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To Israel Moiseevich Gelfand on his 90^{th} birthday

ON THE EXISTENCE OF $(\mathfrak{g}, \mathfrak{k})$-MODULES OF FINITE TYPE

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Abstract. Let $\mathfrak{g}$ be a reductive Lie algebra over an algebraically closed field of characteristic zero, and $\mathfrak{k}$ be a subalgebra reductive in $\mathfrak{g}$. We prove that $\mathfrak{g}$ admits an irreducible $(\mathfrak{g}, \mathfrak{k})$-module M which has finite $\mathfrak{k}$-multiplicities and which is not a $(\mathfrak{g}, \mathfrak{k}')$-module for any proper inclusion of reductive subalgebras $\mathfrak{k} \subset \mathfrak{k}' \subset \mathfrak{g}$, if and only if $\mathfrak{k}$ contains its centralizer in $\mathfrak{g}$. The main point of the proof is a geometric construction of $(\mathfrak{g}, \mathfrak{k})$-modules which is analogous to cohomological induction. For $\mathfrak{g} = \mathfrak{gl}(n)$ we show that, whenever $\mathfrak{k}$ contains its centralizer, there is an irreducible $(\mathfrak{g}, \mathfrak{k})$-module M of finite type over $\mathfrak{k}$ such that $\mathfrak{k}$ coincides with the subalgebra of all $g \in \mathfrak{g}$ which act locally finitely on M. Finally, for a root subalgebra $\mathfrak{k} \subset \mathfrak{gl}(n)$, we describe all possibilities for the subalgebra $\mathfrak{l} \supset \mathfrak{k}$ of all elements acting locally finitely on some M.

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1. Introduction

Let $\mathfrak{g}$ be a reductive Lie algebra over an algebraically closed field of characteristic zero and $\mathfrak{k} \subset \mathfrak{g}$ be a subalgebra reductive in $\mathfrak{g}$. In his program talk [G], I. Gelfand has introduced the notion of a $(\mathfrak{g}, \mathfrak{k})$-module with finite $\mathfrak{k}$-multiplicities. The present paper focuses on a new notion relevant to Gelfand's program: we call $\mathfrak{k}$ *primal* if $\mathfrak{g}$ admits an irreducible $(\mathfrak{g}, \mathfrak{k})$-module with finite $\mathfrak{k}$-multiplicities which is not a $(\mathfrak{g}, \mathfrak{k}')$-module for any proper inclusion of reductive subalgebras $\mathfrak{k} \subset \mathfrak{k}' \subset \mathfrak{g}$. Our central result is that $\mathfrak{k}$ is primal if and only if $\mathfrak{k}$ contains its centralizer in $\mathfrak{g}$, or equivalently, if and only if $\mathfrak{k}$ is a direct sum of a semisimple subalgebra $\mathfrak{k}'$ in $\mathfrak{g}$ and a Cartan subalgebra of the centralizer $C(\mathfrak{k}')$ in $\mathfrak{g}$. This provides a complete description of all primal subalgebras,

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as the semisimple subalgebras of a reductive Lie algebra have been classified by E. Dynkin, [D].

Here is a brief account of our motivation. It is common wisdom that classifying all irreducible representations of a reductive Lie algebra $\mathfrak{g}$ is not a well-posed problem. In contrast with that, classifying irreducible representations with natural finiteness properties has remained a core problem in representation theory since the work of E. Cartan and H. Weyl. A landmark success has been the celebrated classification of irreducible Harish-Chandra modules (see [V], Ch. 6 and [KV], Ch. 11). The case of $(G_2, \mathfrak{sl}(3))$-modules has been considered by P. Kekäläinen in [Ke], and by G. Savin in [S]. In 1998 O. Mathieu (following up on work of S. Fernando and others) obtained a very important different classification: of irreducible weight modules with finite-dimensional weight spaces, [M].

In [PS] it was noticed that both classifications, of irreducible Harish-Chandra modules and of irreducible weight modules, are particular cases of the problem of classifying irreducible $\mathfrak{g}$-modules which have finite type over their Fernando-Kac subalgebra. The *Fernando-Kac subalgebra* $\mathfrak{g}[M]$ associated to an irreducible $\mathfrak{g}$-module M is by definition the set of all elements in $\mathfrak{g}$ which act locally finitely on M. The fact that $\mathfrak{g}[M]$ is a Lie subalgebra of $\mathfrak{g}$ was discovered independently by S. Fernando, [F], and V. Kac, [K]. Furthermore, M is of *finite type* over a given subalgebra $\mathfrak{l} \subset \mathfrak{g}[M]$ if the multiplicity of an arbitrary fixed irreducible $\mathfrak{l}$-module in any (varying) finite-dimensional $\mathfrak{l}$-submodule of M is bounded. The subalgebra $\mathfrak{l}$ is called a *Fernando-Kac subalgebra of finite type* if $\mathfrak{g}$ admits an irreducible $\mathfrak{g}$-module M with $\mathfrak{g}[M] = \mathfrak{l}$ which is of finite type over $\mathfrak{l}$. The problem of classifying all, not necessarily reductive, Fernando-Kac subalgebras of finite type is of fundamental importance for the structure theory of $\mathfrak{g}$-modules. In this article we classify the reductive parts of Fernando-Kac subalgebras of finite type, as a subalgebra is primal if and only if it is a reductive part of a Fernando-Kac subalgebra of finite type.

A short outline of the paper is as follows. In Section 3 we establish some necessary, (but in general not sufficient) conditions for a subalgebra $\mathfrak{l} \subset \mathfrak{g}$ to be a Fernando-Kac subalgebra of finite type. We show in particular that a Fernando-Kac subalgebra of finite type $\mathfrak{l}$ is algebraic and admits a natural decomposition $\mathfrak{l} = \mathfrak{l}_{red} \ni\!\!\!\!\supset \mathfrak{n}_{\mathfrak{l}}$, where $\mathfrak{l}_{red}$ is a reductive in $\mathfrak{g}$ subalgebra which contains its centralizer, and $\mathfrak{n}_{\mathfrak{l}}$ is a nilpotent ideal in $\mathfrak{l}$. We also characterize completely all solvable Fernando-Kac subalgebras of finite type in $\mathfrak{g}$. In Section 4 we fix an arbitrary algebraic subalgebra $\mathfrak{k}$, reductive in $\mathfrak{g}$, and construct irreducible $(\mathfrak{g}, \mathfrak{k})$-modules M of finite type over $\mathfrak{k}$. The construction of M is a $\mathcal{D}$-module version of cohomological induction: M equals the global sections of a $\mathcal{D}^{\mu}$-module supported on the preimage in G/B of $K \cdot P \subset G/P$ for a suitable parabolic subgroup $P \subset G$. Here G is a connected algebraic group with Lie algebra $\mathfrak{g}$ and K is a connected subgroup with Lie algebra $\mathfrak{k}$. We show then, that if $\mathfrak{k}$ contains its centralizer in $\mathfrak{g}$, $\mathfrak{g}[M]_{red} = \mathfrak{k}$ for some M. Therefore, $\mathfrak{k}$ is primal if and only if it contains its centralizer. Furthermore, as a corollary we obtain that any semisimple subalgebra of $\mathfrak{g}$ is the derived subalgebra of a primal subalgebra, and that any subalgebra which is not a proper subalgebra of a maximal root subalgebra is a Fernando-Kac subalgebra of finite type. In Section 5 we consider in more detail the case $\mathfrak{g} = \mathfrak{gl}(n)$. We prove that here any primal subalgebra $\mathfrak{k}$ is itself a reductive Fernando-Kac subalgebra of finite type, and also give an explicit description of all Fernando-Kac subalgebras of finite type which contain a Cartan subalgebra.

In conclusion, for an arbitrary reductive Lie algebra $\mathfrak{g}$, we give a complete description of all primal subalgebras $\mathfrak{k} \subset \mathfrak{g}$, and for each primal subalgebra $\mathfrak{k}$ we construct certain "series" of irreducible $(\mathfrak{g}, \mathfrak{k})$-modules of finite type over $\mathfrak{k}$. A direct comparison with known results in the case of a symmetric pair $(\mathfrak{g}, \mathfrak{k})$, shows that the $(\mathfrak{g}, \mathfrak{k})$-modules obtained by our construction are only a part of all irreducible $(\mathfrak{g}, \mathfrak{k})$-modules. Consequently the problem of classifying all irreducible $(\mathfrak{g}, \mathfrak{k})$-modules of finite type over an arbitrary primal subalgebra $\mathfrak{k} \subset \mathfrak{g}$ is still open.

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2. General preliminaries

The ground field F is algebraically closed of characteristic zero. If X is a topological space and $\mathcal{F}$ is a sheaf of abelian groups on X, then $\Gamma(\mathcal{F})$ denotes the global sections of $\mathcal{F}$ on X. If $f\colon X \to Y$ is a continuous map of topological spaces, f^{-1} denotes the topological inverse image functor from sheaves on Y to sheaves on X. If X is an algebraic variety, $\mathcal{O}_X$ stands for the structure sheaf of X, and if $f\colon X \to Y$ is a morphism of algebraic varieties, f^* (respectively f_*) denotes the inverse image (resp. direct image) functor of $\mathcal{O}$-modules. A *multiset* is defined as a map from a set Y into $\mathbb{Z}_+ \cup \infty$, where $\mathbb{Z}_+ := \{0, 1, 2, 3, ...\}$, or, more informally, as a set whose elements have finite or infinite multiplicities.

Throughout this paper $\mathfrak{g}$ is a fixed reductive Lie algebra, and G stands for a connected algebraic group with Lie algebra $\mathfrak{g}$. Denote by $C(\mathfrak{l})$ (respectively $N(\mathfrak{l})$) the centralizer (respectively normalizer) of a subalgebra $\mathfrak{l} \subset \mathfrak{g}$. Furthermore, $U(\mathfrak{l})$ stands for the universal enveloping algebra of $\mathfrak{l}$, $Z(\mathfrak{l})$ stands for the center of $\mathfrak{l}$, $\mathfrak{r}_\mathfrak{l}$ stands for the solvable radical of $\mathfrak{l}$, and $\mathfrak{n}_\mathfrak{l}$ stands for the maximal ideal in $\mathfrak{l}$ which acts nilpotently on $\mathfrak{g}$. The sign $\subset\!\!\!\!+$ denotes the semi-direct sum of Lie algebras, and $\mathfrak{l}_{ss}$ is a Levi component of $\mathfrak{l}$. If $\mathfrak{l}$ is reductive, then $\mathfrak{l}_{ss}$ simply equals the derived subalgebra $[\mathfrak{l}, \mathfrak{l}]$. For a Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ which contains a Cartan subalgebra $\mathfrak{h}$, $\rho_\mathfrak{b}$ denotes as usual the half-sum of the roots of $\mathfrak{b}$. In what follows a *root subalgebra* $\mathfrak{l} \subset \mathfrak{g}$ means a subalgebra containing a Cartan subalgebra of $\mathfrak{g}$.

By definition a $\mathfrak{g}$-module M is a $(\mathfrak{g}, \mathfrak{l})$-*module* if $\mathfrak{l} \subset \mathfrak{g}[M]$. M is a *strict* $(\mathfrak{g}, \mathfrak{l})$-module if $\mathfrak{l} = \mathfrak{g}[M]$. We also need the following definition from [PS]: M is an *isotropic* $(\mathfrak{g}, \mathfrak{l})$-module if for each $0 \neq m \in M$ the set of elements $g \in \mathfrak{g}$ acting finitely on m coincides with $\mathfrak{l}$. An irreducible strict $(\mathfrak{g}, \mathfrak{l})$-module is automatically isotropic.

