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ON HOMOMORPHISMS OF DIAGONAL LIE ALGEBRAS

by

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Abstract

On Homomorphisms of
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Diagonal Lie algebras are defined as direct limits of finite-dimensional Lie algebras under diagonal injective homomorphisms. An explicit description of the isomorphism classes of diagonal locally simple Lie algebras is given in the paper [A. A. Baranov, A. G. Zhilinskii, Diagonal direct limits of simple Lie algebras, Comm. Algebra, **27** (1998), 2749-2766]. The three finitary infinite-dimensional Lie algebras sl(∞), so(∞), and sp(∞) are important special cases of diagonal locally simple Lie algebras. Many classical results have been extended to these three infinite-dimensional Lie algebras. In particular, in the paper [I. Dimitrov, I. Penkov, Locally semisimple and maximal subalgebras of the finitary Lie algebras gl(∞), sl(∞), so(∞), and sp(∞), J. Algebra **322** (2009), 2069-2081] all locally semisimple subalgebras of $\mathfrak{g} \cong$ sl(∞), so(∞), and sp(∞) are described, and moreover all injective homomorphisms $\mathfrak{s} \to \mathfrak{g}$ are described in terms of the action of $\mathfrak{s}$ on the natural and the conatural $\mathfrak{g}$-modules. The present dissertation makes a substantial contribution to further extending these results to the class of diagonal locally simple Lie algebras.

In Chapter 3 all locally simple Lie subalgebras of any diagonal locally simple Lie algebra are described up to isomorphism. The main result of the dissertation, Theorem 3.1.11, provides a list of conditions under which there exists an injective homomorphism $\mathfrak{s} \to \mathfrak{g}$ of a locally simple Lie algebra $\mathfrak{s}$ into a diagonal locally simple Lie algebra $\mathfrak{g}$.

In Chapter 4, with Ivan Penkov, we study certain invariants of homomorphisms of diagonal locally simple Lie algebras. The ideas and partial results presented in this Chapter may lead to a description of such homomorphisms in the future.

Declaration

I hereby declare that this submission is my own work and that, to the best of my knowledge and belief, it contains no material previously published or written by another person nor material which to a substantial extent has been accepted for the award of any other degree or diploma of the university or other institute of higher learning, except where due acknowledgment has been made in the text.

Signature: Name: Date:

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Contents

Chapter 1

Introduction

Infinite-dimensional Lie algebras have been studied over half a century, and it is an accepted fact that there is no general structure theory of infinite-dimensional Lie algebras. The reason is that there is a too large variety of types of infinite-dimensional Lie algebras. In this thesis we study locally finite Lie algebras, i.e. direct limits of finite-dimensional Lie algebras. These Lie algebras are a very natural generalization of finite-dimensional Lie algebras, and for a long time their theory has been overshadowed by the theory of affine or general Kac-Moody Lie algebras. Only in the last decade the theory of infinite-dimensional locally finite Lie algebras has started to play a more prominent role within the general subject of infinite-dimensional Lie algebras and their representations.

The purpose of this thesis is to investigate a most natural class of locally simple Lie algebras. This class consists of diagonal direct limits of simple Lie algebras, and includes in particular the three classical locally simple infinite-dimensional Lie algebras sl(∞), so(∞), and sp(∞).

Locally simple Lie algebras has been studied by many authors, in particular by Y. Bahturin, A. Baranov, G. Benkart, E. Dan-Cohen, I. Dimitrov, K.-H. Neeb, I.Penkov, H. Strade, N. Stumme, and A. Zhilinskii. A key role in the subject plays the work of A. Baranov and his collaborators, see [B3], [B4], [BS]. In a series of important papers Baranov classifies all simple finitary infinite-dimensional locally finite Lie algebras over $\mathbb{C}$ and $\mathbb{R}$. Over $\mathbb{C}$ Baranov's result is extremely simple: there are just three such Lie algebras sl(∞), so(∞), and sp(∞). In [B1], [B2] Baranov introduces the class of diagonal locally finite Lie algebras and establishes their general properties. In the important paper [BZ] A. Baranov and A. Zhilinskii classify up to isomorphism the diagonal locally simple Lie algebras over $\mathbb{C}$. The answer here is quite complicated, and the diagonal locally simple Lie algebras provide a rich class of simple locally finite Lie algebras for which there is a hope for a deep structure theory. Nevertheless, the classification of Baranov and Zhilinskii is completely explicit.

There are already many results in the structure theory of the classical infinite-dimensional Lie algebras sl(∞), so(∞), sp(∞), as well as of the wider class of root-reductive Lie algebras. In particular, the Cartan, Borel, and parabolic subalgebras of these Lie algebras have been described. This is the work of I. Penkov and his collaborators, see [DP1], [PStr], [NP], [DaPS], [DP2], [Da], [DaP].

One of the key preliminary results for this thesis, along with the classification of diagonal locally simple Lie algebras [BZ], is the recent paper of I. Dimitrov and I. Penkov [DP3]. In this paper, among the other results, all locally semisimple subalgebras of $\mathfrak{g} \cong$ sl(∞), so(∞), sp(∞), or gl(∞) are described up to isomorphism, and the action of these subalgebras on the natural and conatural modules of $\mathfrak{g}$ is studied. This work is rooted in the classical works of A. Malcev [Mal] and E. Dynkin [Dy], where all homomorphisms of semisimple finite-dimensional Lie algebras are described. These latter works deserve a separate discussion, as they supply the motivation for the work presented in the thesis.

The problem of classifying semisimple subalgebras of semisimple Lie algebras is known from the beginning of the last century, and the solution of this problem is one of the main results in the structure theory of finite-dimensional Lie algebras. This problem is not only interesting by itself, but it has also had important algebraic and geometric applications. In particular, it is shown in [Mal] that the more general problems of classifying semisimple subalgebras of arbitrary Lie algebras and classifying all finite-dimensional Lie algebras with a given radical, can be reduced to the above problem. There are also applications in the theory of Lie groups, for example it was shown by Malcev that the problem of describing compact subgroups of a real Lie group is equivalent to the problem of describing its complex semisimple subgroups.

Malcev shows in [Mal] that the problem of classifying semisimple subalgebras of semisimple Lie algebras reduces to the problem of classifying the semisimple subalgebras of a simple Lie algebra. The study of semisimple Lie algebras of the Lie algebra A_n is equivalent to the study of linear representations of semisimple Lie algebras. The main results in this direction were obtained by E. Cartan and H. Weyl in the first quarter of the last century. Semisimple Lie algebras of the classical Lie algebras B_n, C_n, and D_n have been described by Malcev in [Mal] by studying orthogonal and symplectic representations of semisimple Lie algebras. Malcev also described semisimple subalgebras of the exceptional Lie algebras G_2 and F_4.

In his work [Dy], E. Dynkin introduced new methods into the subject and was able to complete the classification. In particular, he was able to list all semisimple subalgebras of the exceptional Lie algebras E_6, E_7, and E_8. Dynkin also introduced an important invariant, the index of a simple subalgebra in a simple Lie algebra.

In relation to this present work, let us comment that in the finite-dimensional case it is relatively easy to describe all pairs $(\mathfrak{s}, \mathfrak{g})$ consisting of a semisimple Lie algebra $\mathfrak{s}$ and a simple Lie algebra $\mathfrak{g}$ such that there is an injective homomorphism $\mathfrak{s} \to \mathfrak{g}$. In their work, Malcev and Dynkin actually did much more: they described all $\mathfrak{g}$-conjugacy classes of semisimple Lie subalgebras $\mathfrak{s} \subset \mathfrak{g}$. In the infinite-dimensional case, however, the problem of describing all pairs $(\mathfrak{s}, \mathfrak{g})$ such that $\mathfrak{s}$ admits an injective homomorphism into $\mathfrak{g}$, already presents a challenge. The main result of the thesis is the solution of the latter problem for diagonal locally simple $\mathfrak{g}$ and locally simple $\mathfrak{s}$ (Theorem 3.1.11).

We now describe the body of the thesis in detail. In Chapter 2 we discuss some ideas of Malcev's paper [Mal] and recall the notion of index introduced by E. Dynkin in [Dy]. We also discuss in detail some results of the two papers [DP3] and [BZ] mentioned above, together with some general facts about diagonal Lie algebras. We complete Chapter 2 by presenting two useful branching rules for finite-dimensional Lie algebras of type A.

Chapter 3 contains the main result of the thesis, namely the classification of locally simple subalgebras of diagonal locally simple Lie algebras up to isomorphism. The first notable result (see Proposition 3.1.1 and Corollary 3.1.4) is that any simple finitary Lie algebra $\mathfrak{s}$ admits an injective homomorphism into any diagonal Lie algebra $\mathfrak{g}$. For the case when both $\mathfrak{s}$ and $\mathfrak{g}$ are non-finitary, the key statements Proposition 3.1.5 and Proposition 3.1.6, together with some technical lemmas, lead to the main result, Theorem 3.1.11. In this theorem the final description of all pairs $(\mathfrak{s}, \mathfrak{g})$, of a locally simple Lie algebra $\mathfrak{s}$ and a diagonal locally simple Lie algebra $\mathfrak{g}$ such that there is an injective homomorphism $\mathfrak{s} \to \mathfrak{g}$, is given by a list of "if and only if" conditions which are easy to check for concrete Lie algebras. In Section 3.2 we present two further corollaries describing the set of equivalence classes of diagonal locally simple Lie algebras.

In Chapter 4, jointly with Ivan Penkov, we study homomorphisms of diagonal locally simple Lie algebras. In particular, the question of the existence of diagonal and non-diagonal homomorphisms is discussed in Section 4.1. A natural representation of a diagonal locally simple Lie algebra is defined in Section 4.2. We also study the socle filtration of a natural $\mathfrak{g}$-module as an $\mathfrak{s}$-module for a homomorphism $\mathfrak{s} \to \mathfrak{g}$. A similar study plays a key role in the paper [DP3]. In Section 4.3 we introduce the level of a homomorphism $\mathfrak{s} \to \mathfrak{g}$; this is a new invariant which is used in the proofs of Proposition 3.1.6 and Proposition 3.1.8.

Chapter 2

Preliminaries

All vector spaces and Lie algebras are defined over the field of complex numbers $\mathbb{C}$. We assume that all Lie algebras considered are finite dimensional or countable dimensional.

2.1 Semisimple subalgebras of finite-dimensional Lie algebras

Let $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be two subalgebras of a given Lie algebra $\mathfrak{g}$. The subalgebras $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are called Aut $\mathfrak{g}$-conjugate, or simply $\mathfrak{g}$-*conjugate* if there exists an automorphism f of $\mathfrak{g}$ for which $\mathfrak{s}_1 = f(\mathfrak{s}_2)$. Clearly, two $\mathfrak{g}$-conjugate subalgebras of $\mathfrak{g}$ are isomorphic. The problem, solved by A. Malcev and E. Dynkin, is to describe all $\mathfrak{g}$-conjugacy classes of semisimple subalgebras $\mathfrak{s} \subset \mathfrak{g}$ for an arbitrary finite-dimensional Lie algebra $\mathfrak{g}$.

An essential result from [Mal] states that if $\mathfrak{g}'$ is a maximal semisimple subalgebra of a finite-dimensional Lie algebra $\mathfrak{g}$, and $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are two $\mathfrak{g}$-conjugated semisimple subalgebras of $\mathfrak{g}$, then $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are also $\mathfrak{g}'$-conjugated. This result reduces the problem of classifying semisimple subalgebras of an arbitrary finite-dimensional Lie algebra to the study of maximal semisimple subalgebras of a Lie algebra and the study of semisimple subalgebras of semisimple Lie algebras. Both of the latter two problems are solved in [Mal], [Dy].

Furthermore, let $\varepsilon : \mathfrak{s} \to \mathfrak{g}$ be an injective homomorphism of semisimple finite-dimensional Lie algebras. Then $\mathfrak{g}$ is isomorphic to a direct sum of simple Lie algebras, so $\mathfrak{g} \cong \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ with all $\mathfrak{g}_i$ being simple. The homomorphism ε is recovered by the homomorphisms $\pi_i \circ \varepsilon : \mathfrak{s} \to \mathfrak{g}_i$, where π_i are the corresponding projections.

Therefore the problem of classifying the homomorphisms of semisimple Lie algebras reduces to the problem of describing the homomorphisms of semisimple Lie algebras into a simple Lie algebra.

The study of semisimple Lie algebras of the Lie algebra A_n is equivalent to the study of linear representations of semisimple Lie algebras. The main results in this direction were obtained by E. Cartan and H. Weyl in the first quarter of the last century. Let $\mathfrak{g}$ be a simple classical Lie algebra of type other than A, and $\mathfrak{s}$ be a semisimple Lie algebra. Malcev shows that to describe $\mathfrak{g}$-conjugacy classes of $\mathfrak{s}$ in $\mathfrak{g}$ one needs the following two steps. First one finds all representations of $\mathfrak{s}$ which yield a homomorphism $\mathfrak{s} \to \mathfrak{g}$: this is equivalent to finding all orthogonal or symplectic representations of $\mathfrak{s}$ of dimension equal to the dimension of the natural representation of $\mathfrak{g}$, up to isomorphism. As a second step one describes all obtained representations up to $\mathfrak{g}$-equivalence, where $\mathfrak{g}$-*equivalent* representations are the ones which are conjugate by an outer automorphism of $\mathfrak{g}$. It is moreover shown in [Mal] that if $\mathfrak{g}$ is of type B or C, then the first step is sufficient, i.e. the isomorphism classes of orthogonal or symplectic representations of $\mathfrak{s}$ of a given dimension correspond exactly to the $\mathfrak{g}$-conjugacy classes of $\mathfrak{s}$. Malcev further describes all symplectic and orthogonal representations of semisimple subalgebras up to isomorphism, and in this way solves the problem of classifying all semisimple subalgebras of simple classical Lie algebras. He also studies the case when $\mathfrak{g}$ is one of the exceptional Lie algebras G_2 and F_4.

We now recall the definition of index of a simple subalgebra in a simple Lie algebra introduced by Dynkin in [Dy]. For a simple finite-dimensional Lie algebra $\mathfrak{g}$ we denote by $\langle\ ,\ \rangle_{\mathfrak{g}}$ the invariant non-degenerate symmetric bilinear form on $\mathfrak{g}$ normalized so that $\langle \alpha, \alpha \rangle_{\mathfrak{g}} = 2$ for any long root α of $\mathfrak{g}$. If $\varphi : \mathfrak{s} \to \mathfrak{g}$ is an injective homomorphism of simple Lie algebras, then $\langle x, y \rangle_{\varphi} := \langle \varphi(x), \varphi(y) \rangle_{\mathfrak{g}}$ is an invariant non-degenerate symmetric bilinear form on $\mathfrak{s}$. Consequently,

$$\langle x, y \rangle_{\varphi} = I_{\mathfrak{s}}^{\mathfrak{g}}(\varphi) \langle x, y \rangle_{\mathfrak{s}}$$

for some scalar $I_{\mathfrak{s}}^{\mathfrak{g}}(\varphi)$. By definition $I_{\mathfrak{s}}^{\mathfrak{g}}(\varphi)$ is the *Dynkin index* (or simply the *index*) of $\mathfrak{s}$ in $\mathfrak{g}$. If φ is clear from the context, we will simply write $I_{\mathfrak{s}}^{\mathfrak{g}}$. If U is any finite-dimensional $\mathfrak{s}$-module, then the *index* $I_{\mathfrak{s}}(U)$ of U is defined as $I_{\mathfrak{s}}^{\mathrm{sl}(U)}$, where $\mathfrak{s}$ is mapped into $\mathrm{sl}(U)$ through the module U. The following properties of the index are established in [Dy].

Proposition 2.1.1. *(i)* $I_{\mathfrak{s}}^{\mathfrak{g}} \in \mathbb{Z}_{\geq 0}$.

(ii) $I_{\mathfrak{s}}^{\mathfrak{k}} I_{\mathfrak{k}}^{\mathfrak{g}} = I_{\mathfrak{s}}^{\mathfrak{g}}$.

(iii) $I_{\mathfrak{s}}(U_1 \oplus \cdots \oplus U_n) = I_{\mathfrak{s}}(U_1) + \cdots + I_{\mathfrak{s}}(U_n)$.

(iv) If U is an $\mathfrak{s}$-module with highest weight λ (with respect to some Borel subalgebra), then $I_{\mathfrak{s}}(U) = \frac{\dim U}{\dim \mathfrak{s}} \langle \lambda, \lambda + 2\rho \rangle_{\mathfrak{s}}$, *where 2ρ is the sum of all positive roots of $\mathfrak{s}$.*

2.2 Diagonal infinite-dimensional Lie algebras

A Lie algebra $\mathfrak{g}$ is called *locally finite* if any finite subset S of $\mathfrak{g}$ is contained in a finite-dimensional Lie subalgebra $\mathfrak{g}(S)$ of $\mathfrak{g}$. If, for any S, $\mathfrak{g}(S)$ can be chosen simple (semisimple, reductive), $\mathfrak{g}$ is called *locally simple (semisimple, reductive)*. An *exhaustion*

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots$$

of a locally finite Lie algebra $\mathfrak{g}$ is a direct system of finite-dimensional Lie subalgebras of $\mathfrak{g}$ such that the direct limit Lie algebra $\varinjlim \mathfrak{g}_n$ is isomorphic to $\mathfrak{g}$.

The classical locally finite infinite-dimensional Lie algebras $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, and $\mathrm{sp}(\infty)$ are defined as the unions $\cup_{i\in\mathbb{Z}_{>1}}\mathrm{sl}(i)$, $\cup_{i\in\mathbb{Z}_{>1}}\mathrm{o}(i)$, and $\cup_{i\in\mathbb{Z}_{>1}}\mathrm{sp}(2i)$, respectively, for any inclusions $\mathrm{sl}(i) \subset \mathrm{sl}(i+1)$, $\mathrm{o}(i) \subset \mathrm{o}(i+1)$, and $\mathrm{sp}(2i) \subset \mathrm{sp}(2i+2)$, $i > 1$. It is easy to show that each of the above Lie algebras does not depend up to isomorphism on the particular exhaustions chosen for its definition. In particular, the union of even orthogonal Lie algebras and the union of odd orthogonal Lie algebras are isomorphic Lie algebras. We therefore do not distinguish the types B and D in the infinite-dimensional case, and when considering classical simple Lie algebras, we consider three types A, C, and O, where O stands for both types B and D.

A locally reductive Lie algebra $\mathrm{gl}(\infty)$ is defined as the union $\cup_{i\in\mathbb{Z}_{>1}}\mathrm{gl}(i)$ for the standard left-hand corner inclusions $\mathrm{gl}(i) \subset \mathrm{gl}(i+1)$. A Lie algebra isomorphic to a subalgebra of $\mathrm{gl}(\infty)$ is called *finitary*. As it was shown by A. Baranov, up to isomorphism, the Lie algebras $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, and $\mathrm{sp}(\infty)$ are the only countable-dimensional finitary locally simple Lie algebras, see [B2], [B3], [B4], [BS].

The notion of a diagonal locally finite Lie algebra is closely related to the notion of a diagonal injective homomorphism of two finite-dimensional Lie algebras. Let $\mathfrak{g}_1$ and $\mathfrak{g}_2$ be two finite-dimensional perfect Lie algebras (i.e. $[\mathfrak{g}_i, \mathfrak{g}_i] = \mathfrak{g}_i$, $i = 1, 2$). Let $\mathfrak{s}_i = \mathfrak{s}_i^1 \oplus \cdots \oplus \mathfrak{s}_i^{n_i}$ be a Levi subalgebra of $\mathfrak{g}_i$, where $\mathfrak{s}_i^1, \ldots, \mathfrak{s}_i^{n_i}$ are the simple constituents of $\mathfrak{s}_i$, and let V_i^k be the natural $\mathfrak{s}_i^k$-module, $i = 1, 2$. Since $\mathfrak{s}_i$ is perfect, there exists a unique irreducible $\mathfrak{s}_i$-module W_i^k such that the restriction

$W_i^k \downarrow \mathfrak{s}_i^k$ is isomorphic to V_i^k, $i = 1, 2$. An injective homomorphism $\mathfrak{g}_1 \to \mathfrak{g}_2$ is called *diagonal* if all the simple constituents of the $\mathfrak{s}_1$-module $W_2^l \downarrow \mathfrak{s}_1$ belong to the set $\{W_1^1, \ldots, W_1^{n_1}, (W_1^1)^*, \ldots, (W_1^{n_1})^*, T_1\}$, $1 \leq l \leq n_2$, where T_1 is the trivial one-dimensional $\mathfrak{s}_1$-module, and $(W_1^k)^*$ is the dual module to W_1^k. A locally finite Lie algebra $\mathfrak{g}$ is called *diagonal* if it admits an exhaustion by perfect subalgebras $\mathfrak{g}_i$ such that all inclusions $\mathfrak{g}_i \subset \mathfrak{g}_{i+1}$ are diagonal homomorphisms.

In the papers [B1], [B2] some general properties of diagonal locally finite Lie algebras are established. For instance, a criterion proved in [B1] which claims that a simple locally finite Lie algebra is diagonal if and only if it admits an injective homomorphism into a Lie algebra associated with some locally finite associative algebra.

In this thesis we will be interested in the more restrictive class of diagonal locally simple Lie algebras. The definition of these Lie algebras can be rewritten in a simpler way. Indeed, if $\mathfrak{g}_1 \subset \mathfrak{g}_2$ are simple classical Lie algebras, the definition of a diagonal inclusion is equivalent to the requirement that

$$V_2 \downarrow \mathfrak{g}_1 \cong \underbrace{V_1 \oplus \ldots \oplus V_1}_{l} \oplus \underbrace{V_1^* \oplus \ldots \oplus V_1^*}_{r} \oplus \underbrace{T_1 \oplus \ldots \oplus T_1}_{z},$$

where V_i is the natural $\mathfrak{g}_i$-module ($i = 1, 2$), V_1^* is the dual of V_1, and T_1 is the one-dimensional trivial $\mathfrak{g}_1$-module. The triple (l, r, z) is called the *signature* of $\mathfrak{g}_1$ in $\mathfrak{g}_2$. The *signature* of $\varepsilon : \mathfrak{g}_1 \to \mathfrak{g}_2$ is by definition the signature of $\varepsilon(\mathfrak{g}_1)$ in $\mathfrak{g}_2$.

It is a result of Baranov (Corollary 5.9 in [B1]) that for any exhaustion $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots$ of a diagonal locally finite Lie algebra $\mathfrak{g}$, all injective homomorphisms $\mathfrak{g}_i \subset \mathfrak{g}_{i+1}$ are diagonal for large enough i. As a consequence, a diagonal locally simple Lie algebra can be defined as a diagonal Lie algebra which admits an exhaustion

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots \tag{2.1}$$

by classical simple finite-dimensional Lie algebras, or equivalently as the direct limit of a sequence of diagonal inclusions (2.1) of classical simple finite-dimensional Lie algebras.

