Locally semisimple and maximal subalgebras of the finitary Lie algebras gl(∞), sl(∞), so(∞), and sp(∞)

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Abstract

We describe all locally semisimple subalgebras and all maximal subalgebras of the finitary Lie algebras gl(∞), sl(∞), so(∞), and sp(∞). For simple finite–dimensional Lie algebras these classes of subalgebras have been described in the classical works of A. Malcev and E. Dynkin.

Key words (2000 MSC): 17B05 and 17B65.

Introduction

The simple infinite–dimensional finitary Lie algebras have been classified by A. Baranov a decade ago, see [Ba3], [Ba4], and [BS], and since then the study of these Lie algebras sl(∞), so(∞), and sp(∞), as well of the finitary Lie algebra gl(∞), has been underway. So far some notable results on the structure of the subalgebras of gl(∞), sl(∞), so(∞), and sp(∞) concern irreducible, Cartan, and Borel subalgebras, see [LP], [BS], [NP],[DPS], [DP2], and [Da]. The objective of the present paper is to describe the locally semisimple subalgebras of gl(∞), sl(∞), so(∞), and sp(∞) (up to isomorphism, as well as in terms of their action on the natural and conatural modules) and the maximal subalgebras of gl(∞), sl(∞), so(∞),

* Research partially supported by NSERC Discovery Grant and by FAPESP Grant 2007/03735–0
† Research partially supported by FAPESP Grant 2007/54820–8

and sp(∞). Our results extend classical results of A. Malcev, [M], and E. Dynkin, [Dy1], [Dy2], to infinite–dimensional finitary Lie algebras and are related to some earlier results of A. Baranov, A. Baranov and H. Strade, and F. Leinen and O. Puglisi.

A subalgebra $\mathfrak{s}$ of gl(∞), sl(∞), so(∞), or sp(∞) is locally semisimple if it is a union of semisimple finite–dimensional subalgebras. The class of locally semisimple subalgebras is the natural analogue of the class of semisimple subalgebras of simple finite–dimensional Lie algebras. In the absence of Weyl's semisimplicity results for locally finite infinite–dimensional Lie algebras, it is a priori not clear whether a locally semisimple subalgebra of gl(∞), sl(∞), so(∞), and sp(∞) is itself a direct sum of simple constituents, cf. Corollary in [LP]. Theorem 3.1 proves that this is true and, moreover, that each simple constituent of a locally semisimple subalgebra of gl(∞), sl(∞), so(∞), and sp(∞) is either finite–dimensional or is itself isomorphic to gl(∞), sl(∞), so(∞), or sp(∞). The latter fact has been established earlier by A. Baranov.

The method of proof of Theorem 3.1 allows to prove also that if $\mathfrak{g} = \mathrm{sl}(\infty)$ (respectively, $\mathfrak{g} = \mathrm{so}(\infty)$ or $\mathrm{sp}(\infty)$) and $\mathfrak{g} = \varinjlim \mathfrak{s}_n$ is an exhaustion of $\mathfrak{g}$ by semisimple finite–dimensional Lie algebras, then there exist n_0 and nested simple ideals $\mathfrak{k}_n$ of $\mathfrak{s}_n$ for $n > n_0$, such that $\varinjlim \mathfrak{k}_n = \mathfrak{g}$, $\mathfrak{k}_n \cong \mathrm{sl}(k_n)$ (respectively, $\mathfrak{k}_n = \mathrm{so}(k_n)$ or $\mathrm{sp}(k_n)$), and the inclusion $\mathfrak{k}_n \subset \mathfrak{k}_{n+1}$ is simply induced by an inclusion of the natural $\mathfrak{k}_n$–modules $V(\mathfrak{k}_n) \subset V(\mathfrak{k}_{n+1})$ (cf. Corollary 5.9 in [Ba2])..

We then study the natural representation V of $\mathfrak{g} = \mathrm{gl}(\infty), \mathrm{sl}(\infty)$, so($\infty$), and sp($\infty$) as a module over any locally semisimple subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ and show that

- the socle filtration of V has depth at most 2;
- the non–trivial simple direct summands of V are just natural and conatural modules over infinite–dimensional simple ideals of $\mathfrak{s}$, as well as finite–dimensional modules over finite–dimensional ideals of $\mathfrak{s}$; each non–trivial simple constituent of V as module over a simple ideal of $\mathfrak{s}$ occurs with finite multiplicity;

- the module V/V' is trivial.

Similar results hold for the conatural $\mathfrak{g}$–module V_* for $\mathfrak{g} = \mathrm{gl}(\infty)$ and $\mathrm{sl}(\infty)$.

We conclude the paper by a description of maximal proper subalgebras of $\mathfrak{g} = \mathrm{gl}(\infty)$, $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, and $\mathrm{sp}(\infty)$. The maximal subalgebras of $\mathfrak{g} = \mathrm{gl}(\infty)$ are $[\mathfrak{g},\mathfrak{g}] \cong \mathrm{sl}(\infty)$ and the stabilizers of subspaces of V or V_* as follows: $W \subset V$ with $W^{\perp\perp} = W$, or $W \subset V$, $\mathrm{codim}_V W = 1$ and $W^\perp = 0$, or $\tilde{W} \subset V_*$, $\mathrm{codim}_{V_*} \tilde{W} = 1$ and $\tilde{W}^\perp = 0$. The maximal subalgebras of $\mathrm{sl}(\infty)$ are intersections of the maximal subalgebras of $\mathfrak{g} = \mathrm{gl}(\infty)$ with $\mathrm{sl}(\infty) = [\mathfrak{g},\mathfrak{g}]$. For $\mathfrak{g} = \mathrm{so}(\infty)$ and $\mathrm{sp}(\infty)$ any maximal subalgebra is the stabilizer in $\mathfrak{g}$ of an isotropic subspace $W \subset V$ with $W^{\perp\perp} = W$, or of a non–degenerate subspace $W \subset V$ with $W \oplus W^\perp = V$ (where for $\mathrm{so}(\infty)$, $\dim W \neq 2$ and $\dim W^\perp \neq 2$), or of a non–degenerate subspace $W \subset V$ of codimension 1 such that $W^\perp = 0$.

Acknowledgment. We are indebted to Gregg Zuckerman for his long term encouragement to study Dynkin's papers [Dy1] and [Dy2]. We thank A. Baranov for very enlightening discussions and Y. Bahturin for a key reference on irreducible subalgebras. We acknowledge the hospitality of the Mathematisches Forschungsinstitut Oberwolfach where this work was initiated.

1 General preliminaries

The ground field is $\mathbb{C}$. In this paper V is a fixed countable–dimensional vector space with basis $v_1, v_2, \dots$ and V_* is the restricted dual of V, i.e. the span of the dual set $v_1^*, v_2^*, \dots$ ($v_i^*(v_j) = \delta_{ij}$). The space $V \otimes V_*$ ($\otimes$ stands throughout the paper for tensor product over $\mathbb{C}$) has an obvious structure of an associative algebra, and by definition $\mathrm{gl}(V, V_*)$ (or $\mathrm{gl}(\infty)$ for short) is the Lie algebra associated with this associative algebra. The Lie algebra $\mathrm{sl}(V, V_*)$ (or $\mathrm{sl}(\infty)$) is the commutator algebra $[\mathrm{gl}(V, V_*), \mathrm{gl}(V, V_*)]$. Given a symmetric non–degenerate form $V \times V \to \mathbb{C}$, we denote by $\mathrm{so}(V)$ (or $\mathrm{so}(\infty)$) the Lie subalgebra $\Lambda^2(V) \subset \mathrm{sl}(V, V_*)$ (the form $V \times V \to \mathbb{C}$ induces an identification of V with V_* which allows to consider $\Lambda^2(V)$ as a subspace of $V \otimes V_*$). Similarly, given an antisymmetric non–degenerate form $V \times V \to \mathbb{C}$, we denote by $\mathrm{sp}(V)$ (or $\mathrm{sp}(\infty)$) the Lie subalgebra $S^2(V) \subset \mathrm{sl}(V, V_*)$. In what follows $\mathfrak{g}$ always stands for one of the Lie algebras $\mathrm{gl}(V, V_*)$, $\mathrm{sl}(V, V_*)$, $\mathrm{so}(V)$, or $\mathrm{sp}(V)$.

The Lie algebras $\mathrm{gl}(\infty)$, $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, and $\mathrm{sp}(\infty)$ are locally finite (i.e. any finite set of elements generates a finite–dimensional subalgebra) and can be defined alternatively as follows. Recall that if $\varphi : \mathfrak{f} \to \mathfrak{f}'$ is an injective homomorphism of reductive finite–dimensional Lie algebras, φ is a *root injection* if for some (equivalently, for any) Cartan subalgebra $\mathfrak{t}_{\mathfrak{f}}$ of $\mathfrak{f}$, there exists a Cartan subalgebra $\mathfrak{t}_{\mathfrak{f}'}$ such that $\varphi(\mathfrak{t}_{\mathfrak{f}}) \subset \mathfrak{t}_{\mathfrak{f}'}$ and each $\mathfrak{t}_{\mathfrak{f}}$–root space of $\mathfrak{f}$ is mapped under φ into a $\mathfrak{t}_{\mathfrak{f}'}$–root space of $\mathfrak{f}'$. It is a known result that the direct limit $\varinjlim \mathfrak{f}_n$ of any system

$$\mathfrak{f}_1 \to \mathfrak{f}_2 \to \dots$$

of root injections of simple finite–dimensional Lie algebras is isomorphic to $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, or $\mathrm{sp}(\infty)$, see for instance [DP1].