The following statement is a reformulation of Lemma 1 in [PS].

Lemma 2.1. *Let $\mathfrak{h}$ be a Cartan subalgebra in $\mathfrak{g}$, $\mathfrak{l} \supset \mathfrak{h}$ be a solvable subalgebra and M be an isotropic strict $(\mathfrak{g},\mathfrak{l})$-module of finite type over $\mathfrak{h}$. Then there exists a parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}$ with $\mathfrak{g} = \mathfrak{l} + \mathfrak{q}$, $\mathfrak{q} \cap \mathfrak{l} = \mathfrak{h}$, and such that the semisimple part of $\mathfrak{q}$ is a direct sum of simple Lie algebras of types A and C.*

3. Necessary conditions for $\mathfrak{l}$ to be of a Fernando-Kac subalgebra of finite type

Theorem 3.1. *Let $\mathfrak{l} \subset \mathfrak{g}$ be a Fernando-Kac subalgebra of finite type.*

1. *$N(\mathfrak{l}) = \mathfrak{l}$; hence $\mathfrak{l}$ is an algebraic subalgebra of $\mathfrak{g}$.*
2. *There is a decomposition $\mathfrak{l} = \mathfrak{n}_{\mathfrak{l}} \in\!\!\!\!+ \mathfrak{l}_{red}$, unique up to an inner automorphism of $\mathfrak{l}$, where $\mathfrak{l}_{red}$ is a (maximal) subalgebra of $\mathfrak{l}$ reductive in $\mathfrak{g}$.*
3. *Any irreducible $(\mathfrak{g},\mathfrak{l})$-module M of finite type over $\mathfrak{l}$ has finite type over $\mathfrak{l}_{red}$ and $\mathfrak{l}_{red}$ acts semi-simply on M.*
4. *$C(\mathfrak{l}_{red}) = Z(\mathfrak{l}_{red})$, and $Z(\mathfrak{l}_{red})$ is a Cartan subalgebra of $C(\mathfrak{l}_{ss})$.*
5. *$\mathfrak{l} \cap C(\mathfrak{l}_{ss})$ is a solvable Fernando-Kac subalgebra of finite type of $C(\mathfrak{l}_{ss})$.*

Proof. Let M be an irreducible strict $(\mathfrak{g},\mathfrak{l})$-module and $M_0 \subset M$ be an irreducible finite-dimensional $\mathfrak{l}$-submodule. To prove 1, assume that $N(\mathfrak{l}) \neq \mathfrak{l}$. Then one can choose $x \in N(\mathfrak{l}) \setminus \mathfrak{l}$ such that $[x,\mathfrak{l}_{ss}] = 0$ for a fixed Levi decomposition $\mathfrak{l} = \mathfrak{l}_{ss} \ni\!\!\!\!+ \mathfrak{r}_{\mathfrak{l}}$. Since $x \notin \mathfrak{l}$, x acts freely on any non-zero vector in M. Set

$$M_n := M_0 + x \cdot M_0 + x^2 \cdot M_0 + ... + x^n \cdot M_0$$

A simple calculation, using $[x,\mathfrak{l}_{ss}] = 0$ and $[x,\mathfrak{r}_{\mathfrak{l}}] \subset \mathfrak{r}_{\mathfrak{l}}$, shows that M_n is $\mathfrak{l}$-invariant and M_n/M_{n-1} is isomorphic to M_0 as an $\mathfrak{l}$-module. Therefore the multiplicity of M_0 in M is infinite. Contradiction. To show the algebraicity of $\mathfrak{l}$, consider the normalizer J of $\mathfrak{l}$ in G. The Lie subalgebra of $\mathfrak{g}$ corresponding to J is $N(\mathfrak{l})$. Hence, $N(\mathfrak{l}) = \mathfrak{l}$ is an algebraic subalgebra of $\mathfrak{g}$.

Claim 2 follows from 1 via some well known statements. For instance, Corollary 1 in [B], §5 implies that a self-normalizing subalgebra $\mathfrak{l}$ is splittable, i.e. for $y \in \mathfrak{l}$ the semisimple and nilpotent parts of y are contained in $\mathfrak{l}$. Proposition 7 in [B] , §5 claims that any splittable subalgebra has a decomposition as required in 2.

To prove 3 note first that M is a quotient of the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} M_0$. As the adjoint action of $\mathfrak{l}_{red}$ on $U(\mathfrak{g})$ is semisimple, $\mathfrak{l}_{red}$ acts semisimply on $U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} M_0$, and therefore also on M. Now note that there exists $\nu \in \mathfrak{n}_{\mathfrak{l}}^*$ such that

$$x \cdot m = \nu(x)\, m$$

for any $m \in M_0$ and $x \in \mathfrak{n}_{\mathfrak{l}}$. Since the adjoint action of $\mathfrak{n}_{\mathfrak{l}}$ on $U(\mathfrak{g})$ is locally nilpotent, we obtain that, for any $x \in \mathfrak{n}_{\mathfrak{l}}$, $x - \nu(x)$ acts locally nilpotently on $U(\mathfrak{g}) \otimes_{U(\mathfrak{l})} M_0$, and hence on M. Therefore $\mathfrak{n}_{\mathfrak{l}}$ acts via the character ν on any irreducible $\mathfrak{l}$-subquotient of M, and consequently two irreducible $\mathfrak{l}$-subquotients of M are isomorphic if and only if they are isomorphic as $\mathfrak{l}_{red}$-modules. This implies that M has also finite type over $\mathfrak{l}_{red}$, and 3 is proved.

4. By 2 any irreducible strict $(\mathfrak{g},\mathfrak{l})$-module M has an $\mathfrak{l}_{red}$-module decomposition

$$M = \oplus_i M_i'$$

for finite-dimensional isotypic components M_i'. Clearly each M_i' is $C(\mathfrak{l}_{red})$-invariant, and, as it is finite-dimensional, $C(\mathfrak{l}_{red}) \subset \mathfrak{g}[M] = \mathfrak{l}$. Note that $C(\mathfrak{l}_{red}) \cap \mathfrak{l}$ is solvable. Consequently, since $C(\mathfrak{l}_{red}) = C(\mathfrak{l}_{ss}) \cap C(Z(\mathfrak{l}_{red})) \subset \mathfrak{l}$, the centralizer of $Z(\mathfrak{l}_{red})$ in $C(\mathfrak{l}_{ss})$ is solvable. On the other hand, as $C(\mathfrak{l}_{ss})$ is reductive and $Z(\mathfrak{l}_{red})$ is reductive in $C(\mathfrak{l}_{ss})$, the centralizer of $Z(\mathfrak{l}_{red})$ in $C(\mathfrak{l}_{ss})$ is reductive. Therefore $Z(\mathfrak{l}_{red})$ coincides with its centralizer in $C(\mathfrak{l}_{ss})$. This implies that $C(\mathfrak{l}_{red}) = C(\mathfrak{l}_{ss}) \cap C(Z(\mathfrak{l}_{red})) = Z(\mathfrak{l}_{red})$, and that $Z(\mathfrak{l}_{red})$ is a Cartan subalgebra of $C(\mathfrak{l}_{ss})$.

To show 5 decompose M as

$$M = \oplus_i (M_i \otimes V_i),$$

where M_i are pairwise non-isomorphic irreducible $\mathfrak{l}_{ss}$-modules, and V_i are $C(\mathfrak{l}_{ss})$-modules. Then each V_i is a strict isotropic $(C(\mathfrak{l}_{ss}), \mathfrak{l} \cap C(\mathfrak{l}_{ss}))$-module of finite type over $\mathfrak{l} \cap C(\mathfrak{l}_{ss})$. Furthermore $\mathfrak{l} \cap C(\mathfrak{l}_{ss})$ is solvable, and 5 follows from Lemma 2.1. □

The conditions in Theorem 3.1 are not sufficient for $\mathfrak{l}$ to be a Fernando-Kac subalgebra of finite type: see the Example in subsection 5.3. In general, the problem of a complete characterization of a Fernando-Kac subalgebra of finite type is open. However, for a solvable $\mathfrak{l}$ we have the answer.

Proposition 3.2. *A solvable subalgebra $\mathfrak{l} \subset \mathfrak{g}$ is a Fernando-Kac subalgebra of finite type if and only if $\mathfrak{l} = \mathfrak{h} \supsetplus \mathfrak{n}_{\mathfrak{l}}$, where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{n}_{\mathfrak{l}}$ is the nilradical of a parabolic subalgebra of $\mathfrak{g}$ whose simple components are all of types A and C.*

Proof. Here $\mathfrak{l}_{ss} = 0$, $C(\mathfrak{l}_{ss}) = \mathfrak{g}$, and Theorem 3.1 4 implies that $\mathfrak{h} := \mathfrak{l}_{red}$ is a Cartan subalgebra of $\mathfrak{g}$. The claim of the Corollary follows now immediately from [PS], Sect. 3 where a criterion for $\mathfrak{l}$ to be a Fernando-Kac subalgebra of finite type is established under the assumption that $\mathfrak{l} \supset \mathfrak{h}$. □

Note that Theorem 3.1 3 and Proposition 3.2, applied to a solvable $\mathfrak{l}$, yield that any strict irreducible $(\mathfrak{g},\mathfrak{l})$-module of finite type over $\mathfrak{l}$ is a weight module with finite-dimensional weight spaces. Such modules are classified by O. Mathieu in [M]. More precisely, any irreducible weight module M with finite-dimensional weight spaces is the unique irreducible quotient of an induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M^{\mathfrak{n}_{\mathfrak{p}}}$, where $\mathfrak{p}$ is a parabolic subalgebra and $M^{\mathfrak{n}_{\mathfrak{p}}}$ is the $\mathfrak{p}$-submodule of $\mathfrak{n}_{\mathfrak{p}}$-invariants in M. The Fernando-Kac subalgebra $\mathfrak{g}[M]$ of M equals $(\mathfrak{g}[M] \cap \mathfrak{p}_{red}) \supsetplus \mathfrak{n}_{\mathfrak{p}}$, and it is solvable if and only if $\mathfrak{g}[M] \cap \mathfrak{p}_{red}$ is a Cartan subalgebra of $\mathfrak{g}$ (in general $\mathfrak{g}[M] \cap \mathfrak{p}_{red}$ is the sum of a Cartan subalgebra and an ideal in $\mathfrak{p}_{ss}$).