The class of diagonal locally simple Lie algebras contains the three finitary Lie algebras sl(∞), so(∞), sp(∞). It makes sense to consider finitary and non-finitary Lie algebras separately, as in the finitary case many more advanced results are available. In particular, the result of Malcev and Dynkin mentioned above is partly generalized in [DP3]. The ideas of the paper [DP3] play a key role in Chapter 4 of the present thesis.

We now recall an alternative definition of the Lie algebras $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, $\mathrm{sp}(\infty)$, and $\mathrm{gl}(\infty)$. Let V be a fixed countable-dimensional vector space with basis $v_1, v_2, \ldots$ and V_* is the restricted dual of V, i.e. the span of the dual set $v_1^*, v_2^*, \ldots$ $(v_i^*(v_j) = \delta_{ij})$. The space $V \otimes V_*$ has an obvious structure of an associative algebra, and by definition $\mathrm{gl}(V, V_*)$ (or $\mathrm{gl}(\infty)$) is the Lie algebra associated with this associative algebra. The Lie algebra $\mathrm{sl}(V, V_*)$ ($\mathrm{sl}(\infty)$) is the commutator algebra $[\mathrm{gl}(V, V_*), \mathrm{gl}(V, V_*)]$. Given a symmetric non-degenerate form $V \times V \to \mathbb{C}$, we denote by $\mathrm{so}(V)$ ($\mathrm{so}(\infty)$) the subalgebra $\bigwedge^2(V) \subset \mathrm{sl}(V, V_*)$ (the form $V \times V \to \mathbb{C}$ induces an identification of V with V_* which allows to consider $\bigwedge^2(V)$ as a subspace of $V \otimes V_*$). Similarly, given an antisymmetric non-degenerate form $V \times V \to \mathbb{C}$, we denote by $\mathrm{sp}(V)$ ($\mathrm{sp}(\infty)$) the subalgebra $S^2(V) \subset \mathrm{sl}(V, V_*)$. Following these notations, the vector spaces V and V_* are by definition the *natural* and the *conatural* $\mathfrak{g}$-modules for $\mathfrak{g} \cong \mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, $\mathrm{sp}(\infty)$, or $\mathrm{gl}(\infty)$. These modules can be defined alternatively: V (respectively, V_*) is (up to isomorphism) the only simple $\mathfrak{g}$-module which, for any exhaustion $\mathfrak{g} = \cup_i \mathfrak{g}_i$, restricts to a direct sum of the natural (respectively, its dual) representation of $\mathfrak{g}_i$ and a trivial module. Notice that for $\mathfrak{g} \cong \mathrm{so}(\infty)$ or $\mathrm{sp}(\infty)$ the natural $\mathfrak{g}$-module V is isomorphic to the conatural $\mathfrak{g}$-module V_*.

Let $\mathfrak{g} \cong \mathrm{gl}(\infty)$, $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, or $\mathrm{sp}(\infty)$. It is shown in [DP3] that a locally semisimple subalgebra of $\mathfrak{g}$ is isomorphic to a direct sum of simple Lie algebras and, moreover, that each of these simple constituents is either finite-dimensional or is itself isomorphic to $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, or $\mathrm{sp}(\infty)$. In the framework of extending Malcev's approach to homomorphisms of infinite-dimensional Lie algebras, it is furthermore shown that the structures of the natural $\mathfrak{g}$-module V and the conatural $\mathfrak{g}$-module V_* as modules over any locally semisimple subalgebra $\mathfrak{s} \subset \mathfrak{g}$ can be described as follows:

- the socle filtration of V (respectively, V_*) has depth at most 2;
- the non-trivial simple direct summands of the socle V' of V (resp., $(V_*)'$ of V_*) are just natural and conatural modules over infinite-dimensional simple ideals of $\mathfrak{s}$, as well as finite-dimensional modules over finite-dimensional ideals of $\mathfrak{s}$; each non-trivial simple constituent of V (resp., V_*) as a module over a simple ideal of $\mathfrak{s}$ occurs with finite multiplicity;
- the $\mathfrak{s}$-modules V/V' and $V_*/(V_*)'$ are trivial.

In contrast with the finite-dimensional case, V and V_* are not necessarily semisimple $\mathfrak{s}$-modules. Nevertheless, the above three statements show that V and V_* have very "rigid" and completely explicit structures as $\mathfrak{s}$-modules.

As we already mentioned in the first section, it has been shown by A. Malcev in [Mal] that a homomorphism of finite-dimensional Lie algebras $\mathfrak{s} \to \mathfrak{g}$ can be characterized by the induced structure of the natural $\mathfrak{g}$-module V as an $\mathfrak{s}$-module. It is not known whether this is the case for infinite-dimensional Lie algebras as the problem of determining $\mathfrak{g}$-conjugacy classes of locally semisimple subalgebras $\mathfrak{s} \subset \mathfrak{g}$ remains open for $\mathfrak{g} \cong \mathrm{gl}(\infty)$, $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, or $\mathrm{sp}(\infty)$. However, the study of the structures of the natural and the conatural $\mathfrak{g}$-modules as an $\mathfrak{s}$-module is an important step as essential invariants of a homomorphism $\mathfrak{s} \to \mathfrak{g}$ are encoded in these $\mathfrak{s}$-module structures.

We now go back to discussing general diagonal locally simple Lie algebras. These Lie algebras are described by A. Baranov and A. Zhilinskii in [BZ]. Let us recall some notions of this paper and state its main result, namely the classification of diagonal locally simple Lie algebras up to isomorphism. This classification plays a key role in the present thesis.

Let $p_1 = 2, p_2 = 3, \ldots$ be the increasing sequence of all prime numbers. A map from the set $\{p_1, p_2, \ldots\}$ into the set $\{0, 1, 2, \ldots\} \bigcup \{\infty\}$ is called a *Steinitz number*. The Steinitz number which has value α_1 at p_1, α_2 at p_2, etc. will be denoted by $p_1^{\alpha_1} p_2^{\alpha_2} \cdots$. Let $\Pi = p_1^{\alpha_1} p_2^{\alpha_2} \cdots$ and $\Pi' = p_1^{\alpha'_1} p_2^{\alpha'_2} \cdots$ be two Steinitz numbers. We put $\Pi\,\Pi' = p_1^{\alpha_1+\alpha'_1} p_2^{\alpha_2+\alpha'_2} \cdots$, and we say that Π *divides* Π' (or $\Pi|\Pi'$) if and only if $\alpha_1 \le \alpha'_1$, $\alpha_2 \le \alpha'_2$, $\ldots$. In the latter case we write $\div(\Pi', \Pi) = p_1^{\alpha'_1-\alpha_1} p_2^{\alpha'_2-\alpha_2} \cdots$, where by convention $p_i^{\infty-\infty} = 1$ for any i. We also define the greatest common divisor $\mathrm{GCD}(\Pi, \Pi')$ as $p_1^{\min(\alpha_1,\alpha'_1)} p_2^{\min(\alpha_2,\alpha'_2)} \cdots$.

Let $q \in \mathbb{Q}$. We write $\Pi = q\Pi'$ (or $q \in \frac{\Pi}{\Pi'}$) if there exists $n \in \mathbb{N}$ such that $nq \in \mathbb{N}$ and $n\Pi = nq\Pi'$. If there exists $0 \neq q \in \mathbb{Q}$ such that $\Pi = q\Pi'$, then we say that Π and Π' are $\mathbb{Q}$-*equivalent* and denote this relation by $\Pi \overset{\mathbb{Q}}{\sim} \Pi'$. Suppose $q \in \frac{\Pi}{\Pi'}$ for some $0 \neq q \in \mathbb{Q}$. If p^∞ divides Π, then p^∞ also divides Π' and so $\Pi = qp^k\Pi'$ for all $k \in \mathbb{Z}$. Hence in this case $\{qp^k\}_{k\in\mathbb{Z}}$ is a subset of $\frac{\Pi}{\Pi'}$ in our notation. On the other hand, if there is no prime p with p^∞ dividing Π, then the set $\frac{\Pi}{\Pi'}$ consists of the only element q. If $\mathcal{S} = (s_1, s_2, \ldots)$ is a sequence of positive integers, $\mathrm{Stz}(\mathcal{S})$ denotes the infinite product $\prod_{i=1}^{\infty} s_i$ considered as a Steinitz number.

Let $\mathfrak{s}$ be an infinite-dimensional diagonal locally simple Lie algebra, so there is an exhaustion $\mathfrak{s} = \cup_i \mathfrak{s}_i$ with all inclusions $\mathfrak{s}_i \subset \mathfrak{s}_{i+1}$ being diagonal. Without loss of generality we may assume that all $\mathfrak{s}_i$ are of the same type X ($X = A$, C, or O), and we say that $\mathfrak{s}$ is of type X. Note that a diagonal Lie algebra can be of more than one type. The triple (l_i, r_i, z_i) denotes the signature of the homomorphism

$\mathfrak{s}_i \to \mathfrak{s}_{i+1}$ and n_i denotes the dimension of the natural $\mathfrak{s}_i$-module. We assume that $r_i = 0$ if X is not A (for all classical Lie algebras of type other than A the natural representation is isomorphic to its dual). We also assume that $l_i \geq r_i$ for all i for type A algebras. (This does not restrict generality as one can apply outer automorphisms to a suitable subexhaustion if necessary.) Finally, if not stated otherwise, we assume that $n_1 = 1$, $l_1 = n_2$, $r_1 = z_1 = 0$. Denote by $\mathcal{T}$ the sequence of all such triples $\{(l_i, r_i, z_i)\}_{i\in\mathbb{N}}$. We will write $\mathfrak{s} = X(\mathcal{T})$ as a manifestation of the fact that $\mathcal{T}$ determines $\mathfrak{s}$ up to isomorphism.

Set $s_i = l_i + r_i$, $c_i = l_i - r_i$ $(i \geq 1)$, $\mathcal{S} = (s_i)_{i\in\mathbb{N}}$, $\mathcal{C} = (c_i)_{i\in\mathbb{N}}$. Put $\delta_i = \frac{s_1 \cdots s_{n-1}}{n_i}$. Then $\delta_{i+1} = \frac{s_1\cdots s_n}{n_{i+1}} = \frac{s_1\cdots s_{n-1}}{n_i + (z_i/s_i)} \leq \delta_i$. The limit $\delta = \lim_{i\to\infty} \delta_i$ is called the *density index* of $\mathcal{T}$ and is denoted by $\delta(\mathcal{T})$. Since $\delta_2 = s_1/n_2 = 1$, we have $0 \leq \delta \leq 1$. If $\delta = 0$ then the sequence of triples $\mathcal{T}$ is called *sparse*. If there exists i such that $\delta_j = \delta_i \neq 0$ for all $j > i$, the sequence is called *pure*. We say that $\mathcal{T}$ is *dense* if $0 < \delta < \delta_i$ for all i.

If there exists i such that $c_j = s_j$ for all $j \geq i$, then $\mathcal{T}$ is called *one-sided* (in which case we can and will assume that $c_j = s_j$ for all $j \geq 1$). Otherwise it is called *two-sided*. If, for each i, there exists $j > i$ such that $c_j = 0$, then $\mathcal{T}$ is called *symmetric*. Otherwise it is called *non-symmetric*. In the latter case we will assume that $c_i > 0$ for all $i \geq 1$. Set $\sigma_i = \frac{c_1\cdots c_i}{s_1\cdots s_i}$. The limit $\sigma = \lim_{i\to\infty} \sigma_i$ is called the *symmetry index* of $\mathcal{T}$ and is denoted by $\sigma(\mathcal{T})$. Observe that $0 \leq \sigma \leq 1$. Two-sided non-symmetric sequences $\mathcal{T}$ with $\sigma(\mathcal{T}) = 0$ are called *weakly non-symmetric*, and those with $\sigma(\mathcal{T}) \neq 0$ are called *strongly non-symmetric*.

The classification of the infinite-dimensional diagonal locally simple Lie algebras is given by the following two theorems.

Theorem 2.2.1. *[BZ] Let $X = A$, C, or O. Let $\mathcal{T} = \{(l_i, r_i, z_i)\}$ and $\mathcal{T}' = \{(l'_i, r'_i, z'_i)\}$, where $r_i = r'_i = 0$ if $X \neq A$. Set $\delta = \delta(\mathcal{T})$, $\sigma = \sigma(\mathcal{T})$, $\delta' = \delta(\mathcal{T}')$, $\sigma' = \sigma(\mathcal{T}')$. Then $X(\mathcal{T}) \cong X(\mathcal{T}')$ if and only if the following conditions hold.*

($\mathcal{A}_1$) *The sequences $\mathcal{T}$ and $\mathcal{T}'$ have the same density type.*

($\mathcal{A}_2$) $\mathrm{Stz}(\mathcal{S}) \overset{\mathbb{Q}}{\sim} \mathrm{Stz}(\mathcal{S}')$.

($\mathcal{A}_3$) $\frac{\delta}{\delta'} \in \frac{\mathrm{Stz}(\mathcal{S})}{\mathrm{Stz}(\mathcal{S}')}$ *for dense and pure sequences.*

($\mathcal{B}_1$) *The sequences $\mathcal{T}$ and $\mathcal{T}'$ have the same symmetry type.*

($\mathcal{B}_2$) $\mathrm{Stz}(\mathcal{C}) \overset{\mathbb{Q}}{\sim} \mathrm{Stz}(\mathcal{C}')$ *for two-sided non-symmetric sequences.*

($\mathcal{B}_3$) *There exists $\alpha \in \frac{\mathrm{Stz}(\mathcal{S})}{\mathrm{Stz}(\mathcal{S}')}$ such that $\alpha\frac{\sigma}{\sigma'} \in \frac{\mathrm{Stz}(\mathcal{C})}{\mathrm{Stz}(\mathcal{C}')}$ for two-sided strongly non-symmetric sequences. Moreover, $\alpha = \frac{\delta}{\delta'}$ if in addition the triple sequences are* *dense or pure.*

Theorem 2.2.2. *[BZ] Let $\mathcal{T} = \{(l_i, r_i, z_i)\}$, $\mathcal{T}' = \{(l_i', 0, z_i')\}$, and $\mathcal{T}'' = \{(l_i'', 0, z_i'')\}$.*

(i) $A(\mathcal{T}) \cong O(\mathcal{T}')$ (resp., $A(\mathcal{T}) \cong C(\mathcal{T}')$) if and only if $\mathcal{T}$ is two-sided symmetric, 2^∞ divides $\mathrm{Stz}(\mathcal{S}')$, and the conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{A}_3)$ of Theorem 2.2.1 hold.

(ii) $O(\mathcal{T}') \cong C(\mathcal{T}'')$ if and only if 2^∞ divides both $\mathrm{Stz}(\mathcal{S}')$, and $\mathrm{Stz}(\mathcal{S}'')$, and the conditions $(\mathcal{A}_1)$, $(\mathcal{A}_2)$, $(\mathcal{A}_3)$ of Theorem 2.2.1 hold.

It is easy to see from Theorem 2.2.1 that a diagonal locally simple Lie algebra $X(\mathcal{T})$ is finitary (i.e. isomorphic to sl(∞), so(∞), or sp(∞)) if and only if $\mathrm{Stz}(\mathcal{S})$ is finite.

As we see from the above classification, the density type and the symmetry type are well-defined invariants of a diagonal locally simple Lie algebra. We will call such an algebra *pure*, *dense*, or *sparse* if its sequence of triples $\mathcal{T}$ can be chosen pure, dense, or sparse, respectively. We will also call an algebra *one-sided*, *two-sided symmetric*, *two-sided strongly non-symmetric*, or *two-sided weakly non-symmetric* if its sequence of triples $\mathcal{T}$ can be chosen with that respective property.

For an arbitrary sequence $\mathcal{S} = \{s_i\}_{i\geq 1}$ by $\mathrm{sl}(\mathrm{Stz}(\mathcal{S}))$ (respectively, $\mathrm{so}(\mathrm{Stz}(\mathcal{S}))$, $\mathrm{sp}(\mathrm{Stz}(\mathcal{S}))$) we will denote the pure Lie algebra $A(\{(s_i, 0, 0)\}_{i\geq 1})$ (resp., $O(\{(s_i, 0, 0)\}_{i\geq 1})$, $C(\{(s_i, 0, 0)\}_{i\geq 1})$).

The following result is due to A. Baranov.

Proposition 2.2.3. *Any simple subalgebra of a diagonal simple Lie algebra is diagonal.*

Proof. Let $\mathfrak{s}$ be a simple subalgebra of a diagonal Lie algebra $\mathfrak{s}'$. Corollary 5.11 in [B1] claims that a simple locally finite Lie algebra is diagonal if and only if it admits an injective homomorphism into a Lie algebra associated with some locally finite associative algebra. Hence $\mathfrak{s}'$ admits an injective homomorphism into a Lie algebra $\mathfrak{g}$ associated with a locally finite associative algebra. Since $\mathfrak{s}$ is locally finite, the existence of the injective homomorphism $\mathfrak{s} \to \mathfrak{s}' \to \mathfrak{g}$ implies that $\mathfrak{s}$ is diagonal too. □

Proposition 2.2.3 reduces the study of locally simple subalgebras of diagonal Lie algebras to the study of diagonal locally simple subalgebras.

As we have seen from the classification theorems, the isomorphism class of a diagonal locally simple Lie algebra $\mathfrak{s}$ can be characterized in terms of the sequence

$\mathcal{T}$ of signatures of a fixed simple exhaustion $\mathfrak{s} = \cup_i \mathfrak{s}_i$. The following corollary of Proposition 2.1.1 expresses the Dynkin index of a diagonal inclusion of classical simple finite-dimensional Lie algebras in terms of the signature of the inclusion.

Corollary 2.2.4. *Let $\mathfrak{s} \subset \mathfrak{g}$ be a diagonal inclusion of signature (l, r, z), $\mathfrak{s}$ and $\mathfrak{g}$ being finite-dimensional classical simple Lie algebras of the same type (A, C, or O). Then $I_{\mathfrak{s}}^{\mathfrak{g}} = l + r$.*

Proof. Indeed, if V is the natural $\mathfrak{s}$-module then clearly $I_{\mathfrak{s}}(V) = I_{\mathfrak{s}}(V^*)$, and (iii) implies the result for type A algebras. If $\mathfrak{s}$ and $\mathfrak{g}$ are of type O or C then the result follows from the observation in [DP3] that $I_{\mathfrak{s}}^{\mathrm{sp}(U)} = I_{\mathfrak{s}}(U)$ and $I_{\mathfrak{s}}^{\mathrm{so}(U)} = \frac{1}{2} I_{\mathfrak{s}}(U)$ when U admits a corresponding invariant form. This latter observation follows easily from [Dy]. □

If $\mathfrak{s}$ and $\mathfrak{g}$ are two diagonal locally simple Lie algebras, then constructing a homomorphism $\theta : \mathfrak{s} \to \mathfrak{g}$ is equivalent to constructing commutative diagram

$$\begin{array}{ccccc} \mathfrak{s}_1 & \xrightarrow{\varphi_1} & \mathfrak{s}_2 & \xrightarrow{\varphi_2} & \cdots \\ \downarrow{\scriptstyle\theta_1} & & \downarrow{\scriptstyle\theta_2} & & \\ \mathfrak{g}_1 & \xrightarrow{\psi_1} & \mathfrak{g}_2 & \xrightarrow{\psi_2} & \cdots \end{array} \tag{2.2}$$

for some exhaustions $\mathfrak{s}_1 \stackrel{\varphi_1}{\to} \mathfrak{s}_2 \stackrel{\varphi_2}{\to} \dots$ and $\mathfrak{g}_1 \stackrel{\psi_1}{\to} \mathfrak{g}_2 \stackrel{\psi_2}{\to} \dots$ of $\mathfrak{s}$ and $\mathfrak{g}$ respectively. An injective homomorphism θ is called *diagonal* if all θ_i can be chosen diagonal for sufficiently large i.

To deal with diagonal homomorphisms we will need the following result.

Lemma 2.2.5. *Let $\varepsilon_1 : \mathfrak{s}_1 \to \mathfrak{s}_2$ and $\varepsilon_2 : \mathfrak{s}_1 \to \mathfrak{g}$ be diagonal injective homomorphisms of finite-dimensional simple classical Lie algebras of signatures (l, r, z) and (p, q, u) respectively. Let a triple of non-negative integers (p', q', u') satisfy the following conditions:*

$$p + q = (l + r)(p' + q'),\ p - q = (l - r)(p' - q'),\ n = n_2(p' + q') + u',$$

where n and n_2 are the dimensions of the natural $\mathfrak{g}$- and $\mathfrak{s}_2$-modules respectively. Then, under the assumption that $\mathfrak{s}_2$ and $\mathfrak{g}$ are of the same type X, there exists a diagonal injective homomorphism $\theta : \mathfrak{s}_2 \to \mathfrak{g}$ of signature (p', q', u') such that $\varepsilon_2 = \theta \circ \varepsilon_1$. If $\mathfrak{s}_2$ and $\mathfrak{g}$ are of different types X and Y, the statement holds under *the following additional conditions on the triple* (p', q', u')*:*

$$p' = q' \text{ if } (X,Y) = (A,O) \text{ or } (X,Y) = (A,C);$$
$$p' \text{ is even if } (X,Y) = (O,C) \text{ or } (X,Y) = (C,O).$$

Proof. Lemma 2.6 in [BZ] states the same result in case all Lie algebras $\mathfrak{s}_1$, $\mathfrak{s}_2$, $\mathfrak{g}$ are of the same type. The proof of Lemma 2.6 in [BZ] works also when the three algebras are not of the same type, but only if $\mathfrak{s}_2$ can be mapped into $\mathfrak{g}$ by an injective homomorphism of signature (p', q', u'). It is easy to check that the additional conditions guarantee the existence of such a homomorphism. □

Consider the diagram in (2.2) without the commutativity assumption. Lemma 2.2.5 implies that if each θ_i is a diagonal injective homomorphism and, for any $i \geq 1$, the two diagonal injective homomorphisms $\psi_i \circ \theta_i$ and $\theta_{i+1} \circ \varphi_i$ of $\mathfrak{s}_i$ into $\mathfrak{g}_{i+1}$ have the same signature, then there are diagonal injective homomorphisms θ'_i with the same property making the diagram commutative. Later on in the thesis when constructing diagrams as in (2.2) in concrete situations, we will check commutativity by showing only that the signatures of $\psi_i \circ \theta_i$ and $\theta_{i+1} \circ \varphi_i$ coincide for all $i \geq 1$. It will then be assumed that θ_i are replaced by corresponding diagonal injective homomorphisms θ'_i making the diagram commute.

We conclude this section by the result which can be found in [BZ] (see also all references in there, for instance [B2]).

