote by $m_{\mathbf{B}}(w)$ the quantity in the right-hand side of (b). Then

$$\ell_{\mathbf{B}}(w) \geq \lim_{n\to\infty} \ell_{\mathbf{B},n}(w) = m_{\mathbf{B}}(w) \tag{14}$$

(the inequality is a consequence of the definitions of $\ell_{\mathbf{B}}(w)$ and $\ell_{\mathbf{B},n}(w)$ while the equality follows from properties of (finite) Weyl groups).

Let $I_e(w) = \{e' \in E : e \prec_{\mathbf{B}} e' \prec_{\mathbf{B}} w(e)\}$. We claim that

$$m_{\mathbf{B}}(w) = +\infty \ \Leftrightarrow\ \exists e \in E \text{ such that } |I_e(w)| = +\infty. \tag{15}$$

We first check the implication $\Rightarrow$ in (15). The assumption yields an infinite sequence $\{(e_i, e'_i)\}_{i\in\mathbb{N}}$ such that $e_i \prec_{\mathbf{B}} e'_i$ and $w(e_i) \succ_{\mathbf{B}} w(e'_i)$. Since w fixes all but finitely many elements of E, one of the sequences $\{e_i\}_{i\in\mathbb{N}}$ and $\{e'_i\}_{i\in\mathbb{N}}$ has a stationary subsequence, and thus along a relabeled subsequence $\{(e_i, e'_i)\}_{i\in\mathbb{N}}$ we have $e_i = e$ for all $i \in \mathbb{N}$ and some $e \in E$, or $e'_i = e'$ for all $i \in \mathbb{N}$ and some $e' \in E$. In the former case, the set $\{e'_i : w(e'_i) = e'_i\}$ is infinite and contained in $I_e(w)$. In the latter case, we similarly obtain that the set $\{f \in E : w(e') \prec_{\mathbf{B}} f \prec_{\mathbf{B}} e'\}$ is infinite, and since w has finite order, this implies that $I_{w^r(e')}(w)$ is infinite for some $r \geq 1$.

Next we check the implication $\Leftarrow$ in (15). We assume that $|I_e(w)| = +\infty$ for some $e \in E$. (Then, necessarily, $w(e) \neq e$, hence $e \neq i_E(e)$ in the case where $\mathbf{G} = \mathbf{G}^\omega(E)$.) Since w fixes all but finitely many elements of E, the set $\{e' \in I_e(w) : w(e') = e'\}$ is infinite. Therefore, there are infinitely many couples $(e, e') \in \hat{E}$ such that $e \prec_{\mathbf{B}} e'$ and $w(e) \succ_{\mathbf{B}} w(e')$. Moreover, in the case where $\mathbf{G} = \mathbf{G}^\omega(E)$, up to replacing (e, e') by $(i_E(e'), i_E(e))$, we get infinitely many such couples satisfying $e \prec_{\mathbf{B}} i_E(e)$. This implies $m_{\mathbf{B}}(w) = +\infty$, and (15) is proved.

In view of (14) and (15), to complete the proof of the proposition, it remains to show the relation $\ell_{\mathbf{B}}(w) \leq m_{\mathbf{B}}(w)$. We argue by induction on $m_{\mathbf{B}}(w)$.

If $m_{\mathbf{B}}(e) = 0$, we get $w = \mathrm{id}$, and thus $\ell_{\mathbf{B}}(w) = 0$. Now let $w \in \mathbf{W}$ such that $0 < m_{\mathbf{B}}(w) < +\infty$ and assume that $\ell_{\mathbf{B}}(w') \leq m_{\mathbf{B}}(w')$ holds for all $w' \in \mathbf{W}$ such that $m_{\mathbf{B}}(w') < m_{\mathbf{B}}(w)$. Let $e \in E'$ be minimal such that there is $e' \in E'$ with $e \prec_{\mathbf{B}} e'$ and $w(e) \succ_{\mathbf{B}} w(e')$. Choose e' maximal for this property. We claim that

$$\text{the set } \{i \in E : w(e) \succ_{\mathbf{B}} i \succ_{\mathbf{B}} w(e')\} \text{ is finite.} \tag{16}$$

Assume the contrary. Since w fixes all but finitely many elements of E, there are infinitely many $i \in E$ such that $w(e) \succ_{\mathbf{B}} i = w(i) \succ_{\mathbf{B}} w(e')$. Note that we have $e \prec_{\mathbf{B}} i$ by the minimality of e. Thus there are infinitely many elements in the set $I_e(w)$. In view of (15), this is impossible, and (16) is established.

By (16) we can find $i \in E'$ such that $w(e') \prec_{\mathbf{B}} i$ and $w(e'), i$ are consecutive in E'. Choose $e'' \in E'$ such that $i = w(e'')$. By the maximality of e', we have $e'' \prec_{\mathbf{B}} e'$. In the case where $\mathbf{G} = \mathbf{G}^\omega(E)$, up to replacing (e'', e') by $(i_E(e'), i_E(e''))$ if necessary, we may assume that $e'' \prec_{\mathbf{B}} i_E(e'')$. Hence we have found $e'', e' \in E'$ with the following properties:

$$e'' \prec_{\mathbf{B}} e'; \quad w(e') \prec_{\mathbf{B}} w(e'') \text{ are consecutive in } E'; \quad e'' \prec_{\mathbf{B}} i_E(e'') \text{ (in the case where } \mathbf{G} = \mathbf{G}^\omega(E)).$$

It is straightforward to deduce that $m_{\mathbf{B}}(s_{w(e'),w(e'')}w) = m_{\mathbf{B}}(w) - 1$. Using the induction hypothesis, we derive: $\ell_{\mathbf{B}}(w) \leq \ell_{\mathbf{B}}(s_{w(e'),w(e'')}w) + 1 \leq m_{\mathbf{B}}(s_{w(e'),w(e'')}w) + 1 = m_{\mathbf{B}}(w)$. The proof is now complete. □

Corollary 2. *The following conditions are equivalent.*

(i) $\mathcal{S}_{\mathbf{B}}$ *generates* $\mathbf{W}$*;*
(ii) $\ell_{\mathbf{B}}(w) < +\infty$ *for all* $w \in \mathbf{W}$*;*
(iii) $(E, \preceq_{\mathbf{B}})$ *is isomorphic as an ordered set to a subset of* $(\mathbb{Z}, \leq)$*.*

Proof. The equivalence (i)⇔(ii) is immediate. Note that condition (iii) is equivalent to requiring that, for all $e, e' \in E$, the interval $\{e'' \in E : e \prec_{\mathbf{B}} e'' \prec_{\mathbf{B}} e'\}$ is finite. Thus the implication (iii)⇒(ii) is guaranteed by Proposition 8 (c). Conversely, if (ii) holds true, then we get $\ell_{\mathbf{B}}(s_{e,e'}) < +\infty$ for all $(e, e') \in \hat{E}$, whence (by Proposition 8 (c)) the set $\{e'' \in E : e \prec_{\mathbf{B}} e'' \prec_{\mathbf{B}} e'\}$ is finite. This implies (iii). □

5.1.2. *Relation with parabolic subgroups.* In addition to the splitting Borel subgroup $\mathbf{B}$, we consider a splitting parabolic subgroup $\mathbf{P} \subset \mathbf{G}$ containing $\mathbf{H}$. Recall that the subgroup $\mathbf{P}$ gives rise (in fact, is equivalent) to each of the following data:

- an E-compatible generalized flag $\mathcal{F}$ (which is ω-isotropic in the case of $\mathbf{G} = \mathbf{G}^\omega(E)$) such that $\mathbf{P} = \{g \in \mathbf{G} : g(\mathcal{F}) = \mathcal{F}\}$;
- a totally ordered set $(A, \preceq_A)$ and a surjective map $\sigma_0 : E \to A$ such that $\mathcal{F} = \mathcal{F}_{\sigma_0}$ (which is equipped with an anti-automorphism $i_A : A \to A$ satisfying $\sigma_0 \circ i_E = i_A \circ \sigma_0$ in the case of $\mathbf{G} = \mathbf{G}^\omega(E)$);
- a partial order $\preceq_{\mathbf{P}}$ on E satisfying property (11), such that $e \prec_{\mathbf{P}} e'$ if and only if $\sigma_0(e) \prec_A \sigma_0(e')$.

Moreover, $\mathbf{P}$ gives rise to a subgroup of $\mathbf{W}$:

$$\mathbf{W}_{\mathbf{P}} = \{w \in \mathbf{W} : \sigma_0 \circ w^{-1} = \sigma_0\} = \{w \in \mathbf{W} : e \not\prec_{\mathbf{P}} w(e) \text{ and } w(e) \not\prec_{\mathbf{P}} e, \ \forall e \in E\}.$$

Note that we do not assume that $\mathbf{B}$ is contained in $\mathbf{P}$.

Lemma 4. *The following conditions are equivalent:*

(i) $\mathbf{B} \subset \mathbf{P}$*;*
(ii) *for all* $e, e' \in E$, $e \prec_{\mathbf{P}} e' \ \Rightarrow \ e \prec_{\mathbf{B}} e'$*, i.e., the total order* $\preceq_{\mathbf{B}}$ *refines the partial order* $\preceq_{\mathbf{P}}$*;*
(iii) *for all* $e, e' \in E$, $e \preceq_{\mathbf{B}} e' \ \Rightarrow \ \sigma_0(e) \preceq_A \sigma_0(e')$*, i.e., the map* σ_0 *is nondecreasing.*

Proof. By the definition of the generalized flag $\mathcal{F}_{\sigma_0}$, conditions (i) and (iii) are equivalent. Since the relation $e \prec_{\mathbf{P}} e'$ is equivalent to $\sigma_0(e') \not\preceq_A \sigma_0(e)$, we obtain that (ii) and (iii) are equivalent. □

For all $w \in \mathbf{W}$, we let

$$m_{\mathbf{B}}^{\mathbf{P}}(w) = \left\{ \begin{array}{ll} |\{(e, e') \in \hat{E} : e \prec_{\mathbf{B}} e', \ w(e) \succ_{\mathbf{P}} w(e')\}| & \text{if } \mathbf{G} = \mathbf{G}(E) \\ |\{(e, e') \in \hat{E} : e \prec_{\mathbf{B}} e', \ e \prec_{\mathbf{B}} i_E(e), \ w(e) \succ_{\mathbf{P}} w(e')\}| & \text{if } \mathbf{G} = \mathbf{G}^\omega(E). \end{array} \right.$$

Note that

$$m_{\mathbf{B}}^{\mathbf{P}}(w) = \left\{ \begin{array}{ll} n_{\mathrm{inv}}(\sigma_0 \circ w) & \text{if } \mathbf{G} = \mathbf{G}(E) \\ n_{\mathrm{inv}}^\omega(\sigma_0 \circ w) & \text{if } \mathbf{G} = \mathbf{G}^\omega(E) \end{array} \right. \tag{17}$$

(see Sections 4.1–4.2). We also know that $m_{\mathbf{B}}^{\mathbf{B}}(w) = \ell_{\mathbf{B}}(w)$ (see Proposition 8 (b)). In the following proposition, we characterize the property that $\mathbf{B}$ is conjugate to a subgroup of $\mathbf{P}$ in terms of $m_{\mathbf{B}}^{\mathbf{P}}(w)$.