4. A construction of irreducible $(\mathfrak{g}, \mathfrak{k})$-modules of finite type

4.1. A geometric set up. Let $\mathfrak{k} \subset \mathfrak{g}$ be an algebraic subalgebra, reductive in $\mathfrak{g}$ and such that $\mathfrak{k}_{ss}$ is proper in $\mathfrak{g}_{ss}$. Denote by K the connected subgroup of G with Lie algebra $\mathfrak{k}$, and let K_{ss} be the connected subgroup corresponding to $\mathfrak{k}_{ss}$. By H_K we denote a fixed Cartan subgroup of K, with Lie algebra $\mathfrak{h}_\mathfrak{k}$. Fix an element $h \in \mathfrak{h}_\mathfrak{k}$ such that $C(Fh) \subset C(\mathfrak{h}_\mathfrak{k} \cap \mathfrak{k}_{ss})$ and for which the operator $\mathrm{ad}_h : \mathfrak{g} \to \mathfrak{g}$ has rational eigenvalues. The element h defines the parabolic subalgebra

$$\mathfrak{p} := \oplus_{\gamma \geq 0} \mathfrak{g}_h^\gamma, \tag{4.1}$$

where $\mathfrak{g}_h^\gamma$ is the γ-eigenspace of $\mathrm{ad}_h : \mathfrak{g} \to \mathfrak{g}$. Clearly $\mathfrak{b}_\mathfrak{k} := \mathfrak{p} \cap \mathfrak{k}$ is a Borel subalgebra of $\mathfrak{k}$ containing $\mathfrak{h}_\mathfrak{k}$. Notice also that $\mathfrak{p}_{red} := \mathfrak{g}_\mathfrak{h}^0$ is a maximal reductive in $\mathfrak{g}$ subalgebra of $\mathfrak{p}$. Let P be the subgroup of G corresponding to $\mathfrak{p}$, and $P_{ss} \subset P$ be the connected subgroup corresponding to a fixed Levi component $\mathfrak{p}_{ss}$ of $\mathfrak{p}$. Furthermore, let $B \subset P$ be a Borel subgroup of G such that $B_K = B \cap K$ has Lie algebra $\mathfrak{b}_\mathfrak{k}$. Set $X := G/B$, $Y := G/P$ and let $\pi\colon X \to Y$ be the natural projection. Denote by S the K-orbit of the closed point in Y corresponding to P, and put $V := \pi^{-1}(S)$.

Lemma 4.1. *$V \cong S \times T$, where $T := P/B$.*

Proof. V is a relative flag variety over S with fiber $T = P/B \cong P_{ss}/(P_{ss} \cap B)$. Moreover, $V = K_{ss} \times_{K_{ss} \cap P} T$. To be able to conclude that the bundle $V \to S$ is trivial it suffices to check that the action of $K_{ss} \cap P$ on T is trivial. The solvable radical of P lies in B, hence the action of $K_{ss} \cap P$ on T factors through the action of $K_{ss} \cap P_{ss}$ on T. The fact that $K_{ss} \cap P_{ss}$ acts trivially on T follows from the inclusion

$$K_{ss} \cap P_{ss} \subset Z(P_{ss}), \tag{4.2}$$

where $Z(G')$ stands for the center of an algebraic group G'. In the rest of the proof, we establish (4.2).

We show first that $\mathfrak{p}_{ss} \cap \mathfrak{k}_{ss} = 0$. By the definition of $\mathfrak{p}$,

$$\mathfrak{h}_\mathfrak{k} \cap \mathfrak{k}_{ss} \subset \mathfrak{p}_{red} \subset C(\mathfrak{h}_\mathfrak{k} \cap \mathfrak{k}_{ss}).$$

Therefore

$$\mathfrak{h}_\mathfrak{k} \cap \mathfrak{k}_{ss} \subset Z(\mathfrak{p}_{red}) \tag{4.3}$$

and

$$\mathfrak{p}_{ss} \cap \mathfrak{k}_{ss} \subset C(\mathfrak{h}_\mathfrak{k} \cap \mathfrak{k}_{ss}) \cap \mathfrak{k}_{ss}. \tag{4.4}$$

Furthermore, $\mathfrak{p}_{ss} \cap \mathfrak{h}_\mathfrak{k} \cap \mathfrak{k}_{ss} = 0$ as

$$\mathfrak{p}_{red} = \mathfrak{p}_{ss} \oplus Z(\mathfrak{p}_{red}). \tag{4.5}$$

The observation that $C(\mathfrak{h}_\mathfrak{k} \cap \mathfrak{k}_{ss}) \cap \mathfrak{k}_{ss}$ equals the centralizer in $\mathfrak{k}_{ss}$ of $\mathfrak{h}_\mathfrak{k} \cap \mathfrak{k}_{ss}$, together with (4.4) yields

$$\mathfrak{p}_{ss} \cap \mathfrak{k}_{ss} \subset C(\mathfrak{h}_\mathfrak{k} \cap \mathfrak{k}_{ss}) \cap \mathfrak{k}_{ss} = \mathfrak{h}_\mathfrak{k} \cap \mathfrak{k}_{ss} \subset Z(\mathfrak{p}_{red}).$$

This implies $\mathfrak{p}_{ss} \cap (\mathfrak{p}_{ss} \cap \mathfrak{k}_{ss}) = 0$, or equivalently $\mathfrak{p}_{ss} \cap \mathfrak{k}_{ss} = 0$.

On the group level, (4.5) implies $P_{ss} \cap Z(P_{red}) \subset Z(P_{ss})$. Similarly, (4.3) yields $H_K \cap K_{ss} \subset Z(P_{red})$. Hence $P_{ss} \cap H_K \cap K_{ss} \subseteq Z(P_{ss})$. By (4.4),

$$P_{ss} \cap K_{ss} \subset C(H_K \cap K_{ss}) \cap K_{ss}, \tag{4.6}$$

where $C(G')$ now stands for the centralizer in G of a closed subgroup $G' \subset G$. Since $C(H_K \cap K_{ss}) \cap K_{ss}$ is the centralizer of $H_K \cap K_{ss}$ in K_{ss}, the fact that a Cartan subgroup of K_{ss} is self-centralizing yields via (4.6)

$$P_{ss} \cap K_{ss} \subset C(H_K \cap K_{ss}) \cap K_{ss} = H_K \cap K_{ss} \subset Z(P_{red}).$$

Therefore

$$P_{ss} \cap (P_{ss} \cap K_{ss}) \subset P_{ss} \cap Z(P_{red}) = Z(P_{ss}),$$

or equivalently $P_{ss} \cap K_{ss} \subset Z(P_{ss})$. This completes the proof. □

4.2. $\mathcal{D}$-module preliminaries. For any $\mu \in \mathfrak{h}^*$ let $\mathcal{D}^\mu$ denote the twisted sheaf of differential operators on X defined in [BB]. A *$\mathcal{D}^\mu$-module* is by convention a sheaf $\mathcal{F}$ of $\mathcal{D}^\mu$-modules on X which is quasicoherent as a sheaf of $\mathcal{O}_X$-modules. The support of $\mathcal{F}$ is the closure of the subvariety of all closed points for which the sheaf-theoretic fiber of $\mathcal{F}$ is non-zero. A weight $\mu \in \mathfrak{h}^*$ defines the character θ^μ of the center of $U(\mathfrak{g})$ via the Harish-Chandra map (see [B], §6).

When the ground field F is not $\mathbf{C}$, by a *dominant* weight we mean an element $\mu \in \mathfrak{h}^*$ whose value on all B-positive co-roots is a nonnegative rational number. For $F = \mathbb{C}$ it suffices that the value has nonnegative real part. The Beilinson-Bernstein localization theorem claims that, for a regular dominant μ, the functor of global sections

$$\Gamma \colon \mathcal{D}^\mu\text{-mod} \to U(\mathfrak{g}) / (\ker \theta^\mu)\text{-mod}$$

is an equivalence between the category of $\mathcal{D}^\mu$-modules and the category of $U(\mathfrak{g}) / (\ker \theta^\mu)$-modules, where $(\ker \theta^\mu)$ stands for the two-sided ideal in $U(\mathfrak{g})$ generated by the kernel of the central character θ^μ. The inverse equivalence (usually referred to as localization) is given by the functor

$$R \mapsto \mathcal{D}^\mu \otimes_{\Gamma(\mathcal{D}^\mu)} R,$$

where the $U(\mathfrak{g}) / (\ker \theta^\mu)$-module R is endowed with a $\Gamma(\mathcal{D}^\mu)$-module structure via the natural isomorphism $U(\mathfrak{g}) / (\ker \theta^\mu) \to \Gamma(\mathcal{D}^\mu)$, see [BB].

Let $i : W \to X$ define a non-singular locally closed subvariety of X; we denote by $\mathcal{D}^\mu_W$ the sheaf of right $i^*\mathcal{D}^\mu$-module endomorphisms of the inverse image sheaf $i^*\mathcal{D}^\mu$ which are left $\mathcal{O}_W$-module differential operators. Furthermore, the inverse image functor i^* of $\mathcal{O}$-modules yields a functor

$$i^\bigstar \colon \mathcal{D}^\mu\text{-mod} \to \mathcal{D}^\mu_W\text{-mod}.$$

If W is a closed subvariety we will also consider the direct image functor

$$i_\bigstar : \mathcal{D}^\mu_W\text{-mod} \to \mathcal{D}^\mu\text{-mod},$$

$$\mathcal{F} \mapsto \mathcal{D}^\mu_{\leftarrow W} \otimes_{\mathcal{D}^\mu_W} \mathcal{F},$$

where $\mathcal{D}^{\mu}_{\leftarrow W} := i^{\bigstar}\left(\mathcal{D}^{\mu} \otimes_{\mathcal{O}_X} \Omega^*_X\right) \otimes_{\mathcal{O}_W} \Omega_W$ and Ω stands for volume forms. Kashiwara's theorem claims that $i_{\bigstar}$ is an equivalence between the category of $\mathcal{D}^{\mu}_W$-modules and the category of $\mathcal{D}^{\mu}$-modules supported in W. It will also be important for us that the sheaf $i^{-1}i_{\bigstar}\mathcal{F}$ has a natural $\mathcal{O}_W$-module filtration with successive quotients

$$\Lambda^{\max}\left(\mathcal{N}_{W|X}\right) \otimes_{\mathcal{O}_W} S^i\left(\mathcal{N}_{W|X}\right) \otimes_{\mathcal{O}_W} \mathcal{F}, \tag{4.7}$$

where $i \in \mathbb{Z}_+$, $\mathcal{N}_{W|X}$ denotes the normal bundle of W in X, S^i stands for i-th symmetric power and $\Lambda^{\max}$ stands for maximal exterior power.

In [PS] the following lemma is proven.

Lemma 4.2. *If Q is the support of a $\mathcal{D}^{\mu}$-module $\mathcal{F}$, then $\mathfrak{g}\left[\Gamma\left(\mathcal{F}\right)\right] \subset \mathrm{Stab}_{\mathfrak{g}} Q$, where $\mathrm{Stab}_{\mathfrak{g}} Q$ is the Lie algebra of the subgroup of G which stabilizes Q.*

4.3. The construction. Let L be an irreducible $(\mathfrak{p}, \mathfrak{h}_{\mathfrak{k}})$-module of finite type over $\mathfrak{h}_{\mathfrak{k}}$ with trivial action of $\mathfrak{n}_{\mathfrak{p}} + (Z(\mathfrak{p}_{red}) \cap \mathfrak{k}_{ss})$ and with $\mathfrak{p}_{red}$-central character $\theta^{\nu}_{\mathfrak{p}_{red}}$ for some $P_{ss} \cap B$-dominant weight $\nu \in \mathfrak{h}^*$. Consider $T = P/B$ as a (non-singular) closed subvariety of $X = G/B$. Set $\mathcal{L} := \mathcal{D}^{\eta}_T \otimes_{\Gamma\left(\mathcal{D}^{\eta}_T\right)} L$, where $\eta = \nu + \rho_{\mathfrak{b}\cap\mathfrak{p}_{red}} - \rho_{\mathfrak{b}}$. Let, furthermore, $\mathcal{O}_S\left(\zeta\right)$ be the invertible K_{ss}-sheaf of local sections on S of the line bundle $K \times_{K\cap P}\left(F_{w(\zeta)}\right)$, where w is the longest element in the Weyl group of $\mathfrak{k}_{ss}$, ζ is a $\mathfrak{k}_{ss}$-integral weight in $\mathfrak{h}^*$ and F_{ξ} stands for the one-dimensional $\mathfrak{h}$-module of weight ξ. Then $\mathcal{F} := \mathcal{O}_S\left(\zeta\right) \boxtimes \mathcal{L}$ is a $\mathcal{D}^{\mu}_V$-module for $\mu = \zeta + \eta$, and $\mathcal{M} = i_{\bigstar}\mathcal{F}$ is a $\mathcal{D}^{\mu}$-module. Finally, set $M = \Gamma\left(\mathcal{M}\right)$.