Lemma 2.2.6. *Let* $\mathfrak{h} \subset \mathfrak{g} \subset \mathfrak{s}$ *be finite-dimensional classical simple Lie algebras,* $\operatorname{rk} \mathfrak{h} > 10$*. Assume that the inclusion* $\mathfrak{h} \subset \mathfrak{s}$ *is diagonal. Then the inclusions* $\mathfrak{h} \subset \mathfrak{g}$ *and* $\mathfrak{g} \subset \mathfrak{s}$ *are also diagonal.*

Corollary 2.2.7. *Let* $\mathfrak{h} \subset \mathfrak{g} \subset \mathfrak{s}$ *be infinite-dimensional diagonal locally simple Lie algebras. Assume that the inclusion* $\mathfrak{h} \subset \mathfrak{s}$ *is diagonal. Then the inclusions* $\mathfrak{h} \subset \mathfrak{g}$ *and* $\mathfrak{g} \subset \mathfrak{s}$ *are also diagonal.*

2.3 Branching rules

For a given injective homomorphism $\varepsilon : \mathfrak{s}_1 \to \mathfrak{s}_2$ of Lie algebras, a branching rule is a rule which allows to decompose an arbitrary $\mathfrak{s}_2$-module as a $\mathfrak{s}_1$-module through the homomorphism ε. In this thesis we use branching rules for two types of homomorphisms. The first one is the so-called standard homomorphisms, i.e. homomorphisms of signature $(1, 0, 1)$ which are used to define the classical infinite-dimensional Lie algebras. The second type are the homomorphisms of signature $(k, 0, 0)$ ("proper" diagonal homomorphisms). We now present these two branching rules for Lie algebras of type A.

Throughout the thesis F_n^λ denotes an irreducible $\mathrm{sl}(n)$-module with highest weight $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_i \in \mathbb{Z}_{\geq 0}$. Note that the isomorphism class of F_n^λ is determined by the differences $\lambda_1 - \lambda_2, \ldots, \lambda_{n-1} - \lambda_n$.

Theorem 2.3.1. *(Gelfand-Tsetlin rule [Z]) Consider a subalgebra* $\mathrm{sl}(n) \subset \mathrm{sl}(n+1)$ *of signature* $(1, 0, 1)$*. Then, there is an isomorphism of* $\mathrm{sl}(n)$*-modules*

$$F_{n+1}^\lambda \downarrow \mathrm{sl}(n) \cong \bigoplus_\mu F_n^\mu, \tag{2.3}$$

where the summation runs over all integral weights $\mu = (\mu_1, \ldots, \mu_n)$ *satisfying* $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \cdots \geq \mu_n \geq \lambda_{n+1}$.

Consider the $\mathrm{sl}(n) \oplus \mathrm{sl}(n)$-module $F_n^\mu \otimes F_n^\nu$. By Theorem 2.1.1 of [HTW] its restriction to $\mathrm{sl}(n) := \{x \oplus x,\, x \in \mathrm{sl}(n)\}$ decomposes as $\bigoplus_\lambda c_{\mu\nu}^\lambda F_n^\lambda$, where $c_{\mu\nu}^\lambda$ is the Littlewood-Richardson coefficient. One can iterate this branching rule to obtain the decomposition for higher tensor products. Let $c_{\mu_1 \ldots \mu_k}^\lambda$ denote the coefficient obtained in this manner, so,

$$F_n^{\mu_1} \otimes \cdots \otimes F_n^{\mu_k} \downarrow \mathrm{sl}(n) \cong \bigoplus_\lambda c_{\mu_1 \ldots \mu_k}^\lambda F_n^\lambda, \tag{2.4}$$

where the summation runs over all integral dominant weights λ with $\lambda_i \geq 0$. We will call the numbers $c_{\mu_1 \ldots \mu_k}^\lambda$ *generalized Littlewood-Richardson coefficients.*

The following branching rule was communicated to us by J. Willenbring.

Proposition 2.3.2. *Consider a diagonal subalgebra* $\mathrm{sl}(n) \subset \mathrm{sl}(kn)$ *of signature* $(k, 0, 0)$*. Then, there is an isomorphism of* $\mathrm{sl}(n)$*-modules*

$$F_{kn}^\lambda \downarrow \mathrm{sl}(n) \cong \bigoplus_\nu \Big(\sum_{\mu_1, \ldots, \mu_k} c_{\mu_1 \ldots \mu_k}^\lambda c_{\mu_1 \ldots \mu_k}^\nu \Big) F_n^\nu, \tag{2.5}$$

where one summation runs over all integral dominant weights ν *with* $\nu_i \geq 0$ *for all* i *and the other summation runs over all sets of integral dominant weights* $\mu_1, \ldots, \mu_k$ *with* $(\mu_j)_i \geq 0$ *for all* i, j.

Proof. Consider the block-diagonal subalgebra $\mathrm{sl}(l) \oplus \mathrm{sl}(m) \subset \mathrm{sl}(n)$ $(n = l + m)$. By Theorem 2.2.1 of [HTW] $F_n^\lambda \downarrow \mathrm{sl}(l) \oplus \mathrm{sl}(m)$ decomposes as $\bigoplus_{\mu\nu} c_{\mu\nu}^\lambda F_l^\mu \otimes F_m^\nu$. Let now the direct sum of k copies of $\mathrm{sl}(n)$ be a subalgebra $\mathrm{sl}(kn)$ with block diagonal inclusion. By iteration of this branching rule we see that the decomposition of $F_{kn}^{\lambda} \downarrow \mathrm{sl}(n) \oplus \cdots \oplus \mathrm{sl}(n)$ is determined by the generalized Littlewood-Richardson coefficients:

$$F_{kn}^{\lambda} \downarrow \mathrm{sl}(n) \oplus \cdots \oplus \mathrm{sl}(n) \cong \bigoplus_{\mu_1 \ldots \mu_k} c_{\mu_1 \ldots \mu_k}^{\lambda} F_n^{\mu_1} \otimes \cdots \otimes F_n^{\mu_k}, \tag{2.6}$$

where $sl(n) \oplus \cdots \oplus \mathrm{sl}(n)$ is the block-diagonal subalgebra of $\mathrm{sl}(kn)$, and the summation runs over all integral dominant weights $\mu_1, \ldots, \mu_k$ with $(\mu_j)_i \geq 0$.

Consider now a subalgebra $\mathrm{sl}(n) \subset \mathrm{sl}(kn)$ of signature $(k, 0, 0)$. One can obtain (2.5) as a combination of the two branching rules (2.4) and (2.6). □

In Proposition 2.3.2 the sum is taken over all integral dominant weights ν with $\nu_i \in \mathbb{Z}_{\geq 0}$ for all i. In order for F_n^{ν} to have a non-zero coefficient in (2.5) both Littlewood-Richardson coefficients $c_{\mu_1 \ldots \mu_k}^{\lambda}$ and $c_{\mu_1 \ldots \mu_k}^{\nu}$ must be non-zero for some $\mu_1, \ldots, \mu_k$. But for that we must have $\sum_{i=1}^{kn} \lambda_i = \sum_{i=1}^{n} \nu_i$. Therefore the summation in (2.5) may be taken to run over only those weights ν with fixed $\sum_{i=1}^{n} \nu_i$. Hence all modules F_n^{ν} which are present in (2.5) with non-zero coefficients are pairwise non-isomorphic. Indeed, if $F_n^{\nu'} \cong F_n^{\nu}$ both have non-zero coefficients in (2.5), then the weight ν' can be obtained by shifting the weight ν by an integer, so $\sum_{i=1}^{n} \nu_i = \sum_{i=1}^{n} \nu_i'$ implies $\nu' = \nu$. This argument allows us to refer to a non-zero coefficient $(\sum_{\mu_1, \ldots, \mu_k} c_{\mu_1 \ldots \mu_k}^{\lambda} c_{\mu_1 \ldots \mu_k}^{\nu})$ as the multiplicity of F_n^{ν} in (2.5).

Corollary 2.3.3. *For a diagonal subalgebra* $\mathrm{sl}(n) \subset \mathrm{sl}(kn)$ *of signature* $(k, 0, 0)$ *the restriction* $F_{kn}^{\lambda} \downarrow \mathrm{sl}(n)$ *has a submodule with highest weight*

$$(\nu_1, \ldots, \nu_n) = (\lambda_1 + \cdots + \lambda_k, \lambda_{k+1} + \cdots + \lambda_{2k}, \ldots, \lambda_{kn-k+1} + \cdots + \lambda_{kn}).$$

Proof. Indeed, if we set $\mu_i = (\lambda_i, \lambda_{k+i}, \ldots, \lambda_{kn-k+i})$ for $i \in \{1, \ldots, k\}$, then it easy to check that both coefficients $c_{\mu_1 \ldots \mu_k}^{\lambda}$ and $c_{\mu_1 \ldots \mu_k}^{\nu}$ are non-zero, and therefore the highest weight module F_n^{ν} is present in (2.5) with non-zero multiplicity. □

Chapter 3

Locally simple subalgebras of diagonal Lie algebras

In this chapter we classify up to isomorphism the locally simple subalgebras of diagonal locally simple Lie algebra. These results are presented in the article [Mar]. Throughout the rest of the dissertation all diagonal Lie algebras considered will be assumed to be infinite dimensional diagonal locally simple.

3.1 The main classification

We begin the classification by asking whether sl(∞) admits an injective homomorphism into any non-finitary diagonal Lie algebra. The answer turns out to be positive, and the following construction was suggested to us by I. Dimitrov.

Let F_n be the natural representation of sl(n). Note that under the injective homomorphism $\mathrm{sl}(n) \to \mathrm{sl}(n+1)$ of signature $(1, 0, 1)$, the exterior algebra $\bigwedge^{\cdot}(F_{n+1})$ decomposes as two copies of $\bigwedge^{\cdot}(F_n)$ as an sl(n)-module. Fix a map $\theta_n : \mathrm{sl}(n) \to \mathrm{sl}(2^n)$ such that the natural representation of $\mathrm{sl}(2^n)$ decomposes as $\bigwedge^{\cdot}(F_n)$ as an sl(n)-module. Then there exists a map $\theta_{n+1} : \mathrm{sl}(n+1) \to \mathrm{sl}(2^{n+1})$ such that the natural representation of $\mathrm{sl}(2^{n+1})$ decomposes as $\bigwedge^{\cdot}(F_{n+1})$ as an $\mathrm{sl}(n+1)$-module and the following diagram commutes:

$$\begin{array}{ccccccccc} \mathrm{sl}(2) & \longrightarrow & \cdots & \longrightarrow & \mathrm{sl}(n) & \longrightarrow & \mathrm{sl}(n+1) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle \theta_2} & & & & \downarrow{\scriptstyle \theta_n} & & \downarrow{\scriptstyle \theta_{n+1}} & & \\ \mathrm{sl}(2^2) & \longrightarrow & \cdots & \longrightarrow & \mathrm{sl}(2^n) & \longrightarrow & \mathrm{sl}(2^{n+1}) & \longrightarrow & \cdots . \end{array} \tag{3.1}$$

Here the lower row is assumed to consist of injective homomorphisms of signature $(2, 0, 0)$. By induction, the diagram in (3.1) yields an injective homomorphism of $\mathrm{sl}(\infty)$ into $\mathrm{sl}(2^\infty)$.

We will prove now that similar injective homomorphisms exist in a more general setting. The following result will be used later to prove that any finitary diagonal Lie algebra admits an injective homomorphism into any diagonal Lie algebra.

Proposition 3.1.1. $\mathrm{sl}(\infty)$ *admits an injective homomorphism into any pure one-sided Lie algebra* $\mathfrak{s}$ *of type A.*

Proof. By Theorem 2.2.1 $\mathfrak{s}$ is isomorphic to $\mathrm{sl}(\Pi)$ for some infinite Steinitz number Π. Therefore it is sufficient to show the existence of a commutative diagram

$$\begin{array}{ccccccccccc} \mathrm{sl}(2) & \longrightarrow & \mathrm{sl}(3) & \longrightarrow & \cdots & \longrightarrow & \mathrm{sl}(k) & \longrightarrow & \mathrm{sl}(k+1) & \longrightarrow & \cdots \\ \downarrow \theta_2 & & \downarrow \theta_3 & & & & \downarrow \theta_k & & \downarrow \theta_{k+1} & & \\ \mathrm{sl}(n_1 n_2) & \longrightarrow & \mathrm{sl}(n_1 n_2 n_3) & \longrightarrow & \cdots & \longrightarrow & \mathrm{sl}(n_1 \cdots n_k) & \longrightarrow & \mathrm{sl}(n_1 \cdots n_{k+1}) & \longrightarrow & \cdots \end{array} \tag{3.2}$$

for suitable $\{n_i\}$, where θ_i are injective homomorphisms and $n_1, n_2, \ldots$ are chosen so that $\prod_{i=1}^{\infty} n_i = \Pi$. Indeed, the diagram in (3.2) yields an injective homomorphism $\mathrm{sl}(\infty) \to \mathrm{sl}(n_1 n_2 \cdots)$, and $\mathrm{sl}(n_1 n_2 \cdots)$ is isomorphic to $\mathfrak{s}$ by Theorem 2.2.1. We will choose the homomorphisms θ_k so that there is an isomorphism of $\mathrm{sl}(k)$-modules

$$V_k \downarrow \mathrm{sl}(k) \cong a_0^k \bigwedge^0(F_k) \oplus a_1^k \bigwedge^1(F_k) \oplus \cdots \oplus a_k^k \bigwedge^k(F_k).$$

Here V_k stands for the natural $\mathrm{sl}(n_1 \cdots n_k)$-module, F_k is the natural $\mathrm{sl}(k)$-module and the coefficients a_i^k, $i = 0, \ldots, k$, are non-negative integers. The above injective homomorphism of $\mathrm{sl}(\infty)$ into $\mathrm{sl}(2^\infty)$ corresponds to the particular case $n_k = 2$ and $a_i^k = 1$ for all $k \geq 2$, $i = 0, \ldots, k$.

We see that if the numbers $\{a_i^k\}$ satisfy the conditions $a_i^k + a_{i+1}^k = n_k a_i^{k-1}$, $k \geq 3$, $i = 0, \ldots, k-1$ and $a_0^2 + 2a_1^2 + a_2^2 = n_1 n_2$, then the homomorphisms θ_k can be chosen so that the diagram in (3.2) commutes.

We will add numbers a_0^1, a_1^1, a_0^0 to the set of coefficients $\{a_i^k\}$ and will require $a_0^2 + a_1^2 = n_2 a_0^1$, $a_1^2 + a_2^2 = n_2 a_1^1$, $a_0^1 + a_1^1 = n_1$, and $a_0^0 = 1$. Then the numbers $\{a_i^k\}$

will form an infinite triangle

$$\begin{matrix} a_0^0 \\ a_0^1 \; a_1^1 \\ a_0^2 \; a_1^2 \; a_2^2 \\ \cdots \end{matrix}$$

such that

$$a_i^k + a_{i+1}^k = n_k a_i^{k-1},\ k \geq 1 \text{ and } a_0^0 = 1. \tag{3.3}$$

It is enough to prove that a triangle of non-negative integers satisfying (3.3) exists for a suitable choice of n_i. Set $b_k := \frac{a_{k-1}^k}{n_1 \cdots n_k}$ for $k \geq 1$. A simple calculation shows that $a_k^k = n_1 \cdots n_k (a_0^0 - b_1 - b_2 - \cdots - b_k)$. Notice that since $a_0^0 = 1$, the numbers $b_1, b_2, \ldots$ uniquely determine the entire triangle, as the l^{th} "diagonal" $\{a_k^{k+l}\}_{k \geq 0}$ of the triangle is determined by the previous diagonal $\{a_k^{k+l-1}\}_{k \geq 0}$ and the sequence $n_1, n_2, \ldots$.

Now we will find conditions on b_k which ensure that all a_i^k are non-negative. Since $a_k^{k+1} \geq 0$, the numbers b_k should be non-negative. In order for a_k^k to be non-negative we should have $\sum_{i=1}^{k} b_i < a_0^0$ for all k (since b_i are non-negative, we can rewrite these conditions as $\sum_{i=1}^{\infty} b_i \leq 1$). The entries of the diagonal $\{a_k^{k+2}\}_{k \geq 0}$ can be found from (3.3): $a_k^{k+2} = n_1 \cdots n_{k+2}(b_{k+1} - b_{k+2})$ for $k \geq 0$. This requires the sequence $\{b_k - b_{k+1}\}$ to be non-negative. If we set $b_k^{(1)} := b_k - b_{k+1}$ for $k \geq 1$, then in a similar way we obtain $a_k^{k+3} = n_1 \cdots n_{k+3}(b_{k+1}^{(1)} - b_{k+2}^{(1)})$. This requires the sequence $\{b_k^{(2)} := b_k^{(1)} - b_{k+1}^{(1)}\}$ to be non-negative. Continuing this procedure, we get $a_k^{k+l} = n_1 \cdots n_{k+l} b_{k+1}^{(l-1)}$ for all $l \geq 3$, where by definition $b_k^{(l+1)} = b_k^{(l)} - b_{k+1}^{(l)}$. Now we see that the non-negative integers a_i^k satisfying (3.3) exist if there exists a non-negative sequence $\{b_k\}_{k \geq 1}$ with $b_k \in \frac{1}{n_1 \cdots n_k}\mathbb{Z}_{\geq 0}$ and $\sum_{k=1}^{\infty} b_k \leq 1$ such that

$$\text{all iterated sequences of differences } \{b_k^{(l)}\}_{k \geq 1} \text{ are non-negative.} \tag{3.4}$$

Note that the sequence $\{b_k = \frac{1}{q^k}\}$, $q > 1$ satisfies (3.4) as $b_k^{(l)} = \frac{1}{q^k}(1 - \frac{1}{q})^l > 0$ for all $k, l \geq 1$. (In the case $n_k = n$ for all k, taking $q = n$ yields an injective homomorphism $\mathrm{sl}(\infty) \hookrightarrow \mathrm{sl}(n^\infty)$.) We will find the desired sequence $\{b_k\}$ as a convergent infinite linear combination of geometric sequences.

Fix $q \geq 4$ and let $\Pi = m_1 m_2 \cdots$. Choose a strictly increasing sequence of integers $\{l_k\}_{k \geq 0}$ so that $l_0 = 0$ and $m_1 m_2 \cdots m_{l_k} > \frac{(q-1)q^{k^2+1}}{q-2}$ for $k \geq 1$, which is possible as Π is infinite. Take $n_k = m_{l_{k-1}+1} \cdots m_{l_k}$ for $k \geq 1$. Then clearly $n_1 n_2 \cdots = \Pi$.

Let us now construct the sequence $\{b_k\}$ for the chosen $n_1, n_2, \ldots$. For $i \geq 1$ we denote $c_i = 1 + \sum_{j=i}^{\infty} \frac{\varepsilon_j}{\frac{1}{q^i}(\frac{1}{q^i} - \frac{1}{q}) \cdots (\frac{1}{q^i} - \frac{1}{q^{i-1}})(\frac{1}{q^i} - \frac{1}{q^{i+1}}) \cdots (\frac{1}{q^i} - \frac{1}{q^j})}$, where the numbers ε_j, satisfying

$$0 \leq \varepsilon_j < \frac{q-2}{(q-1)q^{j^2+1}}, \tag{3.5}$$

are to be chosen later, and put $b_k = \sum_{i=1}^{\infty} c_i \left(\frac{1}{q^i}\right)^k$. We will show that, if the numbers ε_j satisfy (3.5), then the series for c_i converges to a positive number for $i \geq 1$, the series for b_k converges for $k \geq 1$, and $\sum_{k=1}^{\infty} b_k \leq 1$. Moreover, we will show that by varying ε_j inside certain intervals we can make each b_k to be of the form $\frac{1}{n_1 \cdots n_k}\mathbb{Z}_{\geq 0}$. We will have then $b_k^{(l)} = \sum_{i=1}^{\infty} c_i \left(\frac{1}{q^i}\right)^k \left(1 - \frac{1}{q^i}\right)^l \geq 0$, so $\{b_k^{(l)}\}$ will be a sequence of non-negative numbers for any l. Hence the final condition in (3.4) will be satisfied.

As a matter of convenience we denote $q_i = \frac{1}{q^i}$. Then let $c_{ij} = \frac{\varepsilon_j}{q_i(q_i-q_1)\cdots(q_i-q_{i-1})(q_i-q_{i+1})\cdots(q_i-q_j)}$ for $i \leq j$. We see that $c_i = 1 + \sum_{j=i}^{\infty} c_{ij}$. Let us prove that this series converges absolutely. We have

$$\begin{aligned}
|c_i - 1| &= \left| \sum_{j=i}^{\infty} \frac{\varepsilon_j}{(\frac{1}{q^i})^j (1 - q^{i-1}) \cdots (1-q)(1-\frac{1}{q}) \cdots (1 - \frac{1}{q^{j-i}})} \right| \\
&\leq \sum_{j=i}^{\infty} \frac{\varepsilon_j}{(\frac{1}{q^i})^j (q^{i-1} - 1) \cdots (q-1)(1-\frac{1}{q}) \cdots (1 - \frac{1}{q^{j-i}})} \\
&\leq \sum_{j=i}^{\infty} \frac{\varepsilon_j q^{ij}}{(1-\frac{1}{q})(1-\frac{1}{q^2})\cdots} \leq \sum_{j=i}^{\infty} \frac{\varepsilon_j q^{ij}}{(1 - \frac{1}{q} - \frac{1}{q^2} - \cdots)} = \sum_{j=i}^{\infty} \frac{\varepsilon_j q^{ij}(q-1)}{q-2}.
\end{aligned}$$

Then, using (3.5), we obtain $|c_i - 1| \leq \sum_{j=i}^{\infty} \frac{q^{ij}}{q^{j^2+1}} = \frac{1}{q} + \frac{1}{q^{i+2}} + \frac{1}{q^{2i+5}} + \cdots < \frac{1}{q} + \frac{1}{q^2} + \cdots = \frac{1}{q-1}$. Thus, the series $1 + \sum_{j=i}^{\infty} c_{ij}$ converges absolutely and its sum c_i is a number from the interval $\left(\frac{q-2}{q-1}, \frac{q}{q-1}\right)$ (in particular, c_i is positive) for all i.

Furthermore,

$$\begin{aligned}\sum_{k=1}^{\infty} b_k &= \sum_{i=1}^{\infty} \frac{c_i}{q^i} + \sum_{i=1}^{\infty} \frac{c_i}{(q^2)^i} + \cdots < \frac{q}{q-1}\left(\sum_{i=1}^{\infty} \frac{1}{q^i} + \sum_{i=1}^{\infty} \frac{1}{(q^2)^i} + \cdots\right) \\ &= \frac{q}{q-1}\left(\frac{1}{q-1} + \frac{1}{q^2-1} + \frac{1}{q^3-1} + \cdots\right) \\ &< \frac{q}{q-1}\left(\frac{1}{q-1} + \frac{1}{(q-1)^2} + \cdots\right) = \frac{q}{q-1} \cdot \frac{1}{q-2} < 1 \text{ because } q \geq 4.\end{aligned}$$

Since every term in these expressions is non-negative, the convergence of each series $b_k = \sum_{i=1}^{\infty} c_i \left(\frac{1}{q^i}\right)^k$ follows.