Proposition 9. *For* $w \in \mathbf{W}$*, recall that* $\hat{w} \in \mathbf{G}$ *is a representative of* w *in* $N_{\mathbf{G}}(\mathbf{H})$*.*

(a) *We have* $\mathbf{B} \subset \hat{w}\mathbf{P}\hat{w}^{-1}$ *if and only if* $m_{\mathbf{B}}^{\mathbf{P}}(w^{-1}) = 0$*.*
(b) *The following conditions are equivalent:*
   (i) *there is* $w \in \mathbf{W}$ *such that* $\mathbf{B} \subset \hat{w}\mathbf{P}\hat{w}^{-1}$*;*
   (ii) *there is* $w \in \mathbf{W}$ *such that* $m_{\mathbf{B}}^{\mathbf{P}}(w) < +\infty$*.*

Proof. Note that $\hat{w}\mathbf{P}\hat{w}^{-1} \subset \mathbf{G}$ is the isotropy subgroup of the generalized flag $\mathcal{F}_{\sigma_0\circ w^{-1}}$. Thus part (a) follows from Lemma 4 and the definition of $m_{\mathbf{B}}^{\mathbf{P}}(w^{-1})$.
(b) The implication (i)⇒(ii) follows from part (a). Now assume that (ii) holds. Choose $w \in \mathbf{W}$ such that $m_{\mathbf{B}}^{\mathbf{P}}(w)$ is minimal. By (a), it suffices to show that $m_{\mathbf{B}}^{\mathbf{P}}(w) = 0$. Assume, to the contrary, that $m_{\mathbf{B}}^{\mathbf{P}}(w) > 0$. Hence there is a couple $(e,e') \in \hat{E}$ satisfying $e \prec_{\mathbf{B}} e'$, $w(e) \succ_{\mathbf{P}} w(e')$. We can assume that e is minimal such that there is e' with this property, and that e' is maximal possible. We claim that

$$\text{the set } \{i \in E' : w(e) \succ_{\mathbf{P}} i \succ_{\mathbf{P}} w(e')\} \text{ is finite.} \tag{18}$$

Otherwise, there are infinitely many $i \in E$ for which $w(e) \succ_{\mathbf{P}} i = w(i) \succ_{\mathbf{P}} w(e')$. By the minimality of e, we have $e \prec_{\mathbf{B}} i$. Whence there are infinitely many couples $(e,i) \in \hat{E}$ with $e \prec_{\mathbf{B}} i$ and $w(e) \succ_{\mathbf{P}} w(i)$ (in the case of $\mathbf{G} = \mathbf{G}^\omega(E)$, up to replacing (e,i) by $(i_E(i), i_E(e))$, we may also assume that $e \prec_{\mathbf{B}} i_E(e)$). Consequently, $m_{\mathbf{B}}^{\mathbf{P}}(w) = +\infty$, a contradiction. This establishes (18).

By (18) we can find $i \in E'$ minimal (with respect to the order $\preceq_{\mathbf{P}}$) such that $w(e) \succeq_{\mathbf{P}} i \succ_{\mathbf{P}} w(e')$. Let $e'' \in E$ with $w(e'') = i$. The maximality of e' forces $e'' \prec_{\mathbf{B}} e'$. Altogether, we have found a couple $(e'',e') \in \hat{E}$ such that $e'' \prec_{\mathbf{B}} e'$, $w(e'') \succ_{\mathbf{P}} w(e')$, and $w(e'')$ is minimal (with respect to the order $\preceq_{\mathbf{P}}$). For $f \in E$, let $C_{\mathbf{P}}(f)$ denote the class of f for the equivalence relation defined in (11). We may assume that e'' and e' are respectively a minimal element of $w^{-1}(C_{\mathbf{P}}(w(e'')))$ and a maximal element of $w^{-1}(C_{\mathbf{P}}(w(e')))$ (with respect to the order $\preceq_{\mathbf{B}}$). Moreover, in the case of $\mathbf{G} = \mathbf{G}^\omega(E)$, up to replacing (e'',e') by $(i_E(e'), i_E(e''))$, we may assume that $e'' \prec_{\mathbf{B}} i_E(e'')$. Then it is straightforward to check that

$$\begin{aligned} &\{(f,f') \in \hat{E} : f \prec_{\mathbf{B}} f',\ s_{w(e'),w(e'')}w(f) \succ_{\mathbf{P}} s_{w(e'),w(e'')}w(f')\} \\ &\qquad \subset \{(f,f') \in \hat{E} : f \prec_{\mathbf{B}} f',\ w(f) \succ_{\mathbf{P}} w(f')\} \setminus \{(e'',e')\}. \end{aligned}$$

Whence $m_{\mathbf{B}}^{\mathbf{P}}(s_{w(e'),w(e'')}w) < m_{\mathbf{B}}^{\mathbf{P}}(w)$, which contradicts the minimality of $m_{\mathbf{B}}^{\mathbf{P}}(w)$. ☐

Finally, the following proposition points out the relation between $m_{\mathbf{B}}^{\mathbf{P}}(w)$ and $\ell_{\mathbf{B}}(w)$.

Proposition 10. *Assume that there is $w_0 \in \mathbf{W}$ such that $m_{\mathbf{B}}^{\mathbf{P}}(w_0^{-1}) = 0$. Then, for all $w \in \mathbf{W}$, we have*

$$m_{\mathbf{B}}^{\mathbf{P}}(w) = \inf\{\ell_{\mathbf{B}}(w_0 w' w) : w' \in \mathbf{W}_{\mathbf{P}}\}.$$

Proof. Note that, for all $e,e' \in E'$, we have $e \prec_{\mathbf{P}} e'$ if and only if $w_0(e) \prec_{\hat{w}_0\mathbf{P}\hat{w}_0^{-1}} w_0(e')$. This yields $m_{\mathbf{B}}^{\mathbf{P}}(w) = m_{\mathbf{B}}^{\hat{w}_0\mathbf{P}\hat{w}_0^{-1}}(w_0 w)$ and $w_0 \mathbf{W}_{\mathbf{P}} w_0^{-1} = \mathbf{W}_{\hat{w}_0\mathbf{P}\hat{w}_0^{-1}}$. Thus, invoking also Proposition 9 (a), up to replacing $\mathbf{P}$ by $\hat{w}_0\mathbf{P}\hat{w}_0^{-1}$, we may suppose that $\mathbf{B} \subset \mathbf{P}$ and $w_0 = \mathrm{id}$.

By the definition of $\mathbf{W}_{\mathbf{P}}$, Lemma 4, and Proposition 8 (b), for every $w' \in \mathbf{W}_{\mathbf{P}}$ we obtain

$$\begin{aligned} m_{\mathbf{B}}^{\mathbf{P}}(w) &= |\{(e,e') \in \hat{E}_{\mathbf{B}} : \sigma_0(w(e)) \succ_A \sigma_0(w(e'))\}| \\ &= |\{(e,e') \in \hat{E}_{\mathbf{B}} : \sigma_0(w'w(e)) \succ_A \sigma_0(w'w(e))\}| \\ &\leq |\{(e,e') \in \hat{E}_{\mathbf{B}} : w'w(e) \succ_{\mathbf{B}} w'w(e')\}| = \ell_{\mathbf{B}}(w'w), \end{aligned}$$

where $\hat{E}_{\mathbf{B}} = \{(e,e') \in \hat{E} : e \prec_{\mathbf{B}} e'\}$ if $\mathbf{G} = \mathbf{G}(E)$, and $\hat{E}_{\mathbf{B}} = \{(e,e') \in \hat{E} : e \prec_{\mathbf{B}} e',\ e \prec_{\mathbf{B}} i_E(e)\}$ if $\mathbf{G} = \mathbf{G}^\omega(E)$. If $m_{\mathbf{B}}^{\mathbf{P}}(w) = +\infty$, the result is established. So we assume next that $m_{\mathbf{B}}^{\mathbf{P}}(w) < +\infty$.
Claim 1: There is $w' \in \mathbf{W}_{\mathbf{P}}$ such that the set $\mathcal{I}(w'w) := \{e \in E : \sigma_0(e) = \sigma_0(w'w(e)) \text{ and } w'w(e) \neq e\}$ is empty.

For any $w' \in \mathbf{W}_{\mathbf{P}}$, the set $\mathcal{I}(w'w)$ is finite. Let $w' \in \mathbf{W}_{\mathbf{P}}$ such that $|\mathcal{I}(w'w)|$ is minimal. We claim that $\mathcal{I}(w'w) = \emptyset$. For otherwise, assume that there is $e \in \mathcal{I}(w'w)$. Thus $\sigma_0(w'w(e)) = e$. Either $\sigma_0((w'w)^\ell(e)) = \sigma_0(e)$ for all $\ell \in \mathbb{Z}$, or there is $\ell \in \mathbb{Z}$ such that $\sigma_0((w'w)^{\ell-1}(e)) \neq \sigma_0((w'w)^\ell(e)) = \sigma_0((w'w)^{\ell+1}(e))$. In the former case we set $w'' = s_{(w'w)^{m-2}(e),(w'w)^{m-1}(e)} \cdots s_{(w'w)(e),(w'w)^2(e)} s_{e,(w'w)(e)}$, where $m \geq 2$ is minimal such that $(w'w)^m(e) = e$. In the latter case we set $w'' = s_{(w'w)^\ell(e),(w'w)^{\ell+1}(e)}$. In both cases one has $w'' \in \mathbf{W}_{\mathbf{P}}$, and it easy to check that $\mathcal{I}(w''w'w) \subsetneq \mathcal{I}(w'w)$, a contradiction. Hence Claim 1 holds.

