EXAMPLES OF AUTOMORPHISM GROUPS OF IND-VARIETIES OF GENERALIZED FLAGS

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To the memory of Vasil Tsanov

Abstract. We compute the automorphism groups of finite and cofinite ind-grassmannians, as well as of the ind-variety of maximal flags indexed by $\mathbb{Z}_{>0}$. We pay special attention to differences with the case of ordinary flag varieties.
MSC: 14L30, 14M15, 14M17.
Keywords: ind-grassmannian, automorphism group, flag ind-variety.

1. Introduction

The flag varieties of the classical Lie groups are central objects of study both in geometry and representation theory. In a sense, they are a hub for many directions of research in both fields. Several different infinite-dimensional analogues of the ordinary flag varieties have been studied in the literature, one such analogue being the ind-varieties of generalized flags introduced in [1] and further investigated in [2], [3], [4], [5]; see also the survey [6]. The latter ind-varieties are direct limits of classical flag varieties and are homogeneous ind-spaces for the simple ind-groups $SL(\infty)$, $SO(\infty)$, $Sp(\infty)$. Without doubt, some of these ind-varieties, in particular the ind-grassmannians, have been known long before the paper [1].

A natural question of obvious importance is the question of finding the automorphism groups of the ind-varieties of generalized flags. The purpose of the present note is to initiate a discussion in this direction and to point out some differences with the case of ordinary flag varieties: see Section 4. The topic is very close to Vasil's interests and expertise, and for sure I would have discussed it with him if he were still alive.

Acknowledgments. This work has been supported in part by DFG grant PE 980/7-1.

2. Automorphisms of finite and cofinite ind-grassmannians

The base field is $\mathbb{C}$. Let V be a fixed countable-dimensional complex vector space. We fix a basis $E = \{e_1, \dots, e_n, \dots\}$ of V and set $V_n := \mathrm{span}_{\mathbb{C}}\{e_1, \dots, e_n\}$. Then $V = \cup_n V_n$. Fix $k \in \mathbb{Z}_{>0}$. By definition, $\mathrm{Gr}(k, V)$ is the set of all k-dimensional subspaces in V and has an obvious ind-variety structure:

$$\mathrm{Gr}(k, V) = \varinjlim \mathrm{Gr}(k, V_n).$$

The projective ind-space $\mathbb{P}(V)$ equals $\mathrm{Gr}(1, V)$. Note that the basis E plays no role in this construction. We think of the ind-varieties $\mathrm{Gr}(k, E)$ for $k \in \mathbb{Z}_{>0}$ as the "finite ind-grassmannians."

The basis E plays a role when defining the "cofinite" ind-grassmannians. Fix a subspace $W \subset V$ of finite codimension in V and such that $E \cap W$ is a basis of W. Let $\mathrm{Gr}(W, E, V)$ be the set of all subspaces $W' \subset V$ which have the same codimension in V as W and in addition contain almost all elements of E. Then $\mathrm{Gr}(W, E, V)$ has the following ind-variety structure:

$$\mathrm{Gr}(W, E, V) = \varinjlim \mathrm{Gr}(\mathrm{codim}_V W, \bar{V}_n)$$

where $\{\bar{V}_n\}$ is any set of finite-dimensional spaces with the properties that $\bar{V}_n \supset \mathrm{span}\{E \backslash \{E \cap W\}\}$, $\dim \bar{V}_n = n > \mathrm{codim}_V W$, $E \cap \bar{V}_n$ is a basis of $\bar{V}_n$, and $\cup \bar{V}_n = V$. The map identifying the direct limit of $\mathrm{Gr}(\mathrm{codim}_V W, \bar{V}_n)$ with $\mathrm{Gr}(W, E, V)$ is

$$W'' \mapsto W'' \oplus \mathrm{span}\{E \backslash (E \cap \bar{V}_n)\}$$

for $W'' \in \mathrm{Gr}(\mathrm{codim}_V W, \bar{V}_n)$.

