ON BOUNDED GENERALIZED HARISH-CHANDRA MODULES

IVAN PENKOV AND VERA SERGANOVA

ABSTRACT. Let $\mathfrak{g}$ be a complex reductive Lie algebra and $\mathfrak{k} \subset \mathfrak{g}$ be any reductive in $\mathfrak{g}$ subalgebra. We call a $(\mathfrak{g}, \mathfrak{k})$-module M bounded if the $\mathfrak{k}$-multiplicities of M are uniformly bounded. In this paper we initiate a general study of simple bounded $(\mathfrak{g}, \mathfrak{k})$-modules. We prove a strong necessary condition for a subalgebra $\mathfrak{k}$ to be bounded (Corollary 4.6), i.e. to admit an infinite-dimensional simple bounded $(\mathfrak{g}, \mathfrak{k})$-module, and then establish a sufficient condition for a subalgebra $\mathfrak{k}$ to be bounded (Theorem 5.1). As a result we are able to classify the maximal bounded reductive subalgebras of $\mathfrak{g} = \mathrm{sl}(n)$.

1. INTRODUCTION

In recent years several constructions of generalized Harish-Chandra modules have been given, [PS1], [PSZ], [PZ1], [PZ2], [PZ3], and a classification of such modules with generic minimal $\mathfrak{k}$-type has emerged, [PZ2]. Recall that if $\mathfrak{g}$ is a finite-dimensional Lie algebra and $\mathfrak{k} \subset \mathfrak{g}$ is a reductive in $\mathfrak{g}$ subalgebra, a $\mathfrak{g}$-module M is a $(\mathfrak{g}, \mathfrak{k})$-*module* if $\mathfrak{k}$ acts finitely on each vector $m \in M$. In the present paper we study $\mathfrak{k}$-semisimple $(\mathfrak{g}, \mathfrak{k})$-modules with bounded $\mathfrak{k}$-multiplicities, or as we call them, *bounded generalized Harish-Chandra modules*; all necessary definitions are given in sections 3 and 4 below.

There are two important cases of generalized Harish-Chandra modules on which there is extensive literature: the case when $\mathfrak{k}$ is a symmetric subalgebra (Harish-Chandra modules) and the case when $\mathfrak{h}$ is a Cartan subalgebra (weight modules). In the latter case there is a complete description of simple bounded modules, [M]. In the former case several constructions of simple bounded modules are known, but there is still no complete description of all such modules in the literature, see the discussion in Section 6 below.

Our main interest in this paper is the case when $\mathfrak{k}$ is neither a symmetric nor a Cartan subalgebra, and our first main result is that, if there exists an infinite-dimensional simple bounded $(\mathfrak{g}, \mathfrak{k})$-module, then

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$r_{\mathfrak{g}} \leq b_{\mathfrak{k}}$, where $b_{\mathfrak{k}}$ is the dimension of a Borel subalgebra of $\mathfrak{k}$ and $r_{\mathfrak{g}}$ is the half-dimension of a nilpotent orbit of minimal positive dimension in the adjoint representation of $\mathfrak{g}$. This limits severely the possibilities for $\mathfrak{k}$. Our second main result is an explicit geometric construction of simple bounded generalized Harish-Chandra modules, which in particular gives a sufficient condition for a subalgebra $\mathfrak{k} \subset \mathfrak{g}$ with $r_{\mathfrak{g}} \leq b_{\mathfrak{k}}$ to be bounded. As an application we clasify all bounded reductive maximal subalgebras $\mathfrak{k}$ in $\mathfrak{g} = \mathrm{sl}(n)$ and give examples of non-maximal reductive bounded subalgebras of $\mathrm{sl}(n)$.

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2. Notation

All vector spaces, Lie algebras and algebraic groups are defined over $\mathbb{C}$. The sign $\otimes$ stands for $\otimes_{\mathbb{C}}$. S_n is the symmetric group of order n, and $S^{\cdot}(\cdot)$ and $\Lambda^{\cdot}(\cdot)$ denote respectively symmetric and exterior algebra. By $\mathfrak{g}$ we denote a finite-dimensional Lie algebra, subject to further conditions; $U = U(\mathfrak{g})$ denotes the enveloping algebra of $\mathfrak{g}$, and Z_U stands for the center of U. The filtration $(\mathbb{C} = U(\mathfrak{g})_0) \subset U(\mathfrak{g})_1 \subset U(\mathfrak{g})_2 \subset \dots$ is the standard filtration on $U = U(\mathfrak{g})$. If M is a $\mathfrak{g}$-module, then

$$\mathfrak{g}[M] := \left\{ g \in \mathfrak{g} \middle| \dim \mathrm{span}\{m, g \cdot m, g^2 \cdot m, \dots\} < \infty \ \forall m \in M \right\}.$$

It is essential that $\mathfrak{g}[M]$ is in fact a Lie subalgebra of $\mathfrak{g}$. As was pointed out by a referee, this is an unpublished result of B. Kostant and its proof is presented in [GQS, Thm. 8.1]. This result is also proved by V. Kac [K2] and by S. Fernando [F].