We need to recall also two other types of injections of simple finite–dimensional Lie algebras. Let $\mathfrak{f}$ and $\mathfrak{f}'$ be classical simple Lie algebras. We call an injective homomorphism $\varphi : \mathfrak{f} \to \mathfrak{f}'$ a *standard injection* if the natural representation $\omega_{\mathfrak{f}'}$ of $\mathfrak{f}'$ decomposes as an $\mathfrak{f}$–module (via φ) as a direct sum of one copy of a representation which is conjugated by an automorphism of $\mathfrak{f}$ to the natural representation $\omega_{\mathfrak{f}}$ of $\mathfrak{f}$, and of a trivial $\mathfrak{f}$–module. Any root injection of classical Lie algebras is standard, but the converse is not true: an injection $\mathrm{so}(2k+1) \hookrightarrow \mathrm{so}(2k+2)$ is standard without being a root injection. An injective homomorphism of classical Lie algebras $\varphi : \mathfrak{f} \to \mathfrak{f}'$ is *diagonal* if $\omega_{\mathfrak{f}'}$ decomposes as an $\mathfrak{f}$–module as a direct sum of copies of $\omega_{\mathfrak{f}}$, of the dual module $\omega_{\mathfrak{f}}^*$, and of the 1–dimensional trivial $\mathfrak{f}$–module. This definition is a special case of a more general definition of A. Baranov, [Ba2], [BZh].

An *exhaustion* $\varinjlim \mathfrak{g}_n$ of $\mathfrak{g}$ is a system of injections of finite–dimensional Lie algebras

$$\mathfrak{g}_1 \xrightarrow{\varphi_1} \mathfrak{g}_2 \xrightarrow{\varphi_2} \dots$$

such that the direct limit Lie algebra $\varinjlim \mathfrak{g}_n$ is isomorphic to $\mathfrak{g}$. A *standard exhaustion* is an exhaustion $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ such that $\mathfrak{g}_n \to \mathfrak{g}_{n+1}$ is a standard injection of classical simple

Lie algebras for all n. In a standard exhaustion, for large enough n, $\mathfrak{g}_n$ is of type A for $\mathfrak{g} = \mathrm{sl}(\infty)$, $\mathfrak{g}_n$ is of type B or D for $\mathfrak{g} = \mathrm{so}(\infty)$, and $\mathfrak{g}_n$ is of type C for $\mathfrak{g} \cong \mathrm{sp}(\infty)$.

A subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ is *locally semisimple* if it admits an exhaustion $\mathfrak{s} = \varinjlim \mathfrak{s}_n$ by injective homomorphisms $\mathfrak{s}_n \to \mathfrak{s}_{n+1}$ of semisimple finite–dimensional Lie algebras $\mathfrak{s}_n$.

For $\mathfrak{g} \cong \mathrm{gl}(\infty)$ or $\mathrm{sl}(\infty)$ the vector spaces V and V_* are by definition the *natural* and *conatural* $\mathrm{sl}(\infty)$–modules. They are characterized by the following property: V (respectively, V_*) is the only simple $\mathfrak{g}$–module which, for any standard exhaustion $\mathfrak{g} = \varinjlim \mathfrak{g}_n$, restricts to one copy of the natural (respectively, its dual) representation of $\mathfrak{g}_n$ plus a trivial module. For $\mathfrak{g} \cong \mathrm{so}(\infty)$ or $\mathrm{sp}(\infty)$, V is characterized by the same property (here $V \cong V_*$ as $\mathfrak{g}$–modules).

2 Index of a subalgebra

For a simple finite–dimensional Lie algebra $\mathfrak{f}$ we denote by $\langle \cdot, \cdot \rangle_{\mathfrak{f}}$ the invariant non–degenerate symmetric bilinear form on $\mathfrak{f}$ for which $\langle \alpha^\vee, \alpha^\vee \rangle_{\mathfrak{f}} = 2$ for any long root α of $\mathfrak{f}$. (By convention the roots of a simply–laced Lie algebra are long.) If $\varphi : \mathfrak{f} \to \mathfrak{f}'$ is a homomorphism of a simple Lie algebra $\mathfrak{f}$ into the simple Lie algebra $\mathfrak{f}'$, then $\langle x, y \rangle_\varphi := \langle \varphi(x), \varphi(y) \rangle_{\mathfrak{f}}$ is an invariant symmetric bilinear form on $\mathfrak{f}$. Consequently,

$$\langle x, y \rangle_\varphi = I_{\mathfrak{f}}^{\mathfrak{f}'} \langle x, y \rangle_{\mathfrak{f}}$$

for some scalar $I_{\mathfrak{f}}^{\mathfrak{f}'}$. E. Dynkin, [Dy2], calls $I_{\mathfrak{f}}^{\mathfrak{f}'}$ the *index of* φ. The homomorphism φ is determined (up to an automorphism of $\mathfrak{f}'$) by the pull–back of any nontrivial representation of $\mathfrak{f}'$ of minimal dimension. Such a representation is unique unless $\mathfrak{f}'$ is isomorphic to $\mathrm{sl}(n)$, to D_4, or to E_6. In the rest of the paper we fix a non–trivial representation $\omega_{\mathfrak{f}'}$ of $\mathfrak{f}'$ of minimal dimension. If $\mathfrak{f}$ is classical, $\omega_{\mathfrak{f}}$ stands as above for the natural module. If U is any finite dimensional $\mathfrak{f}$–module, then the *index* $I_{\mathfrak{f}}(U)$ *of* U is defined as $I_{\mathfrak{f}}^{\mathrm{sl}(U)}$ where $\mathfrak{f}$ is mapped into $\mathrm{sl}(U)$ through the module U, see [Dy2]. The following properties are established in [Dy2, § 2].

Proposition 2.1

(i) $I_{\mathfrak{f}}^{\mathfrak{f}'} \in \mathbb{Z}_{\geq 0}$.

(ii) $I_{\mathfrak{f}}^{\mathfrak{f}'} I_{\mathfrak{f}'}^{\mathfrak{f}''} = I_{\mathfrak{f}}^{\mathfrak{f}''}$.

(iii) $I_{\mathfrak{f}}(U_1 \oplus \cdots \oplus U_l) = I_{\mathfrak{f}}(U_1) + \cdots + I_{\mathfrak{f}}(U_l)$.

(iv) $I_{\mathfrak{f}}(U_1 \otimes \cdots \otimes U_l) = \dim U_1 \ldots \dim U_l (\frac{1}{\dim U_1} I_{\mathfrak{f}}(U_1) + \cdots + \frac{1}{\dim U_l} I_{\mathfrak{f}}(U_l))$.

(v) *If* $I_{\mathfrak{f}}^{\mathfrak{f}'} = 1$, *then the root spaces of* $\mathfrak{f}$ *corresponding to long roots are mapped into root spaces of* $\mathfrak{f}'$ *corresponding to long roots.*

In particular, (ii) implies that $I_{\mathfrak{f}}(\omega_{\mathfrak{f}'}) = I_{\mathfrak{f}}^{\mathfrak{f}'} I_{\mathfrak{f}'}(\omega_{\mathfrak{f}'})$. Furthermore, a combination of (ii) and the information from Table 5 in [Dy2] shows that $I_{\mathfrak{f}}^{\mathrm{sp}(U)} = I_{\mathfrak{f}}(U)$ and $I_{\mathfrak{f}}^{\mathrm{so}(U)} = \frac{1}{2} I_{\mathfrak{f}}(U)$ when U admits a corresponding invariant form, see [Dy2].

We need an extension of Proposition 2.1. Let $\varphi : \mathfrak{f} \to \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_l$ and $\eta : \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_l \to \mathfrak{f}'$ be homomorphisms of Lie algebras, where $\mathfrak{k}_1, \ldots, \mathfrak{k}_l$ are simple Lie algebras.

Proposition 2.2 *We have*

$$I_{\mathfrak{f}}^{\mathfrak{f}'} = \sum_{j=1}^{l} I_{\mathfrak{f}}^{\mathfrak{k}_j} I_{\mathfrak{k}_j}^{\mathfrak{f}'}, \tag{1}$$

where $\mathfrak{f} \to \mathfrak{f}'$ *is the homomorphism* $\eta \circ \varphi$, *and the homomorphisms* $\mathfrak{f} \to \mathfrak{k}_i$ *and* $\mathfrak{k}_i \to \mathfrak{f}'$ *are determined by* φ *and* η *in the obvious way.*

Proof. Multiplying by $I_{\mathfrak{f}'}(\omega_{\mathfrak{f}'})$ we see that (1) is equivalent to

$$I_{\mathfrak{f}}(\omega_{\mathfrak{f}'}) = \sum_{j=1}^{l} I_{\mathfrak{f}}^{\mathfrak{k}_j} I_{\mathfrak{k}_j}(\omega_{\mathfrak{f}'}). \tag{2}$$

In the case when $\omega_{\mathfrak{f}'}$ is a reducible $(\mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_l)$–module we use Proposition 2.1(iii) to prove (2) by induction on the length of $\omega_{\mathfrak{f}'}$. Now assume that $\omega_{\mathfrak{f}'}$ is an irreducible $\mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_l$–module.