It is clear that the ind-varieties $\mathrm{Gr}(W, E, V)$ and $\mathrm{Gr}(k, V)$ are isomorphic: the isomorphism is given by

$$\mathrm{Gr}(W, E, V) \ni W' \to \mathrm{Ann}\, W' \subset V_* := \mathrm{span}\{E^*\}, \tag{1}$$

where $E^* = \{e_1^*, e_2^*, \dots\}$ is the system of linear functionals dual to the basis E: i.e. $e_i^*(e_j) = \delta_{ij}$. The map (1) is an obvious analogue of finite-dimensional duality. Therefore the automorphism groups $\mathrm{Aut}\, \mathrm{Gr}(k, V)$ and $\mathrm{Aut}\, \mathrm{Gr}(W, E, V)$ for $\mathrm{codim}_W V = k$ are isomorphic; by an automorphism we mean of course an automorphism of ind-varieties.

The following result should in principle be known. We present a proof which shows a connection with the work [2].

Proposition 1 $\mathrm{Aut}\,\mathrm{Gr}(k,V) = PGL(V)$ *where* $GL(V)$ *denotes the group of all invertible linear operators on* V *and* $PGL(V) := GL(V)/\mathbb{C}_{mult}Id$ *(where* $\mathbb{C}_{mult}$ *is the multiplicative group of* $\mathbb{C}$*).*

Proof: An automorphism $\phi : \mathrm{Gr}(k,V) \to \mathrm{Gr}(k,V)$ induces embeddings $\phi_n : \mathrm{Gr}(k,V_n) \hookrightarrow \mathrm{Gr}(k,V_{N(n)})$ for appropriate $N(n) \geq n$. These embeddings are linear in the sense that
$\phi_n^*(\mathcal{O}_{\mathrm{Gr}(k,V_{N(n)})}(1))$ is isomorphic to $\mathcal{O}_{\mathrm{Gr}(k,V_n)}(1)$, where by $\mathcal{O}_{\cdot}(1)$ we denote the positive generator of the respective Picard group. According to Theorem 1 in [2], ϕ_n is one of the following:

(i) an embedding induced by the choice of an n-dimensional subspace $W_n \subset V_{N(n)}$ for some $N(n) \geq n$,

(ii) an embedding factoring through a linearly embedded projective space $\mathbb{P}^{M(n)} \subset \mathrm{Gr}(k,V_{N(n)})$ for some $M(n) < N(n)$.

If $k > 2$, option (ii) may hold only for finitely many n as the contrary implies that the image of ϕ_n is contained in a projective ind-subspace

$$\mathbb{P} := \varinjlim \mathbb{P}^{M(n)} \subset \mathrm{Gr}(k,V).$$

Then, since $\mathbb{P}$ is not isomorphic to $\mathrm{Gr}(k,V)$ by Theorem 2 in [2], the image of ϕ_n would necessarily be a proper ind-subvariety of $\mathrm{Gr}(k,V)$, which is a contradiction.

For $k = 1$, options (i) and (ii) are the same, and therefore without loss of generality we can now assume that for our fixed k option (i) holds for all n. The embeddings $\phi_n : \mathrm{Gr}(k,V_n) \hookrightarrow \mathrm{Gr}(k,V_{N(n)})$ determine injective linear operators $\tilde{\phi}_n : V_n \to V_{N(n)}$. Moreover, the operators $\tilde{\phi}_n$ are defined up to multiplicative constants which can be chosen so that $\tilde{\phi}_n|_{V_{n-1}} = \tilde{\phi}_{n-1}$ for any n. Therefore, we obtain a well-defined linear operator

$$\tilde{\phi} : V = \varinjlim V_n \to V = \varinjlim V_{N(n)}$$

which induces our automorphism ϕ. Since ϕ is invertible, $\tilde{\phi}$ is also invertible, and since $\tilde{\phi}$ depends on a multiplicative constant, we conclude that ϕ determines a unique element $\bar{\phi} \in PGL(V)$.