Finally, let us show that the numbers ε_j, satisfying (3.5), can be chosen so that $b_k \in \frac{1}{n_1 \cdots n_k}\mathbb{Z}_{\geq 0}$. We know that $b_k = \sum_{i=1}^{\infty} c_i q_i^k = \sum_{i=1}^{\infty} q_i^k + \sum_{i=1}^{\infty} \sum_{j=i}^{\infty} c_{ij} q_i^k$. From what we proved it follows that the latter sum is absolutely convergent. Therefore we can rewrite it as $b_k = \sum_{i=1}^{\infty} q_i^k + \sum_{j=1}^{\infty} \sum_{i=1}^{j} c_{ij} q_i^k$. Note that the numbers c_{ij} were defined as solutions of the equation $\begin{pmatrix} q_1 & \dots & q_j \\ \vdots & \ddots & \vdots \\ q_1^{j-1} & \dots & q_j^{j-1} \\ q_1^j & \dots & q_j^j \end{pmatrix} \begin{pmatrix} c_{1j} \\ \vdots \\ c_{jj} \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \varepsilon_j \end{pmatrix}$ using the well-known formula for inverting a Vandermonde matrix. Thus, $\sum_{i=1}^{j} q_i^k c_{ij} = 0$ for $k < j$ and $\sum_{i=1}^{j} q_i^j c_{ij} = \varepsilon_j$. Hence, $b_k = \sum_{i=1}^{\infty} q_i^k + \sum_{j=1}^{k-1} \sum_{i=1}^{j} c_{ij} q_i^k + \varepsilon_k$, so $b_k - \varepsilon_k$ depends only on $\varepsilon_1, \dots, \varepsilon_{k-1}$. Let us introduce the notation $f_k(\varepsilon_1, \dots, \varepsilon_{k-1}) = \sum_{i=1}^{\infty} q_i^k + \sum_{j=1}^{k-1} \sum_{i=1}^{j} c_{ij} q_i^k$ for $k \geq 2$ and $f_1 = \sum_{i=1}^{\infty} q_i = \sum_{i=1}^{\infty} \frac{1}{q^i} = \frac{1}{q-1}$.

Now we define inductively the numbers ε_k. We choose ε_1 in such a way that b_1 is the smallest number of the form $\frac{1}{n_1}\mathbb{Z}_{\geq 0}$ which is not less than f_1. Then we have $0 \leq \varepsilon_1 = b_1 - f_1 < \frac{1}{n_1} < \frac{q-2}{(q-1)q^2}$ (because of the choice of n_1), so ε_1 lies inside the corresponding interval of (3.5). Assuming that $\varepsilon_1, \dots, \varepsilon_{k-1}$ are already chosen, we choose ε_k to make b_k the smallest number of the form $\frac{1}{n_1 \cdots n_k}\mathbb{Z}_{\geq 0}$ which is not less than $f_k(\varepsilon_1, \dots, \varepsilon_{k-1})$. Then $0 \leq \varepsilon_k = b_k - f_k(\varepsilon_1, \dots, \varepsilon_{k-1}) < \frac{1}{n_1 \cdots n_k} < \frac{q-2}{(q-1)q^{k^2+1}}$ (again, because of the choice of $n_1, \dots, n_k$), so ε_k satisfies (3.5). Therefore the sequence

$\{b_k\}$ satisfies all the required conditions, and the statement follows. □

Remark. Since so(∞) and sp(∞) are subalgebras of sl(∞), each of them admits also an injective homomorphism into any one-sided pure Lie algebra of type A.

The following two lemmas show that certain conditions guarantee the existence of injective homomorphisms of non-finitary diagonal Lie algebras.

Lemma 3.1.2. *Let $\mathfrak{s}_1 = X(\mathcal{T}_1)$ and $\mathfrak{s}_2 = X(\mathcal{T}_2)$ be diagonal Lie algebras of the same type ($X = A$, C, or O), neither of them finitary. Set $S_i = \mathrm{Stz}(\mathcal{S}_i)$, $S = \mathrm{GCD}(S_1, S_2)$, $R_i = \div(S_i, S)$, $\delta_i = \delta(\mathcal{T}_i)$, $C_i = \mathrm{Stz}(\mathcal{C}_i)$, $C = \mathrm{GCD}(C_1, C_2)$, $B_i = \div(C_i, C)$, and $\sigma_i = \sigma(\mathcal{T}_i)$ for $i = 1, 2$. We assume that R_1 is finite.*

(i) Assume that $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are non-sparse of type A, both R_1 and R_2 are finite, and S is not divisible by an infinite power of any prime number. If $2\frac{R_1}{\delta_1} < \frac{R_2}{\delta_2}$, then $\mathfrak{s}_1$ admits an injective homomorphism into $\mathfrak{s}_2$. If $2\frac{R_1}{\delta_1} = \frac{R_2}{\delta_2}$, then $\mathfrak{s}_1$ admits an injective homomorphism into $\mathfrak{s}_2$ unless $\mathfrak{s}_1$ is pure and $\mathfrak{s}_2$ is dense.

(ii) Assume that $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are non-sparse, both R_1 and R_2 are finite, and S is not divisible by an infinite power of any prime number. In addition, assume that one of the following is true:

- *both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are one-sided;*
- *B_1 is finite, either $\mathfrak{s}_1$ is one-sided and $\mathfrak{s}_2$ is two-sided non-symmetric or $\mathfrak{s}_2$ is two-sided weakly non-symmetric and $\mathfrak{s}_1$ is two-sided non-symmetric;*
- *B_1 is finite, both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are two-sided strongly non-symmetric, either B_2 is infinite or C is divisible by an infinite power of some prime number;*
- *both B_1 and B_2 are finite, both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are two-sided strongly non-symmetric, C is not divisible by an infinite power of a prime number, and $\frac{R_1\sigma_1}{B_1} \geq \frac{R_2\sigma_2}{B_2}$.*

Under these assumptions $\frac{R_1}{\delta_1} < \frac{R_2}{\delta_2}$ implies that $\mathfrak{s}_1$ admits an injective homomorphism into $\mathfrak{s}_2$. If $\frac{R_1}{\delta_1} = \frac{R_2}{\delta_2}$, $\mathfrak{s}_1$ admits an injective homomorphism into $\mathfrak{s}_2$ unless $\mathfrak{s}_1$ is pure and $\mathfrak{s}_2$ is dense.

(iii) Assume that $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are non-sparse. If R_2 is infinite or S is divisible by an infinite power of some prime number, then $\mathfrak{s}_1$ admits an injective homomorphism into $\mathfrak{s}_2$.

(iv) If $\mathfrak{s}_2$ is sparse, then $\mathfrak{s}_1$ admits an injective homomorphism into $\mathfrak{s}_2$.

Proof. The Steinitz numbers S_1, C_1 and the indices δ_1, σ_1 are in general not well-defined for a Lie algebra $\mathfrak{s}_1$: these values characterize a given exhaustion of $\mathfrak{s}_1$. However, if $\mathfrak{s}_1$ is non-sparse and S_1 is not divisible by an infinite power of any prime number, then the number $\frac{R_1}{\delta_1}$ does not depend on the exhaustion of $\mathfrak{s}_1$ (because then by condition $\mathcal{A}_2$ of Theorem 2.2.1 $\frac{\mathrm{Stz}(\mathcal{S}_1)}{\mathrm{Stz}(\mathcal{S}_1')}$ is a set containing exactly one element for S_1' corresponding to any other exhaustion of $\mathfrak{s}_1$, and therefore $\frac{R_1}{\delta_1}$ is well-defined by condition $\mathcal{A}_3$). Also, under the assumptions made in the last statement of (ii), the number $\frac{\sigma_1 R_1}{B_1}$ does not depend on the exhaustion of $\mathfrak{s}_1$ (this follows from condition $\mathcal{B}_3$ of Theorem 2.2.1). The finiteness of R_1, R_2, B_1, B_2 does not depend on the exhaustion either, so in the proofs of all four statements we can exhaust $\mathfrak{s}_1$ in any convenient way. The same applies to $\mathfrak{s}_2$.

We will assume that $X = A$ and prove all statements for type A Lie algebras. If $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are of type O or C, then both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are one-sided and the proof is analogous to the proof in the type A case when $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are one-sided.

Let us now set up some notations. Let $\mathfrak{s}_1$ be exhausted as $\mathrm{sl}(n_0) \subset \mathrm{sl}(n_1) \subset \cdots$, each inclusion $\mathrm{sl}(n_i) \to \mathrm{sl}(n_{i+1})$ being of signature (l_i, r_i, z_i), $i \geq 0$. By possibly changing some first terms of the exhaustion, we can choose n_0 to be divisible by R_1. Similarly, let $\mathrm{sl}(m_0) \subset \mathrm{sl}(m_1) \subset \cdots$ be the exhaustion of $\mathfrak{s}_2$, each inclusion $\mathrm{sl}(m_i) \to \mathrm{sl}(m_{i+1})$ being of signature (l_i', r_i', z_i'), $i \geq 0$. Set $s_i = l_i + r_i$, $c_i = l_i - r_i$, $s_i' = l_i' + r_i'$, and $c_i' = l_i' - r_i'$ for $i \geq 0$. Then $S_1 = n_0 s_0 s_1 \cdots$, $C_1 = n_0 c_0 c_1 \cdots$, $S_2 = m_0 s_0' s_1' \cdots$, $C_2 = m_0 c_0' c_1' \cdots$, $\delta_1 = \lim\limits_{i\to\infty} \dfrac{n_0 s_0 \cdots s_{i-1}}{n_i}$, $\delta_2 = \lim\limits_{i\to\infty} \dfrac{m_0 s_0' \cdots s_{i-1}'}{m_i}$, $\sigma_1 = \lim\limits_{i\to\infty} \dfrac{c_0 \cdots c_i}{s_0 \cdots s_i}$, and $\sigma_2 = \lim\limits_{i\to\infty} \dfrac{c_0' \cdots c_i'}{s_0' \cdots s_i'}$.

Consider a diagram

$$\begin{array}{ccccccccccc} \mathrm{sl}(n_0) & \longrightarrow & \mathrm{sl}(n_1) & \longrightarrow & \cdots & \longrightarrow & \mathrm{sl}(n_i) & \longrightarrow & \mathrm{sl}(n_{i+1}) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle \theta_0} & & \downarrow{\scriptstyle \theta_1} & & & & \downarrow{\scriptstyle \theta_i} & & \downarrow{\scriptstyle \theta_{i+1}} & & \\ \mathrm{sl}(m_{k_0}) & \longrightarrow & \mathrm{sl}(m_{k_1}) & \longrightarrow & \cdots & \longrightarrow & \mathrm{sl}(m_{k_i}) & \longrightarrow & \mathrm{sl}(m_{k_{i+1}}) & \longrightarrow & \cdots, \end{array} \tag{3.6}$$

where θ_i is a diagonal homomorphism of signature $(x_i, y_i, m_{k_i} - (x_i + y_i)n_i)$, $i \geq 0$. Taking into consideration the argument given after Lemma 2.2.5, we see that to make such a diagram well-defined and commutative it is enough to have

$$s_i(x_{i+1} + y_{i+1}) = (x_i + y_i)s_{k_i}' \cdots s_{k_{i+1}-1}', \tag{3.7}$$

$$c_i(x_{i+1} - y_{i+1}) = (x_i - y_i)c_{k_i}' \cdots c_{k_{i+1}-1}', \tag{3.8}$$

and

$$m_{k_i} \geq (x_i + y_i)n_i \tag{3.9}$$

for $i \geq 0$. Finally, we set $p_0 = \frac{n_0}{R_1}$ and $p_i = p_0 s_0 \cdots s_{i-1}$ for $i \geq 1$. We are now ready to prove that there exist numbers x_i, y_i, $i \geq 0$ satisfying (3.7) $-$ (3.9) in all four cases.

(i) The Steinitz number R_2 is finite in this case. Possibly by changing the exhaustion of $\mathfrak{s}_2$ we can choose m_0 to be divisible by R_2. Choose also each k_i large enough so that $m_0 s'_0 \cdots s'_{k_i-1}$ is divisible by $R_2 p_i$ (this is possible since p_i divides S) and put $q_i = \frac{m_0 s'_0 \cdots s'_{k_i-1}}{R_2 p_i}$ for $i \geq 0$. Put also $x_i = y_i = q_i$. Then it is easy to verify that (3.7) and (3.8) hold, and (3.9) is equivalent to $\frac{m_0 s'_0 \cdots s'_{k_i-1}}{R_2 m_{k_i}} \leq \frac{n_0 s_0 \cdots s_{i-1}}{2R_1 n_i}$.

Suppose that $\frac{\delta_2}{R_2} < \frac{\delta_1}{2R_1}$. Pick $\alpha \in (\frac{\delta_2}{R_2}, \frac{\delta_1}{2R_1})$. Since $\delta_1 = \lim\limits_{i\to\infty} \dfrac{n_0 s_0 \cdots s_{i-1}}{n_i}$ and $\delta_2 = \lim\limits_{i\to\infty} \dfrac{m_0 s'_0 \cdots s'_i}{m_i}$ we have $\frac{m_0 s'_0 \cdots s'_{k_i-1}}{R_2 m_{k_i}} \leq \alpha \leq \frac{n_0 s_0 \cdots s_{i-1}}{2R_1 n_i}$ for $i \geq i_0$, $k_i \geq j_0$. Obviously we can choose each k_i greater than j_0. Also we can construct θ_i only for $i \geq i_0$ and the diagram in (3.6) will still give us an injective homomorphism of $\mathfrak{s}_1$ into $\mathfrak{s}_2$.

Let now $\frac{\delta_2}{R_2} = \frac{\delta_1}{2R_1}$. If $\mathfrak{s}_2$ is pure then $\frac{m_0 s'_0 \cdots s'_{k_i-1}}{R_2 m_{k_i}} = \frac{\delta_2}{R_2} = \frac{\delta_1}{2R_1} \leq \frac{n_0 s_0 \cdots s_{i-1}}{2R_1 n_i}$, where the latter inequality holds because the sequence $\frac{n_0 s_0 \cdots s_{i-1}}{n_i}$ is non-increasing. Finally, if both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are dense, then for each i we have $\frac{\delta_2}{R_2} = \frac{\delta_1}{2R_1} < \frac{n_0 s_0 \cdots s_{i-1}}{2R_1 n_i}$, so to make $\frac{m_0 s'_0 \cdots s'_{k_i-1}}{R_2 m_{k_i}} \leq \frac{n_0 s_0 \cdots s_{i-1}}{2R_1 n_i}$ we choose k_i sufficiently large.

(ii) Possibly by changing the exhaustions of $\mathfrak{s}_1$ and $\mathfrak{s}_2$ we choose n_0 to be divisible by $R_1 2^u$ and m_0 to be divisible by $R_2 2^u$, where u is the maximal power of 2 dividing S (u is finite because 2^∞ does not divide S). We also choose m_0 large enough so that $\frac{m_0}{R_2} \geq \frac{n_0}{R_1}$. Denote again $q_i = \frac{m_0 s'_0 \cdots s'_{k_i-1}}{R_2 p_i}$, $i \geq 0$ (k_i is chosen large enough to make $R_2 p_i$ divide $m_0 s'_0 \cdots s'_{k_i-1}$).

If both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are one-sided, we put $x_i = q_i, y_i = 0$. In the other three cases B_1 is finite, so $c_0 c_1 \cdots$ divides $M c'_0 c'_1 \cdots$ for some finite M. By changing the exhaustion of $\mathfrak{s}_1$ we can make $c_0 c_1 \cdots$ divide $c'_0 c'_1 \cdots$. For that we replace the signature (l_i, r_i, z_i) with $((l_i + r_i + 1)/2, (l_i + r_i - 1)/2, z_i)$ for finitely many i ($l_i + r_i$ is odd for all $i \geq 0$ because $s_0 s_1 \cdots = \frac{R_1 S}{n_0}$ is not divisible by 2). Now we can choose each k_i large enough so that $c_0 \cdots c_{i-1}$ divides $c'_0 \cdots c'_{k_i-1}$. Then denote $t_i = \frac{c'_0 \cdots c'_{k_i-1}}{c_0 \cdots c_{i-1}}$ for $i \geq 1$ and $t_0 = 1$. Notice that for each $i \geq 0$ the numbers c_i and c'_i have the same parities as the numbers s_i and s'_i respectively. But all s_i and s'_i are odd, so c_i and c'_i are odd as well. Hence t_i and q_i are odd, and we put $x_i = (q_i + t_i)/2$ and $y_i = (q_i - t_i)/2$. Let us check that $y_i \geq 0$ (or $q_i \geq t_i$). This is obvious for $i = 0$. For

$i \geq 1$ the inequality $y_i \geq 0$ is equivalent to $\frac{R_2}{m_0} \cdot \frac{c'_0 \cdots c'_{k_i-1}}{s'_0 \cdots s'_{k_i-1}} \leq \frac{R_1}{n_0} \cdot \frac{c_0 \cdots c_{i-1}}{s_0 \cdots s_{i-1}}$, or

$$\frac{R_2}{m_0}(\sigma_2)_{k_i} \leq \frac{R_1}{n_0}(\sigma_1)_i, \tag{3.10}$$

where $(\sigma_1)_i = \frac{c_0 \cdots c_{i-1}}{s_0 \cdots s_{i-1}}$ is a non-increasing sequence which tends to σ_1 and $(\sigma_2)_i = \frac{c'_0 \cdots c'_{i-1}}{s'_0 \cdots s'_{i-1}}$ is a non-increasing sequence which tends to σ_2. Let us verify the inequality in (3.10) case by case.

If $\mathfrak{s}_1$ is one-sided, then $(\sigma_1)_i = 1$ for $i \geq 1$ and our inequality is equivalent to $(\sigma_2)_{k_i} \leq \frac{m_0 R_1}{n_0 R_2}$. This holds in case $\mathfrak{s}_2$ is two-sided non-symmetric because of the assumption $\frac{m_0}{R_2} \geq \frac{n_0}{R_1}$ made at the beginning of the proof. If $\mathfrak{s}_2$ is two-sided weakly non-symmetric, then $\lim_{i\to\infty}(\sigma_2)_{k_i} = \sigma_2 = 0$, and therefore $(\sigma_2)_{k_i} \leq \frac{m_0 R_1}{n_0 R_2}(\sigma_1)_i$ for large enough k_i in case $\mathfrak{s}_1$ is two-sided non-symmetric.

Let now both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be two-sided strongly non-symmetric, B_2 be infinite or C be divisible by an infinite power of some prime number. In this case there exists an infinite Steinitz number C' such that $c_0 c_1 \cdots$ divides $\frac{1}{C'} c'_0 c'_1 \cdots$. Since $\sigma_1 = \lim_{i\to\infty}(\sigma_1)_i > 0$ and the sequence $(\sigma_1)_i$ decreases, to verify (3.10) it suffices to prove that $(\sigma_2)_{k_i} \leq \frac{m_0 R_1}{n_0 R_2}\sigma_1$. We have $\frac{m_0}{R_2} \geq \frac{n_0}{R_1}$, therefore it is enough to prove that $(\sigma_2)_{k_i} \leq \sigma_1$. This clearly holds for large enough k_i if $\sigma_2 < \sigma_1$. Otherwise we change the exhaustion of $\mathfrak{s}_2$ such that the new symmetry index $\tilde{\sigma}_2 = \sigma_2/N$ is less than σ_1 for a finite $N|C'$ (we replace l'_i, r'_i by $(s'_i+u)/2, (s'_i-u)/2$ respectively, where $c'_i = uv$ and $v|N$ for finitely many i) and repeat the same construction of x_i, y_i. Then σ_1 stays the same and in the new construction the inequality $(\tilde{\sigma}_2)_{k_i} \leq \sigma_1$ holds for large enough k_i.

Finally, let both B_1 and B_2 be finite, both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be two-sided strongly non-symmetric, C be not divisible by an infinite power of a prime number, and $\frac{R_1\sigma_1}{B_1} \geq \frac{R_2\sigma_2}{B_2}$. Then $c'_0 c'_1 \cdots = N c_0 c_1 \cdots$ for an odd number N, and by possibly changing the exhaustion of $\mathfrak{s}_2$ we can make $c'_0 c'_1 \cdots = c_0 c_1 \cdots$ and repeat the same construction. Then $\frac{B_1}{B_2} = \frac{n_0}{m_0}$, and therefore $\frac{R_1\sigma_1}{R_2\sigma_2} \geq \frac{B_1}{B_2} = \frac{n_0}{m_0}$. Then $\lim_{i\to\infty}(\sigma_2)_{k_i} = \sigma_2 < \dfrac{m_0 R_1}{n_0 R_2}(\sigma_1)_i$ for all i, since $(\sigma_1)_i$ is a non-increasing sequence which does not stabilize. Now clearly (3.10) holds for large enough k_i.

So far we have proven that in all cases we can choose exhaustions of $\mathfrak{s}_1$ and $\mathfrak{s}_2$ such that $x_i = \frac{1}{2}(q_i + t_i)$ and $y_i = \frac{1}{2}(q_i - t_i)$ are non-negative integers (in the first case, where both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are one-sided, we just put $t_i = q_i$, so $x_i = q_i$, $y_i = 0$). Since we have $x_i + y_i = q_i$ and $x_i - y_i = t_i$, it is easy to check (3.7) and (3.8). The condition in (3.9) is equivalent to $\frac{m_0 s'_0 \cdots s'_{k_i-1}}{R_2 m_{k_i}} \leq \frac{n_0 s_0 \cdots s_{i-1}}{R_1 n_i}$, and under the assumption

$\frac{\delta_2}{R_2} < \frac{\delta_1}{R_1}$ or $\frac{\delta_2}{R_2} = \frac{\delta_1}{R_1}$ its proof is analogous to that in (i).

(iii) Let us fix an exhaustion of $\mathfrak{s}_1$ and choose m_0 in the exhaustion of $\mathfrak{s}_2$ such that $R_2' p_0 | m_0$ and $\frac{m_0}{R_2'} s_0' s_1' \cdots$ is divisible by S for some finite R_2'. Moreover, we can choose R_2' to be arbitrary large (if R_2 is infinite, then R_2' can be any divisor of R_2; if $p^\infty | S$, then R_2' can be p^N for any $N \geq 1$). Denote $q_i = \frac{m_0 s_0' \cdots s_{k_i-1}'}{R_2' p_i}$ and put $x_i = y_i = q_i$ ($x_i = 2q_i$, $y_i = 0$ for types O and C). Similar to the proof of (i), the conditions in (3.7) and (3.8) are satisfied, and (3.9) is equivalent to the inequality $\frac{m_0 s_0' \cdots s_{k_i-1}'}{R_2' m_{k_i}} \leq \frac{n_0 s_0 \cdots s_{i-1}}{2R_1 n_i}$. Since the exhaustion of $\mathfrak{s}_1$ is fixed, the right-hand side is bounded by $\frac{\delta_1}{2R_1}$ from below. But $\frac{m_0 s_0' \cdots s_{k_i-1}'}{R_2' m_{k_i}} \leq \frac{1}{R_2'}$, and therefore it is enough to choose R_2' to be greater than $\frac{2R_1}{\delta_1}$.