Then $\omega_{\mathfrak{f}'} = U_1 \otimes \cdots \otimes U_l$ for some irreducible $\mathfrak{k}_j$–modules U_j. Note that if $U_j = \omega_{\mathfrak{k}_j}$ for every j, identity (2) follows from Proposition 2.1. Indeed, in this case $I_{\mathfrak{k}_j}(\omega_{\mathfrak{f}'}) = \frac{\dim(U_1 \otimes \cdots \otimes U_l)}{\dim U_j} I_{\mathfrak{k}_j}(U_j)$ by (iii), and applying (iv) we obtain

$$I_{\mathfrak{f}}(\omega_{\mathfrak{f}'}) = \sum_j \frac{\dim(U_1 \otimes \cdots \otimes U_l)}{\dim U_j} I_{\mathfrak{f}}(U_j) = \sum_j \frac{I_{\mathfrak{k}_j}(\omega_{\mathfrak{f}'})}{I_{\mathfrak{k}_j}(U_j)} I_{\mathfrak{f}}(U_j) = \sum_j I_{\mathfrak{f}}^{\mathfrak{k}_j} I_{\mathfrak{k}_j}(\omega_{\mathfrak{f}'}).$$

To prove (2) for general irreducible $\mathfrak{k}_j$–modules U_j we consider the diagram

$$\begin{tikzcd}
 & & & & \mathfrak{f} \arrow[dllll] \arrow[dll] \arrow[drr] \arrow[drrrr] & & & & \\
\mathfrak{k}_1 \arrow[d] & \oplus & \mathfrak{k}_2 \arrow[d] & \oplus & \cdots & \oplus & \mathfrak{k}_{l-1} \arrow[d] & \oplus & \mathfrak{k}_l \arrow[d] \\
\mathrm{sl}(U_1) \arrow[drrrr] & \oplus & \mathrm{sl}(U_2) \arrow[drr] & \oplus & \cdots & \oplus & \mathrm{sl}(U_{l-1}) \arrow[dll] & \oplus & \mathrm{sl}(U_l) \arrow[dllll] \\
 & & & & \mathrm{sl}(\omega_{\mathfrak{f}'}) & & & &
\end{tikzcd}$$

This diagram enables us to first apply (2) to $\mathfrak{f} \to \mathrm{sl}(U_1) \oplus \cdots \oplus \mathrm{sl}(U_l) \to \mathrm{sl}(\omega_{\mathfrak{f}'})$ and then use $I_{\mathfrak{f}}^{\mathrm{sl}(U_j)} = I_{\mathfrak{f}}^{\mathfrak{k}_j} I_{\mathfrak{k}_j}^{\mathrm{sl}(U_j)}$ to get

$$I_{\mathfrak{f}}(\omega_{\mathfrak{f}'}) = \sum_j I_{\mathfrak{f}}^{\mathrm{sl}(U_j)} I_{\mathrm{sl}(U_j)}(\omega_{\mathfrak{f}'}) = \sum_j I_{\mathfrak{f}}^{\mathfrak{k}_j} I_{\mathfrak{k}_j}^{\mathrm{sl}(U_j)} I_{\mathrm{sl}(U_j)}(\omega_{\mathfrak{f}'}) = \sum_j I_{\mathfrak{f}}^{\mathfrak{k}_j} I_{\mathfrak{k}_j}(\omega_{\mathfrak{f}'}).$$

This completes the proof. $\square$

Proposition 2.3 *Let $\varphi : \mathfrak{f} \to \mathfrak{f}'$ denote an injective homomorphism of classical simple Lie algebras.*

(i) *Assume that* $\mathrm{rk}\,\mathfrak{f} > 4$*. If $\mathfrak{f}'$ is not of type B or D and $I_{\mathfrak{f}}^{\mathfrak{f}'} = 1$, then φ is a standard injection. Similarly, if $\mathfrak{f}$ is of type B or D and $I_{\mathfrak{f}}^{\mathfrak{f}'} = 1$, then φ is a standard injection.*

(ii) *For any n there exists a constant c_n depending on n only, such that* $\mathrm{rk}\,\mathfrak{f} = n$ *and $I_{\mathfrak{f}}^{\mathfrak{f}'} \le c_n$ imply that φ is diagonal. Furthermore, $\lim_{n\to\infty} c_n = \infty$.*

Proof. (i) Assume first that $\mathfrak{f}'$ is not of type B or D. Then $I_{\mathfrak{f}}^{\mathfrak{f}'} = I_{\mathfrak{f}}(\omega_{\mathfrak{f}'}) = 1$. Proposition 2.1(iii) implies that $\omega_{\mathfrak{f}'}$ considered as an $\mathfrak{f}$–module has exactly one non–trivial irreducible constituent U with $I_{\mathfrak{f}}(U) = 1$. We show now that U is isomorphic to $\omega_{\mathfrak{f}}$ or to $\omega_{\mathfrak{f}}^*$. Theorem 2.5 of [Dy2] states that

$$I_{\mathfrak{f}}(U) = \frac{\dim U}{\dim \mathfrak{f}}\langle \lambda, \lambda + 2\rho\rangle, \tag{3}$$

where $\langle\cdot,\cdot\rangle$ is the form induced on $\mathfrak{f}^*$ by $\langle\cdot,\cdot\rangle_{\mathfrak{f}}$, λ is the highest weight of U, and ρ is the half–sum of the positive roots of $\mathfrak{f}$. Since both $\dim U$ and $\langle\lambda, \lambda+2\rho\rangle$ are increasing functions of λ (with respect to the order: $\lambda' > \lambda''$ if $\lambda' - \lambda''$ is a non–negative combination of fundamental weights), so is $I_{\mathfrak{f}}(U)$. Table 5 in [Dy2] shows that, for $\operatorname{rk} \mathfrak{f} > 4$, a fundamental representation U of $\mathfrak{f}$ with $I_{\mathfrak{f}}(U) = 1$ is isomorphic to $w_{\mathfrak{f}}$ or $\omega_{\mathfrak{f}}^*$. The monotonicity of $I_{\mathfrak{f}}(U)$ now shows that $I_{\mathfrak{f}}(U) = 1$ implies $U \cong w_{\mathfrak{f}}$ or $U \cong \omega_{\mathfrak{f}}^*$. Since for $\operatorname{rk} \mathfrak{f} > 4$ every $\mathfrak{f}$–module conjugate to $\omega_{\mathfrak{f}}$ is isomorphic to $\omega_{\mathfrak{f}}$ or $\omega_{\mathfrak{f}}^*$, φ is a standard injection.

If $\mathfrak{f}$ is of type B or D, an argument similar to the one above shows that $I_{\mathfrak{f}}(\omega_{\mathfrak{f}'}) \geq 2$. Consequently, formula $I_{\mathfrak{f}}^{\mathfrak{f}'} = I_{\mathfrak{f}}(\omega_{\mathfrak{f}'})/I_{\mathfrak{f}'}(\omega_{\mathfrak{f}'}) = 1$ forces $I_{\mathfrak{f}}(\omega_{\mathfrak{f}'}) = 2$. Going back to the argument above we see that $I_{\mathfrak{f}}(\omega_{\mathfrak{f}'}) = 2$ implies that the homomorphism φ is a standard embedding.

(ii) Every simple Lie algebra of rank $n \geq 9$ contains a root subalgebra isomorphic to $\mathrm{sl}(n)$. Moreover, $I_{\mathrm{sl}(n)}^{\mathfrak{f}'} = \frac{I_{\mathrm{sl}(n)}(\omega_{\mathfrak{f}'})}{I_{\mathfrak{f}'}(\omega_{\mathfrak{f}'})} \geq \frac{1}{2} I_{\mathrm{sl}(n)}(\omega_{\mathfrak{f}'})$. Hence, it is enough to show that there exist constants d_n with $\lim d_n = \infty$ such that $I_{\mathrm{sl}(n)}(U) \geq d_n$ for any $\mathrm{sl}(n)$–module U which has a simple constituent not isomorphic to $\omega_{\mathrm{sl}(n)}$ or $\omega_{\mathrm{sl}(n)}^*$. To prove the existence of the constants d_n we first observe that Weyl's dimension formula implies the existence a constant $a_1 > 0$, such that $\dim U \geq a_1 n^2$. Next, a direct computation gives a constant $a_2 > 0$, such that $\langle\lambda, \lambda+2\rho\rangle \geq a_2 n$. Substituting these estimates into (3) implies the existence of the constants d_n with the desired properties. □

Corollary 2.4 *Let*

$$\mathfrak{f}_1 \to \mathfrak{f}_2 \to \dots$$

be a system of injective homomorphisms of simple finite–dimensional Lie algebras such that $I_{\mathfrak{f}_n}^{\mathfrak{f}_{n+1}} = 1$ *for all* n *and* $\lim(\mathrm{rk}\, \mathfrak{f}_n) = \infty$. *Then there exists* n_0 *such that, for* $n > n_0$, *all homomorphisms* $\mathfrak{f}_n \to \mathfrak{f}_{n+1}$ *are standard injections and all* $\mathfrak{f}_n$ *are of type* A, *or all* $\mathfrak{f}_n$ *are of type* C, *or each* $\mathfrak{f}_n$ *is of type* B *or* D.