In this way we have constructed an injective homomorphism

$$\mathrm{Aut}\,\mathrm{Gr}(k,V) \to PGL(V),\ \phi \mapsto \bar{\phi}.$$

The inverse homomorphism

$$PGL(V) \to \mathrm{Aut}\,\mathrm{Gr}(k,V)$$

is obvious because of the natural action of $PGL(V)$ on $\mathrm{Gr}(k, V)$. The statement follows. ∎

3. Ind-variety of maximal ascending flags

We now consider a particular ind-variety of maximal generalized flags, in fact the simplest case of maximal generalized flags. Let V and E be as above. Define $Fl(F_E, E, V)$ as the set of all infinite chains F'_E of subspaces of V

$$0 \subset (F'_E)^1 \subset \cdots \subset (F'_E)^k \subset \dots$$

where $\dim(F'_E)^k = k$ and $(F'_E)^n = F^n_E := \mathrm{span}\{e_1, \dots, e_n\}$ for large enough n. This set has an obvious structure of ind-variety as

$$Fl(F_E, E, V) = \varinjlim Fl(F^n_E)$$

where $Fl(F^n_E)$ stands for the variety of maximal flags in the finite-dimensional vector space F^n_E.

Denote by $GL(E, V)$ the subgroup of $GL(V)$ of automorphisms of V which keep all but finitely many elements of E fixed. The elements of $GL(E, V)$ are the E-*finitary* automorphisms of V.

Proposition 2

$$\mathrm{Aut}\, Fl(F_E, E, V) = P(GL(E, V) \cdot B_E)$$

where $B_E \subset GL(V)$ is the stabilizer of the chain F_E in $GL(V)$ and $GL(E, V) \cdot B_E$ is the subgroup of $GL(V)$ generated by $GL(E, V)$ and B_E.

We start with a lemma.

Lemma 3 *Fix $k \geq 2$. Let $\psi_{k-1}, \psi_k : V \to V$ be invertible linear operators such that $\psi_{k-1}(W_{k-1}) \subset \psi_k(W_k)$ for any pair of subspaces $W_{k-1} \subset W_k$ of V with $\dim W_{k-1} = k - 1$, $\dim W_k = k$. Then $\psi_{k-1} = c\psi_k$ for some $0 \neq c \in \mathbb{C}$.*

Proof: Assume the contrary. Let v be a vector in V such that the space $Z := \mathrm{span}_{\mathbb{C}}\{\psi_{k-1}(v), \psi_k(v)\}$ has dimension 2. Extend v to a basis $v = v_1, v_2, \dots$ of V. Then, setting $W_k = \mathrm{span}_{\mathbb{C}}\{v_1, \dots, v_k\}$ and $W_{k-1} = \mathrm{span}_{\mathbb{C}}\{v_1, \dots, v_{k-1}\}$, we see that the condition $\psi_{k-1}(W_{k-1}) \subset \psi_k(W_k)$ implies $Z \subset \psi_k(W_k)$. Similarly, setting $W'_k = \mathrm{span}_{\mathbb{C}}\{v_1, v_{k+1}, v_{k+2} \dots, v_{2k-1}\}$ and $W'_{k-1} = \mathrm{span}_{\mathbb{C}}\{v_1, v_{k+1}, v_{k+2} \dots, v_{2k-2}\}$ we have $Z \subset \psi_k(W'_k)$. However clearly

$$\dim(W_k \cap W'_k) = 1,$$

hence the dimension of the intersection $\psi_k(W_k) \cap \psi_k(W_k')$ must also be 1 due to the invertibility of ψ_k. Contradiction. ∎

Proof of Proposition 2: We first embed $A := \mathrm{Aut}\, Fl(F_E, E, V)$ into the group $PGL(V)$. For this we consider the obvious embedding

$$A \hookrightarrow \Pi_{i=1}^{\infty} \mathrm{Aut}\, \mathrm{Gr}(i, V)$$

arising from the diagram of surjective morphisms of ind-varieties

$$\begin{array}{ccccccc} & & & Fl(F_E, E, V) & & & \\ & \swarrow & & \downarrow & & \searrow & \\ \mathbb{P}(V) = \mathrm{Gr}(1, V) & & \mathrm{Gr}(2, V) & & \ldots & & \mathrm{Gr}(k, V) \ldots \quad . \end{array}$$

By Proposition 1, the groups $\mathrm{Aut}\, \mathrm{Gr}(k, V)$ are isomorphic to $PGL(V)$ for all $k \in \mathbb{Z}_{>0}$. Moreover, it is clear that the homomorphism $A \to \Pi_k PGL(V)$ is injective as the ind-varieties $\mathrm{Gr}(k, V)$ are pairwise nonisomorphic for $k \geq 1$ [2] (this argument is false in the finite-dimensional case). It is also clear that this homomorphism factors through the diagonal of $\Pi_k PGL(V)$ since Lemma 1 shows that an automorphism from A induces necessarily the same element in $PGL(V)$ via any projection $Fl(F_E, E, V) \to \mathrm{Gr}(k, V)$.