Theorem 4.3. *Assume that ζ is dominant and μ is regular and dominant. Then*

(1) *M is an infinite-dimensional irreducible $\mathfrak{g}$-module;*
(2) *$\mathfrak{g}\left[M\right] = \mathfrak{k}_{ss} \ni \mathfrak{m}_L$, where $\mathfrak{m}_L$ is the maximal $\mathfrak{k}_{ss}$-invariant subspace in $\mathfrak{p}\left[L\right]$; moreover $\mathfrak{g}\left[M\right]$ is the unique maximal subalgebra in $\mathfrak{p}\left[L\right] + \mathfrak{k}$ which contains $\mathfrak{k}$;*
(3) *M is a $(\mathfrak{g}, \mathfrak{k})$-module of finite type over $\mathfrak{k}$.*

Proof. $\mathcal{D}^{\eta}_T$ is a sheaf of twisted differential operators on the flag variety T. By the Beilinson-Bernstein theorem applied to T, $\mathcal{L}$ is an irreducible $\mathcal{D}^{\eta}_T$-module. Furthermore, $\mathcal{F}$ is an irreducible $\mathcal{D}^{\mu}_V$-module. Since V is a non-singular closed subvariety, $\mathcal{M}$ is an irreducible $\mathcal{D}^{\mu}$-module by Kashiwara's theorem. Finally, by the Beilinson-Bernstein theorem applied to X, $M = \Gamma\left(\mathcal{M}\right)$ is an irreducible $\mathfrak{g}$-module. 1 is proven.

To prove 2 consider the subalgebra $\mathrm{Stab}_{\mathfrak{g}} Q$, where Q is the support of the $\mathcal{D}^{\mu}$-module $\mathcal{M}$. Note that $Q \subset V$ and that $V = \pi^{-1}(\pi(Q))$. Hence, $\mathrm{Stab}_{\mathfrak{g}} Q$ is a subalgebra of $\mathfrak{st} := \mathrm{Stab}_{\mathfrak{g}} V$. One can check easily that

$$\mathfrak{st} = \mathfrak{k}_{ss} \ni \mathfrak{m}, \tag{4.8}$$

where $\mathfrak{m}$ is the maximal $\mathfrak{k}_{ss}$-invariant subspace in $\mathfrak{p}$. Thus $\mathfrak{st}$ is a maximal subalgebra in $\mathfrak{k} + \mathfrak{p}$ containing $\mathfrak{k}$. By Lemma 4.2, $\mathfrak{g}\left[M\right] \subset \mathrm{Stab}_{\mathfrak{g}} Q \subset \mathfrak{st}$ and therefore $\mathfrak{g}\left[M\right] = \mathfrak{st}\left[M\right]$.

Recall now that by (4.7) $i^{-1}\mathcal{M} = i^{-1}i_{\bigstar}\mathcal{F}$ (considered as an $\mathfrak{st}$-sheaf) has a natural $\mathfrak{st}$-sheaf filtration with successive quotients

$$\Lambda^{\max}\left(\mathcal{N}_{V|X}\right) \otimes_{\mathcal{O}_V} S^i\left(\mathcal{N}_{V|X}\right) \otimes_{\mathcal{O}_V} \mathcal{F}.$$

In particular, $\mathcal{M}_0 := \Lambda^{\max}\left(\mathcal{N}_{V|X}\right) \otimes_{\mathcal{O}_V} \mathcal{F}$ is a subsheaf of $i^{-1}\mathcal{M}$. As $\mathcal{N}_{V|X} \cong \mathcal{N}_{S|Y} \boxtimes \mathcal{O}_Z$, $\Lambda^{\max}\left(\mathcal{N}_{V|X}\right) \cong \mathcal{O}_S(\tau) \boxtimes \mathcal{O}_Z$, where $\tau = -w\left(\sum_{\alpha\in\Delta(\mathfrak{n}_{\mathfrak{p}})}\alpha\right) - 2\rho_{\mathfrak{b}\cap\mathfrak{k}_{ss}}$. Therefore $\mathcal{M}_0 \cong \mathcal{O}_S(\tau+\zeta) \boxtimes \mathcal{L}$ and

$$M_0 := \Gamma(\mathcal{M}_0) \cong \Gamma(\pi_*\mathcal{M}_0) \cong \Gamma(\mathcal{O}_S(\tau+\zeta)) \otimes L.$$

Both weights τ and ζ are dominant. Hence $\tau+\zeta$ is $\mathfrak{k}_{ss}$-dominant, $M_0 \neq 0$, and by the irreducibility of M,

$$\mathfrak{g}[M] = \mathfrak{st}[M] = \mathfrak{st}[M_0]. \tag{4.9}$$

To calculate $\mathfrak{st}[M_0]$ we use that $\Gamma(\mathcal{M}_0) \cong \Gamma(\pi_*\mathcal{M}_0)$. Observe that $\pi_*\mathcal{M}_0$ is the sheaf of sections of the induced vector bundle $K_{ss} \times_{K_{ss}\cap P} \left(F_{w(\zeta+\tau)} \otimes L\right)$. The latter is a K_{ss}-sheaf, hence $\mathfrak{k}_{ss} \subset \mathfrak{st}[M_0]$. By (4.8) and (4.9), $\mathfrak{g}[M] = \mathfrak{k}_{ss} \ni \mathfrak{m}_L$, where $\mathfrak{m}_L = \mathfrak{g}[M] \cap \mathfrak{m}$. To calculate $\mathfrak{m}_L$, let's write down the action of $\mathfrak{m}$ on $\Gamma(\pi_*\mathcal{M}_0)$. An element of $\Gamma(\pi_*\mathcal{M}_0)$ is a function $\phi : K_{ss} \to F_{w(\zeta+\tau)} \otimes L$ satisfying the condition $\phi(ab) = b^{-1}\phi(a)$ for all $a \in K_{ss}, b \in K_{ss} \cap P$. For $x \in \mathfrak{m}$ and $a \in K_{ss}$ we have

$$(L_x\phi)(a) = \mathrm{Ad}_a^{-1}(x)(\phi(a)), \tag{4.10}$$

where $L_x\phi$ stands for the action of x on ϕ. This formula immediately implies that

$$\mathfrak{m}_L \subset \left\{x \in \mathfrak{m} \mid \mathrm{Ad}_{K_{ss}}(x) \subset \mathfrak{m}\left[F_{w(\zeta+\tau)} \otimes L\right] = \mathfrak{m}[L]\right\}.$$

To see that $\mathfrak{m}_L$ is equal to the right hand side, let U be a unipotent subgroup of K_{ss} complementary to $K_{ss} \cap P$. U acts simply transitively on an open dense subset of S. Consider a U-invariant function $f : K_{ss} \to L$. For any $a \in U$ we have $f(a) = f(1)$. Let x be in $\mathfrak{m}[L]$ and assume x is $\mathrm{Ad}\,U$-invariant. Then by (4.10), x acts locally finitely on f, and, by the irreducibility of M, x acts locally finitely on M. Finally, any y obtained from x by the action of K_{ss} also acts locally finitely on M. Hence

$$\mathfrak{m}_L = \left\{x \in \mathfrak{m} \mid \mathrm{Ad}_{K_{ss}}(x) \subset \mathfrak{m}\left[F_{w(\zeta+\tau)} \otimes L\right] = \mathfrak{m}[L]\right\}. \tag{4.11}$$

In other words, $\mathfrak{m}_L$ is the maximal $\mathfrak{k}_{ss}$-invariant subspace in $\mathfrak{m}[L]$, or equivalently in $\mathfrak{p}[L]$. Consequently $\mathfrak{k}_{ss} \ni \mathfrak{m}_L$ is the maximal subalgebra in $\mathfrak{k} + \mathfrak{p}[L]$ containing $\mathfrak{k}$, and 2 is proven.

It remains to prove 3. Let $j : S \to Y$ be the natural embedding. Observe that the isomorphism $\mathcal{N}_{V|X} \cong \mathcal{N}_{S|Y} \boxtimes \mathcal{O}_Z$ yields an isomorphism of $\mathfrak{k}$-sheaves

$$j^{-1}j_{\bigstar}\mathcal{O}_S(\zeta) \boxtimes \mathcal{L} \cong i^{-1}i_{\bigstar}(\mathcal{O}_S(\zeta) \boxtimes \mathcal{L}) \cong i^{-1}\mathcal{M}.$$

Therefore we have an isomorphism of $\mathfrak{k}$-modules

$$\Gamma(\mathcal{M}) \cong \Gamma(\pi_*\mathcal{M}) \cong \Gamma(j_{\bigstar}\mathcal{O}_S(\zeta)) \otimes L,$$

where the action of $\mathfrak{k}_{ss}$ on L is trivial and the action of $Z(\mathfrak{k})$ is induced by the embedding $Z(\mathfrak{k}) \subset \mathfrak{p}_{red}$. By (4.7) $j^{-1}j_\star \mathcal{O}_S(\zeta)$ has a filtration by $\mathfrak{k}$-sheaves with successive quotients

$$S^i\left(\mathcal{N}_{S|Y}\right) \otimes_{\mathcal{O}_S} \mathcal{O}_S(\zeta+\tau).$$

Consequently, M has a $\mathfrak{k}$-module filtration whose associated graded $\mathfrak{k}$-module is a submodule of

$$\Gamma\left(S^{\cdot}\left(\mathcal{N}_{S|Y}\right) \otimes_{\mathcal{O}_S} \mathcal{O}_S(\zeta+\tau)\right) \otimes L.$$

The sheaf $S^{\cdot}\left(\mathcal{N}_{S|Y}\right) \otimes_{\mathcal{O}_S} \mathcal{O}_S(\zeta+\tau)$ is locally free on S, and has a filtration with invertible successive quotients $\mathcal{O}_S(\kappa)$, where κ runs over the multiset Θ of weights in $\mathfrak{h}_{\mathfrak{k}}^*$

$$\Theta = \{\zeta+\tau+\sum_{n_\alpha\in\mathbb{N}} n_\alpha\alpha \ \mid n_\alpha \in \mathbb{Z}_+\}.$$

Here we take the summation over all weights α of the $\mathfrak{h}_{\mathfrak{k}}$-module $\mathfrak{n}_{\mathfrak{p}}/\left(\mathfrak{n}_{\mathfrak{p}}\cap\mathfrak{k}\right)$. Thus the multiplicity of the irreducible $\mathfrak{k}$-module with the highest weight κ in M is majorized by the multiplicity of κ in $\Theta+\Theta_L$, where Θ_L is the multiset of $\mathfrak{h}_{\mathfrak{k}}$-weights of L. Our goal is to show that the multiset $\Theta+\Theta_L$ has finite multiplicities. For any multiset $C \subset \mathfrak{h}_{\mathfrak{k}}^*$ and $t \in F$, put $C^t := \{\kappa \in C \mid \kappa(h) = t\}$. Then $\Theta_L = \Theta_L^{t_0}$ for some $t_0 \in F$, and $(\Theta+\Theta_L)^t = \Theta^{t-t_0}+\Theta_L$. As L has finite type over $\mathfrak{h}_{\mathfrak{k}}$, Θ_L has finite multiplicities. Furthermore, Θ^{t-t_0} is a finite multiset as $\alpha(h)$ are all positive. Therefore $(\Theta+\Theta_L)^t$ has finite multiplicities, and thus $\Theta+\Theta_L$ also has finite multiplicities. Theorem 4.3 is proven. □

The construction in Theorem 4.3 does not provide all irreducible $(\mathfrak{g},\mathfrak{k})$-modules of finite type over $\mathfrak{k}$. Consider for instance the case when $\mathfrak{k}$ is symmetric, i.e. $\mathfrak{k}$ is stable under an involution of $\mathfrak{g}$. Here irreducible $(\mathfrak{g},\mathfrak{k})$-modules of finite type over $\mathfrak{k}$ are nothing but irreducible Harish-Chandra modules. The Beilinson-Bernstein classification of irreducible Harish-Chandra modules implies that the supports of their corresponding localizations (the latter are $\mathcal{D}^\mu$-modules on $X = G/B$) run over the closures of all K-orbits in X. In particular, there are infinite-dimensional irreducible Harish-Chandra modules whose localizations are supported on the closure X of the open orbit of K in G/B. These latter modules do not appear among the modules constructed in Theorem 4.3, as all $\mathcal{D}^\mu$-modules $\mathcal{M}$ considered above are supported on a closed proper subvariety of X.