Proof. The statement follows directly from Proposition 2.3(ii). $\square$

3 Locally semisimple subalgebras

Theorem 3.1 *A subalgebra* $\mathfrak{s} \subset \mathfrak{g}$ *is locally semisimple if and only if it is isomorphic to* $\oplus_{\alpha \in A} \mathfrak{s}^\alpha$, *where each* $\mathfrak{s}^\alpha$ *is a finite–dimensional simple Lie algebra or is isomorphic to* $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, *or* $\mathrm{sp}(\infty)$, *and* A *is a finite or countable set.*

Proof. In one direction the statement is obvious: if $\mathfrak{s} \cong \oplus_{\alpha \in A} \mathfrak{s}^\alpha$, then by identifying A with a subset of $\mathbb{Z}_{>0}$ and exhausting each $\mathfrak{s}^\alpha$ as $\varinjlim \mathfrak{s}_n^\alpha$ for some classical simple Lie algebras $\mathfrak{s}_n^\alpha$, one exhausts $\mathfrak{s}$ via the semisimple Lie algebras $\oplus_{\alpha=1}^n \mathfrak{s}_n^\alpha$.

Let now $\mathfrak{s}$ be locally semisimple, $\mathfrak{s} = \varinjlim \mathfrak{s}_n$. Write $\mathfrak{s}_n = \oplus_{j=1}^{l_n} \mathfrak{s}_n^j$, where each $\mathfrak{s}_n^j$ is a simple finite–dimensional Lie algebra. Fix a standard exhaustion $\mathfrak{g} = \varinjlim \mathfrak{g}_n$ of $\mathfrak{g}$ such that the diagram

(4)
$$\begin{array}{ccccccccc} \cdots & \longrightarrow & \mathfrak{s}_n & \xrightarrow{\varphi_n} & \mathfrak{s}_{n+1} & \longrightarrow & \cdots & \longrightarrow & \mathfrak{s} \\ & & \downarrow{\scriptstyle \theta_n} & & \downarrow{\scriptstyle \theta_{n+1}} & & & & \\ \cdots & \longrightarrow & \mathfrak{g}_n & \xrightarrow[\psi_n]{} & \mathfrak{g}_{n+1} & \longrightarrow & \cdots & \longrightarrow & \mathfrak{g} \end{array}$$

is commutative. In particular, $I_{\mathfrak{g}_n}^{\mathfrak{g}_{n+1}} = 1$ for every n.

For each $1 \leq j \leq l_n$ let

$$i_n^j : \mathfrak{s}_n^j \to \mathfrak{s}_n \quad \text{and} \quad \pi_n^j : \mathfrak{s}_n \to \mathfrak{s}_n^j$$

be the natural injection and projection respectively. Set $\theta_n^j = \theta_n \circ i_n^j : \mathfrak{s}_n^j \to \mathfrak{g}_n$ and let $\varphi_n^{j,k} = \pi_{n+1}^k \circ \varphi_n \circ i_n^j : \mathfrak{s}_n^j \to \mathfrak{s}_{n+1}^k$. Then $\varphi_n^{i,k}$ is a homomorphism of simple Lie algebras. Set also

$$\alpha_n^j := I_{\mathfrak{s}_n^j}^{\mathfrak{g}_n}, \qquad \beta_n^{j,k} := I_{\mathfrak{s}_n^j}^{\mathfrak{s}_{n+1}^k}.$$

By Proposition 2.2 we have

$$\alpha_n^j = \sum_{k=1}^{l_{n+1}} \beta_n^{j,k} \alpha_{n+1}^k. \tag{5}$$

We now assign an oriented graph Γ (a Bratteli diagram) to the direct system $\{\mathfrak{s}_n\}$. The vertices of Γ are the pairs (n, j) with $1 \le j \le l_n$. A vertex (n, j) has *level* n. An arrow points from (n, j) to $(n+1, k)$ if and only if $\varphi_n^{j,k}$ is not trivial. A *path* γ in Γ is a sequence of vertices $(n, j_n), (n+1, j_{n+1}), \ldots, (m, j_m)$ such that, for every i with $n \le i \le m-1$, an arrow points from (i, j_i) to $(i+1, j_{i+1})$. We label the vertices and arrows of Γ as follows: the vertex (n, j) is labeled by α_n^j and the arrow from (n, j) to $(n+1, k)$ is labeled by $\beta_n^{j,k}$. For the path γ above we set $\gamma(i) := j_i$ for $n \le i \le m$ and define $\beta(\gamma)$ as the product $\beta_n^{j_n, j_{n+1}} \beta_{n+1}^{j_{n+1}, j_{n+2}} \ldots \beta_{m-1}^{j_{m-1}, j_m}$ of the labels of all arrows of γ. Formula (5) generalizes to

$$\alpha_n^j = \sum_{\gamma} \beta(\gamma) \alpha_m^{\gamma(m)}, \tag{6}$$

where the summation is over all paths starting at (n, j) and ending at (m, k) for some $1 \le k \le l_k$.

For each vertex (n, j), let $\Gamma(n, j)$ denote the full subgraph of Γ whose vertices appear in paths starting at (n, j). Let $a_m(n, j)$ be the sum of the labels of all vertices of $\Gamma(n, j)$ of level m, i.e. $a_m(n, j) := \sum_{(m,k) \in \Gamma(n,j)} \alpha_m^k$. Then

$$a_m(n, j) = \sum_{(m,k) \in \Gamma(n,j)} \alpha_m^k = \sum_{(m+1,t) \in \Gamma(n,j)} \left(\sum_{(m,k) \in \Gamma(n,j)} \beta_m^{k,t} \right) \alpha_{m+1}^t \ge a_{m+1}(n, j). \tag{7}$$

This implies that the sequence $\{a_m(n, j)\}$ stabilizes, i.e. $a_m(n, j) = a(n, j)$ for m large enough. Furthermore, (7) shows that if $a_m(n, j) = a_{m+1}(n, j) = a(n, j)$, then each vertex of $\Gamma(n, j)$ of level m points to exactly one vertex of $\Gamma(n, j)$ of level $(m+1)$. In other words, the graph $\Gamma(n, j)$ is nothing but several disjoint strings from some level on. More precisely, there exist m_0 and t such that, for $m \ge m_0$, $\Gamma(n, j)$ has exactly t vertices $(m, j_{m,1}), \ldots, (m, j_{m,t})$

of level m and the arrows pointing from vertices of level m to vertices of level $m+1$, after a possible relabeling of the vertices of level $m+1$, are

$$\begin{array}{rcl}(m, j_{m,1}) & \rightarrow & (m+1, j_{m+1,1}) \\ & \vdots & \\ (m, j_{m,t}) & \rightarrow & (m+1, j_{m+1,t}).\end{array}$$

Finally, formula (7) implies $\beta_m^{j_{m,i}, j_{m+1,i}} = 1$ for every $1 \leq i \leq t$.

Let $\mathfrak{s}_m(n,j) := \oplus_{(m,k)\in\Gamma(n,j)}\mathfrak{s}_m^k$. Clearly $\varphi_m(\mathfrak{s}_m(n,j)) \subset \mathfrak{s}_{m+1}(n,j)$, hence $\mathfrak{s}(n,j) = \varinjlim \mathfrak{s}_m(n,j)$ is a well–defined Lie subalgebra of $\mathfrak{g}$. The fact that $\Gamma(n,j)$ splits into t disjoint strings for $m \geq m_0$ implies that

$$\mathfrak{s}(n,j) = \oplus_{i=1}^t \mathfrak{s}^i(n,j),$$

where $\mathfrak{s}^i(n,j) := \varinjlim_{m\geq m_0} \mathfrak{s}_m^{j_{m,i}}$. The equality $\beta_m^{j_{m,i}, j_{m+1,i}} = 1$ implies via Corollary 2.4 that $\mathfrak{s}^i(n,j)$ is a finite–dimensional simple Lie algebra or is a Lie algebra isomorphic to $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, or $\mathrm{sp}(\infty)$.