(iv) Choose each k_i large enough so that $m_0 s_0' \cdots s_{k_i-1}'$ is divisible by p_i and denote $q_i = \frac{m_0 s_0' \cdots s_{k_i-1}'}{p_i}$, $i \geq 0$. Then put $x_i = y_i = q_i$ ($x_i = 2q_i$, $y_i = 0$ for types O and C). The conditions in (3.7) and (3.8) are again satisfied, and (3.9) is equivalent to the inequality $\frac{m_0 s_0' \cdots s_{k_i-1}'}{m_{k_i}} \leq \frac{n_0 s_0 \cdots s_{i-1}}{2R_1 n_i}$. But $\mathfrak{s}_2$ is sparse, therefore $\lim\limits_{i\to\infty} \frac{m_0 s_0' \cdots s_i'}{m_i} = 0$, so the inequality holds for large enough k_i. □

Lemma 3.1.3. *Let $\mathfrak{s}_1 = X_1(\mathcal{T}_1)$ and $\mathfrak{s}_2 = X_2(\mathcal{T}_2)$ be diagonal Lie algebras, neither of them finitary. Set $S_i = \mathrm{Stz}(\mathcal{S}_i)$, $S = \mathrm{GCD}(S_1, S_2)$, $R_i = \div(S_i, S)$, and $\delta_i = \delta(\mathcal{T}_i)$ for $i = 1, 2$. We assume that R_1 is finite.*

(i) Assume that $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are non-sparse, both R_1 and R_2 are finite, and S is not divisible by an infinite power of any prime number. In addition, let $(X_1, X_2) = (A, C)$, (A, O), (O, C), or (C, O). If $2\frac{R_1}{\delta_1} < \frac{R_2}{\delta_2}$, then $\mathfrak{s}_1$ admits an injective homomorphism into $\mathfrak{s}_2$. If $2\frac{R_1}{\delta_1} = \frac{R_2}{\delta_2}$, then $\mathfrak{s}_1$ admits an injective homomorphism into $\mathfrak{s}_2$ unless $\mathfrak{s}_1$ is pure and $\mathfrak{s}_2$ is dense.

(ii) Assume that $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are non-sparse, both R_1 and R_2 are finite, and S is not divisible by an infinite power of any prime number. In addition, assume that $(X_1, X_2) = (C, A)$ or (O, A). If $\frac{R_1}{\delta_1} < \frac{R_2}{\delta_2}$, then $\mathfrak{s}_1$ admits an injective homomorphism into $\mathfrak{s}_2$. If $\frac{R_1}{\delta_1} = \frac{R_2}{\delta_2}$, then $\mathfrak{s}_1$ admits an injective homomorphism into $\mathfrak{s}_2$ unless $\mathfrak{s}_1$ is pure and $\mathfrak{s}_2$ is dense.

(iii) Assume that $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are non-sparse. If R_2 is infinite or S is divisible by an infinite power of some prime number, then $\mathfrak{s}_1$ admits an injective homomorphism into $\mathfrak{s}_2$.

(iv) If $\mathfrak{s}_2$ is sparse, then $\mathfrak{s}_1$ admits an injective homomorphism into $\mathfrak{s}_2$.

Proof. The proofs of all four statements in the lemma are analogous to the corresponding proofs of Lemma 3.1.2. We will point out only the essential differences.

(i) If $(X_1, X_2) = (A, C)$ or (A, O), we put $x_i = y_i = q_i$ as in the proof of Lemma 3.1.2 (i). If $(X_1, X_2) = (O, C)$ or (C, O), we put $x_i = 2q_i$, $y_i = 0$. Since we are dealing with Lie algebras of different types we have to pay attention to the additional conditions of Lemma 2.2.5. These conditions are obviously satisfied. The rest of the proof is the same and the diagram in (3.6) (with Lie algebras of corresponding types) yields an injective homomorphism of $\mathfrak{s}_1$ into $\mathfrak{s}_2$.

(ii) Since $\mathfrak{s}_1$ is of type O or C, $\mathfrak{s}_1$ is one-sided. The Lie algebra $\mathfrak{s}_2$ is not two-sided symmetric because 2^∞ does not divide S_2. Thus $\mathfrak{s}_2$ is either one-sided or two-sided non-symmetric. Both cases were considered in Lemma 3.1.2 (ii) for type A Lie algebras. The construction of an injective homomorphism of $\mathfrak{s}_1$ into $\mathfrak{s}_2$ is the same in the case we now consider.

(iii), (iv) If $(X_1, X_2) = (A, C)$ or (A, O), we put $x_i = y_i = q_i$, and if $(X_1, X_2) = (C, A)$, (O, A), (O, C), or (C, O), we put $x_i = 2q_i$, $y_i = 0$. The proofs of (iii) and (iv) are completed in a way similar to the proofs of Lemma 3.1.2 (iii) and (iv). □

Corollary 3.1.4. *The three finitary Lie algebras* $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, *and* $\mathrm{sp}(\infty)$ *admit an injective homomorphism into any diagonal Lie algebra.*

Proof. Let $\mathfrak{s}$ be a diagonal Lie algebra. If $\mathfrak{s}$ is finitary, then $\mathfrak{s}$ is isomorphic to one of the three Lie algebras $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, $\mathrm{sp}(\infty)$. Hence $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, admit $\mathrm{sp}(\infty)$ admit an injective homomorphism into $\mathfrak{s}$. If $\mathfrak{s}$ is not finitary, then (by an easy corollary from Lemma 3.1.3 (iii), (iv)) there exists a pure one-sided Lie algebra $\mathfrak{s}'$ of type A which admits an injective homomorphism into $\mathfrak{s}$. Then each of the Lie algebras $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, $\mathrm{sp}(\infty)$ can be mapped by an injective homomorphism into $\mathfrak{s}'$ by Proposition 3.1.1, and the statement follows. □

Proposition 3.1.5. *Let* $\mathfrak{s}_1 = X_1(\mathcal{T}_1)$ *be a subalgebra of* $\mathfrak{s}_2 = X_2(\mathcal{T}_2)$. *Set* $S_1 = \mathrm{Stz}(\mathcal{S}_1)$, $S_2 = \mathrm{Stz}(\mathcal{S}_2)$. *Then* $S_1 | S_2 N$ *for some* $N \in \mathbb{Z}_{>0}$.

Proof. We take $\mathfrak{s} := \mathfrak{s}_1$ and $\mathfrak{g} := \mathfrak{s}_2$, in order to use the notation $\mathfrak{s}_i$ for an exhaustion of $\mathfrak{s}$. Since $\mathfrak{s}$ admits an injective homomorphism into $\mathfrak{g}$ there is a commutative diagram

$$\begin{array}{ccccccc} \mathfrak{s}_1 & \longrightarrow & \cdots & \longrightarrow & \mathfrak{s}_i & \longrightarrow & \cdots \\ \downarrow{\scriptstyle \theta_1} & & & & \downarrow{\scriptstyle \theta_i} & & \\ \mathfrak{g}_{k_1} & \longrightarrow & \cdots & \longrightarrow & \mathfrak{g}_{k_i} & \longrightarrow & \cdots . \end{array}$$

Set $M = I_{\mathfrak{s}_1}^{\mathfrak{g}_{k_1}}(\theta_1)$. Then, by Proposition 2.1.1 (ii), we have $I_{\mathfrak{g}_{k_1}}^{\mathfrak{g}_{k_i}} M = I_{\mathfrak{s}_1}^{\mathfrak{s}_i} I_{\mathfrak{s}_i}^{\mathfrak{g}_{k_i}}(\theta_i)$ for $i \geq 1$. Then $\prod_{j=1}^{i-1} I_{\mathfrak{s}_j}^{\mathfrak{s}_{j+1}} | M \prod_{j=k_1}^{k_i-1} I_{\mathfrak{g}_j}^{\mathfrak{g}_{j+1}}$ for $i \geq 1$. Thus, $S_1|S_2Mn_1$, where n_1 is the dimension of the natural representation of $\mathfrak{s}_1$. □

Proposition 3.1.6. *Let $\mathfrak{s}$ be a sparse one-sided Lie algebra of type A not isomorphic to* $\mathrm{sl}(\infty)$*. Then $\mathfrak{s}$ admits no non-trivial homomorphism into a pure one-sided Lie algebra of type A.*

Proof. Assume for the sake of a contradiction that there is an injective homomorphism of $\mathfrak{s}$ into some pure one-sided Lie algebra of type A. Let $\mathfrak{s}$ be exhausted as $\mathrm{sl}(n_1) \subset \mathrm{sl}(n_2) \subset \cdots$, each inclusion $\mathrm{sl}(n_i) \to \mathrm{sl}(n_{i+1})$ being of signature $(l_i, 0, z_i)$. Recall that by the definition of a sparse Lie algebra, $\lim_{i\to\infty} \frac{n_1 l_1 \cdots l_{i-1}}{n_i} = 0$. Then there is a commutative diagram

$$\begin{array}{ccccccccc}
\mathrm{sl}(n_1) & \longrightarrow \cdots \longrightarrow & \mathrm{sl}(n_i) & \xrightarrow{(l_i,0,z_i)} & \mathrm{sl}(n_{i+1}) & \longrightarrow \cdots \\
\downarrow{\scriptstyle\theta_1} & & \downarrow{\scriptstyle\theta_i} & & \downarrow{\scriptstyle\theta_{i+1}} & \\
\mathrm{sl}(m_1) & \longrightarrow \cdots \longrightarrow & \mathrm{sl}(m_1\cdots m_i) & \xrightarrow{(m_{i+1},0,0)} & \mathrm{sl}(m_1\cdots m_{i+1}) & \longrightarrow \cdots .
\end{array} \tag{3.11}$$

The lower row constitutes an exhaustion of the pure Lie algebra $\mathrm{sl}(m_1 m_2 \cdots)$.

Denote by V_i the natural $\mathrm{sl}(m_1 \cdots m_i)$-module for $i \geq 1$. Note that θ_i makes V_i into an $\mathrm{sl}(n_i)$-module. Let

$$V_i \downarrow \mathrm{sl}(n_i) \cong \bigoplus_{\lambda \in H_i} T_\lambda \otimes F_{n_i}^{\lambda} \tag{3.12}$$

be the decomposition into a direct sum of isotypic components. Here $T_\lambda = \mathrm{Hom}_{\mathrm{sl}(n_i)}(F_{n_i}^{\lambda}, V_i \downarrow \mathrm{sl}(n_i))$ is a trivial $\mathrm{sl}(n_i)$-module, and H_i is the set of all highest weights appearing in this decomposition. We can rewrite (3.12) (non-canonically) as

$$V_i \downarrow \mathrm{sl}(n_i) \cong \bigoplus_{\lambda \in H_i} \underbrace{F_{n_i}^{\lambda} \oplus \cdots \oplus F_{n_i}^{\lambda}}_{t_\lambda}, \tag{3.13}$$

where $t_\lambda = \dim T_\lambda$. Since all weights $\lambda \in H_i$ are dominant, for each $\lambda = (\lambda_1, \ldots, \lambda_{n_i})$, $\lambda_1 - \lambda_{n_i}$ is a non-negative integer. Set $d_i = \max_{\lambda \in H_i}(\lambda_1 - \lambda_{n_i})$. We define $H(\varphi)$ and $d(\varphi)$ in a similar way for an arbitrary injective homomorphism φ of finite-dimensional classical simple Lie algebras of type A, so that $H(\theta_i) = H_i$ and $d(\theta_i) = d_i$.

Let us show that $d_i \geq d_{i+1}$ for $i \geq 1$. By φ_i we denote the injective homomorphism $\mathrm{sl}(m_1 \cdots m_i) \overset{(m_{i+1},0,0)}{\longrightarrow} \mathrm{sl}(m_1 \cdots m_{i+1})$ as in (3.11). Notice first that

$H(\varphi_i \circ \theta_i) = H(\theta_i) = H_i$ and $\dim \mathrm{Hom}_{\mathrm{sl}(n_i)}(F^\lambda_{n_i}, V_{i+1}) = m_{i+1} \dim \mathrm{Hom}_{\mathrm{sl}(n_i)}(F^\lambda_{n_i}, V_i)$ for all $\lambda \in H_i$. Furthermore, $d(\varphi_i \circ \theta_i) = d(\theta_i) = d_i$.

Let $\lambda \in H_{i+1}$ be a weight such that $\lambda_1 - \lambda_{n_{i+1}} = d_{i+1}$. Since $(l_i, 0, z_i)$ is the signature of the diagonal injective homomorphism $\mathrm{sl}(n_i) \to \mathrm{sl}(n_{i+1})$, there is a chain of inclusions $\mathrm{sl}(n_i) \subset \mathrm{sl}(l_i n_i) \subset \mathrm{sl}(l_i n_i + 1) \subset \cdots \subset \mathrm{sl}(l_i n_i + z_i) = \mathrm{sl}(n_{i+1})$ such that their composition is the original map in (3.11). Applying the Gelfand-Tsetlin rule (see Theorem 2.3.1) repeatedly, we obtain that $F^\lambda_{n_{i+1}} \downarrow \mathrm{sl}(l_i n_i + z_i - j)$ has a submodule with highest weight $(\lambda_1, \lambda_2, \ldots, \lambda_{l_i n_i + z_i - j - 2}, \lambda_{l_i n_i + z_i - j - 1}, \lambda_{n_{i+1}})$ for $j = 1, \ldots, z_i$. We then apply Corollary 2.3.3 to the submodule of $F^\lambda_{n_{i+1}} \downarrow \mathrm{sl}(l_i n_i)$ with highest weight $(\lambda_1, \ldots, \lambda_{l_i n_i - 1}, \lambda_{n_{i+1}})$ and see $\hat{\lambda} := (\lambda_1 + \cdots + \lambda_{l_i}, \lambda_{l_i+1} + \cdots + \lambda_{2l_i}, \ldots, \lambda_{l_i n_i - l_i + 1} + \cdots + \lambda_{l_i n_i - 1} + \lambda_{n_{i+1}}) \in H(\varphi_i \circ \theta_i)$, i.e. the $\mathrm{sl}(n_i)$-module with highest weight $\hat{\lambda}$ is a constituent of $F^\lambda_{n_{i+1}} \downarrow \mathrm{sl}(n_i)$. Hence, $d(\varphi_i \circ \theta_i) \geq (\hat{\lambda}_1 - \hat{\lambda}_{n_i}) = (\lambda_1 + \cdots + \lambda_{l_i}) - (\lambda_{l_i n_i - l_i + 1} + \cdots + \lambda_{l_i n_i - 1} + \lambda_{n_{i+1}}) \geq \lambda_1 - \lambda_{n_{i+1}} = d_{i+1}$, where the latter inequality holds because λ is dominant. Since $d(\varphi_i \circ \theta_i) = d_i$, we have the desired inequality $d_i \geq d_{i+1}$.

Since $\{d_i\}$ is a non-increasing sequence of positive integers, it stabilizes, so there exists $d \in \mathbb{Z}_{>0}$ such that $d_i = d$ for all $i \geq J$. Pick K such that $l_J \cdots l_{K-1} > d$ (this is possible since $\mathfrak{s}$ is not isomorphic to $\mathrm{sl}(\infty)$, and therefore $\prod_{i=1}^{\infty} l_i$ is infinite). Consider now the following part of the diagram in (3.11):

$$\begin{array}{ccccc} \mathrm{sl}(n_J) & \longrightarrow & \cdots & \longrightarrow & \mathrm{sl}(n_K) \\ {\scriptstyle\theta_J}\downarrow & & & & {\scriptstyle\theta_K}\downarrow \\ \mathrm{sl}(m_1 \cdots m_J) & \longrightarrow & \cdots & \longrightarrow & \mathrm{sl}(m_1 \cdots m_K). \end{array}$$

The injective homomorphism $\mathrm{sl}(n_J) \to \mathrm{sl}(n_K)$ is diagonal of signature $(l, 0, z)$, where $l = l_J \cdots l_{K-1}$ and $z = n_K - l n_J$. Using similar arguments as above we obtain that $\hat{\lambda} = (\lambda_1 + \cdots + \lambda_l, \lambda_{l+1} + \cdots + \lambda_{2l}, \ldots, \lambda_{n_K - l + 1} + \cdots + \lambda_{n_K - 1} + \lambda_{n_K}) \in H_J$ for any $\lambda \in H_K$. This shows that $\lambda_1 + \cdots + \lambda_l - (\lambda_{n_K - l + 1} + \cdots + \lambda_{n_K}) \leq d$. If $\lambda_{d+1} \neq \lambda_{n_K - d}$, then $\lambda_{d+1} \geq \lambda_{n_K - d} + 1$, in which case $\lambda_1 + \cdots + \lambda_l - (\lambda_{n_K - l + 1} + \cdots + \lambda_{n_K}) \geq (\lambda_1 + \cdots + \lambda_{d+1}) - (\lambda_{n_K - d} + \cdots + \lambda_{n_K}) \geq d + 1$ as $l > d$. Hence, $\lambda_{d+1} = \lambda_{n_K - d}$ which yields $\lambda_{d+1} = \lambda_{d+2} = \cdots = \lambda_{n_K - d}$. We thus conclude that for $i \geq K$ each integral dominant weight appearing in H_i has the property that all its marks apart from the first d and the last d must be equal.

Let us calculate the index $I^{\mathrm{sl}(m_1 \cdots m_i)}_{\mathrm{sl}(n_1)}$ of the corresponding composition of homomorphisms in (3.11). Using Proposition 2.1.1 (ii) and Corollary 2.2.4, we compute $I^{\mathrm{sl}(m_1 \cdots m_i)}_{\mathrm{sl}(n_1)} = I(\theta_1) m_2 \cdots m_i$ by following down θ_1 and to the right; similarly we compute $I_{\mathrm{sl}(n_1)}^{\mathrm{sl}(m_1\cdots m_i)} = l_1 \cdots l_{i-1} I(\theta_i)$ by going to the right and then down θ_i. By Proposition 2.1.1 (iii), (iv) we have

$$I(\theta_i) = \sum_{\lambda \in H_i} t_\lambda I(F_{n_i}^\lambda) = \frac{1}{n_i^2 - 1} \sum_{\lambda \in H_i} t_\lambda \dim F_{n_i}^\lambda \langle \lambda, \lambda + 2\rho \rangle_{\mathrm{sl}(n_i)}, \tag{3.14}$$

where 2ρ is the sum of all the positive roots of $\mathrm{sl}(n_i)$.

Note that $\langle \lambda, \lambda + 2\rho \rangle_{\mathrm{sl}(n_i)} = (\tilde{\lambda}, \tilde{\lambda} + 2\rho)$, where $\tilde{\lambda}_j = \lambda_j - \frac{1}{n_i} \sum_{k=1}^{n_i} \lambda_k$ for $j = 1, \ldots, n_i$, $2\rho = (n_i - 1, n_i - 3, \ldots, -(n_i - 1))$, and (,) is the usual scalar product on $\mathbb{C}^{n_i}$.