If $M' \subset M$ is any subspace of a $\mathfrak{g}$-module M, by $\mathrm{Ann} M'$ we denote the annihilator of M' in $U(\mathfrak{g})$. If $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{g}$, we put $M^{\mathfrak{k}} := \{m \in M | g \cdot m = 0 \quad \forall g \in \mathfrak{k}\}$.

If σ is an automorphism of $\mathfrak{g}$ and M is a $\mathfrak{g}$-module, M^{σ} stands for the $\mathfrak{g}$-module twisted by σ. If $\mathfrak{g}$ is a reductive Lie algebra, $(\ ,\ )$ stands for any non-degenerate invariant form on $\mathfrak{g}^*$.

If X is an algebraic variety, $\mathscr{O}_X$ is the sheaf of regular functions on X, $\mathcal{T}_X$ is the tangent and cotangent bundle on X, Ω_X is the bundle

of forms of maximal degree on X, and $\mathscr{D}_X$ denotes the sheaf of linear differential operators on X with coefficients in $\mathscr{O}_X$.

3. Preliminary Results

We start we the following well-known result.

Lemma 3.1. *Let $\{V_i\}$ be a family of vector spaces whose dimension is bounded by a positive integer C, and let R be any associative subalgebra of $\prod_i \mathrm{End}V_i$. Then any simple R-module has dimension less than or equal to C.*

Proof. The Amitsur - Levitzki Theorem, [AL], yields the equality

$$\sum_{s\in S_{2C}} \mathrm{sign}(s) x_{s(1)} \dots x_{s(2C)} = 0$$

for any $x_1, \dots, x_{2C} \in R$. Let W be a simple R-module. Assume $\dim W \geq C+1$, fix a subspace $W' \subset W$ with $\dim W' = C+1$, and choose $y_1, \dots, y_{2C} \in \mathrm{End}(W')$, such that $\sum_{s\in S_{2C}} \mathrm{sign}(s) y_{s(1)} \dots y_{s(2C)} \neq 0$. By the Chevalley-Jacobson density theorem [Fa], there exist $x_1, \dots, x_{2C} \in R$ such that

$$x_i \cdot w = y_i(w)$$

for all i and any $w \in W'$. Hence

$$\sum_{s\in S_{2C}} \mathrm{sign}(s) y_{s(1)} \dots y_{s(2C)} = 0.$$

Contradiction. □

Lemma 3.2. *Let $\mathfrak{k}$ be a semisimple Lie algebra and C be a positive integer. There are finitely many non-isomorphic finite-dimensional $\mathfrak{k}$-modules of dimension less or equal than C.*

Proof. Let M_μ be a simple finite-dimensional $\mathfrak{k}$-module with highest weight μ with respect to a fixed Borel subalgebra $\mathfrak{b}_\mathfrak{k} \subset \mathfrak{k}$. Recall that

$$\dim M_\mu = \Pi_{\alpha\in\Delta_+} \frac{(\mu+\rho,\alpha)}{(\alpha,\rho)},$$

where Δ_+ is the set of roots of $\mathfrak{b}_\mathfrak{k}$ and $\rho := \frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha$. If $\dfrac{(\mu+\rho,\alpha)}{(\alpha,\rho)} > C$ at least for one α, then $\dim M_\mu > C$. But the number of all weights μ such that $\dfrac{(\mu+\rho,\alpha)}{(\alpha,\rho)} < C$ for all $\alpha \in \Delta_+$ is finite. Hence the number of modules M_μ of dimension less or equal than C is finite. Therefore the number of all finite-dimensional $\mathfrak{k}$-modules with dimension less or equal than C is finite. □