It remains to determine which elements of the group $PGL(V)$ arise as images of elements of A. It is clear that this image contains both $PGL(E, V)$ and PB_E as each of these groups acts faithfully on $Fl(F_E, E, V)$. Indeed, the fact that $PGL(E, V)$ acts on $Fl(F_E, E, V)$ is clear. To see that PB_E acts on $Fl(F_E, E, V)$ one notices that for any $F_E' \in Fl(F_E, E, V)$ and any $\gamma \in PB_E$, the flag $\gamma(F_E')$ differs from F_E only in finitely many positions, hence is a point on $Fl(F_E, E, V)$.

On the other hand, it is clear that the image $\bar{\phi} \in PGL(V)$ of $\phi \in A$ is contained in $P(GL(E, V) \cdot B_E)$. Indeed, the composition $\psi \circ \bar{\phi}$ with a suitable element of $PGL(E, V)$ will fix the point F_E on $Fl(F_E, E, V)$. This means that $\psi \circ \bar{\phi} \in PB_E$. Therefore the image of A in $PGL(V)$ is contained in $P(GL(E, V) \cdot B_E)$, and we are done.

∎

4. Discussion

First, Proposition 1 can be generalized to ind-varieties of the form $Fl(F, E, V)$ where F is a finite chain consisting only of finite-dimensional subspaces of V, or only of subspaces of finite codimension of V. The precise definition of the ind-varieties $Fl(F,E,V)$ is given in [1]. In these cases, the respective automorphism groups are always isomorphic to $PGL(V)$, however in the case of finite codimension there is a natural isomorphism with $PGL(V_*)$.

We now point out some differences with the case of ordinary flag varieties. A first obvious difference is the following. Despite the fact that $\mathrm{Gr}(k,V) = PGL(E,V)/P_k$, where P_k is the stabilizer in $PGL(E,V)$ of a k-dimensional subspace of V, the automorphism group of $\mathrm{Gr}(k,V)$ is much larger than $PGL(E,V)$. Therefore $\mathrm{Gr}(k,V)$ is a quotient of any subgroup G satisfying $PGL(E,V) \subset G \subset PGL(V)$, and there is quite a variety of such subgroups. Similar comments apply to the other examples we consider.

Next, we note that the automorphism group of an ind-grassmannian is in general not naturally embedded into $PGL(V)$. Indeed, the case of the cofinite ind-grassmannian $\mathrm{Gr}(W,E,V)$ shows that the natural isomorphism $\mathrm{Aut}\,\mathrm{Gr}(W,E,V) = PGL(V_*)$ does not embed $\mathrm{Aut}\,\mathrm{Gr}(W,E,V)$ into $PGL(V)$ by duality, but only embeds $\mathrm{Aut}\,\mathrm{Gr}(W,E,V)$ into the much larger group $PGL((V_*)^*)$ in a way that its image does not keep the subspace $V \subset (V_*)^*$ invariant. This is clearly an infinite-dimensional phenomenon.

Finally, recall that the group of automorphisms of a finite-dimensional grassmannian is naturally a subgroup of the automorphism group of the corresponding full flag variety; more precisely, the former group is the connected component of unity of the latter group. This note shows that the situation in the infinite-dimensional case essentially different: indeed, the injection $\mathrm{Aut}\,\mathrm{Fl}(F_E,E,V) \hookrightarrow \mathrm{Aut}\,\mathrm{Gr}(k,V)$ constructed in the proof of Proposition 2 is proper.

We hope that the above differences motivate a more detailed future study of the automorphism groups of arbitrary ind-varieties of generalized flags.

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