Fix $i \geq K$, using the notation from above, so that $\lambda_1 - \lambda_{n_i} \leq d$ and $\lambda_{d+1} = \lambda_{d+1} = \cdots = \lambda_{n_i - d}$. Set $\alpha = \tilde{\lambda}_{d+1}$, so that $|\tilde{\lambda}_j - \alpha| = 0$ for $j = d+1, d+2, \ldots, n_i - d$. Then $|\tilde{\lambda}_j - \alpha| = |\lambda_j - \lambda_{d+1}| \leq d$ for all j. Since $\sum_{j=1}^{n_i} \tilde{\lambda}_j = 0$ and $\tilde{\lambda}_1 - \tilde{\lambda}_{n_i} = \lambda_1 - \lambda_{n_i} \leq d$, we have $|\tilde{\lambda}_j| \leq d$ for all j. Hence,

$$\begin{aligned}
|\langle \lambda, \lambda + 2\rho \rangle_{\mathrm{sl}(n_i)}| = |(\tilde{\lambda}, \tilde{\lambda} + 2\rho)| &= \left| \sum_{j=1}^{n_i} \tilde{\lambda}_j (\tilde{\lambda}_j + n_i - 2j + 1) \right| \\
&= \left| \sum_{j=1}^{n_i} \tilde{\lambda}_j (\tilde{\lambda}_j - \alpha - 2j) + (n_i + 1 + \alpha) \sum_{j=1}^{n_i} \tilde{\lambda}_j \right| \\
&= \left| \sum_{j=1}^{n_i} (\tilde{\lambda}_j - \alpha + \alpha)(\tilde{\lambda}_j - \alpha - 2j) \right| \\
&= \left| \sum_{j=1}^{n_i} (\tilde{\lambda}_j - \alpha)^2 - 2 \sum_{j=1}^{n_i} (\tilde{\lambda}_j - \alpha) j + \sum_{i=1}^{n_i} (\alpha(\tilde{\lambda}_j - \alpha) - 2\alpha j) \right| \\
&= \left| \sum_{j=1}^{n_i} (\tilde{\lambda}_j - \alpha)^2 - 2 \sum_{j=1}^{d} (\tilde{\lambda}_j - \alpha) j - 2 \sum_{j=n_i-d+1}^{n_i} (\tilde{\lambda}_j - \alpha) j - n_i \alpha^2 - n_i(n_i + 1)\alpha \right| \\
&\leq \sum_{j=1}^{n_i} d^2 + 2 \sum_{j=1}^{d} jd + 2 \sum_{j=n_i-d+1}^{n_i} jd + n_i \alpha^2 + n_i(n_i + 1)|\alpha| \\
&= 2n_i d^2 + 2(n_i + 1)d^2 + n_i \alpha^2 + n_i(n_i + 1)|\alpha|.
\end{aligned}$$

Since $\tilde{\lambda}_1 + \cdots + \tilde{\lambda}_d + \alpha(n_i - 2d) + \tilde{\lambda}_{n_i-d+1} + \cdots + \tilde{\lambda}_{n_i} = 0$ (which implies $|\alpha| \leq \frac{2d^2}{n_i - 2d}$), we obtain the following inequality:

$$|\langle \lambda, \lambda + 2\rho \rangle_{\mathrm{sl}(n_i)}| \leq 2d^2 n_i + 2d^2(n_i + 1) + \frac{4d^4 n_i}{(n_i - 2d)^2} + \frac{2d^2 n_i(n_i + 1)}{n_i - 2d} \leq c_0 n_i$$

for all $i \geq K$, where c_0 is some positive constant. Then from (3.14) we have $I(\theta_i) \leq \dfrac{c_0 n_i}{n_i^2 - 1} \sum_{\lambda \in H_i} t_\lambda \dim F_{n_i}^{\lambda} = \dfrac{c_0 n_i}{n_i^2 - 1} m_1 \cdots m_i$. Hence, $I(\theta_1) m_2 \cdots m_i = I_{\mathrm{sl}(n_1)}^{\mathrm{sl}(m_1 \cdots m_i)} = l_1 \cdots l_{i-1} I(\theta_i) \leq l_1 \cdots l_{i-1} \frac{c_0 n_i}{n_i^2-1} m_1 \cdots m_i$. This implies $\frac{I(\theta_1)}{c_0 m_1} \leq l_1 \cdots l_{i-1} \frac{n_i}{n_i^2-1}$, so $\frac{l_1 \cdots l_{i-1}}{n_i} \geq c_1$ for some positive constant c_1. The last inequality contradicts the fact that $\lim_{i \to \infty} \dfrac{n_1 l_1 \cdots l_{i-1}}{n_i} = 0$, so the proposition follows. □

Corollary 3.1.7. *Let $\mathfrak{s}_1$, $\mathfrak{s}_2$ be non-finitary diagonal Lie algebras. Assume that $\mathfrak{s}_1$ is sparse and there is an injective homomorphism of $\mathfrak{s}_1$ into $\mathfrak{s}_2$. Then $\mathfrak{s}_2$ must be sparse as well.*

Proof. Suppose, on the contrary, that $\mathfrak{s}_2$ is pure or dense. Lemma 3.1.3 (iv) implies that there exists a sparse one-sided Lie algebra $\mathfrak{s}_1'$ of type A which admits an injective homomorphism into $\mathfrak{s}_1$. By Lemma 3.1.3 (iii) there exists a pure one-sided Lie algebra $\mathfrak{s}_2'$ of type A such that $\mathfrak{s}_2$ admits an injective homomorphism into $\mathfrak{s}_2'$. If $\mathfrak{s}_1$ would admit an injective homomorphism into $\mathfrak{s}_2$, then $\mathfrak{s}_1'$ would admit an injective homomorphism into $\mathfrak{s}_2'$ through the chain $\mathfrak{s}_1' \subset \mathfrak{s}_1 \subset \mathfrak{s}_2 \subset \mathfrak{s}_2'$, which would contradict Proposition 3.1.6. Hence the statement holds. □

Proposition 3.1.8. *Let $\mathfrak{s}_1 = A(\mathcal{T}_1)$ and $\mathfrak{s}_2 = A(\mathcal{T}_2)$ be pure one-sided Lie algebras, neither of them finitary. Set $S_i = \mathrm{Stz}(\mathcal{S}_i)$ for $i = 1, 2$, and $S = \mathrm{GCD}(S_1, S_2)$. Assume that both Steinitz numbers $\div(S_1, S)$ and $\div(S_2, S)$ are finite and S is not divisible by an infinite power of any prime number. An injective homomorphism of $\mathfrak{s}_1$ into $\mathfrak{s}_2$ is necessarily diagonal.*

Proof. Let $S = p_1^{l_1} p_2^{l_2} \cdots$ for the increasing sequence $\{p_i\}$ of all prime numbers dividing S. Denote $n_i = \frac{S_1}{S}(p_1)^{l_1} \cdots (p_{N+i})^{l_{N+i}}$ for $i \geq 0$, where the integer N is to be fixed later. Suppose that there is an injective homomorphism $\theta : \mathfrak{s}_1 \to \mathfrak{s}_2$. Then it is given by a commutative diagram

$$\begin{array}{ccccccccc}
\mathrm{sl}(n_0) & \longrightarrow & \cdots & \longrightarrow & \mathrm{sl}(n_i) & \longrightarrow & \mathrm{sl}(n_{i+1}) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \theta_0} & & & & \downarrow{\scriptstyle \theta_i} & & \downarrow{\scriptstyle \theta_{i+1}} & & \\
\mathrm{sl}(m_0) & \longrightarrow & \cdots & \longrightarrow & \mathrm{sl}(m_i) & \longrightarrow & \mathrm{sl}(m_{i+1}) & \longrightarrow & \cdots,
\end{array} \tag{3.15}$$

where $m_i = \frac{S_2}{S}(p_1)^{l_1} \cdots (p_{N+k_i})^{l_{N+k_i}}$ for $i \geq 0$ for some $k_0, k_1, \ldots$. By possibly shifting the bottom row of the diagram we may assume that $k_i \geq i + 1$ for each $i \geq 0$.

Denote by W_i the natural $\mathrm{sl}(m_i)$-module. Let $H(\varphi)$ and $d(\varphi)$ be as in the proof of Proposition 3.1.5 for an arbitrary injective homomorphism φ of finite-dimensional classical simple Lie algebras of type A. Set $H_i = H(\theta_i)$ and $d_i = d(\theta_i)$ for $i \geq 0$. Similarly to (3.13) we then have

$$W_i \downarrow \mathrm{sl}(n_i) \cong \bigoplus_{\lambda \in H_i} \underbrace{F^\lambda_{n_i} \oplus \cdots \oplus F^\lambda_{n_i}}_{t_{\lambda,i}},$$

where $t_{\lambda,i} = \dim \mathrm{Hom}_{\mathrm{sl}(n_i)}(F^\lambda_{n_i}, W_i \downarrow \mathrm{sl}(n_i))$.

As in the proof of Proposition 3.1.5, $\{d_i\}$ is a non-increasing sequence, and therefore $d_i = d$ for $i \geq i_0$. By choosing N large enough we make $d_i = d$ and $p_{N+i} > d+1$ for all $i \geq 0$. Take now $0 \leq i < j \leq k_i$ and consider the following piece of the diagram in (3.15):

$$\begin{array}{ccc} \mathrm{sl}(n_i) & \longrightarrow \cdots \longrightarrow & \mathrm{sl}(n_j) \\ \theta_i \downarrow & & \theta_j \downarrow \\ \mathrm{sl}(m_i) & \longrightarrow \cdots \longrightarrow & \mathrm{sl}(m_j). \end{array} \tag{3.16}$$

Here the injective homomorphism $\mathrm{sl}(n_i) \to \mathrm{sl}(n_j)$ is of signature $(q, 0, 0)$, where $q = (p_{N+i+1})^{l_{N+i+1}} \cdots (p_{N+j})^{l_{N+j}}$. Take an arbitrary non-zero highest weight λ in H_j, yielding the $\mathrm{sl}(n_j)$-module $F^\lambda_{n_j}$. Since $n_j = qn_i$, by Proposition 2.3.2 we have

$$F^\lambda_{qn_i} \downarrow \mathrm{sl}(n_i) \cong \bigoplus_\nu \Big(\sum_{\mu_1,\ldots,\mu_q} c^\lambda_{\mu_1\ldots\mu_q} c^\nu_{\mu_1\ldots\mu_q}\Big) F^\nu_{n_i}.$$

Since the coefficients $c^\lambda_{\mu_1\ldots\mu_q}$ and $c^\nu_{\mu_1\ldots\mu_q}$ are independent of the order of $\mu_1, \ldots, \mu_q$, we can rewrite this as

$$F^\lambda_{qn_i} \downarrow \mathrm{sl}(n_i) \cong \bigoplus_\nu \Big(\sum_{[\mu_1,\ldots,\mu_q]} C^{q_1,\ldots,q_r}_q c^\lambda_{\mu_1\ldots\mu_q} c^\nu_{\mu_1\ldots\mu_q}\Big) F^\nu_{n_i}. \tag{3.17}$$

Here $[\mu_1, \ldots, \mu_q]$ stands for the multiset with elements $\mu_1, \ldots, \mu_q$, and $q_1, \ldots, q_r$ is the set of multiplicities of $[\mu_1, \ldots, \mu_q]$. Note that $q_1 + \cdots + q_r = q$.

Fix a highest weight ν such that $F^\nu_{n_i}$ has non-zero multiplicity in (3.17) and fix a multiset of integral dominant weights $[\mu_1, \ldots, \mu_q]$ making both generalized Littlewood-Richardson coefficients $c^\lambda_{\mu_1\ldots\mu_q}$ and $c^\nu_{\mu_1\ldots\mu_q}$ non-zero. We will show that q divides $C^{q_1,\ldots,q_r}_q$ (and hence the contribution from $[\mu_1, \ldots, \mu_q]$ to the multiplicity of $F^\nu_{n_i}$) if the module $F^\nu_{n_i}$ is non-trivial. Suppose that p_l divides all $q_1, \ldots, q_r$ for some $N + i + 1 \leq l \leq N + j$. Note that the $\mathrm{sl}(n_i)$-module $F^{\nu'}_{n_i}$ for $\nu' = \mu_1 + \cdots + \mu_q$ also has non-zero multiplicity in (3.17) because $c^{\nu'}_{\mu_1\ldots\mu_q} \neq 0$. Since all $q_1, \ldots, q_r$ are divisible by p_l, we have $\nu' = p_l\mu'$ for some integral dominant weight μ'. Furthermore, using the path along θ_j in (3.16), we see that $F_{n_i}^{\nu'}$ has non-zero multiplicity in W_j, and since $W_j \downarrow \mathrm{sl}(m_i)$ is a direct sum of copies of W_i, it must be that $F_{n_i}^{\nu'}$ has non-zero multiplicity in $W_i \downarrow \mathrm{sl}(n_i)$, i.e. $\nu' \in H_i$. Since $d_i = d < p_l - 1$ we have $p_l > \nu'_1 - \nu'_{n_i} = p_l(\mu'_1 - \mu'_{n_i})$ which possible only if $\mu'_1 = \mu'_{n_i}$ (equivalently, $\nu'_1 = \nu'_{n_i}$). Therefore ν' is the zero highest weight, and hence all $\mu_1, \ldots, \mu_q$ are zero as well. Then the coefficient $c^{\nu}_{\mu_1 \ldots \mu_q}$ is non-zero only if the weight ν is zero, so $F^{\nu}_{n_i}$ is a trivial module.

Suppose now that p_l does not divide at least one of $q_1, \ldots, q_r$ for each l such that $N + i \leq l \leq N + j$. A combinatorial argument shows that $C_q^{q_1,\ldots,q_r} = \frac{q!}{q_1! \cdots q_r!}$ is divisible by q if each prime divisor of q fails to divide at least one of $q_1, \ldots, q_r$. We thus conclude that each non-trivial $\mathrm{sl}(n_i)$-module $F^{\nu}_{n_i}$ with non-zero multiplicity in (3.17), has multiplicity divisible by q. As a corollary, any non-trivial simple constituent of $W_j \downarrow \mathrm{sl}(n_i)$ appears with multiplicity divisible by q.

By following the diagram in (3.16) down θ_i and then to the right, we get $W_j \downarrow \mathrm{sl}(n_i) \cong \frac{m_j}{m_i} \bigoplus_{\nu \in H_i} t_{\nu,i} F^{\nu}_{n_i}$. Since $q = (p_{N+i+1})^{l_{N+i+1}} \cdots (p_{N+j})^{l_{N+j}}$ is relatively prime to $\frac{m_j}{m_i} = (p_{N+k_i+1})^{l_{N+k_i+1}} \cdots (p_{N+k_j})^{l_{N+k_j}}$ (as $j \leq k_i$), the commutativity of the diagram in (3.16) implies that $t_{\nu,i}$ is divisible by q for any non-zero ν in H_i.

Let us introduce a new notation. For an arbitrary injective homomorphism $\varphi : \mathfrak{g}_1 \to \mathfrak{g}_2$ of finite-dimensional classical simple Lie algebras of type A we denote by $N(\varphi)$ the number (counting multiplicities) of simple non-trivial constituents of the natural representation of $\mathfrak{g}_2$ considered as a $\mathfrak{g}_1$-module via φ. Then $N_i := N(\theta_i)$ is divisible by $q = (p_{N+i+1})^{l_{N+i+1}} \cdots (p_{N+j})^{l_{N+j}}$ by the above argument. Taking $j = k_i$ we obtain that N_i is divisible by $(p_{N+i+1})^{l_{N+i+1}} \cdots (p_{N+k_i})^{l_{N+k_i}}$.

Fix now $j = i + 1$ in the diagram in (3.16), and let $\psi : \mathrm{sl}(n_i) \to \mathrm{sl}(m_{i+1})$ denote the map produced by this diagram. As shown above, each non-zero weight $\lambda \in H_{i+1}$ yields a non-zero weight in $H(\psi) = H_i$ with non-zero multiplicity divisible by $(p_{N+i+1})^{l_{N+i+1}}$, and hence at least $(p_{N+i+1})^{l_{N+i+1}}$. Therefore by following the diagram to the right and then down θ_{i+1}, we obtain $N(\psi) \geq (p_{N+i+1})^{l_{N+i+1}} N_{i+1}$. Note also that equality holds here only if for each non-zero $\lambda \in H_{i+1}$ we have $F^{\lambda}_{qn_i} \downarrow \mathrm{sl}(n_i) \cong qF^{\nu}_{n_i} \oplus T$ for a non-zero $\nu \in H_i$, where T is a trivial (possibly 0-dimensional) module. Meanwhile, by following the diagram down θ_i and to the right we have $N(\psi) = (p_{N+k_i+1})^{l_{N+k_i+1}} \cdots (p_{N+k_{i+1}})^{l_{N+k_{i+1}}} N_i$. As a result we obtain the inequality $(p_{N+k_i+1})^{l_{N+k_i+1}} \cdots (p_{N+k_{i+1}})^{l_{N+k_{i+1}}} N_i \geq (p_{N+i+1})^{l_{N+i+1}} N_{i+1}$, i.e. $\alpha_i \geq \alpha_{i+1}$, where $\alpha_i := \frac{N_i}{(p_{N+i+1})^{l_{N+i+1}} \cdots (p_{N+k_i})^{l_{N+k_i}}}$ are integers for $i \geq 0$. Since $\{\alpha_i\}$ is a non-increasing sequence of positive integers it stabilizes, and by choosing N sufficiently large we can assume that $\alpha_0 = \alpha_1 = \alpha_2 = \cdots$.

Now take an arbitrary non-zero $\lambda \in H_{i+1}$. Since $\alpha_i = \alpha_{i+1}$, the decomposition in (3.17) becomes $F^{\lambda}_{qn_i} \downarrow \mathrm{sl}(n_i) \cong qF^{\nu}_{n_i} \oplus T$ for some non-zero $\nu \in H_i$, where T is some trivial (possibly 0-dimensional) module. Since the contribution from each multiset $[\mu_1, \ldots, \mu_q]$ to the multiplicity of $F^{\nu}_{n_i}$ in (3.17) is divisible by q, there exists exactly one multiset $[\mu_1, \ldots, \mu_q]$ making a non-zero contribution to the multiplicity of $F^{\nu}_{n_i}$. Moreover, the fact that $C^{q_1,\ldots,q_r}_q c^{\lambda}_{\mu_1\ldots\mu_q} c^{\nu}_{\mu_1\ldots\mu_q} = q$ together with the fact that q divides $C^{q_1,\ldots,q_r}_q$ implies $C^{q_1,\ldots,q_r}_q = q$. It is easy to check that $\frac{q!}{q_1!\cdots q_r!} = q$ only if $r = 2$ and $\{q_1, q_2\} = \{1, q-1\}$. Then we safely can assume that $\mu_1 = \mu_2 = \cdots = \mu_{q-1}$. Since $\nu' = \mu_1 + \cdots + \mu_q$ is a non-zero weight satisfying $c^{\nu'}_{\mu_1\ldots\mu_q} \neq 0$, the module $F^{\nu'}_{n_i}$ also has non-zero multiplicity in (3.17), and therefore $\nu = \nu'$. Hence $\nu = (q-1)\mu_1 + \mu_q$, and since $\nu_1 - \nu_{n_i} \leq d < (p_{N+i+1})^{l_{N+i+1}} - 1 = q - 1$, we immediately get that μ_1 is the zero weight. Then the only multiset $[\mu_1, \ldots, \mu_q]$ making $c^{\lambda}_{\mu_1\ldots\mu_q}$ non-zero has $q-1$ zero weights. One can check that this is only possible if λ is either of the form $(c+1, c, \ldots, c, c)$ or $(c, c, \ldots, c, c+1)$. Thus, all non-zero highest weights from H_{i+1} are either those of the natural or of the conatural representation. This means precisely that all homomorphisms θ_i are diagonal. □

Corollary 3.1.9. *Let* $\mathfrak{s}_1 = X_1(\mathcal{T}_1)$ *and* $\mathfrak{s}_2 = X_2(\mathcal{T}_2)$ *be non-sparse Lie algebras, neither of them finitary. Set* $S_i = \mathrm{Stz}(\mathcal{S}_i)$, $S = \mathrm{GCD}(S_1, S_2)$, *and* $R_i = \div(S_i, S)$ *for* $i = 1, 2$. *Assume that* S *is not divisible by an infinite power of any prime number, and that both* R_1 *and* R_2 *are finite. An injective homomorphism of* $\mathfrak{s}_1$ *into* $\mathfrak{s}_2$ *is necessary diagonal.*

Proof. Set $\delta_i = \delta(\mathcal{T}_i)$, $i = 1, 2$. Denote $\mathfrak{s}'_1 = \mathrm{sl}(\div(S_1, R'_1))$, where $R'_1 > 2\delta_1$ is some finite divisor of S_1, and $\mathfrak{s}'_2 = \mathrm{sl}(S_2 R'_2)$, where R'_2 is finite and $R'_2 > \frac{2}{\delta_2}$. Then, by Lemma 3.1.2 (i) and Lemma 3.1.3 (i), (ii), $\mathfrak{s}'_1$ admits an injective homomorphism into $\mathfrak{s}_1$ and $\mathfrak{s}_2$ admits an injective homomorphism into $\mathfrak{s}'_2$. Then there exists an injective homomorphism of $\mathfrak{s}'_1$ into $\mathfrak{s}'_2$ through the chain $\mathfrak{s}'_1 \subset \mathfrak{s}_1 \subset \mathfrak{s}_2 \subset \mathfrak{s}'_2$ and this homomorphism is diagonal because the Lie algebras $\mathfrak{s}'_1$ and $\mathfrak{s}'_2$ satisfy the conditions of Proposition 3.1.8. Finally, it follows from Corollary 2.2.7 that the injective homomorphism of $\mathfrak{s}_1$ into $\mathfrak{s}_2$ has to be diagonal as well. □

Lemma 3.1.10. *Let* $\mathfrak{s}_1 = X_1(\mathcal{T}_1)$ *and* $\mathfrak{s}_2 = X(\mathcal{T}_2)$ *be non-sparse Lie algebras, neither of them finitary. Set* $S_i = \mathrm{Stz}(\mathcal{S}_i)$, $S = \mathrm{GCD}(S_1, S_2)$, $R_i = \div(S_i, S)$, $\delta_i = \delta(\mathcal{T}_i)$, $C_i = \mathrm{Stz}(\mathcal{C}_i)$, $C = \mathrm{GCD}(C_1, C_2)$, $B_i = \div(C_i, C)$, *and* $\sigma_i = \sigma(\mathcal{T}_i)$ *for* $i = 1, 2$. *Assume that* S *is not divisible by an infinite power of any prime, and both* R_1 *and* R_2 *are finite. If* $\mathfrak{s}_1$ *admits a diagonal injective homomorphism into* $\mathfrak{s}_2$, *then the* *following holds.*

(i) $\frac{R_1}{\delta_1} \leq \frac{R_2}{\delta_2}$. *The inequality is strict if* $\mathfrak{s}_1$ *is pure and* $\mathfrak{s}_2$ *is dense.*

(ii) $2\frac{R_1}{\delta_1} \leq \frac{R_2}{\delta_2}$ *when one of the following additional hypotheses holds:*

- $(X_1, X_2) = (A, C)$, (A, O), (O, C), *or* (C, O)*;*
- $(X_1, X_2) = (A, A)$, B_1 *is infinite;*
- $(X_1, X_2) = (A, A)$, B_1 *is finite,* $\mathfrak{s}_1$ *is two-sided weakly non-symmetric,* $\mathfrak{s}_2$ *is either one-sided or two-sided strongly non-symmetric;*
- $(X_1, X_2) = (A, A)$, *both* B_1 *and* B_2 *are finite,* C *is not divisible by an infinite power of a prime number, both* $\mathfrak{s}_1$, $\mathfrak{s}_2$ *are two-sided strongly non-symmetric, and* $\frac{R_1\sigma_1}{B_1} < \frac{R_2\sigma_2}{B_2}$.

Again the inequality is strict if $\mathfrak{s}_1$ *is pure and* $\mathfrak{s}_2$ *is dense.*

Proof. (i) Assume that $(X_1, X_2) = (A, A)$ (the other cases are analogous). Let $\mathfrak{s}_1$ be exhausted as $\mathrm{sl}(n_0) \subset \mathrm{sl}(n_1) \subset \cdots$, each inclusion $\mathrm{sl}(n_i) \to \mathrm{sl}(n_{i+1})$ being of signature (l_i, r_i, z_i), $i \geq 0$ and $\mathfrak{s}_2$ as $\mathrm{sl}(m_0) \subset \mathrm{sl}(m_1) \subset \cdots$ with $\mathrm{sl}(m_i) \to \mathrm{sl}(m_{i+1})$ being of signature (l'_i, r'_i, z'_i), $i \geq 0$. Moreover, we choose n_0 to be divisible by R_1 and m_0 to be divisible by R_2.