Note that $m_{\mathbf{B}}^{\mathbf{P}}(w'w) = m_{\mathbf{B}}^{\mathbf{P}}(w)$. Up to dealing with $w'w$ instead of w, we may assume that $\mathcal{I}(w) = \emptyset$. For $\alpha \in A$, let $I_\alpha(w) = \{e \in \sigma_0^{-1}(\alpha) : w(e) \neq e\}$. Since $\mathcal{I}(w) = \emptyset$, one has $I_\alpha(w) = I_\alpha^+(w) \sqcup I_\alpha^-(w)$ with

$$I_\alpha^+(w) = \{e \in \sigma_0^{-1}(\alpha) : \sigma_0(w^{-1}(e)) \succ_A \alpha\} \quad \text{and} \quad I_\alpha^-(w) = \{e \in \sigma_0^{-1}(\alpha) : \sigma_0(w^{-1}(e)) \prec_A \alpha\}.$$

Claim 2: There is $w' \in \mathbf{W}_{\mathbf{P}}$ with $w'(e) = e$ whenever $w(e) = e$, and satisfying the following property: for every $\alpha \in A$, the set $\{e' \in \sigma_0^{-1}(\alpha) : w'(e) \prec_{\mathbf{B}} e'\}$ is finite whenever $e \in I_\alpha^+(w)$, and the set $\{e' \in \sigma_0^{-1}(\alpha) : w'(e) \succ_{\mathbf{B}} e'\}$ is finite whenever $e \in I_\alpha^-(w)$.

Let $e \in I_\alpha^+(w)$. There is $\ell(e) \geq 2$ minimal such that $\sigma_0(w^{-\ell(e)}(e)) \preceq_A \alpha$. Since $m_{\mathbf{B}}^{\mathbf{P}}(w) < +\infty$, the set $\{e' \in \sigma_0^{-1}(\alpha) : w^{-\ell(e)}(e) \prec_{\mathbf{B}} e'\}$ is finite. Set $w'(e) = w^{-\ell(e)}(e)$. Similarly, given $e \in I_\alpha^-(w)$, there is $m(e) \geq 2$ minimal such that $\sigma_0(w^{-m(e)}(e)) \succeq_A \alpha$, and the set $\{e' \in \sigma_0^{-1}(\alpha) : w^{-m(e)}(e) \succ_{\mathbf{B}} e'\}$ is finite; we set $w'(e) = w^{-m(e)}(e)$ in this case. If $e \in \sigma_0^{-1}(\alpha) \setminus I_\alpha(w)$, we set $w'(e) = e$. It is readily seen that the so-obtained map $w' : \sigma_0^{-1}(\alpha) \to \sigma_0^{-1}(\alpha)$ is bijective. Collecting these maps for all $\alpha \in A$, we obtain an element $w' \in \mathbf{W_P}$ satisfying the desired properties. This shows Claim 2.

Set $\hat{w} = w'w$ with $w' \in \mathbf{W_P}$ as in Claim 2. For every $\alpha \in A$, the set

$$J_\alpha(\hat{w}) = \{e \in \sigma_0^{-1}(\alpha) : (\exists e' \in \sigma_0^{-1}(\alpha) \text{ with } e' \preceq_{\mathbf{B}} e \text{ and } \sigma_0(\hat{w}^{-1}(e')) \succ_A \alpha) \\ \text{or } (\exists e' \in \sigma_0^{-1}(\alpha) \text{ with } e' \succeq_{\mathbf{B}} e \text{ and } \sigma_0(\hat{w}^{-1}(e')) \prec_A \alpha)\}$$

is finite (by Claim 2). We write $J_\alpha(\hat{w}) = \{e_i^\alpha\}_{i=1}^{k_\alpha}$ so that $\hat{w}^{-1}(e_1^\alpha) \prec_{\mathbf{B}} \ldots \prec_{\mathbf{B}} \hat{w}^{-1}(e_{k_\alpha}^\alpha)$. There is $w'' \in \mathbf{W_P}$ with $w''(e) = e$ whenever $e \notin \bigcup_{\alpha\in A} J_\alpha(\hat{w})$ and such that

$$w''(J_\alpha(\hat{w})) = J_\alpha(\hat{w}) \ \text{ and } \ w''(e_1^\alpha) \prec_{\mathbf{B}} \ldots \prec_{\mathbf{B}} w''(e_{k_\alpha}^\alpha) \ \text{ for all } \alpha \in A.$$

Taking the construction of w'' into account, one can check that there is no couple $(e, e') \in \hat{E}$ with $e \prec_{\mathbf{B}} e'$, $w''\hat{w}(e) \succ_{\mathbf{B}} w''\hat{w}(e')$, and $\sigma_0(w''\hat{w}(e)) = \sigma_0(w''\hat{w}(e'))$. Therefore, $m_{\mathbf{B}}^{\mathbf{P}}(w) = \ell_{\mathbf{B}}(w''\hat{w}) = \ell_{\mathbf{B}}((w''w')w)$ with $w''w' \in \mathbf{W_P}$. The proof is complete. ☐

5.2. **Review of (finite-dimensional) flag varieties.** We consider an E-compatible generalized flag $\mathcal{F} = \mathcal{F}_{\sigma_0}$ corresponding to a surjection $\sigma_0 : E \to A$. Let $I \subset E$ be a finite subset (resp., i_E-stable, if the form ω is considered). In this section we recall standard properties of the Schubert decomposition of the flag varieties $\mathrm{Fl}(\mathcal{F}, I)$ and $\mathrm{Fl}(\mathcal{F}, \omega, I)$ (see Section 3.3). We refer to [1, §3.2–3.5], [2, §1.2], [9, §10.8] for more details.

Proposition 11. *Let $\mathbf{G} = \mathbf{G}(E)$. Let $\mathbf{B}$ be a splitting Borel subgroup of $\mathbf{G}$ containing $\mathbf{H}$ and let $B(I) := G(I) \cap \mathbf{B}$ be the corresponding Borel subgroup of the group $G(I)$. Let $H(I) = G(I) \cap \mathbf{H}$. Let $W(I) \subset \mathbf{W}$ be the Weyl group of $G(I)$.*
(a) *We have the decomposition*

$$\mathrm{Fl}(\mathcal{F}, I) = \bigcup_{w\in W(I)} B(I)\mathcal{F}_{\sigma_0\circ w^{-1}}.$$

Moreover, $\mathcal{F}_{\sigma_0\circ w^{-1}}$ is the unique element of $B(I)\mathcal{F}_{\sigma_0\circ w^{-1}}$ fixed by the maximal torus $H(I)$.
(b) *Each subset $B(I)\mathcal{F}_{\sigma_0\circ w^{-1}}$, for $w \in W(I)$, is a locally closed subvariety isomorphic to an affine space of dimension $|\{(e, e') \in I \times I : e \prec_{\mathbf{B}} e',\ \sigma_0 \circ w^{-1}(e') \prec_A \sigma_0 \circ w^{-1}(e)\}|$.*
(c) *Given $w, w' \in W(I)$, the inclusion $B(I)\mathcal{F}_{\sigma_0\circ w^{-1}} \subset \overline{B(I)\mathcal{F}_{\sigma_0\circ w'^{-1}}}$ holds if and only if $\sigma_0\circ w^{-1} \leq \sigma_0\circ w'^{-1}$ for the order $\leq$ defined in Section 4.1.*
(d) *Let $J \subset E$ be another finite subset such that $I \subset J$. Let $\iota_{I,J} : \mathrm{Fl}(\mathcal{F}, I) \hookrightarrow \mathrm{Fl}(\mathcal{F}, J)$ be the embedding constructed in Section 3.3. Then, for all $w \in W(I)$, the image of the Schubert cell $B(I)\mathcal{F}_{\sigma_0\circ w^{-1}}$ by the map $\iota_{I,J}$ is an affine subspace of $B(J)\mathcal{F}_{\sigma_0\circ w^{-1}}$.*

Proposition 12. *Let $\mathbf{G} = \mathbf{G}^\omega(E)$. Let $\mathbf{B}$ be a splitting Borel subgroup of $\mathbf{G}$ containing $\mathbf{H}$ and let $B^\omega(I) := G^\omega(I) \cap \mathbf{B}$ be the corresponding Borel subgroup of the group $G^\omega(I)$. Let $H^\omega(I) = G^\omega(I) \cap \mathbf{H}$. Let $W^\omega(I) \subset \mathbf{W}$ be the Weyl group of $G^\omega(I)$.*
(a) *We have the decomposition*

$$\mathrm{Fl}(\mathcal{F}, \omega, I) = \bigcup_{w\in W^\omega(I)} B^\omega(I)\mathcal{F}_{\sigma_0\circ w^{-1}}.$$

Moreover, $\mathcal{F}_{\sigma_0\circ w^{-1}}$ is the unique element of $B^\omega(I)\mathcal{F}_{\sigma_0\circ w^{-1}}$ fixed by the maximal torus $H^\omega(I)$.
(b) *Each subset $B^\omega(I)\mathcal{F}_{\sigma_0\circ w^{-1}}$, for $w \in W^\omega(I)$, is a locally closed subvariety isomorphic to an affine space of dimension $|\{(e, e') \in I \times I : e \prec_{\mathbf{B}} e',\ e \prec_{\mathbf{B}} i_E(e),\ e' \neq i_E(e'),\ \sigma_0 \circ w^{-1}(e') \prec_A \sigma_0 \circ w^{-1}(e)\}|$.*
(c) *Given $w, w' \in W^\omega(I)$, the inclusion $B^\omega(I)\mathcal{F}_{\sigma_0\circ w^{-1}} \subset \overline{B^\omega(I)\mathcal{F}_{\sigma_0\circ w'^{-1}}}$ holds if and only if $\sigma_0 \circ w^{-1} \leq_\omega \sigma_0 \circ w'^{-1}$, for the order $\leq_\omega$ defined in Section 4.2.*
(d) *Let $J \subset E$ be another i_E-stable finite subset such that $I \subset J$. Let $\iota_{I,J}^\omega : \mathrm{Fl}(\mathcal{F}, \omega, I) \hookrightarrow \mathrm{Fl}(\mathcal{F}, \omega, J)$ be the embedding constructed in Section 3.3. Then, for all $w \in W^\omega(I)$, the image of the Schubert cell $B^\omega(I)\mathcal{F}_{\sigma_0\circ w^{-1}}$ by the map $\iota_{I,J}^\omega$ is an affine subspace of $B^\omega(J)\mathcal{F}_{\sigma_0\circ w^{-1}}$.*

5.3. **Proof of Lemmas 2 and 3.** We consider the map

$$\phi : \mathbf{W}(E) \to \mathbf{Fl}_A(V),\ w \mapsto \mathcal{F}_{\sigma_0 \circ w^{-1}}$$

and, in the proof of Lemma 3, we also consider its restriction $\phi^\omega : \mathbf{W}^\omega(E) \to \mathbf{Fl}_A^\omega(V)$.