We are now ready to construct a decomposition $\mathfrak{s} = \oplus_{\alpha\in A}\mathfrak{s}^\alpha$ as required. Notice first that $\Gamma(n,j) \cap \Gamma(n',j')$ is either empty or consists of several disjoint strings from some level on. Hence $\mathfrak{s}(n,j)$ and $\mathfrak{s}(n',j')$ intersect in subsums of the direct sums $\mathfrak{s}(n,j) = \oplus_{i=1}^t \mathfrak{s}^i(n,j)$ and $\mathfrak{s}(n',j') = \oplus_{i'=1}^{t'} \mathfrak{s}^{i'}(n',j')$. Consequently,

$$\mathfrak{s} = \sum_{(n,j)\in\Gamma} \mathfrak{s}(n,j). \tag{8}$$

Let $A(n,j)$ denote set of paths of $\Gamma(n,j)$ and let $\sim$ be the following equivalence relation on the set $\cup_{(n,j)\in\Gamma} A(n,j)$: $a \in A(n,j) \sim a' \in A(n',j')$ if a and a' coincide for large enough m. Define $A := \left(\cup_{(n,j)\in\Gamma} A(n,j)\right) / \sim$ and, for every $\alpha \in A$, set $\mathfrak{s}^\alpha := \mathfrak{s}^i(n,j)$, where $(m, j_{m,i}), (m+1, j_{m+1,i}), \ldots$ is a representative of α. Equation (8) implies that $\mathfrak{s} = \oplus_{\alpha\in A}\mathfrak{s}^\alpha$ and this completes the proof. $\square$

We will illustrate the results of this paper in a series of examples built on the same set–up, cf. Theorem 5.8 in [Ba1].

Example 1. Set $\tilde{V} := V \oplus \mathbb{C}\tilde{v}$ with $\langle \tilde{v}, v_j^* \rangle = 1$ for every j. Both couples V, V_* and $\tilde{V}, V_*$ are non–degenerately paired and both Lie algebras $\mathfrak{g} = [V \otimes V_*, V \otimes V_*]$ and $\tilde{\mathfrak{g}} = [\tilde{V} \otimes V_*, \tilde{V} \otimes V_*]$ are isomorphic to $\mathrm{sl}(\infty)$. Any partition $\mathbb{Z}_{>0} = \sqcup_{\alpha \in A} I^\alpha$ defines a locally semisimple subalgebra $\mathfrak{s}$ of both $\mathfrak{g}$ and $\tilde{\mathfrak{g}}$ in the following way. Set $V^\alpha := \mathrm{Span}\{v_j\}_{j \in I^\alpha}$, $(V^\alpha)_* := \mathrm{Span}\{v_j^*\}_{j \in I^\alpha}$, and $\mathfrak{s}^\alpha := \mathfrak{g} \cap (V^\alpha \otimes (V^\alpha)_*)$. Define $\mathfrak{s}$ as $\oplus_{\alpha \in A} \mathfrak{s}^\alpha$. In particular, $\mathfrak{g}$ itself is a locally semisimple subalgebra of $\tilde{\mathfrak{g}}$.

A corollary of Theorem 3.1 concerns the structure of an arbitrary exhaustion of $\mathfrak{g}$ by semisimple Lie algebras, cf. Corollary 5.9 in [Ba2].

Corollary 3.2 *Let $\mathfrak{g} = \varinjlim \mathfrak{s}_n$, where each $\mathfrak{s}_n$ is semisimple. There exist n_0 and simple ideals $\mathfrak{k}_n \subset \mathfrak{s}_n$ for $n \geq n_0$, such that $\mathfrak{k}_n \subset \mathfrak{k}_{n+1}$ and $\mathfrak{g} = \varinjlim \mathfrak{k}_n$. Furthermore, the system $\{\mathfrak{k}_n\}$ admits a refinement $\{\mathfrak{g}_s\}$ with*

$$\mathfrak{g}_s \cong \begin{cases} \mathrm{sl}(s) & \text{if } \mathfrak{g} = \mathrm{sl}(\infty) \\ \mathrm{so}(s) & \text{if } \mathfrak{g} = \mathrm{so}(\infty) \\ \mathrm{sp}(2s) & \text{if } \mathfrak{g} = \mathrm{sp}(\infty). \end{cases}$$

Proof. By Theorem 3.1 $\mathfrak{g} = \oplus_{\alpha \in A} \mathfrak{s}^\alpha$. Since $\mathfrak{g}$ is simple, A consists of a single element, i.e. there exists m such that, for $n \geq m$, $\Gamma(n, j)$ is a single string

$$(m, j_m), (m+1, j_{m+1}), \ldots \quad .$$

Set $\mathfrak{k}_n := \mathfrak{s}_n^{j_n}$ for $n \geq m$. Clearly $\mathfrak{g} = \varinjlim \mathfrak{k}_n$. Note that, as $I_{\mathfrak{k}_n}^{\mathfrak{k}_{n+1}} = 1$ for large enough n, Corollary 2.4 implies that there exists $n_0 \geq m$ such that all injections $\mathfrak{k}_n \to \mathfrak{k}_{n+1}$ are standard for $n \geq n_0$. The fact that a standard exhaustion of $\mathfrak{g}$ admits a refinement as in the statement of the corollary is obvious. $\square$

In the special case when $\mathfrak{g}$ is exhausted by simple Lie algebras $\mathfrak{g}_n$, Corollary 3.2 implies that, for large enough n, all injections $\mathfrak{g}_n \to \mathfrak{g}_{n+1}$ are standard. Furthermore, by Corollary 2.4 all $\mathfrak{g}_n$ are of type A, or all $\mathfrak{g}_n$ are of type C, or each $\mathfrak{g}_n$ is of type B or D.

Here is an example showing that there exist interesting exhaustions of $\mathrm{sl}(\infty)$ by non–reductive Lie algebras.

Example 2. We build on Example 1. Put $V_n := \mathrm{Span}\{v_1, v_2, \dots, v_n\} \subset V$, $\tilde{V}_n := V_n \oplus \mathbb{C}\tilde{v} \subset \tilde{V}$, and $(V_n)_* := \mathrm{Span}\{v_1^*, v_2^*, \dots, v_n^*\} \subset V_*$. Set also $\mathfrak{g}_n = \mathfrak{g} \cap (V_n \otimes (V_n)_*)$ and $\tilde{\mathfrak{g}}_n := \mathfrak{g} \cap (\tilde{V}_n \otimes (V_n)_*)$. Then $\mathbb{C}(\tilde{v} - v_1 - \dots - v_n) \otimes (V_n)_*$ is the radical of $\tilde{\mathfrak{g}}_n$ and $\varinjlim \tilde{\mathfrak{g}}_n$ is an exhaustion of $\tilde{\mathfrak{g}}$ with non–reductive finite dimensional Lie algebras. Note that the Levi components $\mathfrak{g}_n$ of $\tilde{\mathfrak{g}}_n$ are nested and their direct limit $\varinjlim \mathfrak{g}_n$ is nothing but the proper subalgebra $\mathfrak{g}$ of $\tilde{\mathfrak{g}}$. On the other hand, a different choice of Levi components of $\tilde{\mathfrak{g}}_n$ yields an exhaustion of $\tilde{\mathfrak{g}}$. Indeed, the Lie algebras $\mathfrak{k}_n := \tilde{\mathfrak{g}} \cap (\tilde{V}_{n-1} \otimes (V_n)_*)$ are also nested and their direct limit $\varinjlim \mathfrak{k}_n$ is the entire Lie algebra $\tilde{\mathfrak{g}}$. Moreover, since $\tilde{V}_{n-1}$ and $(V_n)_*$ are non–degenerately paired, we have $\mathfrak{k}_n \cong \mathrm{sl}(n)$, which means that $\mathfrak{k}_n$ is a Levi component of $\tilde{\mathfrak{g}}_n$ for every n.

We conclude this section by another corollary of Theorem 3.1.

Corollary 3.3 *Let $\mathfrak{a}$ be a Lie algebra isomorphic to a finite or countable direct sum of finite–dimensional simple Lie algebras and of copies of* $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, *and* $\mathrm{sp}(\infty)$. *Then a subalgebra $\mathfrak{s} \subset \mathfrak{a}$ is locally semisimple if and only if $\mathfrak{s}$ itself is isomorphic to a finite or countable direct sum of finite–dimensional simple Lie algebras and of copies of* $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, *and* $\mathrm{sp}(\infty)$.

Proof. Since $\mathfrak{a}$ admits an obvious injective homomorphism into $\mathrm{sl}(\infty)$, the statement follows directly from Theorem 3.1. □

4 V and V_* as modules over a locally semisimple subalgebra $\mathfrak{s} \subset \mathfrak{g}$

Fix a locally semisimple subalgebra $\mathfrak{s} \subset \mathfrak{g}$. In this section we describe the structure of V and V_* as $\mathfrak{s}$–modules. Let $\mathfrak{s} = \oplus_{\alpha \in A} \mathfrak{s}^\alpha$ where $\mathfrak{s}^\alpha$ are the simple constituents of $\mathfrak{s}$ according to Theorem 3.1. Set

$$\begin{aligned} A^f &:= \{\alpha \in A \,|\, \mathfrak{s}^\alpha \text{ is finite–dimensional}\}, \\ A^{inf} &:= \{\alpha \in A \,|\, \mathfrak{s}^\alpha \text{ is infinite–dimensional}\}, \\ \mathfrak{s}^f &:= \oplus_{\alpha \in A^f} \mathfrak{s}^\alpha. \end{aligned}$$

We start by describing the structure of V and V_* as modules over $\mathfrak{s}^f$.