4.4. Description of primal subalgebras.

Theorem 4.4. *Let $\mathfrak{k}$ be a reductive in $\mathfrak{g}$ subalgebra with $C(\mathfrak{k}) = Z(\mathfrak{k})$. Then $\mathfrak{k}$ is primal, i.e. there exists a Fernando-Kac subalgebra of finite type $\mathfrak{l} \subset \mathfrak{g}$ such that $\mathfrak{l}_{red} = \mathfrak{k}$. In addition, $\mathfrak{l}$ can be chosen so that $\mathfrak{n}_{\mathfrak{l}}$ is the nilradical of a Borel subalgebra of $C(\mathfrak{k}_{ss})$.*

Proof. The assumption $C(\mathfrak{k}) = Z(\mathfrak{k})$ implies that $Z(\mathfrak{k})$ is a Cartan subalgebra of $C(\mathfrak{k}_{ss})$. Let h' be a semisimple element in $\mathfrak{h}_{\mathfrak{k}} \cap \mathfrak{k}_{ss}$ such that $C(Fh') = C(\mathfrak{h}_{\mathfrak{k}} \cap \mathfrak{k}_{ss})$

and $\mathrm{ad}_{h'} : \mathfrak{g} \to \mathfrak{g}$ has rational eigenvalues γ'_i. Let furthermore $h'' \in \mathfrak{h}_{\mathfrak{k}} \cap Z(\mathfrak{k})$ be a regular element in $C(\mathfrak{k}_{ss})$ for which $\mathrm{ad}_{h''} : \mathfrak{g} \to \mathfrak{g}$ has rational eigenvalues γ''_j, and such that

$$|\gamma''_j| < \min_{\gamma'_i \neq 0} |\gamma'_i| \tag{4.12}$$

for all j. Denote by $\mathfrak{p}$ the parabolic subalgebra of $\mathfrak{g}$ defined by the element $h := h' + h''$ and let L be a 1-dimensional $\mathfrak{p}$-module. Theorem 4.3 applies to the triple $(\mathfrak{k}, \mathfrak{p}, L)$ (as $\mathfrak{k}$ is automatically algebraic) and hence yields an irreducible $(\mathfrak{g}, \mathfrak{k})$-module M of finite type over $\mathfrak{k}$. Put $\mathfrak{l} := \mathfrak{g}[M]$. Then $\mathfrak{l} = \mathfrak{k}_{ss} \ni \mathfrak{m}$, where as in the proof of Theorem 4.3 $\mathfrak{m}$ is the maximal $\mathfrak{k}_{ss}$-invariant subspace in $\mathfrak{p}$. Let κ be the $\mathfrak{p} \cap \mathfrak{k}_{ss}$-lowest weight of an irreducible $\mathfrak{k}_{ss}$-submodule in $\mathfrak{m}$. We have $\kappa(h') = \gamma'_i \leq 0$ for some i. On the other hand, as $\mathfrak{m} \subset \mathfrak{p}$, $\kappa(h' + h'') \geq 0$. Condition (4.12) gives $\kappa(h') = 0$, i.e. $\mathfrak{m} = C(\mathfrak{k}_{ss}) \cap \mathfrak{p}$. As h'' is regular in $C(\mathfrak{k}_{ss})$, $\mathfrak{m}$ is a Borel subalgebra in $C(\mathfrak{k}_{ss})$, hence $\mathfrak{m}$ is solvable and $\mathfrak{m} = Z(\mathfrak{k}_{ss}) + [\mathfrak{m}, \mathfrak{m}]$. Therefore $\mathfrak{l}_{red} = \mathfrak{k}$, $\mathfrak{n}_{\mathfrak{l}} = [\mathfrak{m}, \mathfrak{m}]$ and $[\mathfrak{n}_{\mathfrak{l}}, \mathfrak{k}_{ss}] = 0$. □

Corollary 4.5. *A reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k}$ is primal if and only if $C(\mathfrak{k}) = Z(\mathfrak{k})$.*

Proof. The statement follows directly from Theorems 4.4 and 3.1 4. □

Corollary 4.5, together with the remark that $\mathfrak{k}$ is primal if and only if $\mathfrak{k} = \mathfrak{l}_{red}$ for a Fernando-Kac subalgebra of finite type, reduces the problem of classifying all Fernando-Kac subalgebras of finite type to the problem of describing all nilpotent subalgebras $\mathfrak{n}$ such that $\mathfrak{k} \ni \mathfrak{n}$ is a Fernando-Kac subalgebra of finite type, $\mathfrak{k}$ being a fixed primal subalgebra of $\mathfrak{g}$. The latter problem is open. In the next section we solve this problem in the case when $\mathfrak{g} = \mathfrak{gl}(n)$ and $\mathfrak{k}$ is a root subalgebra, and show also that every primal subalgebra of $\mathfrak{gl}(n)$ is a Fernando-Kac subalgebra of finite type.

For simple Lie algebras not of type A, it is not true that any primal subalgebra is itself a Fernando-Kac subalgebra of finite type. Indeed, Proposition 3.2 implies that a Cartan subalgebra of a simple Lie algebra $\mathfrak{g}$ (which is always primal) is a Fernando-Kac subalgebra of finite type if and only if $\mathfrak{g}$ is of type A or C. (This was proved first by S. Fernando in [F].) An important particular case of the above open problem is the problem of characterizing all primal subalgebras which are Fernando-Kac subalgebras of finite type.

We conclude this section with an application to the classical theory of subalgebras of a semisimple Lie algebra. In the fundamental paper [D], an important role is played by subalgebras $\mathfrak{s} \subset \mathfrak{g}$ which are not contained in a proper root subalgebra. We propose the term *stem subalgebra*. By Theorem 7.3 and 7.4 of [D], a stem subalgebra is necessarily semisimple with zero centralizer. Here are some well known examples of stem subalgebras.

(1) A principal $\mathfrak{sl}(2)$-subalgebra is a stem subalgebra.
(2) If $\mathfrak{g} = \mathfrak{sl}(n)$, a proper subalgebra $\mathfrak{s} \in \mathfrak{g}$ is a stem subalgebra if and only if the defining representation of $\mathfrak{g}$ is irreducible over $\mathfrak{s}$.

(3) In general, any semisimple maximal subalgebra is either a stem subalgebra or a root subalgebra. For instance, if $n \geq 3$ is odd, $\mathfrak{o}(n) \oplus \mathfrak{o}(n)$ is a stem subalgebra of $\mathfrak{o}(2n)$. (This is moreover a symmetric pair.) Another well known example of a stem subalgebra is $G_2 \oplus F_4$ in E_8.
(4) If $\mathfrak{g}$ is an exceptional simple Lie algebra over $\mathbb{C}$, Table 39 of [D] gives a complete catalog of the stem subalgebras of $\mathfrak{g}$.

Theorem 4.4 combined with Theorems 7.3 and 7.4 in [D] imply

Corollary 4.6. *If $\mathfrak{g} = \mathfrak{g}_{ss}$, any stem subalgebra is a Fernando-Kac subalgebra of finite type.*

Finally, we have

Corollary 4.7. *If $\mathfrak{g} = \mathfrak{g}_{ss}$, every maximal proper subalgebra $\mathfrak{l} \subset \mathfrak{g}$ is a Fernando-Kac subalgebra of finite type.*

Proof. By a theorem of F. Karpelevic, [Kar], $\mathfrak{l}$ is a parabolic subalgebra or a semisimple subalgebra. If $\mathfrak{l}$ is parabolic the statement is obvious as any module induced from a finite-dimensional $\mathfrak{l}$-module has finite $\mathfrak{l}$-multiplicities. Let $\mathfrak{l}$ be semisimple. Then $C(\mathfrak{l}) = 0$, and thus $\mathfrak{l}$ is primal by Corollary 4.5. But as $\mathfrak{l}$ is maximal, any irreducible infinite-dimensional $(\mathfrak{g},\mathfrak{l})$-module of finite type is strict, i.e. $\mathfrak{l}$ is a Fernando-Kac subalgebra of finite type. □

5. The case $\mathfrak{g} = \mathfrak{gl}(n)$

5.1. Description of reductive Fernando-Kac subalgebras of finite type.

Theorem 5.1. *A reductive in $\mathfrak{g} = \mathfrak{gl}(n)$ subalgebra $\mathfrak{k}$ is a Fernando-Kac subalgebra of finite type if and only if it is primal, or equivalently, if and only if $C(\mathfrak{k}) = Z(\mathfrak{k})$.*

Proof. By Theorem 3.1 it suffices to prove that if $C(\mathfrak{k}) = Z(\mathfrak{k})$, then $\mathfrak{k}$ is a Fernando-Kac subalgebra of finite type. We will modify the argument in the proof of Theorem 4.4 under the assumption that $\mathfrak{g} = \mathfrak{gl}(n)$.

Let $h = h' + h''$, $\mathfrak{p}$ and $\mathfrak{m}$ be as in the proof of Theorem 4.4. In particular $\mathfrak{m} = C(\mathfrak{k}_{ss}) \cap \mathfrak{p}$. We claim that h'' can be chosen so that, in addition, there is a decomposition $\mathfrak{p} = \mathfrak{a}' \mathbin{\ooplus} \mathfrak{a}$ and an isomorphism $p : C(\mathfrak{k}_{ss}) \to \mathfrak{a}$. Here is how this claim implies the Theorem. Note that $C(\mathfrak{k}_{ss})$ is a direct sum of an abelian ideal and simple ideals of type A. Choose now L to be a strict irreducible $(C(\mathfrak{k}_{ss}), Z(\mathfrak{k}))$-module of finite type over $Z(\mathfrak{k})$. Define a $\mathfrak{p}$-module structure on L by putting $\mathfrak{a}' \cdot L = 0$ and letting $\mathfrak{a}$ act on L via the isomorphism p. One can see immediately that L is an irreducible $(\mathfrak{p},\mathfrak{h}_{\mathfrak{k}})$-module of finite type over $\mathfrak{h}_{\mathfrak{k}}$ with $\mathfrak{p}[L] = \mathfrak{a}' + \mathfrak{h}$. Apply the construction in Theorem 4.3 to the triple $(\mathfrak{k},\mathfrak{p},L)$ to obtain a $(\mathfrak{g},\mathfrak{k})$-module M of finite type over $\mathfrak{k}$. As $\mathfrak{m} \subset C(\mathfrak{k}_{ss})$, we have $\mathfrak{m}_L = Z(\mathfrak{k})$ and consequently $\mathfrak{g}[M] = \mathfrak{k}$.