Proof of Lemma 2. Let $\mathbf{Fl}'(\mathcal{F}, E) \subset \mathbf{Fl}(\mathcal{F}, E)$ denote the subset of E-compatible generalized flags. By definition the generalized flag $\phi(w)$ is E-compatible for all $w \in \mathbf{W}(E)$. Moreover, it is easily seen that $\phi(w) = \hat{w}(\mathcal{F}_{\sigma_0})$ where $\hat{w} \in \mathbf{G}(E)$ is the element for which $\hat{w}(e) = w(e)$ for all $e \in E$. Thus $\phi(w)$ is E-commensurable with $\mathcal{F} = \mathcal{F}_{\sigma_0}$ (see Proposition 2). Consequently, $\phi(w) \in \mathbf{Fl}'(\mathcal{F}, E)$ for all $w \in \mathbf{W}(E)$.

Conversely, let $\mathcal{G} \in \mathbf{Fl}'(\mathcal{F}, E)$. Choosing n such that $\mathcal{G} \in \mathrm{Fl}(\mathcal{F}, E_n)$, we have that $\mathcal{G}$ is fixed by the maximal torus $H(E_n) \subset G(E_n)$. Using Proposition 11 (a), we find $w \in W(E_n) \subset \mathbf{W}(E)$ such that $\mathcal{G} = \mathcal{F}_{\sigma_0 \circ w^{-1}} = \phi(w)$.

Finally, for $w, w' \in \mathbf{W}(E)$, we have $\phi(w) = \phi(w')$ if and only if $\sigma_0 \circ w^{-1} = \sigma_0 \circ w'^{-1}$, and the latter condition reads as $w'^{-1}w \in \mathbf{W_P}(E)$. Therefore, ϕ induces a bijection $\mathbf{W}(E)/\mathbf{W_P}(E) \to \mathbf{Fl}'(\mathcal{F}, E)$. ☐

Proof of Lemma 3. Let $\mathbf{Fl}'(\mathcal{F}, \omega, E) = \mathbf{Fl}'(\mathcal{F}, E) \cap \mathbf{Fl}(\mathcal{F}, \omega, E)$. From Lemma 2 we know that $\phi^\omega(w)$ is E-compatible and E-commensurable with $\mathcal{F} = \mathcal{F}_{\sigma_0}$, whence $\phi^\omega(w) \in \mathbf{Fl}'(\mathcal{F}, \omega, E)$ for all $w \in \mathbf{W}^\omega(E)$.

Let $\mathcal{G} \in \mathbf{Fl}'(\mathcal{F}, \omega, E)$. Choosing n such that $\mathcal{G} \in \mathrm{Fl}(\mathcal{F}, \omega, E_n)$, we have that $\mathcal{G}$ is a fixed point of the maximal torus $H^\omega(E_n) \subset G^\omega(E_n)$, hence we can find $w \in W^\omega(E_n)$ such that $\mathcal{G} = \mathcal{F}_{\sigma_0 \circ w^{-1}} = \phi^\omega(w)$.

As in the proof of Lemma 2 it is easy to conclude that ϕ^ω induces a bijection $\mathbf{W}^\omega(E)/\mathbf{W}^\omega_{\mathbf{P}}(E) \to \mathbf{Fl}'(\mathcal{F}, \omega, E)$. ☐

5.4. **Proof of Theorems 1 and 2.**

Proof of Theorem 1. Recall the exhaustions (3) and (8) of the ind-group $\mathbf{G}(E)$ and the ind-variety $\mathbf{Fl}(\mathcal{F}, E)$. For all $n \geq 1$, the subgroups $H(E_n) := G(E_n) \cap \mathbf{H}(E)$, $B_n := G(E_n) \cap \mathbf{B}$, and $P_n := G(E_n) \cap \mathbf{P}$ are respectively a maximal torus, a Borel subgroup, and a parabolic subgroup of $G(E_n)$.

(a) Let $\mathcal{G} \in \mathbf{Fl}(\mathcal{F}, E)$. By Proposition 11 (a), for any $n \geq 1$ large enough so that $\mathcal{G} \in \mathrm{Fl}(\mathcal{F}, E_n)$, the B_n-orbit of $\mathcal{G}$ contains a unique element of the form $\mathcal{F}_{\sigma_0 \circ w^{-1}}$ with $w \in W(E_n)$. Therefore, every element $\mathcal{G} \in \mathbf{Fl}(\mathcal{F}, E)$ lies in the $\mathbf{B}$-orbit of $\mathcal{F}_\sigma$ for a unique $\sigma \in \mathbf{W}(E) \cdot \sigma_0$.

(b) Let $\mathcal{G} = \{G'_\alpha, G''_\alpha : \alpha \in A\} \in \mathbf{Fl}(\mathcal{F}, E)$. According to part (a) of the proof, there is a unique $\sigma \in \mathbf{W}(E) \cdot \sigma_0$ such that $\mathcal{G} \in \mathbf{B}\mathcal{F}_\sigma$, say $\mathcal{G} = b(\mathcal{F}_\sigma)$, where $b \in \mathbf{B}$. Thus

$$G''_\alpha \cap F'_{0,e} = b(F''_{\sigma,\alpha} \cap F'_{0,e}) \quad \text{and} \quad G''_\alpha \cap F''_{0,e} = b(F''_{\sigma,\alpha} \cap F''_{0,e})$$

(because $F'_{0,e}, F''_{0,e}$ are b-stable). This clearly implies that $\sigma_\mathcal{G} = \sigma_{\mathcal{F}_\sigma}$. Moreover, from the definition of $\mathcal{F}_\sigma$ we see that $F''_{\sigma,\alpha} \cap F''_{0,e} \neq F''_{\sigma,\alpha} \cap F'_{0,e}$ if and only if $\sigma(e) \preceq_A \alpha$. Whence $\sigma(e) = \min\{\alpha \in A : F''_{\sigma,\alpha} \cap F''_{0,e} \neq F''_{\sigma,\alpha} \cap F'_{0,e}\} = \sigma_{\mathcal{F}_\sigma}(e)$ for all $e \in E$. Thus $\sigma_\mathcal{G} = \sigma$. Note that the last equality guarantees in particular that $\sigma_\mathcal{G} \in \mathfrak{S}(E, A)$.

(c) follows from Proposition 11 (b) and (d).

(d) We consider $\sigma, \tau \in \mathbf{W}(E) \cdot \sigma_0$ and let $n \geq 1$ be such that $\mathcal{F}_\sigma, \mathcal{F}_\tau \in \mathrm{Fl}(\mathcal{F}, E_n)$. Assume that $\sigma \hat{<} \tau$, i.e., $\tau = \sigma \circ t_{e,e'}$ for a pair $(e, e') \in E \times E$ with $e \prec_\mathbf{B} e'$ and $\sigma(e) \preceq_A \sigma(e')$. Up to choosing n larger if necessary, we may assume that $e, e' \in E_n$. Then, by Proposition 11 (c), we get $B_n\mathcal{F}_\sigma \subset \overline{B_n\mathcal{F}_\tau}$. Whence $\mathbf{B}\mathcal{F}_\sigma \subset \overline{\mathbf{B}\mathcal{F}_\tau}$. This argument also shows that the latter inclusion holds whenever $\sigma \leq \tau$. Conversely, assume that $\mathcal{F}_\sigma \in \overline{\mathbf{B}\mathcal{F}_\tau}$. Hence $\mathcal{F}_\sigma \in \overline{B_n\mathcal{F}_\tau}$ for $n \geq 1$ large enough. Once again, by Proposition 11 (c), this yields $\sigma \leq \tau$. The proof of Theorem 1 is complete. ☐

Proof of Theorem 2. The proof of Theorem 2 follows exactly the same scheme as the proof of Theorem 1, relying this time on Proposition 12 instead of Proposition 11. We skip the details. ☐

5.5. **Proof of Theorem 3.**

Proof of Theorem 3. (a) Condition (i) means that there is $g \in \mathbf{G}$ such that $\mathbf{B} \subset g\mathbf{P}g^{-1}$. This equivalently means that the element $g\mathbf{P} \in \mathbf{G}/\mathbf{P}$ is fixed by $\mathbf{B}$, i.e., that $\mathbf{G}/\mathbf{P}$ comprises a $\mathbf{B}$-orbit reduced to a single point. We have shown the equivalence (i)$\Leftrightarrow$(iii). The implication (iii)$\Rightarrow$(ii) is immediate, while the implication (ii)$\Rightarrow$(i) follows from Proposition 9, relation (17), and Theorems 1(c)–2 (c).

(b) The implication (i)$\Rightarrow$(ii) is a consequence of part (a), Corollary 2, Proposition 10, relation (17), and Theorems 1 (c)–2 (c). Assume that (ii) holds. From part (a), there is $g \in \mathbf{G}$ such that $\mathbf{B} \subset g\mathbf{P}g^{-1}$. Up to dealing with $g\mathbf{P}g^{-1}$ instead of $\mathbf{P}$, we may assume that $\mathbf{B} \subset \mathbf{P}$. Arguing by contradiction, say that $(E, \preceq_\mathbf{B})$ is not isomorphic to a subset of $(\mathbb{Z}, \leq)$. Thus there are $e, e' \in E$ such that the set

$\{e'' \in E : e \prec_{\mathbf{B}} e'' \prec_{\mathbf{B}} e'\}$ is infinite. Since the surjective map $\sigma_0 : E \to A$, corresponding to $\mathbf{P}$, is nondecreasing (by Lemma 4) and nonconstant (because $\mathbf{P} \neq \mathbf{G}$), we find $\hat{e}, \hat{e}'$ with $\hat{e} \preceq_{\mathbf{B}} e \prec_{\mathbf{B}} e' \preceq_{\mathbf{B}} \hat{e}'$ such that $\sigma_0(\hat{e}) \prec_A \sigma_0(\hat{e}')$. Then, $\dim \mathbf{B}\mathcal{F}_{\sigma_0 \circ s_{\hat{e},\hat{e}'}} = +\infty$ (by Theorems 1 (c)–2 (c)), a contradiction. ☐

6. Smoothness of Schubert ind-varieties

In this section $\mathbf{G}$ is one of the ind-groups $\mathbf{G}(E)$ or $\mathbf{G}^\omega(E)$ and $\mathbf{B}$ is a splitting Borel subgroup of $\mathbf{G}$ which contains the splitting Cartan subgroup $\mathbf{H} = \mathbf{H}(E)$ or $\mathbf{H}^\omega(E)$. We consider the *Schubert ind-varieties* defined as the closures of the Schubert cells $\mathbf{B}\mathcal{F}_\sigma$ in the ind-varieties of generalized flags $\mathbf{Fl}(\mathcal{F}, E)$ or $\mathbf{Fl}(\mathcal{F}, \omega, E)$. Specifically, we study the smoothness of Schubert ind-varieties. The general principle (Theorem 4) is straightforward: the ind-variety $\overline{\mathbf{B}\mathcal{F}_\sigma}$ is smooth if and only if its intersections with suitable finite-dimensional flag subvarieties of $\mathbf{Fl}(\mathcal{F}, E)$ or $\mathbf{Fl}(\mathcal{F}, \omega, E)$ are smooth. Note however that this fact is not immediate: see Remark 9 below. As an example, in Section 6.3 we give a combinatorial interpretation of this result in the case of ind-varieties of maximal generalized flags and in the case of ind-grassmannians.