Proposition 4.1 *Let W be an at most countable–dimensional $\mathfrak{s}^f$–module with the property that, for every $x \in \mathfrak{s}^f$, the image of x, considered as an endomorphism of W, is finite–dimensional. Then*

(i) *every simple $\mathfrak{s}^f$–submodule of W is finite–dimensional;*

(ii) *W has non–zero socle W', hence by* (i) *W' is a direct sum of simple finite–dimensional $\mathfrak{s}^f$–modules.*

(iii) *W/W' is a trivial $\mathfrak{s}^f$–module.*

Proof. The set A^f is finite or countable. If A^f is finite, $\mathfrak{s}^f$ is a finite–dimensional semisimple Lie algebra and, by the required property on W, the $\mathfrak{s}^f$–module W is integrable. Hence (by a well–known extension of Weyl's semisimplicity theorem to integrable modules) W is semisimple and all of its simple constituents are finite–dimensional.

Assume that A^f is countable and put $A^f := \{1, 2, \dots\}$. Fix an exhaustion of $\mathfrak{s}^f$ of the form $\mathfrak{s}^f_n = \mathfrak{s}^1 \oplus \dots \oplus \mathfrak{s}^n$, $\mathfrak{s}^n$ being the simple constituents of $\mathfrak{s}^f$. If W is trivial there is nothing to prove. Assume that W is non–trivial. Then W is a non–trivial $\mathfrak{s}^n$–module for some n. Let W^n_κ be a non–trivial isotypic component of the $\mathfrak{s}^n$–module W, i.e. an isotypic component of W corresponding to a non–trivial simple finite–dimensional $\mathfrak{s}^n$–module. The condition on W implies that W^n_κ is finite–dimensional as otherwise the image in W of any root vector of $\mathfrak{s}^n$ would be infinite–dimensional. Notice that W^n_κ is actually an $\mathfrak{s}^f$–submodule of W since W^n_κ is $\mathfrak{s}^m$–stable for all m. Furthermore, as every non–trivial simple $\mathfrak{s}^f$–submodule $\tilde{W}$ of W

contains a non–trivial $\mathfrak{s}^n$–submodule for some n, $\tilde{W}$ is necessarily contained in W^n_κ for some κ. This proves (i) and (ii).

To prove (iii) we observe that the socle W' of W is the direct sum of a trivial module and the sum of W^n_κ as above for all n and all κ. □

Example 3. This example shows that W is not necessarily semisimple as an $\mathfrak{s}^f$–module, i.e. that W' does not necessarily equal W. In the set–up of Example 1 consider a partition of $\mathbb{Z}_{>0}$ into two–element subsets. The corresponding locally semisimple subalgebra $\mathfrak{s}$ of $\tilde{\mathfrak{g}}$ is a direct sum of infinitely many copies of sl(2) and hence $\mathfrak{s}^f = \mathfrak{s}$. One checks immediately that for $W = \tilde{V}$, we have $W' = V$.

As a next step we describe the $\mathfrak{s}^\alpha$–module structures of V and V_* for $\alpha \in A^{inf}$.

Proposition 4.2

(i) *For any $\alpha \in A^{inf}$, the socle V'_α of V as an $\mathfrak{s}^\alpha$–module is isomorphic to $k_\alpha V^\alpha \oplus l_\alpha V^\alpha_* \oplus N^\alpha$, where $k_\alpha, l_\alpha \in \mathbb{Z}_{>0}$, V^α and V^α_* are respectively the natural and conatural representation of $\mathfrak{s}^\alpha$ (here $l_\alpha = 0$ for $\mathfrak{s}^\alpha \ncong \mathrm{sl}(\infty)$) and N^α is a trivial $\mathfrak{s}^\alpha$–module of finite or countable dimension. Similarly, for $\mathfrak{g} \cong \mathrm{gl}(\infty)$ or $\mathrm{sl}(\infty)$, the socle $(V_*)'_\alpha$ of V_* as an $\mathfrak{s}^\alpha$–module is isomorphic to $k_\alpha V^\alpha_* \oplus l_\alpha V^\alpha \oplus N^\alpha_*$, where N^α_* is a trivial $\mathfrak{s}^\alpha$–module of finite or countable dimension, not necessarily equal to the dimension of N^α.*

(ii) *V/V'_α and $V_*/(V_*)'_\alpha$ are trivial $\mathfrak{s}^\alpha$–modules.*

Proof. Fix standard exhaustions of $\mathfrak{s}^\alpha$ and $\mathfrak{g}$ such that the diagram

$$\begin{tikzcd} \cdots \arrow[r] & \mathfrak{s}^\alpha_{n-1} \arrow[r] \arrow[d] & \mathfrak{s}^\alpha_n \arrow[r] \arrow[d] & \mathfrak{s}^\alpha_{n+1} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathfrak{s} \\ \cdots \arrow[r] & \mathfrak{g}_{n-1} \arrow[r] & \mathfrak{g}_n \arrow[r] & \mathfrak{g}_{n+1} \arrow[r] & \cdots \arrow[r] & \mathfrak{g} \end{tikzcd}$$

commutes.. As in the proof of Theorem 3.1 we see that, for large enough n, $I^{\mathfrak{g}_n}_{\mathfrak{s}^\alpha_n}$ is a constant, i.e. does not depend on n. Therefore, by Proposition 2.3 each injective homomorphism

$\mathfrak{s}_n^\alpha \hookrightarrow \mathfrak{g}_n$ is diagonal injection for large n, i.e.

$$V(\mathfrak{g}_n) = k_\alpha V(\mathfrak{s}_n^\alpha) \oplus l_\alpha V(\mathfrak{s}_n^\alpha)^* \oplus N_\alpha^n, \tag{9}$$

where $V(\mathfrak{g}_n)$ and $V(\mathfrak{s}_n^\alpha)$ are the natural representation of $\mathfrak{g}_n$ and $\mathfrak{s}_n^\alpha$ respectively, the superscript * stands for dual space, $k_\alpha + l_\alpha = I_{\mathfrak{s}_n^\alpha}^{\mathfrak{g}_n}$, and N_α^n is a trivial $\mathfrak{s}_n^\alpha$–module. Furthermore

$$V(\mathfrak{g}_n)^* = k_\alpha V(\mathfrak{s}_n^\alpha)^* \oplus l_\alpha V(\mathfrak{s}_n^\alpha) \oplus N_\alpha^n. \tag{10}$$

Since $\mathrm{Hom}_{\mathfrak{s}_n^\alpha}(V(\mathfrak{s}_n^\alpha), V(\mathfrak{s}_{n+1}^\alpha)^*) = \mathrm{Hom}_{\mathfrak{s}_n^\alpha}(V(\mathfrak{s}_n^\alpha), N_\alpha^{n+1}) = \mathrm{Hom}_{\mathfrak{s}_n^\alpha}(V(\mathfrak{s}_n^\alpha)^*, V(\mathfrak{s}_{n+1}^\alpha)) = \mathrm{Hom}_{\mathfrak{s}_n^\alpha}(V(\mathfrak{s}_n^\alpha)^*, N_\alpha^{n+1}) = 0$ and $\dim \mathrm{Hom}_{\mathfrak{s}_n^\alpha}(V(\mathfrak{s}_n^\alpha), V(\mathfrak{s}_{n+1}^\alpha)) = \dim \mathrm{Hom}_{\mathfrak{s}_n^\alpha}(V(\mathfrak{s}_n^\alpha)^*, V(\mathfrak{s}_{n+1}^\alpha)^*) = 1$, the fact that $V = \varinjlim V(\mathfrak{g}_n)$ and $V_* = \varinjlim V(\mathfrak{g}_n)^*$ implies $\dim \mathrm{Hom}_{\mathfrak{s}^\alpha}(V^\alpha, V) = k_\alpha$, $\dim \mathrm{Hom}_{\mathfrak{s}^\alpha}(V_*^\alpha, V_*) = l_\alpha$. Therefore $k_\alpha V^\alpha \oplus l_\alpha V_*^\alpha \subset V'_\alpha$, $k_\alpha V_*^\alpha \oplus l_\alpha V^\alpha \subset (V_*)'_\alpha$. Moreover, it follows immediately from (9) and (10) that both V'_α and $(V_*)'_\alpha$ can only have simple constituents isomorphic to V^α, V_*^α and to the 1-dimensional trivial module. This completes the proof of (i).

Claim (ii) follows directly from (i) and from (9) and (10). □

Example 4. This example shows that the socle of the natural representation considered as an $\mathfrak{s}^\alpha$–module can also be a proper subspace. In the notations of Example 1 we can choose the subalgebra $\mathfrak{s}^\alpha$ of $\tilde{\mathfrak{g}}$ to be $\mathfrak{g}$. Then $\tilde{V}' = V$ is a proper subspace of $\tilde{V}$. Note also that the dimensions of the trivial $\mathfrak{s}^\alpha$–modules N^α and N_*^α are different in this case. Indeed, $\dim N^\alpha = 1$ while $\dim N_*^\alpha = 0$.

Put now $\tilde{A} := A^{inf} \cup \{f\}$ and, for every $\alpha \in \tilde{A}$, let $V(\alpha)$ and $V_*(\alpha)$ denote the sum of all non–trivial simple $\mathfrak{s}^\alpha$–submodules of V and V_* respectively.