It remains to prove our claim about the choice of h''. We will consider the parabolic subalgebra $\mathfrak{p}'$ defined via (4.1) by the fixed element h' and then we will choose h'' so that $\mathfrak{p}$ is a certain subalgebra of $\mathfrak{p}'$. Let E be the defining (n-dimensional) $\mathfrak{g}$-module. There is an isomorphism of $\mathfrak{k}_{ss}\oplus C(\mathfrak{k}_{ss})$-modules

$$E\cong\oplus_i\left(E_i\otimes V_i\right), \tag{5.1}$$

where the E_i's are pairwise non-isomorphic irreducible $\mathfrak{k}_{ss}$-modules and the V_i's are irreducible $C(\mathfrak{k}_{ss})$-modules. We have

$$C(\mathfrak{k}_{ss})\cong\oplus_i\operatorname{End}(V_i). \tag{5.2}$$

One can check that

$$\mathfrak{p}'_{red}=C(\mathfrak{h}_{\mathfrak{k}_{ss}})\cong\oplus_{\lambda\in\mathfrak{h}^*_{\mathfrak{k}_{ss}}}\operatorname{End}\left(E^\lambda\right), \tag{5.3}$$

where E^λ denotes the $\mathfrak{h}_{\mathfrak{k}_{ss}}$-weight space of weight λ. Furthermore, by (5.1),

$$E^\lambda\cong\oplus_i\left(E_i^\lambda\otimes V_i\right). \tag{5.4}$$

Put $\mathcal{E}^\lambda_{ij}:=\operatorname{Hom}\left(E_i^\lambda,E_j^\lambda\right)\otimes\operatorname{Hom}(V_i,V_j)$ and $\mathcal{E}^\lambda:=\oplus_{i,j}\mathcal{E}^\lambda_{ij}$. Then combining (5.3) and (5.4) one obtains that $\mathfrak{p}'_{red}\cong\oplus_\lambda\mathcal{E}^\lambda$. Note that $\mathcal{E}^\lambda_+:=\oplus_{i\le j}\mathcal{E}^\lambda_{ij}$ is a parabolic subalgebra of $\mathcal{E}^\lambda$.

We now choose $h''\in Z(\mathfrak{k})$ so that the parabolic subalgebra $\mathfrak{p}$ associated to $h'+h''$ by (4.1) is precisely $(\oplus_\lambda\mathcal{E}^\lambda_+)\supsetplus\mathfrak{n}_{\mathfrak{p}'}$. Note that $\mathfrak{p}_{red}=\oplus_{i,\lambda}\mathcal{E}^\lambda_{i,i}$. For each $\mathfrak{p}\cap\mathfrak{k}_{ss}$-singular weight λ of $\mathfrak{k}$ in E there is a unique index i_λ such that the $\mathfrak{p}\cap\mathfrak{k}_{ss}$-highest weight of E_{i_λ} equals λ. Let $\mathfrak{a}:=\oplus_\lambda\mathcal{E}^\lambda_{i_\lambda i_\lambda}$ and $\mathfrak{a}'$ be the ideal complementary to $\mathfrak{a}$. Since $E^\lambda_{i_\lambda}$ is one-dimensional and $\mathcal{E}^\lambda_{i_\lambda i_\lambda}\cong\operatorname{End}(V_{i_\lambda})$, equation (5.2) enables us to conclude that $C(\mathfrak{k}_{ss})$ is isomorphic to $\mathfrak{a}$. □

Corollary 5.2. *A reductive in $\mathfrak{g}=\mathfrak{gl}(n)$ subalgebra $\mathfrak{k}$ is a Fernando-Kac subalgebra of finite type if and only if the defining $\mathfrak{g}$-module is multiplicity free as a $\mathfrak{k}$-module.*

5.2. A combinatorial set-up. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}=\mathfrak{gl}(n)$ and let $\mathfrak{l}\supset\mathfrak{h}$ be a root subalgebra of $\mathfrak{g}$. The subalgebra $\mathfrak{l}$ is defined by its subset of roots $\Delta(\mathfrak{l})\subset\Delta$, where $\Delta\subset\mathfrak{h}^*$ is the root system of $\mathfrak{g}$. Recall that $\Delta=\{\varepsilon_i-\varepsilon_j\mid 1\le i\ne j\le n\}$ for an orthonormal basis $\varepsilon_1,\dots,\varepsilon_n$ of $\mathfrak{h}^*$. Set $\mathfrak{k}:=\mathfrak{l}_{\mathrm{red}}$ and $\mathfrak{n}:=\mathfrak{n}_{\mathfrak{l}}$. Then $\mathfrak{l}=\mathfrak{k}\supsetplus\mathfrak{n}$. Fix an arbitrary Borel subalgebra $\mathfrak{b}\subset\mathfrak{g}$ containing $\mathfrak{h}$, and let $\mathcal{S}_\mathfrak{k}(\mathfrak{g})\subset\Delta$ be the set of weights of all $\mathfrak{k}\cap\mathfrak{b}$-singular vectors in $\mathfrak{g}$. For any $\alpha\in\mathcal{S}_\mathfrak{k}(\mathfrak{g})$ denote by $\mathfrak{g}(\alpha)$ the irreducible $\mathfrak{k}$-submodule in $\mathfrak{g}$ with highest weight α. Obviously any $\alpha,\beta\in\Delta$ satisfy the condition

$$\alpha+\beta\in\Delta\text{ for }\alpha,\beta\in\mathcal{S}_\mathfrak{k}(\mathfrak{g})\Rightarrow\alpha+\beta\in\mathcal{S}_\mathfrak{k}(\mathfrak{g}). \tag{5.5}$$

More generally, let for any $\mathfrak{k}$-submodule $\mathfrak{f}$ of $\mathfrak{g}$, $\mathcal{S}_\mathfrak{k}(\mathfrak{f})$ denote the set of all weights of $\mathfrak{k}\cap\mathfrak{b}$-singular vectors in $\mathfrak{f}$. As $\mathfrak{k}$ and $\mathfrak{n}$ are subalgebras, $\mathcal{S}_\mathfrak{k}(\mathfrak{n})$ and $\mathcal{S}_\mathfrak{k}(\mathfrak{k})$ satisfy the analog of condition (5.5).

The following lemma is an easy consequence of the description of root subalgebras in $\mathfrak{gl}(n)$ and we leave its proof to the reader.

Lemma 5.3. *There exist pairwise non-intersecting subsets $I, J, K \subset \{1, \dots, n\}$ such that $|I| = |J|$ and*

$$\mathcal{S}_{\mathfrak{k}}(\mathfrak{g}) = \{\varepsilon_i - \varepsilon_j \mid i \in I \cup K, j \in J \cup K\}.$$

Let $\mathcal{C}_{\mathfrak{k}}(\mathfrak{f})$ denote the set of all linear combinations of vectors from $\mathcal{S}_{\mathfrak{k}}(\mathfrak{f})$ with coefficients in $\mathbb{Z}_+$.

Lemma 5.4. *Let $\mathfrak{g} = \mathfrak{gl}(n)$. If $\mathcal{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n}) \cap \mathcal{C}_{\mathfrak{k}}(\mathfrak{n}) \neq \{0\}$, one of the following relations holds*

(1) $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$,
(2) $\alpha_1 + \alpha_2 = \beta$

for some $\alpha_1, \alpha_2 \in \mathcal{S}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l})$, $\beta_1, \beta_2, \beta \in \mathcal{S}_{\mathfrak{k}}(\mathfrak{n})$, where $(\alpha_1, \alpha_2) = (\beta_1, \beta_2) = 0$ in the case of 1.

Proof. If $\mathcal{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n}) \cap \mathcal{C}_{\mathfrak{k}}(\mathfrak{n}) \neq \{0\}$, there is a non-trivial relation

$$\alpha_1 + ... + \alpha_k = \beta_1 + ... + \beta_l \tag{5.6}$$

for $\alpha_i \in S_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n})$ and $\beta_i \in S_{\mathfrak{k}}(\mathfrak{n})$. Among all such relations we fix one with minimal k and minimal l for the fixed k. Consider first the case when $\alpha_1 + \alpha_p \in \mathcal{S}_{\mathfrak{k}}(\mathfrak{g})$ for some $p \leq k$. We claim that then $\alpha_1 + \alpha_p \in \mathcal{S}_{\mathfrak{k}}(\mathfrak{n})$. For, if $\alpha_1 + \alpha_p \in \mathcal{S}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n})$, one can reduce k in (5.6) by the substitution $\beta = \alpha_1 + \alpha_p$, which contradicts our assumption. Thus $\beta := \alpha_1 + \alpha_p \in \mathcal{S}_{\mathfrak{k}}(\mathfrak{n})$, and to show that $\alpha_1 + \alpha_p = \beta$ is a relation of type 2 we need only verify that $\alpha_1, \alpha_p \notin \mathcal{S}_{\mathfrak{k}}(\mathfrak{k})$. But the assumption $\alpha_1 \in \mathcal{S}_{\mathfrak{k}}(\mathfrak{k})$ (and similarly $\alpha_p \in \mathcal{S}_{\mathfrak{k}}(\mathfrak{k})$) is obviously contradictory, as then $-\alpha_1 \in \Delta(\mathfrak{k})$ and $\alpha_p = \alpha_1 + \alpha_p - \alpha_1 = \beta - \alpha_1 \in \Delta(\mathfrak{n})$. Therefore $\alpha_1, \alpha_p \in \mathcal{S}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l})$, $\beta \in \mathcal{S}_{\mathfrak{k}}(\mathfrak{n})$ and $\alpha_1 + \alpha_p = \beta$.

In the remainder of the proof we assume that $\alpha_1 + \alpha_p \notin \mathcal{S}_{\mathfrak{k}}(\mathfrak{g})$ for all $p \leq k$. If $\alpha_1 = \varepsilon_i - \varepsilon_j$, then ε_i and $-\varepsilon_j$ appear in $\alpha_1 + \dots + \alpha_k$ with positive coefficients. Therefore, there exist a and b such that $\beta_a = \varepsilon_i - \varepsilon_r$ and $\beta_b = \varepsilon_s - \varepsilon_j$, $s \neq r$ by minimality. By Lemma 5.3 $\gamma := \varepsilon_s - \varepsilon_r \in \mathcal{S}_{\mathfrak{k}}(\mathfrak{g})$. We claim that $\gamma \in \mathcal{S}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n})$. Indeed, assume to the contrary that $\gamma \in \mathcal{S}_{\mathfrak{k}}(\mathfrak{n})$. Then one can modify (5.6) by removing α_1 and replacing $\beta_a + \beta_b$ by γ. Since (5.6) is minimal, the new relation must be trivial. Thus $\alpha_1 = \beta_1 + \dots + \beta_l$. Since $\beta_1 + \dots + \beta_l \in \Delta$, $\beta := \beta_1 + \dots + \beta_l \in \mathcal{S}_{\mathfrak{k}}(\mathfrak{n})$, and hence $\alpha = \beta \in \mathcal{S}_{\mathfrak{k}}(\mathfrak{n})$. Contradiction. Therefore indeed $\gamma \in \mathcal{S}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n})$, and we have a relation $\alpha_1 + \gamma = \beta_a + \beta_b$, where $\alpha_1, \gamma \in \mathcal{S}(\mathfrak{g}/\mathfrak{n})$, $\beta_a, \beta_b \in \mathcal{S}_{\mathfrak{k}}(\mathfrak{n})$. Obviously, $(\alpha_1, \gamma) = (\beta_a, \beta_b) = 0$. To complete the proof we need to show that $\alpha_1, \gamma \in \mathcal{S}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l})$. But the assumption $\alpha_1 \in \mathcal{S}_{\mathfrak{k}}(\mathfrak{k})$ (and similarly $\gamma \in \mathcal{S}_{\mathfrak{k}}(\mathfrak{k})$) is contradictory as it implies $\beta_b - \alpha_1 \in \Delta(\mathfrak{n})$. Hence $\gamma = \beta_a + (\beta_b - \alpha_1) \in \Delta(\mathfrak{n})$. □

Corollary 5.5. $\mathcal{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n}) \cap \mathcal{C}_{\mathfrak{k}}(\mathfrak{n}) = \{0\}$ *if and only if* $\mathcal{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l}) \cap \mathcal{C}_{\mathfrak{k}}(\mathfrak{n}) = \{0\}$.