6.1. General facts on the smoothness of ind-varieties.

The notion of smooth point of an ind-variety is defined in Section 2.1. We refer to [8, Chapter 4] or [11] for more details. In this section, for later use, we present some general facts regarding the smoothness of ind-varieties.

We start with the following simple smoothness criterion (see [8]).

Lemma 5. *Let $\mathbf{X}$ be an ind-variety with an exhaustion $\mathbf{X} = \bigcup_{n \geq 1} X_n$. Let $x \in \mathbf{X}$. Suppose that there is a subsequence $\{X_{n_k}\}_{k \geq 1}$ such that x is a smooth point of X_{n_k} for all $k \geq 1$. Then x is a smooth point of $\mathbf{X}$. In particular, if $\mathbf{X}$ admits an exhaustion by smooth varieties, then $\mathbf{X}$ is smooth.*

Example 7. It easily follows from Lemma 5 that the infinite-dimensional affine space $\mathbb{A}^\infty$ and the infinite-dimensional projective space $\mathbb{P}^\infty$ are smooth. More generally, it follows from Propositions 4–5 and Lemma 5 that the ind-varieties of the form $\mathbf{Fl}(\mathcal{F}, E)$ and $\mathbf{Fl}(\mathcal{F}, \omega, E)$ are smooth.

Remark 9. The converse of Lemma 5 is clearly false. Consider for instance $\mathbf{X} = \mathbb{A}^\infty = \bigcup_{n \geq 1} \mathbb{A}^n$ and let $x \in \mathbb{A}^1$. For each $n \geq 1$, let $X'_n \subset \mathbb{A}^{n+1}$ be an n-dimensional affine subspace containing x and distinct of $\mathbb{A}^n$, and set $X_n = \mathbb{A}^n \cup X'_n$. The subvarieties X_n exhaust $\mathbb{A}^\infty$. Clearly x is a singular point of every X_n. However x is a smooth point of $\mathbb{A}^\infty$ (which is a smooth ind-variety).

The following partial converse of Lemma 5 is used in Section 6.2 for studying the smoothness of Schubert ind-varieties.

Lemma 6. *Let $\mathbf{X}$ be an ind-variety and let $\mathbf{X} = \bigcup_{n \geq 1} X_n$ be an exhaustion by algebraic varieties. Assume that each inclusion $X_n \subset X_{n+1}$ has a left inverse $r_n : X_{n+1} \to X_n$ in the category of algebraic varieties. Then, if $x \in \mathbf{X}$ is a singular point of X_{n_0} for some $n_0 \geq 1$, x is a singular point of $\mathbf{X}$.*

Proof. We start with a preliminary fact. Let Y be an algebraic variety and $Z \subset Y$ be a subvariety such that there is a retraction $r : Y \to Z$, i.e., a left inverse of the inclusion map $i : Z \hookrightarrow Y$. Let $x \in Z$. We consider the local rings $\mathcal{O}_{Z,x}$, $\mathcal{O}_{Y,x}$ and their maximal ideals $\mathfrak{m}_{Z,x}$, $\mathfrak{m}_{Y,x}$. The map r induces a ring homomorphism $r^\sharp : \mathcal{O}_{Z,x} \to \mathcal{O}_{Y,x}$ such that $r^\sharp(\mathfrak{m}_{Z,x}^k) \subset \mathfrak{m}_{Y,x}^k$ for all $k \geq 1$. Thus $r^\sharp$ induces maps

$$r_k^\sharp : S^k(\mathfrak{m}_{Z,x}/\mathfrak{m}_{Z,x}^2) \to S^k(\mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2) \quad \text{and} \quad \tilde{r}_k^\sharp : \mathfrak{m}_{Z,x}^k/\mathfrak{m}_{Z,x}^{k+1} \to \mathfrak{m}_{Y,x}^k/\mathfrak{m}_{Y,x}^{k+1},$$

which are respective right inverses of the maps $i_k^\sharp : S^k(\mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2) \to S^k(\mathfrak{m}_{Z,x}/\mathfrak{m}_{Z,x}^2)$ and $\tilde{i}_k^\sharp : \mathfrak{m}_{Y,x}^k/\mathfrak{m}_{Y,x}^{k+1} \to \mathfrak{m}_{Z,x}^k/\mathfrak{m}_{Z,x}^{k+1}$ induced by the inclusion $i : Z \hookrightarrow Y$. Moreover the diagrams

$$\begin{array}{ccc} S^k(\mathfrak{m}_{Z,x}/\mathfrak{m}_{Z,x}^2) & \xrightarrow{\alpha_{Z,k}} & \mathfrak{m}_{Z,x}^k/\mathfrak{m}_{Z,x}^{k+1} \\ r_k^\sharp \downarrow & & \downarrow \tilde{r}_k^\sharp \\ S^k(\mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2) & \xrightarrow{\alpha_{Y,k}} & \mathfrak{m}_{Y,x}^k/\mathfrak{m}_{Y,x}^{k+1} \end{array} \quad \text{and} \quad \begin{array}{ccc} S^k(\mathfrak{m}_{Y,x}/\mathfrak{m}_{Y,x}^2) & \xrightarrow{\alpha_{Y,k}} & \mathfrak{m}_{Y,x}^k/\mathfrak{m}_{Y,x}^{k+1} \\ i_k^\sharp \downarrow & & \downarrow \tilde{i}_k^\sharp \\ S^k(\mathfrak{m}_{Z,x}/\mathfrak{m}_{Z,x}^2) & \xrightarrow{\alpha_{Z,k}} & \mathfrak{m}_{Z,x}^k/\mathfrak{m}_{Z,x}^{k+1} \end{array}$$

are commutative, where $\alpha_{Z,k}$ and $\alpha_{Y,k}$ are defined in a natural way.

In the setting of the lemma, for every $n \geq 1$, we denote $\mathfrak{m}_{n,x} := \mathfrak{m}_{X_n,x}$. The retraction $r_n : X_{n+1} \to X_n$ induces maps $(r_n)_k^\sharp : S^k(\mathfrak{m}_{n,x}/\mathfrak{m}_{n,x}^2) \to S^k(\mathfrak{m}_{n+1,x}/\mathfrak{m}_{n+1,x}^2)$ and $(\tilde{r}_n)_k^\sharp : \mathfrak{m}_{n,x}^k/\mathfrak{m}_{n,x}^{k+1} \to \mathfrak{m}_{n+1,x}^k/\mathfrak{m}_{n+1,x}^{k+1}$, which are respective right inverses of the maps $(i_n)^\sharp_k : S^k(\mathfrak{m}_{n+1,x}/\mathfrak{m}^2_{n+1,x}) \to S^k(\mathfrak{m}_{n,x}/\mathfrak{m}^2_{n,x})$ and $(\tilde{i}_n)^\sharp_k : \mathfrak{m}^k_{n+1,x}/\mathfrak{m}^{k+1}_{n+1,x} \to \mathfrak{m}^k_{n,x}/\mathfrak{m}^{k+1}_{n,x}$ induced by the inclusion $X_n \subset X_{n+1}$. Moreover the diagrams

$$\begin{array}{ccc} S^k(\mathfrak{m}_{n,x}/\mathfrak{m}^2_{n,x}) & \xrightarrow{\alpha_{n,k}} & \mathfrak{m}^k_{n,x}/\mathfrak{m}^{k+1}_{n,x} \\ {\scriptstyle (r_n)^\sharp_k} \downarrow & & \downarrow {\scriptstyle (\tilde{r}_n)^\sharp_k} \\ S^k(\mathfrak{m}_{n+1,x}/\mathfrak{m}^2_{n+1,x}) & \xrightarrow{\alpha_{n+1,k}} & \mathfrak{m}^k_{n+1,x}/\mathfrak{m}^{k+1}_{n+1,x} \end{array} \quad \text{and} \quad \begin{array}{ccc} S^k(\mathfrak{m}_{n+1,x}/\mathfrak{m}^2_{n+1,x}) & \xrightarrow{\alpha_{n+1,k}} & \mathfrak{m}^k_{n+1,x}/\mathfrak{m}^{k+1}_{n+1,x} \\ {\scriptstyle (i_n)^\sharp_k} \downarrow & & \downarrow {\scriptstyle (\tilde{i}_n)^\sharp_k} \\ S^k(\mathfrak{m}_{n,x}/\mathfrak{m}^2_{n,x}) & \xrightarrow{\alpha_{n,k}} & \mathfrak{m}^k_{n,x}/\mathfrak{m}^{k+1}_{n,x} \end{array}$$

commute, where $\alpha_{n,k} = \alpha_{X_n,k}$ (see also (2)).