Proposition 4.3 *The sums $\sum_{\alpha\in\tilde{A}} V(\alpha)$ and $\sum_{\alpha\in\tilde{A}} V_*(\alpha)$ are direct in V and V_* respectively. Each $\mathfrak{s}^\alpha$ acts trivially on $V(\beta)$ and $V_*(\beta)$ for $\beta \neq \alpha$. Furthermore, $V/(\oplus_{\alpha\in\tilde{A}} V(\alpha))$ and $V_*/(\oplus_{\alpha\in\tilde{A}} V_*(\alpha))$ are trivial $\mathfrak{s}$–modules.*

Proof. We will prove the proposition for V as the statements for V_* are analogous. Let $\alpha, \beta \in A^{inf}$ and let $\mathfrak{s}^\alpha = \varinjlim \mathfrak{s}_n^\alpha$ and $\mathfrak{s}^\beta = \varinjlim \mathfrak{s}_n^\beta$ be standard exhaustions. Assume that the action of $\mathfrak{s}^\alpha$ on $V(\beta)$ is non–trivial. Then, for some i, V will have simple $\mathfrak{s}_i^\alpha \oplus \mathfrak{s}_n^\beta$–submodules of the form $V_i^\alpha \otimes M_n^\beta$ or $(V_*^\alpha)_i \otimes M_n^\beta$ for some $\mathfrak{s}_\beta^n$–modules M_n^β of unbounded dimension when $n \to \infty$. This would imply that the multiplicity of V_i^α or $(V_*^\alpha)_i$ in V is infinite, which is a contradiction. The case when $\alpha = f$ or $\beta = f$ is dealt with in a similar way.

The fact that $V/\oplus_{\alpha\in\tilde{A}} V(\alpha)$ is a trivial $\mathfrak{s}$–module is obvious.. $\square$

In this way we have proved the following theorem.

Theorem 4.4 *The socle V' of V (respectively, $(V_*)'$ of V_*) considered as an $\mathfrak{s}$–module is isomorphic to the direct sum of all non–trivial $\mathfrak{s}^\alpha$–submodules $V(\alpha)$ (respectively, $V_*(\alpha)$) of V (respectively, V_*), described in Propositions* 4.2 *and* 4.3 *plus a possible trivial $\mathfrak{s}$–submodule. The quotients V/V' and $V_*/(V_*)'$ are trivial $\mathfrak{s}$–modules.*

Proof. By Proposition 4.3, for each $\alpha \in A^{inf}$, the modules $V(\alpha) \subset V$ and $V_*(\alpha) \subset V_*$ are semisimple $\mathfrak{s}$–submodules of finite length. Moreover, the modules $V(f) \subset V$ and $V_*(f) \subset V_*$ are semisimple $\mathfrak{s}$–submodules with finite–dimensional simple constituents. By Proposition 4.3, the quotients $V/\oplus_{\alpha\in\tilde{A}} V(\alpha)$ and $V_*/\oplus_{\alpha\in\tilde{A}} V_*(\alpha)$ are trivial $\mathfrak{s}$–modules, and the statement follows. $\square$

Note that to any locally semisimple subalgebra $\mathfrak{s} \subset \mathfrak{g}$ we can assign some "standard invariants". These are the isomorphism classes of $V(f)$ and $V_*(f)$ as $\mathfrak{s}^f$–modules, the pairs of numbers $\{k_\alpha, l_\alpha\}_{\alpha\in A^{inf}}$, and the dimensions $\{\dim N^J, \dim N_*^J, \dim V/V_J', \dim V_*/(V_*)_J'\}_{J\subset A^{inf}}$, where $N^J := \cap_{\beta\in J} N^\beta$, $N_*^J := \cap_{\beta\in J} N_*^\beta$, and V_J' and $(V_*)_J'$ are the respective socles of V and V_* considered as $(\oplus_{\beta\in J}\mathfrak{s}^\beta)$–modules. Clearly, these invariants are preserved when conjugating by elements of the group $GL(V, V_*)$ of all automorphisms of V under which V_* is stable (respectively, all automorphisms of V preserving the non–degenerate form $V \times V \to \mathbb{C}$ for $\mathfrak{g} = \mathrm{so}(V)$ or $\mathrm{sp}(V)$). In a similar way, when $\mathfrak{s}$ is replaced by a maximal toral subalgebra, it is shown in [DPS] that the analogous invariants are only rather rough invariants of the $GL(V, V_*)$–conjugacy classes of maximal toral subalgebras. The $GL(V, V_*)$–conjugacy classes of locally semisimple subalgebras $\mathfrak{s} \subset \mathfrak{g}$ with fixed "standard invariants" as above remain to be studied.

5 Maximal subalgebras

Theorem 5.1

Let $\mathfrak{m} \subset \mathfrak{g}$ be a proper subalgebra.

(i) *If $\mathfrak{g} = \mathrm{gl}(V, V_*)$, then $\mathfrak{m}$ is maximal if and only if one of the following three mutually exclusive statements holds:*

(ia) $\mathfrak{m} = [\mathfrak{g}, \mathfrak{g}] = \mathrm{sl}(V, V_*)$;

(ib) $\mathfrak{m} = \mathrm{Stab}_{\mathfrak{g}} W$ *or* $\mathfrak{m} = \mathrm{Stab}_{\mathfrak{g}} \tilde{W}$, *where $W \subset V$ (respectively, $\tilde{W} \subset V_*$) is a subspace with the properties* $\mathrm{codim}_V W = 1$, $W^{\perp} = 0$ *(respectively,* $\mathrm{codim}_{V_*} \tilde{W} = 1$, $\tilde{W}^{\perp} = 0$*); in this case* $\mathfrak{m} \cong \mathrm{gl}(\infty)$*;*

(ic) $\mathfrak{m} = \mathrm{Stab}_{\mathfrak{g}} W = \mathrm{Stab}_{\mathfrak{g}} W^{\perp}$, *where $W \subset V$ is a proper subspace with $W^{\perp\perp} = W$.*

(ii) *If $\mathfrak{g} = \mathrm{sl}(V, V_*)$, then $\mathfrak{m}$ is maximal if and only if one of the following three mutually exclusive statements holds:*

(iia) $\mathfrak{m} = \mathrm{so}(V)$ *or* $\mathfrak{m} = \mathrm{sp}(V)$ *for an appropriate non–degenerate symmetric or skew–symmetric form on V;*

(iib) $\mathfrak{m} = \mathrm{Stab}_{\mathfrak{g}} W$ *or* $\mathfrak{m} = \mathrm{Stab}_{\mathfrak{g}} \tilde{W}$, *where $W \subset V$ (respectively, $\tilde{W} \subset V_*$) is a subspace with the properties* $\mathrm{codim}_V W = 1$, $W^{\perp} = 0$ *(respectively,* $\mathrm{codim}_{V_*} \tilde{W} = 1$, $\tilde{W}^{\perp} = 0$*); in this case* $\mathfrak{m} \cong \mathrm{sl}(\infty)$*;*

(iic) $\mathfrak{m} = \mathrm{Stab}_{\mathfrak{g}} W = \mathrm{Stab}_{\mathfrak{g}} W^{\perp}$, *where $W \subset V$ is a proper subspace with $W^{\perp\perp} = W$.*

(iii) *If $\mathfrak{g} = \mathrm{so}(V)$ or $\mathfrak{g} = \mathrm{sp}(V)$, then $\mathfrak{m}$ is maximal if and only if $\mathfrak{m} = \mathrm{Stab}_{\mathfrak{g}} W$ for some subspace $W \subset V$ satisfying one of the following three mutually exclusive conditions:*

(iiia) *W is non–degenerate such that $W \oplus W^\perp = V$ and $\dim W \neq 2$, $\dim W^\perp \neq 2$ for $\mathfrak{g} = \mathrm{so}(V)$; in this case $\mathfrak{m} = \mathrm{so}(W) \oplus \mathrm{so}(W^\perp)$ when $\mathfrak{g} = \mathrm{so}(V)$, and $\mathfrak{m} = \mathrm{sp}(W) \oplus \mathrm{sp}(W^\perp)$ when $\mathfrak{g} = \mathrm{sp}(V)$;*

(iiib) *W is non–degenerate such that $W^\perp = 0$ and $\mathrm{codim}_V W = 1$; in this case $\mathfrak{m} = \mathrm{so}(W)$ when $\mathfrak{g} = \mathrm{so}(V)$, and $\mathfrak{m} = \mathrm{sp}(W)$ when $\mathfrak{g} = \mathrm{sp}(V)$;*

(iiic) *W is isotropic with $W^{\perp\perp} = W$.*

The space W (respectively, $\tilde{W}$) is unique in cases (ib) and (iib)*; the space W is unique in cases* (ic), (iic), (iiib), and (iiic)*; the pair $(W, W^\perp)$ is unique in case* (iiia)*.*