There is a commutative diagram

$$\begin{array}{ccccccccc} \mathrm{sl}(n_0) & \longrightarrow & \mathrm{sl}(n_1) & \longrightarrow & \cdots & \longrightarrow & \mathrm{sl}(n_i) & \longrightarrow & \cdots \\ \downarrow{\scriptstyle \theta_0} & & \downarrow{\scriptstyle \theta_1} & & & & \downarrow{\scriptstyle \theta_i} & & \\ \mathrm{sl}(m_{k_0}) & \longrightarrow & \mathrm{sl}(m_{k_1}) & \longrightarrow & \cdots & \longrightarrow & \mathrm{sl}(m_{k_i}) & \longrightarrow & \cdots, \end{array} \tag{3.18}$$

where each injective homomorphism θ_i is diagonal of signature $(x_i, y_i, m_{k_i} - (x_i + y_i)n_i)$. Denote $q_i = x_i + y_i$. Then, using Corollary 2.5 [BZ], we get

$$q_i s'_{k_i} \cdots s'_{k_j - 1} = s_i \cdots s_{j-1} q_j \text{ for all } j > i \geq 0. \tag{3.19}$$

Hence $s_i s_{i+1} \cdots$ divides $q_i s'_{k_i} s'_{k_i+1} \cdots$ for $i \geq 0$, so $S_1 m_0 s'_0 \cdots s'_{k_i-1}$ divides $q_i S_2 n_0 s_0 \cdots s_{i-1}$. Since S is not divisible by an infinite power of a prime number, the Steinitz number $\div(S_1 m_0 s'_0 \cdots s'_{k_i-1}, S)$ divides $\div(q_i S_2 n_0 s_0 \cdots s_{i-1}, S)$. Therefore we have that $\div(q_i R_2 n_0 s_0 \cdots s_{i-1}, R_1 m_0 s'_0 \cdots s'_{k_i-1})$ is a Steinitz number which is moreover finite, and thus it is a positive integer. So $\frac{m_0 s'_0 \cdots s'_{k_i-1}}{R_2 m_{k_i}} \leq \frac{n_0 s_0 \cdots s_{i-1}}{R_1 n_i}$. Taking the limit of both sides for $i \to \infty$ we get $\frac{\delta_2}{R_2} \leq \frac{\delta_1}{R_1}$. Moreover, if $\mathfrak{s}_1$ is pure and $\mathfrak{s}_2$ is dense, then $\frac{m_0 s'_0 \cdots s'_{k_i-1}}{R_2 m_{k_i}} \leq \frac{\delta_1}{R_1}$ for large enough i. Since the non-increasing sequence $\frac{m_0 s'_0 \cdots s'_{k_i-1}}{m_{k_i}}$ does not stabilize, we obtain the strict inequality $\frac{\delta_2}{R_2} < \frac{\delta_1}{R_1}$.

(ii) We keep the notations from (i). The injective homomorphism of $\mathfrak{s}_1$ into $\mathfrak{s}_2$ is given again by (3.18). If the pair (X_1, X_2) is one of (A, C), (A, O), (O, C), and (C, O), then, by Proposition 2.3 [BZ], for any diagonal injective homomorphism of a type X_1 algebra into a type X_2 algebra of signature (l, r, z) the integer $l + r$ is even. Therefore q_j is divisible by 2 for any j and it follows from (3.19) that $q_i s'_{k_i} s'_{k_i+1} \cdots$ is divisible by $2 s_i s_{i+1} \cdots$. The rest of the proof is analogous to (i).

In the other three cases both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are of type A. Notice that neither $\mathfrak{s}_1$ nor $\mathfrak{s}_2$ is two-sided symmetric (otherwise S would be divisible by 2^∞). Thus we can assume that $c_i > 0$ and $c'_i > 0$ for all $i \geq 0$. Denote $t_i = x_i - y_i$. It is enough to prove that $t_i = 0$ for infinitely many i (because then q_i is even for infinitely many i and the statement can be proven similarly to the first case). Assume the contrary, i.e. let $t_i > 0$ for $i \geq i_0$. Without loss of generality we can assume that $t_i > 0$ for all $i \geq 0$. Let us show that this contradicts the assumptions of the lemma in all three cases.

Let B_1 be infinite. By Corollary 2.5 in [BZ],

$$t_0 c'_{k_0} \cdots c'_{k_i - 1} = c_0 \cdots c_{i-1} t_i \text{ for } i \geq 1. \tag{3.20}$$

Then clearly $c_0 c_1 \cdots$ divides $t_0 c'_{k_0} c'_{k_0+1} \cdots$, and therefore B_1 divides $n_0 t_0$. This contradicts B_1 being infinite.

For the next case, combining (3.19) and (3.20), we obtain $\frac{t_0}{q_0} \cdot \frac{c'_{k_0} \cdots c'_{k_i-1}}{s'_{k_0} \cdots s'_{k_i-1}} = \frac{t_i}{q_i} \cdot \frac{c_0 \cdots c_{i-1}}{s_0 \cdots s_{i-1}}$. By definition $\sigma_1 = \lim_{i\to\infty} \frac{c_0 \cdots c_i}{s_0 \cdots s_i}$, and since $\mathfrak{s}_1$ is two-sided weakly non-symmetric we have $\lim_{i\to\infty} \frac{t_i}{q_i} \frac{c_0 \cdots c_i}{s_0 \cdots s_i} = 0$. But $\lim_{i\to\infty} \frac{t_0}{q_0} \cdot \frac{c'_{k_0} \cdots c'_{k_i-1}}{s'_{k_0} \cdots s'_{k_i-1}} = u\sigma_2$, where $u = \frac{t_0 s'_0 \cdots s'_{k_0-1}}{q_0 c'_0 \cdots c'_{k_0-1}} > 0$. So $\sigma_2 = 0$, contradicting $\mathfrak{s}_2$ being not two-sided weakly non-symmetric.

Finally, let both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ be two-sided strongly non-symmetric. Since $t_i \leq q_i$ for $i \geq 0$, we have $\frac{t_0}{q_0} \cdot \frac{c'_{k_0} \cdots c'_{k_i-1}}{s'_{k_0} \cdots s'_{k_i-1}} \leq \frac{c_0 \cdots c_{i-1}}{s_0 \cdots s_{i-1}}$. Taking the limit we obtain

$$\frac{t_0}{q_0} \cdot \frac{s'_0 \cdots s'_{k_0-1}}{c'_0 \cdots c'_{k_0-1}} \sigma_2 \leq \sigma_1. \tag{3.21}$$

Let us go back to (3.19). We know that $q_0 s'_{k_0} \cdots s'_{k_i-1} = s_0 \cdots s_{i-1} q_i$. If q_i is divisible by some prime number p for infinitely many i, then by an argument similar to that in (i) one derives the inequality $p\frac{R_1}{\delta_1} \leq \frac{R_2}{\delta_2}$, from which the statement follows. So we can assume that every p divides at most finitely many q_i. Then it is easy to see that the Steinitz numbers $q_0 s'_{k_0} s'_{k_0+1} \cdots$ and $s_0 s_1 \cdots$ have equal values at every prime p, so they coincide. Hence,

$$\frac{R_2}{R_1} = \frac{m_0 s'_0 \cdots s'_{k_0-1}}{q_0 n_0}. \tag{3.22}$$

From (3.20) $c_0 c_1 \cdots$ divides $t_0 c'_{k_0} c'_{k_0+1} \cdots$, and therefore $\frac{B_2}{B_1} \geq \frac{m_0 c'_0 \cdots c'_{k_0-1}}{t_0 n_0}$. Combining the latter inequality with (3.21) and (3.22) we obtain $\frac{\sigma_1}{\sigma_2} \geq \frac{R_2 B_1}{R_1 B_2}$, which contradicts an assumption in the statement of the lemma. □

We are now able to prove the main result of the thesis.

Theorem 3.1.11. *a) The three finitary Lie algebras* sl(∞), so(∞), sp(∞) *admit an injective homomorphism into any infinite-dimensional diagonal locally simple Lie algebra. An infinite-dimensional non-finitary diagonal locally simple Lie algebra admits no injective homomorphism into* sl(∞), so(∞), sp(∞).

b) Let $\mathfrak{s}_1 = X_1(\mathcal{T}_1)$, $\mathfrak{s}_2 = X_2(\mathcal{T}_2)$ be infinite-dimensional non-finitary diagonal locally simple Lie algebras. Set $S_i = \mathrm{Stz}(\mathcal{S}_i)$, $S = \mathrm{GCD}(S_1, S_2)$, $R_i = \div(S_i, S)$, $\delta_i = \delta(\mathcal{T}_i)$, $C_i = \mathrm{Stz}(\mathcal{C}_i)$, $C = \mathrm{GCD}(C_1, C_2)$, $B_i = \div(C_i, C)$, and $\sigma_i = \sigma(\mathcal{T}_i)$ for $i = 1, 2$. Then $\mathfrak{s}_1$ admits an injective homomorphism into $\mathfrak{s}_2$ if and only if the following conditions hold.

1) R_1 is finite.

2) $\mathfrak{s}_2$ is sparse if $\mathfrak{s}_1$ is sparse.

3) If $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are non-sparse, both R_1 and R_2 are finite, and S is not divisible by an infinite power of any prime number, then $\epsilon \frac{R_1}{\delta_1} \leq \frac{R_2}{\delta_2}$ for ϵ as specified below. The inequality is strict if $\mathfrak{s}_1$ is pure and $\mathfrak{s}_2$ is dense. We have $\epsilon = 2$, except in the cases listed below, in which $\epsilon = 1$:

3.1) $(X_1, X_2) = (C, C)$, (O, O), (C, A), (O, A), and $(X_1, X_2) = (A, A)$ with both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ being one-sided;

3.2) $(X_1, X_2) = (A, A)$, B_1 is finite, either $\mathfrak{s}_1$ is one-sided and $\mathfrak{s}_2$ is two-sided non-symmetric or $\mathfrak{s}_2$ is two-sided weakly non-symmetric and $\mathfrak{s}_1$ is two-sided non-symmetric;

3.3) $(X_1, X_2) = (A, A)$, B_1 is finite, both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are two-sided strongly non-symmetric, either B_2 is infinite or C is divisible by an infinite power of any prime number;

3.4) $(X_1, X_2) = (A, A)$, both B_1 and B_2 are finite, both $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are two-sided strongly non-symmetric, C is not divisible by an infinite power of any prime number, and $\frac{R_1 \sigma_1}{B_1} \geq \frac{R_2 \sigma_2}{B_2}$.

Proof. a) The statement follows directly from Corollary 3.1.4 and Proposition 3.1.5.

b) The sufficiency of the conditions follows directly from Lemma 3.1.2 and Lemma 3.1.3.

The necessity of conditions 1 and 2 follows from Proposition 3.1.5 and Corollary 3.1.7 respectively. Let us prove the necessity of condition 3. Note that the assumptions of this condition satisfy Corollary 3.1.9. Hence in this case an injective homomorphism of $\mathfrak{s}_1$ into $\mathfrak{s}_2$, if it exists, has to be diagonal. Therefore we can apply Lemma 3.1.10 and this lemma implies the necessity of condition 3 (it is easy to check that under corresponding assumptions the cases which are not listed in 3.1−3.4 are exactly the cases listed in Lemma 3.1.10 (ii)). □

3.2 Equivalence classes of diagonal locally simple Lie algebras

We now introduce a notion of equivalence of infinite-dimensional Lie algebras. We say that $\mathfrak{g}_1$ is *equivalent* to $\mathfrak{g}_2$ ($\mathfrak{g}_1 \sim \mathfrak{g}_2$) if there exist injective homomorphisms $\mathfrak{g}_1 \to \mathfrak{g}_2$ and $\mathfrak{g}_2 \to \mathfrak{g}_1$. For finite-dimensional Lie algebras, equivalence is the same as isomorphism, but this is no longer true for infinite-dimensional Lie algebras.

The following corollary gives a description of the so defined equivalence classes of diagonal locally simple Lie algebras.

Corollary 3.2.1. *a) The three finitary Lie algebras* $\mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, *and* $\mathrm{sp}(\infty)$ *are pairwise equivalent. None of them is equivalent to any non-finitary diagonal locally simple Lie algebra.*

b) Let $\mathfrak{s}_1 = X_1(\mathcal{T}_1)$ *and* $\mathfrak{s}_2 = X_2(\mathcal{T}_2)$ *be infinite-dimensional non-finitary diagonal locally simple Lie algebras. Set* $S_i = \mathrm{Stz}(\mathcal{S}_i)$, $S = \mathrm{GCD}(S_1, S_2)$, $R_i = \div(S_i, S)$, $\delta_i = \delta(\mathcal{T}_i)$, $C_i = \mathrm{Stz}(\mathcal{C}_i)$, $C = \mathrm{GCD}(C_1, C_2)$, $B_i = \div(C_i, C)$, *and* $\sigma_i = \sigma(\mathcal{T}_i)$ *for* $i = 1, 2$. *Then* $\mathfrak{s}_1$ *is equivalent to* $\mathfrak{s}_2$ *if and only if the following conditions hold.*

1) $S_1 \overset{\mathbb{Q}}{\sim} S_2$.

2) Both $\mathfrak{s}_1$ *and* $\mathfrak{s}_2$ *are either sparse or non-sparse.*

3) If $\mathfrak{s}_1$ *and* $\mathfrak{s}_2$ *are non-sparse and* S *is not divisible by an infinite power of any prime number, then:*

3.1) $\frac{R_1}{\delta_1} = \frac{R_2}{\delta_2}$;

3.2) $\mathfrak{s}_1$ *and* $\mathfrak{s}_2$ *have the same density type;*

3.3) $\mathfrak{s}_1$ *and* $\mathfrak{s}_2$ *are of the same type* ($X_1 = X_2$);

3.4) $\mathfrak{s}_1$ *and* $\mathfrak{s}_2$ *have the same symmetry type;*

3.5) $C_1 \overset{\mathbb{Q}}{\sim} C_2$ *if* $\mathfrak{s}_1$ *and* $\mathfrak{s}_2$ *are two-sided non-symmetric;*

3.6) $\frac{R_1\sigma_1}{B_1} = \frac{R_2\sigma_2}{B_2}$ *if* $\mathfrak{s}_1$ *and* $\mathfrak{s}_2$ *are two-sided strongly non-symmetric and* C *is not divisible by an infinite power of any prime number.*

Proof. a) The statement follows directly from Theorem 3.1.11 a).

b) To prove sufficiency it is easy to check case by case that all the conditions of Theorem 3.1.11 b) are satisfied for both pairs $\mathfrak{s}_1 \subset \mathfrak{s}_2$ and $\mathfrak{s}_2 \subset \mathfrak{s}_1$.

Let us prove necessity. Assume that there exist injective homomorphisms $\mathfrak{s}_1 \to \mathfrak{s}_2$ and $\mathfrak{s}_2 \to \mathfrak{s}_1$. Conditions 1 and 2 are obviously satisfied. Suppose that $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are both non-sparse and S is not divisible by an infinite power of any prime number. Then $\epsilon_1\frac{R_1}{\delta_1} \leq \frac{R_2}{\delta_2}$ and $\epsilon_2\frac{R_2}{\delta_2} \leq \frac{R_1}{\delta_1}$ by Theorem 3.1.11 b). Clearly, this is only possible if $\epsilon_1 = \epsilon_2 = 1$ and $\frac{R_1}{\delta_1} = \frac{R_2}{\delta_2}$. Then $\mathfrak{s}_1$ and $\mathfrak{s}_2$ have the same density type (otherwise one of the inequalities would be strict). Conditions 3.3−3.6 follow from conditions 3.1−3.4 of Theorem 3.1.11 b) for both pairs $(\mathfrak{s}_1, \mathfrak{s}_2)$ and $(\mathfrak{s}_2, \mathfrak{s}_1)$. □

Remark. Isomorphic Lie algebras are clearly equivalent. If two Lie algebras satisfy Theorem 2.2.1 (or Theorem 2.2.2), then they satisfy also Corollary 3.2.1. One can check that conditions $\mathcal{A}_3$ and $\mathcal{B}_3$ of Theorem 2.2.1 correspond respectively to conditions 3.1 and 3.6 of Corollary 3.2.1.

Let $\mathbb{D}$ denote the set of equivalence classes of infinite-dimensional diagonal locally simple Lie algebras. If we write $\mathfrak{s}_1 \to \mathfrak{s}_2$ in case there exists an injective homomorphism from $\mathfrak{s}_1$ into $\mathfrak{s}_2$, then the relation $\to$ induces a partial order on $\mathbb{D}$. It follows from Theorem 3.1.11 a) that $\mathbb{D}$ has a unique minimal element (which also is the least element) with respect to the partial order $\to$: this is the equivalence class consisting of the three finitary Lie algebras sl(∞), so(∞), sp(∞). The following statement shows that there exists precisely one maximal element of $\mathbb{D}$ (which also is the greatest element) and describes the corresponding equivalence class.

Corollary 3.2.2. *Let* $\mathfrak{s} = X(\mathcal{T})$ *be a diagonal locally simple Lie algebra. The following are equivalent.*

1) Any diagonal locally simple Lie algebra admits an injective homomorphism into $\mathfrak{s}$.

2) $\mathfrak{s}$ *is sparse and* $\mathrm{Stz}(\mathcal{S}) = p_1^\infty p_2^\infty \cdots$, *where* $p_1, p_2, \ldots$ *is the increasing sequence of all prime numbers.*

Proof. 1)⇒2): Consider a Lie algebra $\mathfrak{s}' = A(\mathcal{T}')$, where $\mathcal{T}'$ is sparse and $\mathrm{Stz}(\mathcal{S}') =$

$p_1^\infty p_2^\infty \cdots$. Since $\mathfrak{s}'$ admits an injective homomorphism into $\mathfrak{s}$, the Steinitz number $\div(p_1^\infty p_2^\infty \cdots, S)$ is finite and $\mathfrak{s}$ is sparse by Theorem 3.1.11 b). Then clearly $S = p_1^\infty p_2^\infty \cdots$.

2)⇒1): This follows immediately from Theorem 3.1.11. □

The equivalence class corresponding to the maximal element of $\mathbb{D}$ consists of infinitely many pairwise non-isomorphic Lie algebras. Indeed, by Theorem 2.2.1 there is only one, up to isomorphism, sparse one-sided Lie algebra of type A satisfying property 2 of Corollary 3.2.2, but there are infinitely many sparse two-sided Lie algebras of type A with this property. In addition, by Theorem 2.2.2, any Lie algebra of type other than A satisfying property 2 of Corollary 3.2.2 is isomorphic to the sparse two-sided symmetric Lie algebra of type A with $\mathrm{Stz}(\mathcal{S}) = p_1^\infty p_2^\infty \cdots$.

Chapter 4

Homomorphisms of diagonal Lie algebras

This chapter is joint work with Ivan Penkov. It contains some ideas and partial results on injective homomorphisms of diagonal locally simple Lie algebras.

4.1 Diagonal and non-diagonal homomorphisms

Let $\mathfrak{s}$ and $\mathfrak{g}$ be two diagonal Lie algebras admitting an injective homomorphism $\theta : \mathfrak{s} \to \mathfrak{g}$. Then θ is given by a commutative diagram

$$\begin{array}{ccccccc} \mathfrak{s}_1 & \xrightarrow{\varphi_1} & \cdots & \xrightarrow{\varphi_{n-1}} & \mathfrak{s}_n & \xrightarrow{\varphi_n} & \cdots \\ \downarrow{\scriptstyle\theta_1} & & & & \downarrow{\scriptstyle\theta_n} & & \\ \mathfrak{g}_1 & \xrightarrow{\psi_1} & \cdots & \xrightarrow{\psi_{n-1}} & \mathfrak{g}_n & \xrightarrow{\psi_n} & \cdots \end{array} \tag{4.1}$$

for some exhaustions $\mathfrak{s}_1 \overset{\varphi_1}{\to} \mathfrak{s}_2 \overset{\varphi_2}{\to} \dots$ and $\mathfrak{g}_1 \overset{\psi_1}{\to} \mathfrak{g}_2 \overset{\psi_2}{\to} \dots$ of $\mathfrak{s}$ and $\mathfrak{g}$ respectively. Diagonal homomorphisms (i.e. homomorphisms for which all θ_n can be chosen diagonal for large enough n) seem to be easier to study. Indeed, a diagonal homomorphism θ can be studied in terms of the signatures (p_n, q_n, u_n) of the respective θ_n; only elementary methods are required to determine necessary and sufficient conditions on sequences $\{p_n\}$, $\{q_n\}$, $\{u_n\}$ so that they define a homomorphism $\theta : \mathfrak{s} \to \mathfrak{g}$. Moreover, we believe that for a homomorphism given by a fixed set of signatures (p_n, q_n, u_n), a study similar to the study of homomorphisms in [DP3] can be carried out (in particular, it follows from [DP3] that all homomorphisms of finitary Lie algebras are necessarily diagonal). This is why it is natural to ask when there is a diagonal homomorphism $\mathfrak{s} \to \mathfrak{g}$ and when there is a non-diagonal homomorphism

$\mathfrak{s} \to \mathfrak{g}$. The following proposition partly answers this question.

Proposition 4.1.1. *Let* $(\mathfrak{s}, \mathfrak{g})$ *be a pair of diagonal Lie algebras such that* $\mathfrak{s}$ *admits an injective homomorphism into* $\mathfrak{g}$.

(a) If both $\mathfrak{s}$ *and* $\mathfrak{g}$ *are finitary, then there exists an injective diagonal homomorphism* $\mathfrak{s} \to \mathfrak{g}$ *and there is no non-zero non-diagonal homomorphisms* $\mathfrak{s} \to \mathfrak{g}$.

(b) If $\mathfrak{s}$ *is finitary and* $\mathfrak{g}$ *is non-finitary, then there exists an injective non-diagonal homomorphism* $\mathfrak{s} \to \mathfrak{g}$. *Moreover, there exists an injective diagonal homomorphism* $\mathfrak{s} \to \mathfrak{g}$ *if and only if* $\mathfrak{g}$ *is sparse.*

(c) If both $\mathfrak{s} = X_1(\mathcal{T}_1)$ *and* $\mathfrak{g} = X_2(\mathcal{T}_2)$ *are non-finitary, set* $S_i = \mathrm{Stz}(\mathcal{S}_i)$, *and let* A_i *be the set of all prime divisors* p *of* S_i *for which* p^∞ *does not divide* S_i, $i = 1, 2$. *There exists an injective diagonal homomorphism* $\mathfrak{s} \to \mathfrak{g}$. *Moreover, there exists an injective non-diagonal homomorphism* $\mathfrak{s} \to \mathfrak{g}$ *if at least one of the sets* A_1, A_2 *is finite, and there exists no non-zero non-diagonal homomorphism* $\mathfrak{s} \to \mathfrak{g}$ *if* $\div(S_2, GCS(S_1, S_2))$ *is finite and all prime divisors of* S_2 *are contained in* A_2.

Proof. (a) The statement follows from the results of [DP3].

(b) The existence of an injective non-diagonal homomorphism in this case is proven in Corollary 3.1.4. If $\mathfrak{g}$ is sparse, one can prove the existence of a diagonal homomorphism similarly to the proof of Lemma 3.1.2 (iv). Let now $\mathfrak{g}$ be pure or dense and let us prove that there is no non-zero diagonal homomorphisms $\mathfrak{s} \to \mathfrak{g}$. Assume the contrary. By Lemma 3.1.3 (i), (ii), $\mathfrak{g}$ admits a diagonal homomorphism into some pure Lie algebra of type A. Since $\mathrm{sl}(\infty)$ can be mapped into $\mathfrak{s}$ by a diagonal homomorphism, we have a diagonal homomorphism of $\mathrm{sl}(\infty)$ into a pure Lie algebra of type A. Then we have a commutative diagram

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathrm{sl}(k) & \longrightarrow & \mathrm{sl}(k+1) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \theta_k} & & \downarrow{\scriptstyle \theta_{k+1}} & & \\ \cdots & \longrightarrow & \mathrm{sl}(n_1 \cdots n_k) & \longrightarrow & \mathrm{sl}(n_1 \cdots n_k n_{k+1}) & \longrightarrow & \cdots \end{array} \quad (4.2)$$

for some integers $n_1, n_2, \ldots$, where θ_k are diagonal homomorphisms of signatures (p_k, q_k, u_k) for all $k \in \mathbb{N}$ and the lower row consists of homomorphisms of signatures $(n_k, 0, 0)$. Therefore $n_1 \cdots n_m = (p_m + q_m)m + u_m \geq (p_m + q_m)m$ for $m \geq 1$. By Proposition 2.1.1 (ii) and Corollary 2.2.4 we also have $p_m + q_m = (p_k + q_k)n_{k+1} \cdots n_m$ for $m > k$. Hence, $p_k + q_k \leq \frac{n_1 \cdots n_k}{m}$ for $m > k$. The latter inequality implies $p_k + q_k = 0$ since m can be chosen arbitrary large. This contradicts the fact that

θ_k is a homomorphism. Therefore there is no diagonal embeddings of $\mathrm{sl}(\infty)$ into a pure Lie algebra of type A, and the statement follows.