Since $x \in X_{n_0}$ is singular, there is $k \geq 2$ such that the map $\alpha_{n_0,k} : S^k(\mathfrak{m}_{n_0,x}/\mathfrak{m}^2_{n_0,x}) \to \mathfrak{m}^k_{n_0,x}/\mathfrak{m}^{k+1}_{n_0,x}$ is not injective, i.e., there is $a_{n_0} \in \ker \alpha_{n_0,k} \setminus \{0\}$. We define the sequence $\{a_n\}$ by letting

$$a_n = \begin{cases} (i_n)^\sharp_k \circ \cdots \circ (i_{n_0-2})^\sharp_k \circ (i_{n_0-1})^\sharp_k(a_{n_0}) & \text{if } 1 \leq n \leq n_0 \\ (r_{n-1})^\sharp_k \circ \cdots \circ (r_{n_0+1})^\sharp_k \circ (r_{n_0})^\sharp_k(a_{n_0}) & \text{if } n \geq n_0. \end{cases}$$

Then $a_n \in S^k(\mathfrak{m}_{n,x}/\mathfrak{m}^2_{n,x})$ and $(i_n)^\sharp_k(a_{n+1}) = a_n$ for all $n \geq 1$. Thus the sequence $a := \{a_n\}$ is an element of the inverse limit $\varprojlim S^k(\mathfrak{m}_{n,x}/\mathfrak{m}^2_{n,x})$. Moreover, we have $a \in \ker \hat{\alpha}_k \setminus \{0\}$, where $\hat{\alpha}_k := \varprojlim \alpha_{n,k}$. Therefore $\hat{\alpha}_k$ is not injective, and so x is a singular point of $\mathbf{X}$. □

6.2. **Smoothness criterion for Schubert ind-varieties.** Let $\mathbf{G} = \mathbf{G}(E)$ (resp., $\mathbf{G} = \mathbf{G}^\omega(E)$).

Let $(A, \preceq_A)$ be a totally ordered set (resp., equipped with an anti-automorphism i_A). A surjective map $\sigma : E \to A$ (resp., such that $i_A \circ \sigma = \sigma \circ i_E$) gives rise to an E-compatible generalized flag $\mathcal{F}_\sigma = \{F'_{\sigma,\alpha}, F''_{\sigma,\alpha}\}_{\alpha \in A}$ (see (10)) and to the corresponding ind-variety $\mathbf{X} = \mathbf{Fl}(\mathcal{F}_\sigma, E)$ (resp., $\mathbf{X} = \mathbf{Fl}(\mathcal{F}_\sigma, \omega, E)$) (see Section 3). We consider the Schubert cell $\mathbf{B}\mathcal{F}_\sigma \subset \mathbf{X}$. We denote its closure in $\mathbf{X}$ by $\mathbf{X}_\sigma$ (resp., $\mathbf{X}^\omega_\sigma$) and call it *Schubert ind-variety.* Note that $\mathbf{X}_\sigma$ and $\mathbf{X}^\omega_\sigma$ depend on the choice of the splitting Borel subgroup $\mathbf{B} \subset \mathbf{G}$.

By Theorems 1 (c), (d) and 2 (c), (d), the Schubert ind-variety $\mathbf{X}_\sigma$ (resp., $\mathbf{X}^\omega_\sigma$) admits a cell decomposition into Schubert cells $\mathbf{B}\mathcal{F}_\tau$ for $\tau \leq \sigma$ (resp., $\tau \leq_\omega \sigma$).

If $I \subset E$ is a finite subset, then the (finite-dimensional) flag variety $\mathrm{Fl}(\mathcal{F}_\sigma, I)$ (defined in Section 3.3) embeds in a natural way in the ind-variety $\mathbf{Fl}(\mathcal{F}_\sigma, E)$. The intersection $X_{\sigma,I} := \mathbf{X}_\sigma \cap \mathrm{Fl}(\mathcal{F}_\sigma, I)$ is a Schubert variety in the usual sense. In the case of $\mathbf{G} = \mathbf{G}^\omega(E)$, if the subset $I \subset E$ is i_E-stable, the flag variety $\mathrm{Fl}(\mathcal{F}_\sigma, \omega, I)$ embeds in the ind-variety $\mathbf{Fl}(\mathcal{F}_\sigma, \omega, E)$. Again, the intersection $X^\omega_{\sigma,I} := \mathbf{X}^\omega_\sigma \cap \mathrm{Fl}(\mathcal{F}_\sigma, \omega, I)$ is a Schubert variety in the usual sense.

Note that the Schubert ind-variety $\mathbf{X}_\sigma$ depends on the generalized flag $\mathcal{F}_\sigma$ and on the splitting Borel subgroup $\mathbf{B}$. Recall that $\mathbf{B}$ is the stabilizer of a maximal generalized flag $\mathcal{F}_0$ (see Propositions 1, 3). Our singularity criterion (Theorem 4 below) requires a technical assumption on $\mathbf{B}$ and $\mathcal{F}_\sigma$:

(H) At least one of the following conditions holds:
   (i) $\mathcal{F}_0$ is a flag (i.e., $(\mathcal{F}_0, \subset)$ is isomorphic as ordered set to a subset of $(\mathbb{Z}, \leq)$);
   (ii) $\mathcal{F}_\sigma$ is a flag, and $\dim F''_{\sigma,\alpha}/F'_{\sigma,\alpha}$ is finite whenever $0 \neq F'_{\sigma,\alpha} \subset F''_{\sigma,\alpha} \neq V$.

By $\mathrm{Sing}(X)$ we denote the set of singular points of a variety or an ind-variety X.

Theorem 4. *Let $\mathbf{G} = \mathbf{G}(E)$ (resp., $\mathbf{G} = \mathbf{G}^\omega(E)$). Let $\sigma, \mathbf{X}_\sigma, \mathbf{X}^\omega_\sigma, X_{\sigma,I}, X^\omega_{\sigma,I}$ be as above. Assume that hypothesis* (H) *holds. The following alternative holds: either*

(i) *the variety $X_{\sigma,I}$ (resp., $X^\omega_{\sigma,I}$) is smooth for all (resp., i_E-stable) finite subsets $I \subset E$; then the ind-variety $\mathbf{X}_\sigma$ (resp., $\mathbf{X}^\omega_\sigma$) is smooth;*

or

(ii) *there is a finite subset $I_0 \subset E$ such that, for every (resp., i_E-stable) finite subset $I \subset E$ with $I \supset I_0$, the variety $X_{\sigma,I}$ (resp., $X^\omega_{\sigma,I}$) is singular; then $\mathbf{X}_\sigma$ (resp., $\mathbf{X}^\omega_\sigma$) is singular and*

$$\mathrm{Sing}(\mathbf{X}_\sigma) = \bigcup_{I \supset I_0} \mathrm{Sing}(X_{\sigma,I}) \qquad \textit{(resp., } \mathrm{Sing}(\mathbf{X}^\omega_\sigma) = \bigcup_{I \supset I_0,\, i_E\text{-stable}} \mathrm{Sing}(X^\omega_{\sigma,I})\textit{).}$$

Proof. We provide the proof only for the case $\mathbf{G} = \mathbf{G}(E)$ (the proof in the case of $\mathbf{G} = \mathbf{G}^\omega(E)$ follows the same scheme).

We need preliminary constructions and notation. For a finite subset $I \subset E$ and an element $\tau \in W(I)\cdot\sigma$, we define closed subgroups of $G(I)$ and $B(I)$ by letting

$$G_\tau(I) := \{g \in G(I) : g(e) - e \in \langle e' \in I : \tau(e') \succ_A \tau(e)\rangle \ \ \forall e \in E\}$$
$$\text{and} \quad B_\tau(I) := \{g \in G(I) : g(e) - e \in \langle e' \in I : e' \prec_{\mathbf{B}} e,\ \tau(e') \succ_A \tau(e)\rangle \ \ \forall e \in E\} = B(I) \cap G_\tau(I).$$

It is well known that the set

$$U_\tau(I) := \{g\mathcal{F}_\tau : g \in G_\tau(I)\}$$

is an open subvariety of $\mathrm{Fl}(\mathcal{F}_\sigma, I)$, and the maps

$$\Phi_\tau : G_\tau(I) \to U_\tau(I),\ g \mapsto g\mathcal{F}_\tau \quad \text{and} \quad \Phi'_\tau = \Phi_\tau|_{B_\tau(I)} : B_\tau(I) \to B(I)\mathcal{F}_\tau$$

are isomorphisms of algebraic varieties. Thus, for every $\tau \in \mathbf{W}(E)\cdot\sigma$, we obtain an open ind-subvariety of $\mathbf{Fl}(\mathcal{F}_\sigma, E)$ by letting

$$\mathbf{U}_\tau := \bigcup_I U_\tau(I),$$

where the union is taken over finite subsets $I \subset E$ such that $\tau \in W(I)\cdot\sigma$. Clearly $\mathbf{B}\mathcal{F}_\tau \subset \mathbf{U}_\tau$, hence by Theorem 1 (a) the open subsets $\mathbf{U}_\tau$ (for $\tau \in \mathbf{W}(E)\cdot\sigma$) cover the ind-variety $\mathbf{Fl}(\mathcal{F}_\sigma, E)$.

Let $I, J \subset E$ be finite subsets such that $I \subset J$. Let $\mathrm{Fl}(\mathcal{F}_\sigma, I)$, $\mathrm{Fl}(\mathcal{F}_\sigma, J)$ be corresponding finite-dimensional flag varieties, and let $\iota_{I,J} : \mathrm{Fl}(\mathcal{F}_\sigma, I) \to \mathrm{Fl}(\mathcal{F}_\sigma, J)$ be the embedding defined in Section 3.3. As noted in Proposition 11, we have $\iota_{I,J}(B(I)\mathcal{F}_\sigma) \subset B(J)\mathcal{F}_\sigma$, hence $\iota_{I,J}(X_{\sigma,I}) \subset X_{\sigma,J}$.

Let $\tau \in W(I)\cdot\sigma$. The inclusion $G_\tau(I) \subset G_\tau(J)$ holds. Moreover, using that $g(e) = e$ for all $g \in G_\tau(I)$, all $e \in J\setminus I$, in view of the definition of the map $\iota_{I,J}$, we have $\iota_{I,J}(g\mathcal{F}_\tau) = g\mathcal{F}_\tau \in U_\tau(J)$ for all $g \in G_\tau(I)$. Hence the map $\iota_{I,J}$ restricts to an embedding $\iota'_{I,J} : U_\tau(I) \cap X_{\sigma,I} \to U_\tau(J) \cap X_{\sigma,J}$.

Claim 1. Let $I, J \subset E$ be finite subsets such that $I \subset J$ and let $\tau \in W(I)\cdot\sigma$. Then, $\iota'_{I,J}$ restricts to an embedding $U_\tau(I) \cap \mathrm{Sing}(X_{\sigma,I}) \to U_\tau(J) \cap \mathrm{Sing}(X_{\sigma,J})$.