Proof. As $\mathcal{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l}) \subset \mathcal{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n})$, $\mathcal{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n}) \cap \mathcal{C}_{\mathfrak{k}}(\mathfrak{n}) = \{0\}$ implies $\mathcal{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l}) \cap \mathcal{C}_{\mathfrak{k}}(\mathfrak{n}) = \{0\}$. To prove the converse assume that $\mathcal{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l}) \cap \mathcal{C}_{\mathfrak{k}}(\mathfrak{n}) = \{0\}$ but $\mathcal{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{n}) \cap \mathcal{C}_{\mathfrak{k}}(\mathfrak{n}) \neq \{0\}$. Then by Lemma 5.4 one has a relation 1 or 2 with $\alpha_1, \alpha_2 \in \mathcal{S}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l})$. Hence $\mathcal{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l}) \cap \mathcal{C}_{\mathfrak{k}}(\mathfrak{n}) \neq \{0\}$. Contradiction. □

Lemma 5.6. *Let $\mathfrak{s} = \mathfrak{gl}(m)$, $\mathfrak{q} \subset \mathfrak{s}$ be a maximal parabolic subalgebra, and $\mathfrak{k} := \mathfrak{q}_{red}$. Let V_κ be the irreducible $\mathfrak{s}$-module with highest weight κ and $V_\mu(\mathfrak{k})$ be the irreducible $\mathfrak{k}$-module with highest weight μ. If λ is a dominant $\mathfrak{k}$-integral weight and β is the highest root of $\mathfrak{s}$, then, for large enough $q \in \mathbb{Z}_+$, the multiplicity of $V_{\lambda+q\beta}(\mathfrak{k})$ in $V_{\lambda+p\beta}$ is one for any $p \in \mathbb{Z}_+, p \geq q$.*

Proof. Set $\mu := \lambda + p\beta$, $\nu := \lambda + q\beta$ for fixed $p \geq q \in \mathbb{Z}_+$. Note that μ and ν are $\mathfrak{k}$-dominant and hence $V_\nu(\mathfrak{k})$ is finite-dimensional. If M_μ denotes the Verma module over $\mathfrak{s}$ and $M_\mu(\mathfrak{k})$ denotes the Verma module over $\mathfrak{k}$, then M_μ is isomorphic to $M_\mu(\mathfrak{k}) \otimes S(\mathfrak{q}/\mathfrak{k})^*$ as a $\mathfrak{k}$-module. Thus M_μ admits a filtration by $\mathfrak{k}$-submodules such that the associated graded $\mathfrak{k}$-module is a direct sum of Verma modules over $\mathfrak{k}$, each appearing with finite multiplicity. As the multiplicity of the weight $(q-p)\beta$ in $S(\mathfrak{q}/\mathfrak{k})^*$ is one, the multiplicity of $M_\nu(\mathfrak{k})$ in M_μ is one. Therefore the multiplicity of $V_\nu(\mathfrak{k})$ in M_μ is also one.

Now let $N \neq V_\mu$ be an irreducible subquotient of M_μ. We will show that, for q large, the multiplicity of $V_\nu(\mathfrak{k})$ in N is zero. It is known (see for example Theorem 7.6.23 [Dix]) that N is a subquotient of $M_{w_\alpha(\mu+\rho)-\rho}$ for some positive root α such that $(\mu, \alpha) \in \mathbb{Z}_+$. Therefore it suffices to prove that the multiplicity of $M_\nu(\mathfrak{k})$ in $M_{w_\alpha(\mu+\rho)-\rho}$ is zero. This is equivalent to showing that $w_\alpha(\mu+\rho) - \rho - \nu$ is not a weight of $S(\mathfrak{q}/\mathfrak{k})$, i.e. that $w_\alpha(\mu+\rho) - \rho - \nu$ does not belong to the convex hull $\mathcal{C}$ of $\Delta(\mathfrak{q}/\mathfrak{k})$.

Choose q so that $(\nu, \alpha) > 0$ for any positive α satisfying $(\alpha, \beta) = 1$, and assume $p \geq q$. First consider the case when $(\alpha, \beta) = 0$. Here $w_\alpha(\mu+\rho) - \rho - \nu = w_\alpha(\nu+\rho) - \rho - \nu + (p-q)\beta$. But $w_\alpha(\nu+\rho) - \rho - \nu = a\alpha$ for some negative a, which implies that $w_\alpha(\mu+\rho) - \rho - \nu = (p-q)\beta + a\alpha$ does not belong to $\mathcal{C}$. Next consider the case when $(\alpha, \beta) = 1$. Here $w_\alpha(\mu+\rho) - \rho - \nu = w_\alpha(\nu+\rho) - \rho - \nu + (p-q)w_\alpha(\beta) = -(b+1+p-q)\alpha + (p-q)\beta$, where $b = (\nu, \alpha)$ is positive by our choice of q. One can see that $-(b+1+p-q)\alpha + (p-q)\beta$ is not in $\mathcal{C}$. Finally, the case $\alpha = \beta$ is obvious. ☐

Corollary 5.7. *Let $\mathfrak{s} = \mathfrak{s}_1 \oplus \cdots \oplus \mathfrak{s}_j$ where each $\mathfrak{s}_i$ is isomorphic to $\mathfrak{gl}(m_i)$, $\mathfrak{q}_i$ be a maximal parabolic subalgebra of $\mathfrak{s}_i$ and β_i be the highest root of $\mathfrak{s}_i$. Let $\mathfrak{q} = \mathfrak{q}_1 \oplus \ldots \oplus \mathfrak{q}_l \oplus \mathfrak{s}_{l+1} \oplus \ldots \oplus \mathfrak{s}_j$ for $l \leq j$, $\mathfrak{k}$ be the reductive part of $\mathfrak{q}$, and $\beta = \beta_1 + \ldots + \beta_l$. If λ is a dominant $\mathfrak{k}$-integral weight, then there is a positive integer q such that the multiplicity of $V_{\lambda+q\beta}(\mathfrak{k})$ in $V_{\lambda+p\beta}$ is one for any $p \geq q$.*

Proof. The statement follows easily from Lemma 5.6 as an irreducible highest weight module over a direct sum of reductive Lie algebras is isomorphic to the tensor product of irreducible modules over the components. ☐

5.3. Description of Fernando-Kac root subalgebras of finite type.

Theorem 5.8. *A root subalgebra $\mathfrak{l} = (\mathfrak{k} \ni \mathfrak{n}) \subset \mathfrak{g} = \mathfrak{gl}(n)$ is a Fernando-Kac subalgebra of finite type if and only if $\mathcal{C}_\mathfrak{k}(\mathfrak{g}/\mathfrak{l}) \cap \mathcal{C}_\mathfrak{k}(\mathfrak{n}) = \{0\}$.*

Proof. First, we will show that if $\mathcal{C}_\mathfrak{k}(\mathfrak{g}/\mathfrak{l}) \cap \mathcal{C}_\mathfrak{k}(\mathfrak{n}) \neq \{0\}$, then $\mathfrak{l}$ is not a Fernando-Kac subalgebra of finite type. If $\mathcal{C}_\mathfrak{k}(\mathfrak{g}/\mathfrak{l}) \cap \mathcal{C}_\mathfrak{k}(\mathfrak{n}) \neq 0$, Lemma 5.4 provides us with a relation of type 1 or 2. Assume that the relation is of type 2, i.e. $\alpha_1 + \alpha_2 = \beta$ for some $\alpha_1, \alpha_2 \in \mathcal{S}_\mathfrak{k}(\mathfrak{g}/\mathfrak{l})$, $\beta \in \mathcal{S}_\mathfrak{k}(\mathfrak{n})$. Let $\mathfrak{s}$ be the subalgebra generated by $\mathfrak{k}$ and $\mathfrak{g}^{\pm\beta}$, and $\mathfrak{q}$ be the subalgebra generated by $\mathfrak{k}$ and $\mathfrak{g}^{\beta}$. Then the triple $(\mathfrak{s}, \mathfrak{q}, \beta)$ satisfies the hypothesis of Corollary 5.7 with $l = 1$. Moreover, $\mathfrak{g}(\beta)$ commutes with $\mathfrak{g}^{\alpha_i}$.

Let M be an irreducible strict $(\mathfrak{g}, \mathfrak{l})$-module. There exists a $\mathfrak{b} \cap \mathfrak{k}$-singular vector $v \in M$ such that $\mathfrak{g}(\beta)v = 0$. Let λ denote the weight of v. For any positive integer t, set $v_t = (g^{\alpha_1})^t (g^{\alpha_2})^t v$ for $0 \neq g^{\alpha_i} \in \mathfrak{g}^{\alpha_i}$. As g^{α_i} acts freely on M, we have $v_t \neq 0$. Furthermore v_t is $\mathfrak{b} \cap \mathfrak{k}$-singular and $\mathfrak{g}(\beta)v_t = 0$. Hence v_t generates an $\mathfrak{s}$-submodule $M(v_t) \subset M$ of highest weight $\lambda + t\beta$. By Corollary 5.7, for a fixed large $r \in \mathbb{Z}_+$, the multiplicity of $V_{\lambda+r\beta}$ in $M(v_t)$ is not zero for any $t \geq r$. Therefore the multiplicity of $V_{\lambda+r\beta}$ in M is infinite. Contradiction.

In the case of a relation of type 1, $\alpha_1 + \alpha_2 = \beta_1 + \beta_2$, let $\mathfrak{s} \subset \mathfrak{g}$ be the subalgebra generated by $\mathfrak{k}$, $\mathfrak{g}^{\pm\beta_1}$ and $\mathfrak{g}^{\pm\beta_2}$, and $\mathfrak{q} \subset \mathfrak{s}$ be the subalgebra generated by $\mathfrak{k}$, $\mathfrak{g}^{\beta_1}$ and $\mathfrak{g}^{\beta_2}$. The reader can check that the triple $(\mathfrak{s}, \mathfrak{q}, \beta)$ satisfies the conditions of Corollary 5.7 with $l = 2$, and moreover that $\mathfrak{g}(\beta_1) \oplus \mathfrak{g}(\beta_2)$ commutes with $\mathfrak{g}^{\alpha_i}$. Therefore, an argument similar to that in the case of a relation of type 2, leads to a contradiction.

It remains to prove that $\mathfrak{l}$ is a Fernando-Kac subalgebra of finite type whenever $\mathcal{C}_\mathfrak{k}(\mathfrak{g}/\mathfrak{l}) \cap \mathcal{C}_\mathfrak{k}(\mathfrak{n}) = \{0\}$. Using Theorem 4.3 we will construct an irreducible strict $(\mathfrak{g}, \mathfrak{l})$-module M of finite type over $\mathfrak{l}$.