(c) All the injective homomorphisms of diagonal non-finitary Lie algebras constructed in Section 3.1 are diagonal, therefore there always exists an injective diagonal homomorphism $\mathfrak{s} \to \mathfrak{g}$. Corollary 3.1.9 proves the last statement of (c). It is left to prove that if at least one of the sets A_1, A_2 is finite, then there exists a non-diagonal injective homomorphism of $\mathfrak{s}$ into $\mathfrak{g}$.

Suppose that the set A_1 is finite. Then, since the Steinitz number S_1 is infinite, there exists at least one prime p for which p^∞ divides S_1. Hence $S_1 = kS_1'$, where k is finite and, for any prime divisor p of S_1', p^∞ divides S_1'. Then by Lemma 3.1.3 (iii) there exist injective homomorphisms $\mathfrak{s} \to \mathrm{sl}(S_1')$ and $\mathrm{sl}(S_1') \to \mathfrak{g}$. Therefore to show the existence of an injective non-diagonal homomorphism of $\mathfrak{s}$ into $\mathfrak{g}$ it is enough to show that there is an injective non-diagonal homomorphism of $\mathrm{sl}(S_1')$ into itself. Let us prove the latter fact.

Let $S_1' = n_1 n_2 \cdots$, where n_i are integers. From the definition of S_1' it is clear that $(S_1')^2 = S_1'$. We now construct an injective homomorphism of $\mathrm{sl}(n_1 n_2 \cdots)$ into $\mathrm{sl}((n_1)^2 (n_2)^2 \cdots)$. Fix an injective homomorphism θ_k of $\mathrm{sl}(n_1 \cdots n_k)$ into $\mathrm{sl}((n_1)^2 \cdots (n_k)^2)$ such that the natural $\mathrm{sl}((n_1)^2 \cdots (n_k)^2)$-module decomposes as an $\mathrm{sl}(n_1 \cdots n_k)$-module as the second tensor power of the natural $\mathrm{sl}(n_1 \cdots n_k)$-module. One checks that it is possible to define θ_{k+1} by a similar procedure so that the following diagram is commutative:

$$\begin{array}{ccccccccc}
\mathrm{sl}(n_1) & \longrightarrow & \cdots & \longrightarrow & \mathrm{sl}(n_1 \cdots n_k) & \longrightarrow & \mathrm{sl}(n_1 \cdots n_k n_{k+1}) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \theta_1} & & & & \downarrow{\scriptstyle \theta_n} & & \downarrow{\scriptstyle \theta_{n+1}} & & \\
\mathrm{sl}((n_1)^2) & \longrightarrow & \cdots & \longrightarrow & \mathrm{sl}((n_1)^2 \cdots (n_k)^2)) & \longrightarrow & \mathrm{sl}((n_1)^2 \cdots (n_k)^2 (n_{k+1})^2)) & \longrightarrow & \cdots .
\end{array} \tag{4.3}$$

Here the upper row consists of homomorphisms of signatures $(n_k, 0, 0)$, and the lower rows consist of homomorphisms of signatures $((n_k)^2, 0, 0)$ respectively. By defining such homomorphisms θ_k for every k we obtain an injective homomorphism $\mathrm{sl}(S_1') \to \mathrm{sl}(S_1')$, and this homomorphism is clearly not diagonal.

In case A_2 is infinite we prove the statement in a similar way, only now we construct an injective non-diagonal homomorphism of $\mathrm{sl}(S_2')$ into itself, where $S_2 = kS_2'$ for finite k. □

4.2 Natural representations of diagonal Lie algebras

In [DP3] the $\mathfrak{s}$-module structures of the natural and conatural $\mathfrak{g}$-modules are studied as natural invariants of an injective homomorphism of finitary Lie algebras $\mathfrak{s} \to \mathfrak{g}$. In order to generalize the results of [DP3] to diagonal Lie algebras, we should first define natural and conatural modules over a diagonal Lie algebra. Let $\mathfrak{g}$ be a diagonal Lie algebra and let $\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \cdots$ be an exhaustion of $\mathfrak{g}$. By definition, a *natural* $\mathfrak{g}$*-module* (respectively, *conatural* $\mathfrak{g}$*-module*) is any non-zero $\mathfrak{g}$-module which can be constructed as a direct limit $V = \varinjlim V_n$ (respectively, $V_* = \varinjlim V_n^*$), where V_n is the natural $\mathfrak{g}_n$-module. One can check that this definition agrees with the definition given in Section 2.2 for $\mathfrak{g} \cong \mathrm{sl}(\infty)$, $\mathrm{so}(\infty)$, and $\mathrm{sp}(\infty)$. However, a natural $\mathfrak{g}$-module is not uniquely defined if $\mathfrak{g}$ is non-finitary, and moreover there exists an infinite family of non-isomorphic natural representations of $\mathfrak{g}$ in this case. Consider for example the simplest case of a non-finitary diagonal Lie algebra: set $\mathfrak{g} = \mathrm{sl}(2^\infty) = \varinjlim(\mathfrak{g}_n = \mathrm{sl}(2^n))$, where each injective homomorphism $\mathrm{sl}(2^n) \to \mathrm{sl}(2^{n+1})$ is of signature $(2, 0, 0)$. Notice that for each choice of nested Cartan subalgebras $\mathfrak{h}_n \subset \mathfrak{g}_n$, $\mathfrak{h}_n \subset \mathfrak{h}_{n+1}$, there exists a natural $\mathfrak{g}$-module $V_\mathfrak{h}$ which is an $\mathfrak{h}$-weight module (i.e. $V_\mathfrak{h}$ equals the sum of its $\mathfrak{h}$-eigenspaces; $\mathfrak{h} = \cup_n \mathfrak{h}_n$). On the other hand, there clearly exists a direct limit $\widetilde{V} = \varinjlim V_n$ such that $\mathfrak{h}$ does not act on $\widetilde{V}$ via weight spaces: to obtain such a direct limit it suffices to fix a decomposition of $\mathfrak{g}_n$-modules $V_{n+1} = V_n \oplus V_n$ such that each $\mathfrak{h}_{n+1}$-weight space of V_{n+1} lies in one of the two copies of V_n, then map V_n diagonally into V_{n+1}, and after that pass to the direct limit. As $V_\mathfrak{h}$ is an $\mathfrak{h}$-weight module, while $\widetilde{V}$ is not, there is no $\mathfrak{g}$-isomorphism between $V_\mathfrak{h}$ and $\widetilde{V}$. This construction produces two non-isomorphic natural representations of $\mathfrak{g}$, but it is easy to see that in fact there are infinitely many isomorphism classes of such representations. In a similar way one shows that there are infinitely many non-isomorphic conatural $\mathfrak{g}$-modules of $\mathfrak{g} = \mathrm{sl}(2^\infty)$.

Let us go back to the study of homomorphisms $\mathfrak{s} \to \mathfrak{g}$. Following the ideas of [DP3], we investigate the socle filtrations of natural representations of $\mathfrak{g}$ as $\mathfrak{s}$-modules. Consider an injective homomorphism $\theta : \mathfrak{s} \to \mathfrak{g}$ of non-finitary diagonal Lie algebras. Then we have the following.

Proposition 4.2.1. *Let $\mathfrak{g}$ be one-sided, and let θ be an injective diagonal homomorphism given by a commutative diagram*

$$\begin{array}{ccccccccc} \mathfrak{s}_1 & \longrightarrow \cdots \longrightarrow & \mathfrak{s}_n & \longrightarrow & \mathfrak{s}_{n+1} & \longrightarrow \cdots & & \mathfrak{s} \\ \downarrow{\scriptstyle \theta_1} & & \downarrow{\scriptstyle \theta_n} & & \downarrow{\scriptstyle \theta_{n+1}} & & & \downarrow{\scriptstyle \theta} \\ \mathfrak{g}_1 & \longrightarrow \cdots \longrightarrow & \mathfrak{g}_n & \longrightarrow & \mathfrak{g}_{n+1} & \longrightarrow \cdots & & \mathfrak{g}, \end{array} \tag{4.4}$$

where θ_n is of signature $(1, 0, z_n)$ (respectively, $(1, 1, z_n)$) for all n. Then any natural $\mathfrak{g}$-module V, considered as an $\mathfrak{s}$-module, has a non-zero socle V' which is isomorphic to a direct sum of a natural (resp., a natural and a conatural) $\mathfrak{s}$-module and possibly a trivial $\mathfrak{s}$-module. Moreover, V/V' is a trivial $\mathfrak{s}$-module.

Proof. Let $V = \varinjlim V_n$ where V_n is the natural $\mathfrak{g}_n$-module. Consider first the case when θ_n is of signature $(1, 0, z_n)$ for all n. Then we have $V_n \downarrow \mathfrak{s}_n = F_n \oplus z_n T_n$, where F_n is the natural $\mathfrak{s}_n$-module and T_n is the trivial one-dimensional $\mathfrak{s}_n$-module. Let $v \in V$ be any element on which $\mathfrak{s}$ acts non-trivially. We have $v \in V_n$ for some n. By acting by $\mathfrak{s}_n$ on v we obtain F_n, the only non-trivial submodule of $V_n \downarrow \mathfrak{s}_n$. Applying the same argument to the image of v in V_{n+1}, V_{n+2}, etc. we conclude that a natural $\mathfrak{s}$-module $F = \varinjlim F_n$ is a submodule of V and moreover V/F is a trivial module, so the statement follows.

It remains to notice that in the case when θ_n is of signature $(1, 1, z_n)$ for all n, the homomorphism of V_n into V_{n+1} induced by the diagram in (4.4) maps the $\mathfrak{s}_n$-submodule F_n (respectively, F_n^*) into F_{n+1} (resp., F_{n+1}^*). This is a consequence of the fact that $\mathfrak{g}$ is one-sided. Then as above we can show that the simple direct summands of the socle of $V \downarrow \mathfrak{s}$ are a natural $\mathfrak{s}$-module $F = \varinjlim F_n$ and a conatural $\mathfrak{s}$-module $F_* = \varinjlim F_n^*$, and that the module $V/(F \oplus F_*)$ is trivial. □

Consider now a non-diagonal homomorphism $\mathfrak{s} \to \mathfrak{g}$. In contrast with the case of diagonal homomorphisms, the socle filtration of the module $V \downarrow \mathfrak{s}$ may not be an adequate tool to study the structure of this module. For instance, if $\mathfrak{s} = \mathrm{sl}(\infty)$ is mapped into $\mathfrak{g} = \mathrm{sl}(2^\infty)$ by the injective homomorphism constructed in the beginning of Section 3.1, then $V \downarrow \mathfrak{s}$ has zero socle. Moreover, the following statement holds.

Proposition 4.2.2. *If $\mathfrak{s}$ is diagonal finitary and $\mathfrak{g}$ is one-sided non-finitary with the property $Stz(\mathfrak{g}) = 2^\infty$, then for any injective homomorphism $\mathfrak{s} \to \mathfrak{g}$ constructed in Section 3.1, the socle of any natural representation of $\mathfrak{g}$ considered as an $\mathfrak{s}$-module is a trivial $\mathfrak{s}$-module.*

Proof. Consider first the case $\mathfrak{s} = \mathrm{sl}(\infty)$ and $\mathfrak{g} = \mathrm{sl}(2^\infty)$. An injective homomorphism $\theta : \mathrm{sl}(\infty) \to \mathrm{sl}(2^\infty)$ is constructed explicitly in the beginning of Section 3.1 and is given by the commutative diagram

$$\begin{array}{ccccccccc} \mathrm{sl}(2) & \longrightarrow \cdots \longrightarrow & \mathrm{sl}(n) & \longrightarrow & \mathrm{sl}(n+1) & \longrightarrow \cdots & & \mathrm{sl}(\infty) \\ \downarrow{\scriptstyle\theta_2} & & \downarrow{\scriptstyle\theta_n} & & \downarrow{\scriptstyle\theta_{n+1}} & & & \downarrow{\scriptstyle\theta} \\ \mathrm{sl}(2^2) & \longrightarrow \cdots \longrightarrow & \mathrm{sl}(2^n) & \xrightarrow{\varphi_n} & \mathrm{sl}(2^{n+1}) & \longrightarrow \cdots & & \mathrm{sl}(2^\infty), \end{array} \quad (4.5)$$

where φ_n is a diagonal injective homomorphism of signature $(2,0,0)$ for all n. Moreover, each θ_n is chosen so that

$$V_n \downarrow \mathrm{sl}(n) \cong \bigwedge^0(F_n) \oplus \bigwedge^1(F_n) \oplus \cdots \oplus \bigwedge^n(F_n). \quad (4.6)$$

Here F_n stands for the natural $\mathrm{sl}(n)$-module and V_n is the natural $\mathrm{sl}(2^n)$-module.

For the sake of contradiction assume that a natural representation $V = \varinjlim V_n$ of $\mathfrak{g}$ considered as an $\mathfrak{s}$-module has non-trivial socle. Then there is a simple $\mathfrak{s}$-submodule $V' \subset V \downarrow \mathfrak{s}$. Consider a non-zero element $v \in V'$. We have $v \in V_n \downarrow \mathrm{sl}(n)$ for some $n \in \mathbb{Z}_{\geq 2}$, and hence $\mathrm{sl}(n) \cdot v \subset V'$. Therefore, since in (4.6) the decomposition of $V_n \downarrow \mathrm{sl}(n)$ into direct sum of simple $\mathrm{sl}(n)$-modules is given explicitly, we obtain $\bigwedge^k(F_n) \subset V'$ for some k, $1 \leq k \leq n-1$. We know that under the homomorphism of $\mathrm{sl}(n)$-modules $V_n \to V_{n+1}$ induced by φ_n, $\bigwedge^k(F_n)$ is being mapped into the direct sum $\bigwedge^k(F_{n+1}) \oplus \bigwedge^{k+1}(F_{n+1})$. However, by our construction, the image of this map is not contained in $\bigwedge^k(F_{n+1})$ or $\bigwedge^{k+1}(F_{n+1})$, and hence generates $\bigwedge^k(F_{n+1}) \oplus \bigwedge^{k+1}(F_{n+1})$ when acted upon by $\mathrm{sl}(n+1)$. Therefore, $\bigwedge^k(F_{n+1}) \oplus \bigwedge^{k+1}(F_{n+1}) \subset V'$. Continuing this procedure we get $\bigcup_{m \geq n} \left(\bigoplus_{k \leq i \leq m-n+k} \bigwedge^i(F_m) \right) \subset V'$. One can check that $\bigcup_{m \geq n+1} \left(\bigoplus_{k+1 \leq i \leq m-n+k} \bigwedge^i(F_m) \right)$ is a proper $\mathfrak{s}$-submodule of V', which contradicts with the assumption that V' is a simple $\mathfrak{s}$-module. One can prove in a similar way that the same holds for any conatural $\mathfrak{g}$-module V^*, so the socle of $V^* \downarrow \mathfrak{s}$ (if non-zero) must be a trivial $\mathfrak{s}$-module.

Let now $\mathfrak{s}$ be an arbitrary finitary Lie algebra and $\mathfrak{g}$ be a one-sided non-finitary Lie algebra with $\mathrm{Stz}(\mathfrak{g}) = 2^\infty$. Note that $\mathfrak{s}$ admits an injective homomorphism into $\mathrm{sl}(\infty)$. According to Lemma 3.1.2 and Lemma 3.1.3, there exists an injective homomorphism $\mathrm{sl}(2^\infty) \cong \mathfrak{g}' \to \mathfrak{g}$. Moreover, this homomorphism is given by a commutative diagram

$$\begin{array}{ccccccc} \cdots \longrightarrow & \mathfrak{g}'_n & \longrightarrow & \mathfrak{g}'_{n+1} & \longrightarrow \cdots \\ & \downarrow{\scriptstyle \theta_n} & & \downarrow{\scriptstyle \theta_{n+1}} & \\ \cdots \longrightarrow & \mathfrak{g}_n & \longrightarrow & \mathfrak{g}_{n+1} & \longrightarrow \cdots, \end{array} \quad (4.7)$$

where the diagonal homomorphisms θ_n are of signatures $(1,0,z_n)$ for $\mathfrak{g}$ of type A, and $(1,1,z_n)$ otherwise. Let V be again a natural module of $\mathfrak{g}$. Then by Proposition 4.2.1 we get that the socle of $V \downarrow \mathfrak{g}'$ is isomorphic to a direct sum of a natural $\mathfrak{g}'$-module and possibly a trivial module for $\mathfrak{g}$ of type A, and to the direct sum of a natural $\mathfrak{g}'$-module, a conatural $\mathfrak{g}'$-module, and possibly a trivial module for $\mathfrak{g}$ of type O or C. According to Corollary 3.1.4, a homomorphism of $\mathfrak{s}$ into $\mathfrak{g}$ is given by the following chain of inclusions: $\mathfrak{s} \to \mathrm{sl}(\infty) \to \mathfrak{g}' \to \mathfrak{g}$. We thus obtain that the socle of $V \downarrow \mathrm{sl}(\infty)$ is necessarily a trivial $\mathrm{sl}(\infty)$-module, and therefore the socle of $V \downarrow \mathfrak{s}$ is a trivial $\mathfrak{s}$-module. □

We complete this section by the remark that, in addition to the socle filtration of $V \downarrow \mathfrak{s}$, the radical filtration of $V \downarrow \mathfrak{s}$ provides also essential information about the homomorphism $\mathfrak{s} \to \mathfrak{g}$. Note that for the injective homomorphism $\mathrm{sl}(\infty) \to \mathrm{sl}(2^\infty)$ constructed via the diagram in (4.5), the socle of $V \downarrow \mathfrak{s}$ is a trivial module, but the radical filtration of $V \downarrow \mathfrak{s}$ can easily be checked to be exhaustive.

4.3 The level of a homomorphism

In Propositions 3.1.6 and 3.1.8 we used a certain invariant of an injective homomorphism $\mathfrak{s} \to \mathfrak{g}$. We now recall this invariant and define it in greater generality.

Let $\mathfrak{s}_1$, $\mathfrak{s}_2$ be finite-dimensional classical simple Lie algebras of type A, and $\psi : \mathfrak{s}_1 \to \mathfrak{s}_2$ be an injective homomorphism. Consider the decomposition

$$V_2 \downarrow \mathfrak{s}_1 \cong \bigoplus_{\lambda \in H(\psi)} \underbrace{V_1^\lambda \oplus \cdots \oplus V_1^\lambda}_{t_\lambda},$$

where V_2 is the natural $\mathfrak{s}_2$-module, V_1^λ is an $\mathfrak{s}_1$-module with highest weight λ, $H(\psi)$ is the set of all weights appearing in this decomposition. Since all the weights considered are dominant, for each $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\lambda_1 - \lambda_n$ is a non-negative integer. Set $d(\psi) = \max\limits_{\lambda \in H(\psi)} (\lambda_1 - \lambda_n)$.

Let now the homomorphism $\theta : \mathfrak{s} \to \mathfrak{g}$ be given by the diagram in (4.1). It was shown in the proof of Proposition 3.1.6, that if $\mathfrak{s}$ is a sparse one-sided Lie algebra of type A and $\mathfrak{g}$ is a pure one-sided Lie algebra of type A, then the sequence $\{d_n = d(\theta_n)\}$ is non-increasing. One can check that the same argument works under the assumption that both $\mathfrak{s}$ and $\mathfrak{g}$ are diagonal Lie algebras of type A, and the sequence $\{d_n = d(\theta_n)\}$ is again non-increasing. We call $\lim\limits_{n \to \infty} d_n$ the *level* of the homomorphism θ. In this way the level of an injective homomorphism $\mathfrak{s} \to \mathfrak{g}$ of diagonal Lie algebras of type A is defined. A similar invariant exists most likely when $\mathfrak{s}$ and/or $\mathfrak{g}$ are of arbitrary types, and this question needs to be studied in the future.

The level of an injective homomorphism is a positive integer. Diagonal homomorphisms are of level 1. The homomorphism given in (4.5) is non-diagonal, but it is also of level 1. The diagram in (4.3) gives an example of a non-diagonal homomorphism of level 2. Non-diagonal homomorphisms in general can be of any positive integer level. Moreover, the following stronger statement holds.

Proposition 4.3.1. *If* $\mathfrak{s} = \mathrm{sl}(\infty)$ *and* $\mathfrak{g}$ *is any diagonal Lie algebra of type* A*, then there exist homomorphisms of* $\mathfrak{s}$ *into* $\mathfrak{g}$ *of any positive integer level.*

Proof. In the proof of Proposition 3.1.1, which was later generalized to Corollary 3.1.4, it was shown that there exists a non-diagonal homomorphism of $\mathfrak{s}$ into $\mathfrak{g}$ of level 1. Denote this homomorphism by θ and consider the composition $\mathfrak{s} \stackrel{\theta'}{\to} \mathfrak{s} \stackrel{\theta}{\to} \mathfrak{g}$, where θ' maps any $\mathrm{sl}(n)$ into $\mathrm{sl}(nl)$ diagonally of signature $(l, 0, 0)$ for a fixed $l > 0$. One can check that this composition is a homomorphism of level l. □

Note that the same result would hold for any diagonal Lie algebra $\mathfrak{s}$ of type A with $n^\infty | \mathrm{Stz}(\mathfrak{s})$ for some n, under the condition that there exists a non-diagonal homomorphism of $\mathfrak{s}$ into $\mathfrak{g}$ of level 1. Indeed, it would be enough to consider an endomorphism θ' of $\mathfrak{s}$ which maps any $\mathrm{sl}(x)$ into $\mathrm{sl}(n^k x)$ diagonally of signature $(l, 0, (n^k - l)x)$ (k is chosen large enough so that $n^k > l$).

It seems reasonable to study separately non-diagonal homomorphisms of different levels. The hope is that the study of general non-diagonal homomorphism will be reduced to the study of homomorphisms of level 1, which appears to be a much simpler problem.

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