Let $H \subset G(J)$ be the torus formed by the elements $h \in G(J)$ such that $h(e) = e$ for all $e \in I$ and $h(e) \in \mathbb{K}^*e$ for all $e \in J \setminus I$. The torus H acts on $X_{\sigma,J}$. From [7], it follows that $\mathrm{Sing}((X_{\sigma,J})^H) \subset \mathrm{Sing}(X_{\sigma,J})$, where $(X_{\sigma,J})^H \subset X_{\sigma,J}$ stands for the subset of H-fixed points. On the other hand, it is easy to see that the equality $\iota'_{I,J}(U_\tau(I) \cap X_{\sigma,I}) = U_\tau(J) \cap (X_{\sigma,J})^H$ holds. Thereby,

$$\iota'_{I,J}(U_\tau(I) \cap \mathrm{Sing}(X_{\sigma,I})) = U_\tau(J) \cap \mathrm{Sing}((X_{\sigma,J})^H) \subset \mathrm{Sing}(X_{\sigma,J}).$$

This shows Claim 1.

Claim 2. Let $I, J \subset E$ be finite subsets such that $J = I \cup \{e_J\}$ and let $\tau \in W(I)\cdot\sigma$. Assume that at least one of the following conditions holds:

(i) $e_J \prec_{\mathbf{B}} e$ for all $e \in I$;
(ii) $e_J \succ_{\mathbf{B}} e$ for all $e \in I$;
(iii) $\tau(e_J) \preceq_A \tau(e)$ for all $e \in I$;
(iv) $\tau(e_J) \succeq_A \tau(e)$ for all $e \in I$.

Then the map $\iota'_{I,J} : U_\tau(I) \cap X_{\sigma,I} \to U_\tau(J) \cap X_{\sigma,J}$ admits a left inverse $r'_{I,J} : U_\tau(J) \cap X_{\sigma,J} \to U_\tau(I) \cap X_{\sigma,I}$.

We write an element $g \in \mathbf{G}(E)$ as a matrix $(g_{e',e})_{e',e\in E}$ such that $g(e) = \sum_{e'\in E} g_{e',e}e'$. Let $G_\tau(J) \to G_\tau(J)$, $g \mapsto g'$ and $R_{I,J} : G_\tau(J) \to G_\tau(I)$, $g \mapsto \tilde{g}$ be the maps defined by

$$g'_{e',e} = \begin{cases} 0 & \text{if } e \neq e' = e_J \\ g_{e',e} & \text{otherwise,} \end{cases} \quad \text{and} \quad \tilde{g}_{e',e} = \begin{cases} 0 & \text{if } e \neq e' \text{ and } e_J \in \{e, e'\} \\ g_{e',e} & \text{otherwise.} \end{cases} \tag{19}$$

The map $R_{I,J}$ induces a morphism of algebraic varieties $r_{I,J} : U_\tau(J) \to U_\tau(I)$, $g\mathcal{F}_\tau \mapsto \tilde{g}\mathcal{F}_\tau$. It is clear that $\tilde{g} = g$ whenever $g \in G_\tau(I)$, hence $r_{I,J}(\iota_{I,J}(\mathcal{G})) = \mathcal{G}$ whenever $\mathcal{G} \in U_\tau(I)$.

We claim that

$$\mathcal{G} \in U_\tau(J) \cap \mathbf{X}_\sigma \Rightarrow r_{I,J}(\mathcal{G}) \in \mathbf{X}_\sigma. \tag{20}$$

Let $\mathcal{G} = g\mathcal{F}_\tau$ with $g \in G_\tau(J)$. Assume that $\mathcal{G} \in \mathbf{X}_\sigma$. We first check that

$$\mathcal{G}' := g'\mathcal{F}_\tau \in \mathbf{X}_\sigma \tag{21}$$

with g' as in (19). We distinguish four cases depending on the conditions (i)–(iv) of Claim 2.

- Assume that condition (i) holds. Let $\mathcal{F}_0 = \{F'_{0,e}, F''_{0,e} : e \in E\}$ be the maximal generalized flag corresponding to $\mathbf{B}$, i.e., $F'_{0,e} = \langle e' : e' \prec_{\mathbf{B}} e\rangle$ and $F''_{0,e} = \langle e' : e' \preceq_{\mathbf{B}} e\rangle$ (see Section 4.1). In view of condition (i) and the definition of the map $g \mapsto g'$, for any $F \in \mathcal{F}_0$ and any linear combination $\sum_{e\in J} \lambda_e e \in \langle J\rangle$, we have

$$\sum_{e\in J} \lambda_e g(e) \in F \Rightarrow \sum_{e\in J} \lambda_e g'(e) \in F.$$

This implication yields $\dim g'(M) \cap \langle J\rangle \cap F \geq \dim g(M) \cap \langle J\rangle \cap F$ for all $M \in \mathcal{F}_\tau$, all $F \in \mathcal{F}_0$. It is well known that this property implies $g'\mathcal{F}_\tau \in \overline{B(J)g\mathcal{F}_\tau} \subset \mathbf{X}_\sigma$ (see, e.g., [1, §3.2.9]).
- Assume that condition (ii) holds. Then every $\mathcal{F} = \{F'_\alpha, F''_\alpha\}_{\alpha\in A} \in B(J)\mathcal{F}_\sigma$ satisfies $F''_\alpha \subset \langle E \setminus \{e_J\}\rangle$ whenever $\alpha \prec_A \sigma(e_J)$. The same property holds whenever $\mathcal{F} \in \overline{B(J)\mathcal{F}_\sigma} = \mathrm{Fl}(\mathcal{F}_\sigma, J) \cap \mathbf{X}_\sigma$. Applying this observation to $\mathcal{F} = g\mathcal{F}_\tau$ (and noting that $\tau(e_J) = \sigma(e_J)$ because $\tau \in W(I) \cdot \sigma$), we deduce that $g_{e_J,e} = 0$ for all $e \neq e_J$, whence $g' = g$. This clearly yields (21) in this case.
- Assume that condition (iii) holds. Then the definition of $G_\tau(J)$ yields $g_{e_J,e} = 0$ for all $e \in I$, whence $g' = g$. This implies (21).
- Finally, assume that condition (iv) holds. Then the definition of $G_\tau(J)$ implies that $g(e_J) = e_J$. For $t \in \mathbb{K}^*$, let $\tilde{h}_t \in \mathbf{H}(E)$ be defined by

$$\tilde{h}_t(e) = \begin{cases} e & \text{if } e \neq e_J \\ te_J & \text{if } e = e_J \end{cases} \quad \text{for all } e \in E. \tag{22}$$

We have $g'\mathcal{F}_\tau = \lim_{t\to 0} \tilde{h}_t g\mathcal{F}_\tau$. Since $\tilde{h}_t g\mathcal{F}_\tau \in \mathbf{X}_\sigma$ for all $t \in \mathbb{K}^*$, we get $g'\mathcal{F}_\tau \in \mathbf{X}_\sigma$, whence (21).

Therefore (21) holds true in all the cases. Moreover, we have

$$\tilde{g}\mathcal{F}_\tau = \lim_{t\to\infty} \tilde{h}_t g'\mathcal{F}_\tau$$

with $\tilde{h}_t$ as in (22). Since $g'\mathcal{F}_\tau \in \mathbf{X}_\sigma$ (by (21)) and $\tilde{h}_t$ stabilizes $\mathbf{X}_\sigma$, we conclude that $r_{I,J}(\mathcal{G}) = \tilde{g}\mathcal{F}_\tau \in \mathbf{X}_\sigma$. Whence (20).

By (20), the map $r'_{I,J} : U_\tau(J) \cap X_{\sigma,J} \to U_\tau(I) \cap X_{\sigma,I}$ obtained by restriction of $r_{I,J}$ is well defined and fulfills the conditions of Claim 1.

Relying on Claims 1 and 2, the proof of the theorem is carried out as follows. If $X_{\sigma,I}$ is smooth for all finite subsets $I \subset E$, then Lemma 5 guarantees that $\mathbf{X}_\sigma$ is a smooth ind-variety. We now assume that there is a finite subset $I_0 \subset E$ such that X_{σ,I_0} is singular. In this case Lemma 5 yields an inclusion

$$\mathrm{Sing}(\mathbf{X}_\sigma) \subset \bigcup_{I\supset I_0} \mathrm{Sing}(X_{\sigma,I})$$

where the union is taken over all finite subsets $I \subset E$ such that $I \supset I_0$. For completing the proof it is sufficient to prove that

$$\mathrm{Sing}(X_{\sigma,I}) \subset \mathrm{Sing}(\mathbf{X}_\sigma) \tag{23}$$

for each finite subset $I \subset E$ with $I \supset I_0$. To show this, let $\mathcal{G} \in \mathrm{Sing}(X_{\sigma,I})$. There is $\tau \in W(I) \cdot \sigma$ such that $\mathcal{G} \in U_\tau(I)$. We consider the two cases involved in assumption (H).

- If (H) (i) holds, then let $e_0 = \min I$ and $e_1 = \max I$ (for the order $\preceq_{\mathbf{B}}$), and set $I' = \{e \in E : e_0 \preceq_{\mathbf{B}} e \preceq_{\mathbf{B}} e_1\}$. The set I' is finite (by (H) (i)). Moreover, again relying on (H) (i), we can find a filtration $E = \bigcup_{n\geq 1} E_n$ with $E_1 = I'$ and $E_n = E_{n-1} \cup \{e_n\}$ for all $n \geq 2$, where e_n is either the minimum or the maximum of $(E_n, \preceq_{\mathbf{B}})$.
- If (H) (ii) holds, then let $\alpha_0 = \min\{\tau(e) : e \in I\}$ and $\alpha_1 = \max\{\tau(e) : e \in I\}$ (for the order $\preceq_A$), and set $I' = I \cup \{e \in E : \alpha_0 \prec_A \tau(e) \prec_A \alpha_1\}$. The first part of (H) (ii) ensures that there are at most finitely many $\alpha \in A$ such that $\alpha_0 \prec_A \alpha \prec_A \alpha_1$, while the second part of (H) (ii) (together with the fact that $\tau \in \mathbf{W}(E) \cdot \sigma$) implies that $\tau^{-1}(\alpha)$ is finite for each such α, hence the set I' is finite. Again relying on (H) (ii), we can construct a filtration $E = \bigcup_{n\geq 1} E_n$ with $E_1 = I'$ and $E_n = E_{n-1} \cup \{e_n\}$ for all $n \geq 2$, where e_n satisfies either $\tau(e_n) \preceq_A \tau(e)$ for all $e \in E_{n-1}$ or $\tau(e_n) \succeq_A \tau(e)$ for all $e \in E_{n-1}$.