Proof. Let $\mathfrak{g} = \mathrm{gl}(V, V_*)$ and let $\mathfrak{m}$ be maximal. If both V and V_* are irreducible $\mathfrak{m}$–modules, then $\mathfrak{m} = [\mathfrak{g}, \mathfrak{g}]$. This follows from the description of irreducible subalgebras of $\mathfrak{g}$ given in Theorem 1.3 in [BS]. Let V be a reducible $\mathfrak{m}$–module. Then $\mathfrak{m} \subset \mathrm{Stab}_{\mathfrak{g}} W$ for some proper subspace $W \subset V$. Since V is an irreducible $\mathfrak{g}$–module, $\mathrm{Stab}_{\mathfrak{g}} W$ is a proper subalgebra of $\mathfrak{g}$. Therefore the maximality of $\mathfrak{m}$ yields $\mathfrak{m} = \mathrm{Stab}_{\mathfrak{g}} W$. If $W^{\perp\perp} = W$, we are in case (ic). If the inclusion $W \subset W^{\perp\perp}$ is proper, then the inclusion $\mathrm{Stab}_{\mathfrak{g}} W \subset \mathrm{Stab}_{\mathfrak{g}} W^{\perp\perp}$ is also proper since $W^{\perp\perp} \otimes V_* \subset \mathrm{Stab}_{\mathfrak{g}} W^{\perp\perp}$ and $W^{\perp\perp} \otimes V_* \not\subset \mathrm{Stab}_{\mathfrak{g}} W$. Hence we have a contradiction unless $\mathrm{Stab}_{\mathfrak{g}} W^{\perp\perp} = \mathfrak{g}$. In the latter case W must have codimension 1 in V as otherwise $\mathrm{Stab}_{\mathfrak{g}} W$ again would not be maximal. Moreover, $\mathrm{Stab}_{\mathfrak{g}} W = W \otimes V_*$ and, as W and V_* are non–degenerately paired, $\mathfrak{m} = \mathrm{Stab}_{\mathfrak{g}} W \cong \mathrm{gl}(\infty)$.

Finally, if V_* is a reducible $\mathfrak{m}$–module and V is an irreducible $\mathfrak{m}$–module then $\mathfrak{m} = V \otimes \tilde{W}$ for a subspace $\tilde{W} \subset V_*$ as in (ib). This proves (i) in one direction.

For the other direction, one needs to show that if W (respectively, $\tilde{W}$) is a subspace as in (ib) or (ic), $\mathrm{Stab}_{\mathfrak{g}} W$ (respectively, $\mathrm{Stab}_{\mathfrak{g}} \tilde{W}$) is a maximal subalgebra. In case (ic) this follows from the observation that $\mathrm{Stab}_{\mathfrak{g}} W = W \otimes V_* + V \otimes W^\perp$ which shows that $\mathfrak{g}/\mathrm{Stab}_{\mathfrak{g}} W \cong (V/W) \otimes (V_*/W^\perp)$ is an irreducible $\mathrm{Stab}_{\mathfrak{g}} W$–module.. In case (ib) $\mathrm{Stab}_{\mathfrak{g}} W = W \otimes V_*$ (respectively, $\mathrm{Stab}_{\mathfrak{g}} \tilde{W} = V \otimes \tilde{W}$), hence $\mathfrak{g}/\mathrm{Stab}_{\mathfrak{g}} W \cong V_*$ (respectively, $\mathfrak{g}/\mathrm{Stab}_{\mathfrak{g}} \tilde{W} \cong V$) is an irreducible $\mathrm{Stab}_{\mathfrak{g}} W$–module. The proof of (i) is now complete.

Claim (ii) is proved in the same way.

Let $\mathfrak{g} = \mathrm{so}(V)$ or $\mathfrak{g} = \mathrm{sp}(V)$ and let $\mathfrak{m}$ be maximal. Then V must be a reducible $\mathfrak{m}$–module by Theorem 1.3 in [BS]. If W is a proper $\mathfrak{m}$–submodule of V, then $\mathfrak{m}$ stabilizes $W^{\perp\perp}$ as well. If $W^{\perp\perp} = V$, i.e. $W^{\perp} = 0$, the inclusion $\mathrm{Stab}_{\mathfrak{g}}W \subset \mathrm{Stab}_{\mathfrak{g}}W^{\sharp}$ is proper whenever W is a proper subspace of $W^{\sharp}$. The maximality of $\mathfrak{m}$ implies then $\mathrm{codim}_V W = 1$ and we are in case (iiic). If $W^{\perp\perp}$ is a proper subspace of V, the inclusions $\mathfrak{m} \subset \mathrm{Stab}_{\mathfrak{g}}W \subset \mathrm{Stab}_{\mathfrak{g}}W^{\perp\perp}$ and the maximality of $\mathfrak{m}$ imply that $\mathfrak{m} = \mathrm{Stab}_{\mathfrak{g}}W^{\perp\perp}$. Noting that $(W^{\perp\perp})^{\perp\perp} = W^{\perp\perp}$ we may replace W by $W^{\perp\perp}$ and for the rest of the proof assume that $\mathfrak{m} = \mathrm{Stab}_{\mathfrak{g}}W$, where $W^{\perp\perp} = W$.

If W is isotropic or $W^{\perp}$ is isotropic, then $\mathrm{Stab}_{\mathfrak{g}}W = \mathrm{Stab}_{\mathfrak{g}}W^{\perp}$ and we are in case (iiic).

If $W \cap W^{\perp}$ is a proper subspace both of W and $W^{\perp}$, $W \cap W^{\perp}$ is an isotropic space. The inclusion $\mathfrak{m} \subset \mathrm{Stab}_{\mathfrak{g}}(W \cap W^{\perp})$ implies $\mathfrak{m} = \mathrm{Stab}_{\mathfrak{g}}(W \cap W^{\perp})$, and again we are in case (iiic) as $(W \cap W^{\perp})^{\perp\perp} = W \cap W^{\perp}$. Assume $W \cap W^{\perp} = 0$. Then $\mathfrak{m} \subset \mathrm{Stab}_{\mathfrak{g}}(W \oplus W^{\perp})$. If $W \oplus W^{\perp} = V$ and $\dim W \neq 2$ and $\dim W^{\perp} \neq 2$ for $\mathfrak{g} = \mathrm{so}(V)$, then $\mathrm{Stab}_{\mathfrak{g}}W = \mathrm{so}(W) \oplus \mathrm{so}(W^{\perp})$ or $\mathrm{Stab}_{\mathfrak{g}}W = \mathrm{sp}(W) \oplus \mathrm{sp}(W^{\perp})$, and we are in case (iiia). The case when $\mathfrak{g} = \mathrm{so}(V)$ and $\dim W = 2$ or $\dim W^{\perp} = 2$ does not occur as then $\mathrm{Stab}_{\mathfrak{g}}W$ is contained properly in the stabilizer of an isotropic subspace of W or $W^{\perp}$ respectively.

If the inclusion $W \oplus W^{\perp} \subset V$ is proper, then $\mathrm{Stab}_{\mathfrak{g}}(W \oplus W^{\perp})$ is a proper subalgebra of $\mathfrak{g}$ and the the inclusion $\mathrm{Stab}_{\mathfrak{g}}W \subset \mathrm{Stab}_{\mathfrak{g}}(W \oplus W^{\perp})$ is also proper. Indeed, for $\mathfrak{g} = \mathrm{so}(V)$ we have $\Lambda^2(W \oplus W^{\perp}) \subset \mathrm{Stab}_{\mathfrak{g}}(W \oplus W^{\perp})$ and $\Lambda^2(W \oplus W^{\perp}) \not\subset \mathrm{Stab}_{\mathfrak{g}}W$, and for $\mathfrak{g} = \mathrm{sp}(V)$ we have $S^2(W \oplus W^{\perp}) \subset \mathrm{Stab}_{\mathfrak{g}}(W \oplus W^{\perp})$ and $S^2(W \oplus W^{\perp}) \not\subset \mathrm{Stab}_{\mathfrak{g}}W$. Hence the maximality of $\mathfrak{m}$ implies $V = W \oplus W^{\perp}$, and we have proved (iii) in one direction.

We leave it to the reader to verify that, for every W as in (iiia), (iiib), and (iiic), $\mathrm{Stab}_{\mathfrak{g}}W$ is a maximal subalgebra of $\mathfrak{g}$.

To prove the uniqueness of W (respectively, $\tilde{W}$) or of the pair $(W, W^{\perp})$ as stated, it is enough to notice that W (respectively, $\tilde{W}$) is the unique proper $\mathfrak{m}$–submodule of V (respectively, V_*) in cases (ib) and (iib); that W is the unique proper $\mathfrak{m}$–submodule of V in cases (ic), (iic), (iiib), and (iiic); and that W are $W^{\perp}$ are the only proper $\mathfrak{m}$–submodules of V in case (iiia). $\square$

Note that the subalgebra $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ from Example 2 is a maximal simple subalgebra of $\tilde{\mathfrak{g}}$ as in (ib). Furthermore, in all cases but (ic), (iic), and (iiic), a maximal subalgebra $\mathfrak{m}$ is irreducible in the sense of [LP] and [BS], and in all cases but (ib), (iib), and (iiib) $\mathfrak{g}$ admits a standard exhaustion $\varinjlim \mathfrak{g}_n$ such that the Lie algebras $\mathfrak{m} \cap \mathfrak{g}_n$ are maximal subalgebras of $\mathfrak{g}_n$ for all n.

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