Note first that $\mathcal{C}_\mathfrak{k}(\mathfrak{g})$ consist of $\mathfrak{k}$-dominant roots, and therefore $\mathcal{C}_\mathfrak{k}(\mathfrak{g}) \cap -\mathcal{C}_\mathfrak{k}(\mathfrak{g}) = \mathcal{C}_\mathfrak{k}(C(\mathfrak{k}_{ss}))$ and $\mathcal{S}_\mathfrak{k}(\mathfrak{g}) \cap -\mathcal{S}_\mathfrak{k}(\mathfrak{g}) = \mathcal{S}_\mathfrak{k}(C(\mathfrak{k}_{ss}))$. Furthermore, as $\mathfrak{n}$ is nilpotent, $\mathcal{C}_\mathfrak{k}(\mathfrak{n}) \cap -\mathcal{C}_\mathfrak{k}(\mathfrak{n}) = \{0\}$. Let $\mathcal{C}_0 = \mathcal{C}_\mathfrak{k}(\mathfrak{g}/\mathfrak{n}) \cap -\mathcal{C}_\mathfrak{k}(\mathfrak{g}/\mathfrak{n})$, and $\Delta_0 = \mathcal{S}_\mathfrak{k}(\mathfrak{g}/\mathfrak{n}) \cap -\mathcal{S}_\mathfrak{k}(\mathfrak{g}/\mathfrak{n})$. The above implies immediately that $\Delta_0 \subset \mathcal{S}_\mathfrak{k}(C(\mathfrak{k}_{ss}))$ and Δ_0 generates $\mathcal{C}_0$. By Corollary 5.5, $\mathcal{C}_\mathfrak{k}(\mathfrak{g}/\mathfrak{n}) \cap \mathcal{C}_\mathfrak{k}(\mathfrak{n}) = \{0\}$. Therefore one can find $h \in \mathfrak{h}$ such that all eigenvalues of $\mathrm{ad}_h : \mathfrak{g} \to \mathfrak{g}$ are rational and

$$(5.7)\qquad \begin{array}{lll} \alpha(h) > 0 & \text{for} & \alpha \in \mathcal{S}_\mathfrak{k}(\mathfrak{n}), \\ \alpha(h) = 0 & \text{for} & \alpha \in \Delta_0, \\ \alpha(h) < 0 & \text{for} & \alpha \in \mathcal{S}_\mathfrak{k}(\mathfrak{g}/(\mathfrak{n} + C(\mathfrak{k}_{ss}))). \end{array}$$

One can easily verify that, in additon, h can be chosen so that

$$(5.8)\qquad \alpha(h) < 0 \text{ for all } \alpha \in \Delta(\mathfrak{b} \cap \mathfrak{k}_{ss}).$$

Let $\mathfrak{p}$ be defined by (4.1). Then $\Delta(\mathfrak{p}_{red}) = \Delta_0$, and $\mathfrak{n} \subset \mathfrak{n}_\mathfrak{p}$.

Let L be an irreducible $(\mathfrak{p}, \mathfrak{h})$-module of finite type over $\mathfrak{h}$ with trivial action of $\mathfrak{n}_\mathfrak{p}$ and such that $\mathfrak{p}_{red}[L] = \mathfrak{h}$. Such L exists as $\mathfrak{p}_{ss}$ is a sum of ideals of type A. Let M be as in Section 4.3. Then by Theorem 4.3, M is an irreducible $(\mathfrak{g}, \mathfrak{k})$-module of finite type over $\mathfrak{k}$. Let $\mathfrak{g}\,[M] = \mathfrak{k} \ni \mathfrak{n}'$. We claim that $\mathfrak{n}' = \mathfrak{n}$. Indeed $\mathfrak{g}\,(\alpha) \subset \mathfrak{n}'$ if and only if $\mathfrak{g}\,(\alpha) \subset \mathfrak{p}\,[L]$. In particular, $\alpha(h) \geq 0$. If $\alpha\,(h) > 0$, then by (5.7) and (5.8) $\mathfrak{g}\,(\alpha) \subset \mathfrak{n} \subset \mathfrak{n}_\mathfrak{p} \subset \mathfrak{p}\,[L]$. If $\alpha\,(h) = 0$, then $\alpha \in \Delta_0$. As $\mathfrak{p}_{red}\,(L) = \mathfrak{h}$, we have $\mathfrak{g}\,(\alpha) \not\subset \mathfrak{p}\,[L]$. Thus $\mathfrak{n} = \mathfrak{n}'$. Theorem 5.8 is proven. $\square$

Corollary 5.9. *A root subalgebra* $\mathfrak{l} = (\mathfrak{k} \ni \mathfrak{n}) \subset \mathfrak{gl}(n)$ *with* $\mathfrak{n} \subset C(\mathfrak{k}_{ss})$ *is a Fernando-Kac subalgebra of finite type if and only if* $\mathfrak{n}$ *is the nilradical of a parabolic subalgebra in* $C(\mathfrak{k}_{ss})$.

Proof. For the necessity see Theorem 3.1 5. For the sufficiency we use Theorem 5.8. By hypothesis $\mathfrak{n}$ is the nilradical of a parabolic subalgebra in $C(\mathfrak{k}_{ss})$. We will show that $\mathcal{C}_{\mathfrak{k}}(\mathfrak{g}/\mathfrak{l}) \cap \mathcal{C}_{\mathfrak{k}}(\mathfrak{n}) = \{0\}$. Suppose not. Then there exist roots $\alpha_1, ..., \alpha_k \in \mathcal{C}_{\mathfrak{k}}(\mathfrak{n}/\mathfrak{l})$ and roots $\beta_1, ..., \beta_l \in \mathcal{C}_{\mathfrak{k}}(\mathfrak{n})$ such that (5.6) holds. Restrict both sides of (5.6) to $\mathfrak{h}_{\mathfrak{k}_{ss}}$ and write $\widetilde{\gamma}$ for the restriction of a weight γ to $\mathfrak{h}_{\mathfrak{k}_{ss}}$. Because $\mathfrak{n} \subset C(\mathfrak{k}_{ss})$, $\widetilde{\beta}_i = 0$ for all i and hence $\widetilde{\alpha}_1 + ... + \widetilde{\alpha}_l = 0$. But the $\widetilde{\alpha}_j$'s are dominant weights for $\mathfrak{k}_{ss}$. Therefore $\widetilde{\alpha}_j = 0$ for all j, and each $\alpha_j \in \mathcal{C}_{\mathfrak{k}}(C(\mathfrak{k}_{ss})) = \Delta(C(\mathfrak{k}_{ss}))$. Equation (5.6) becomes a nontrivial relation among roots in $\Delta(\mathfrak{n})$ and $\Delta(C(\mathfrak{k}_{ss})) \setminus \Delta(\mathfrak{n})$. Contradiction. □

Example. Let $\mathfrak{g} = \mathfrak{gl}(4)$, $\mathfrak{h}$ be the diagonal subalgebra, and $\mathfrak{l} \supset \mathfrak{h}$ be a root subalgebra of $\mathfrak{g}$. The rank of $\mathfrak{l}_{ss}$ can be $0, 1$ or 2. In the first case $\mathfrak{l}$ is solvable, and, by Proposition 3.2, $\mathfrak{l}$ is of finite type if and only if $\mathfrak{n}_{\mathfrak{l}}$ is the nilradical of a parabolic subalgebra. In the third case $\mathfrak{l}_{\mathrm{red}}$ equals the fixed points of an involution $\theta : \mathfrak{g} \to \mathfrak{g}$ and $\mathfrak{l}$ is always a Fernando subalgebra of finite type: the corresponding strict $(\mathfrak{g}, \mathfrak{l})$-modules are Harish-Chandra modules.

In the case when $\mathfrak{l}_{ss} \cong \mathfrak{sl}(2)$ we can fix the roots of $\mathfrak{l}_{ss}$ to be $\pm(\varepsilon_1 - \varepsilon_2)$. To determine $\mathfrak{l}$ we need to specify the roots of $\mathfrak{n}_{\mathfrak{l}}$. Up to automorphisms of $\mathfrak{g}$ that stabilize $\mathfrak{l}_{ss}$ there are eight choices for $\mathfrak{n}_{\mathfrak{l}}$ (including the possibility $\mathfrak{n}_{\mathfrak{l}} = 0$). A direct checking based on Theorem 5.8 and Corollary 5.9 shows that there is a single choice of $\mathfrak{n}_{\mathfrak{l}}$ for which $\mathfrak{l}$ is not a Fernando-Kac subalgebra of finite type. We may normalize this $\mathfrak{l}$ so that the roots in $\mathfrak{n}_{\mathfrak{l}}$ are $\varepsilon_1 - \varepsilon_3$ and $\varepsilon_2 - \varepsilon_3$. Furthermore, the so defined $\mathfrak{l} = \mathfrak{l}_{\mathrm{red}} \ni \mathfrak{n}_{\mathfrak{l}}$ satisfies conditions 1–5 in Theorem 3.1. This shows in particular that the conditions in Theorem 3.1 are not sufficient for a subalgebra $\mathfrak{l}$ to be a Fernando-Kac subalgebra of finite type.

References

[BB] A. Beilinson, J. Bernstein, A proof of Jantzen conjectures, I. M. Gelfand Seminar, Adv. Soviet Math. **16**, Part 1, Amer. Math. Soc., Providence, 1993, 1-50.

[B] N. Bourbaki, Groupes et algèbres de Lie, Ch VII, Hermann, Paris, 1975.

[D] E. Dynkin, Semisimple subalgebras of semisimple Lie algebras, (Russian) *Mat. Sbornik N.S*, **30(72)** (1952), 349-462. Engl.: *AMS Translations* **6** (1957), 111-244.

[Dix] J. Dixmier, Enveloping algebras, North Holland, Amsterdam, 1977.

[F] S. Fernando, Lie algebra modules with finite-dimensional weight spaces I, *Trans. Amer. Math. Soc.* **322** (1990), 757-781.

[G] I. M. Gelfand, The cohomology of infinite dimensional Lie algebras: some questions of integral geometry. Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, Gauthier-Villars, Paris, 1971, 95-111.

[K] V. Kac, Constructing groups associated to infinite-dimensional Lie algebras, Infinite-dimensional groups with applications (Berkeley, 1984), *Math. Sci. Res. Inst. Publ.*, **4**, 167-216.

[Kar] F. Karpelevic, On nonsemisimple maximal subalgebras of semisimple Lie algebras. (Russian) *Doklady Akad. Nauk SSSR (N.S.)* **76** (1951), 775-778.

[Ke] P. Kekäläinen, On irreducible A_2-finite G_2-modules, *J. Algebra* **117** (1988), 72-80.

[KV] A. Knapp, D. Vogan, Cohomological induction and unitary representations, *Princeton Mathematical Series*, **45**, Princeton University Press, Princeton, 1995.

[M] O. Mathieu, Classification of irreducible weight modules, *Ann. Inst. Fourier* **50** (2000), 537-592.

[PS] I. Penkov, V. Serganova, Generalized Harish-Chandra modules, *Moscow Math. J.*, to appear.

[S] G. Savin, Dual pair $PGL(3) \times G_2$ and $(\mathfrak{g}_2, SL(3))$-modules, *IMRN* No. 4 (1994), 177-184.

[V] D. Vogan, Representations of real reductive Lie groups, *Progress in Mathematics*, **15**, Birkhäuser, Boston, 1981.

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