In both cases, we get a filtration $\{E_n\}_{n\geq 1}$ of E by finite subsets such that $I \subset E_1$ and, for every $n \geq 2$, the pair (E_{n-1}, E_n) satisfies one of the conditions (i)–(iv) of Claim 2. We obtain an exhaustion of the open subset $\mathbf{U}_\tau \cap \mathbf{X}_\sigma$ of $\mathbf{X}_\sigma$ given by the chain

$$U_{\sigma,\tau,1} \stackrel{\iota_1}{\hookrightarrow} U_{\sigma,\tau,2} \stackrel{\iota_2}{\hookrightarrow} U_{\sigma,\tau,3} \hookrightarrow \ldots \hookrightarrow U_{\sigma,\tau,n} \stackrel{\iota_n}{\hookrightarrow} \ldots$$

where $U_{\sigma,\tau,n} = U_\tau(E_n) \cap X_{\sigma,E_n}$ and $\iota_n = \iota'_{E_n,E_{n+1}}$. Claim 1 implies that $\mathcal{G}$ is a singular point of $U_{\sigma,\tau,1}$. By Claim 2, we can apply Lemma 6 which implies that $\mathcal{G}$ is a singular point of $\mathbf{U}_\tau \cap \mathbf{X}_\sigma$, hence of $\mathbf{X}_\sigma$. Therefore the inclusion (23) holds. The proof is complete. □

Remark 10. (a) Note that hypothesis (H) is valid in the case where $\mathbf{Fl}(\mathcal{F}_\sigma, E)$ is an ind-grassmannian.
(b) Hypothesis (H) is needed in the proof of Theorem 4 for showing Claim 2 which is necessary for applying Lemma 6. We have no indication whatsoever that Theorem 4 is not valid in general (without hypothesis (H)).

Remark 11. The Schubert ind-varieties $\mathbf{X}_\sigma$ considered in this paper form a narrower class than the ones considered by H. Salmasian [12]. Indeed, a closed ind-subvariety $\mathbf{X} \subset \mathbf{Fl}(\mathcal{F}, E)$ such that $\mathbf{X} \cap \mathrm{Fl}(\mathcal{F}, I)$ is a Schubert variety for all finite subsets $I \subset E$ is a Schubert ind-variety in the sense of [12], and it may happen that $\mathbf{X}$ has no open $\mathbf{B}$-orbit and admits no smooth point in this case (see [12, Section 2]). On the other hand, the ind-variety $\mathbf{X}_\sigma$ defined in Section 6.2 always contains the open $\mathbf{B}$-orbit $\mathbf{B}\mathcal{F}_\sigma$, and the points of $\mathbf{B}\mathcal{F}_\sigma$ are smooth in $\mathbf{X}_\sigma$.

6.3. **Examples.** A consequence of Theorem 4 is that the smoothness criteria for Schubert varieties of (finite-dimensional) flag varieties that are expressed in terms of pattern avoidance, may pass to the limit at infinity.

For example, let us apply Theorem 4 to the ind-variety $\mathbf{Fl}(\mathcal{F}, E)$ for an E-compatible maximal generalized flag $\mathcal{F}$. In this case we have two total orders on the basis E: the first one $\preceq_\mathbf{B}$ corresponds to the splitting Borel subgroup $\mathbf{B}$, and the second order $\preceq_\mathcal{F}$ corresponds to the maximal generalized flag $\mathcal{F}$, i.e., $\mathcal{F} = \{F'_e, F''_e : e \in E\}$ is given by

$$F'_e = \langle e' \in E : e' \prec_\mathcal{F} e \rangle, \quad F''_e = \langle e' \in E : e' \preceq_\mathcal{F} e \rangle.$$

By Theorem 1, the Schubert ind-varieties $\mathbf{X}_\sigma$ of $\mathbf{Fl}(\mathcal{F}, E)$ are parametrized by the permutations $\sigma \in \mathbf{W}(E)$, and we have

$$\dim \mathbf{X}_\sigma = n_{\mathrm{inv}}(\sigma) = |\{(e, e') \in E : e \prec_\mathbf{B} e',\ \sigma(e') \prec_\mathcal{F} \sigma(e)\}|.$$

From Theorem 4 and the known characterization of smooth Schubert varieties of full flag varieties in terms of pattern avoidance (see [1, §8]) we obtain the following criterion.

Corollary 3. *Assume that $\mathcal{F}$ or $\mathcal{F}_0$ is a flag, so that hypothesis* (H) *is satisfied. Let $\sigma \in \mathbf{W}(E)$. Then the Schubert ind-variety $\mathbf{X}_\sigma$ is singular if and only if there exist $e_1, e_2, e_3, e_4 \in E$ such that $e_1 \prec_\mathbf{B} e_2 \prec_\mathbf{B} e_3 \prec_\mathbf{B} e_4$ and ($\sigma(e_3) \prec_\mathcal{F} \sigma(e_4) \prec_\mathcal{F} \sigma(e_1) \prec_\mathcal{F} \sigma(e_2)$ or $\sigma(e_4) \prec_\mathcal{F} \sigma(e_2) \prec_\mathcal{F} \sigma(e_3) \prec_\mathcal{F} \sigma(e_1)$).*

Remark 12. (a) Corollary 3 shows in particular that, if the basis E comprises infinitely many pairwise disjoint quadruples (e_1, e_2, e_3, e_4) such that $e_1 \prec_\mathbf{B} e_2 \prec_\mathbf{B} e_3 \prec_\mathbf{B} e_4$ and, say, $e_3 \prec_\mathcal{F} e_4 \prec_\mathcal{F} e_1 \prec_\mathcal{F} e_2$, then for every permutation $\sigma \in \mathbf{W}(E)$, the Schubert ind-variety $\mathbf{X}_\sigma$ is singular. Thus, there exist pairs $(\mathbf{B}, \mathcal{F})$ such that all Schubert ind-varieties of the ind-variety $\mathbf{Fl}(\mathcal{F}, E)$ are singular.
(b) In the case where the ind-variety $\mathbf{Fl}(\mathcal{F}, E)$ has finite-dimensional Schubert cells, it has one cell equal to a single point (see Theorem 3), hence has at least one smooth Schubert ind-variety. Note that $\mathbf{Fl}(\mathcal{F}, E)$ may have smooth Schubert ind-varieties although all its Schubert cells are infinite dimensional. Take for instance $E = \{e_i\}_{i\in\mathbb{Z}}$, let the order $\preceq_\mathbf{B}$ be the natural order on $\mathbb{Z}$, and let the order $\preceq_\mathcal{F}$ be the inverse order, i.e., $i \preceq_\mathcal{F} j$ if and only if $i \geq j$. Then every Schubert cell of $\mathbf{Fl}(\mathcal{F}, E)$ is infinite dimensional, but the permutation $\sigma = \mathrm{id}_E \in \mathbf{W}(E)$ avoids the two forbidden patterns of Corollary 3, hence $\mathbf{X}_\sigma$ is smooth.

As a second example, we apply Theorem 4 to the case of the ind-grassmannian $\mathbf{Gr}(2)$. In this case, for a splitting Borel subgroup $\mathbf{B}$, the Schubert ind-varieties $\mathbf{X}_\sigma$ are parametrized by the surjective maps $E \to \{1, 2\}$ such that $|\sigma^{-1}(1)| = 2$, or equivalently by the pairs of elements $\sigma = \{\sigma_1, \sigma_2\} \subset E$. From Theorem 4 and [1, §9.3.3] we have:

Corollary 4. *Let $\sigma = \{\sigma_1, \sigma_2\} \subset E$ with $\sigma_1 \prec_\mathbf{B} \sigma_2$. The Schubert ind-variety $\mathbf{X}_\sigma$ is smooth if and only if σ_1 is the smallest element of the ordered set $(E, \preceq_\mathbf{B})$ or σ_1, σ_2 are two consecutive elements of $(E, \preceq_\mathbf{B})$.*

Acknowledgement

This project was supported in part by the Priority Program "Representation Theory" of the DFG (SPP 1388). L. Fresse acknowledges partial support through ISF Grant Nr. 882/10 and ANR Grant Nr. ANR-12-PDOC-0031. I. Penkov thanks the Mittag-Leffler Institute in Djursholm for its hospitality.

References

[1] S. Billey and V. Lakshmibai: Singular loci of Schubert varieties. Progress in Mathematics, 182. Birkhäuser Boston, Inc., Boston, MA, 2000.
[2] M. Brion: Lectures on the geometry flag varieties. In: Topics in cohomological studies of algebraic varieties, 33–85, Trends Math., Birkhäuser, Basel, 2005.
[3] I. Dimitrov and I. Penkov: Weight modules of direct limit Lie algebras, Int. Math. Res. Not. 1999, no. 5, 223–249.
[4] I. Dimitrov and I. Penkov: Ind-varieties of generalized flags as homogeneous spaces for classical ind-groups. Int. Math. Res. Not. 2004, no. 55, 2935–2953.
[5] I. Dimitrov, I. Penkov, and J.A. Wolf: A Bott–Borel–Weil theory for direct limits of algebraic groups. Amer. J. Math. 124 (5) (2002), 955–998.
[6] J. Hennig: A generalization of Lie's theorem. Comm. Algebra 42 (2014), 4269–4273.
[7] B. Iversen: "A fixed point formula for action of tori on algebraic varieties". Invent. Math. 16 (1972), 229–236.
[8] S. Kumar: Kac-Moody groups, their flag varieties and representation theory. Progress in Mathematics, 204. Birkhäuser, Boston, MA, 2002.
[9] V. Lakshmibai and J. Brown: Flag varieties. An interplay of geometry, combinatorics, and representation theory. Texts and Readings in Mathematics, 53. Hindustan Book Agency, New Delhi, 2009.
[10] I. Penkov and A. S. Tikhomirov: Linear ind-Grassmannians. Pure and Applied Math. Quarterly 10 (2014), 289–323.
[11] I. R. Šafarevič: On some infinite dimensional groups II. Math. USSR Izvestija 18 (1982), 185–194.
[12] H. Salmasian: Direct limits of Schubert varieties and global sections of line bundles. J. Algebra 320 (2008), 3187–3198.

Université de Lorraine, CNRS, Institut Élie Cartan de Lorraine, UMR 7502, Vandoeuvre-lès-Nancy, F-54506, France
E-mail address: lucas.fresse@univ-lorraine.fr

Jacobs University Bremen, Campus Ring 1, 28759 Bremen, Germany
E-mail address: i.penkov@jacobs-university.de