In what follows, $\mathfrak{k} \subset \mathfrak{g}$ will denote a reductive in $\mathfrak{g}$ subalgebra. By definition, the latter means that $\mathfrak{g}$ is a semisimple $\mathfrak{k}$-module. For the purpose of this paper, we call a $\mathfrak{g}$-module M a $(\mathfrak{g}, \mathfrak{k})$-*module* if $\mathfrak{k} \subset \mathfrak{g}[M]$ and M is a semisimple $\mathfrak{k}$-module. For any $(\mathfrak{g}, \mathfrak{k})$-module M,

$$M = \bigoplus_{r \in R_{\mathfrak{k}}} V^r \otimes M^r,$$

where $R_{\mathfrak{k}}$ is the set of isomorphism classes of simple finite-dimensional $\mathfrak{k}$-modules, V^r denotes a representative of $r \in R_{\mathfrak{k}}$, and $M^r := \mathrm{Hom}_{\mathfrak{k}}(V^r, M)$. We set

$$\mathrm{supp}_{\mathfrak{k}} M := \{V^r \in R_{\mathfrak{k}} | M^r \neq 0\}.$$

In addition, note that each M^r has a natural structure of a $U(\mathfrak{g})^{\mathfrak{k}}$ - module. The following is a well-known statement [Dix, Prop. 9.1.6], whose proof we present for the convenience of the reader.

Lemma 3.3. *If M is a simple $(\mathfrak{g}, \mathfrak{k})$-module, then M^r is a simple $U(\mathfrak{g})^{\mathfrak{k}}$ - module for each r.*

Proof. Let $0 \neq w$, $w' \in M^r$. By the density theorem ([Fa]), for any $v \in V^r$ there exists $x \in U(\mathfrak{g})$ such that $x \cdot (v \otimes w) = v \otimes w'$. If $t \in \mathfrak{k}$, then $xt \cdot (v \otimes w) = t \cdot v \otimes w' = tx \cdot (v \otimes w)$, hence $[\mathfrak{k}, x] \subset \mathrm{Ann}(V^r \otimes w)$. Since $\mathrm{Ann}(V^r \otimes w)$ is $\mathfrak{k}$-invariant under the adjoint action, and since $U(\mathfrak{g})$ is a semisimple $\mathfrak{k}$ -module, we can write $x = y + z$ with $z \in \mathrm{Ann}(V^r \otimes w)$ and $y \in U(\mathfrak{g})^{\mathfrak{k}}$. Therefore $y \cdot w = w'$, i.e. M^r is a simple $U(\mathfrak{g})^{\mathfrak{k}}$ - module. □

Lemma 3.4. *Let M be a $(\mathfrak{g}, \mathfrak{k})$-module such that $\mathrm{supp}_{\mathfrak{k}} M$ is a finite set.*

(a) *Then $\mathfrak{g}[M] + \mathfrak{g}^{\mathfrak{k}} = \mathfrak{g}$.*
(b) *If in addition $\mathfrak{g}$ is simple and M is finitely generated, then M is finite-dimensional.*

Proof. (a)Let $\mathfrak{g} = \bigoplus_i \mathfrak{g}_i$ be a decomposition of $\mathfrak{g}$ into a sum of simple $\mathfrak{k}$-modules. It suffices to prove that $\mathfrak{g}_i \subset \mathfrak{g}[M]$ for every non-trivial $\mathfrak{k}$-module $\mathfrak{g}_i$. Assuming that the Borel subalgebra $\mathfrak{b}_{\mathfrak{k}} \subset \mathfrak{k}$ is fixed, let x_i be a $\mathfrak{b}_{\mathfrak{k}}$- singular vector of $\mathfrak{g}_i$, i.e. let x_i be a generator of a one-dimensional $\mathfrak{b}_{\mathfrak{k}}$-submodule of M. For any $\mathfrak{b}_{\mathfrak{k}}$ - singular vector $m \in M$, $x_i^l \cdot m$ is a $\mathfrak{b}_{\mathfrak{k}}$ - singular vector for any $l \in \mathbb{N}$. If $\mathfrak{g}_i$ is not a trivial $\mathfrak{k}$-module, all non-zero vectors of the form $x_i^l \cdot m$ generate pairwise non-isomorphic simple $\mathfrak{k}$-submodules of M. Hence, $x_i^l \cdot m = 0$ for large l whenever $\mathfrak{g}_i$ is non-trivial. Since M is generated as a $\mathfrak{k}$-module by $\mathfrak{b}_{\mathfrak{k}}$-singular vectors, we have $x_i \in \mathfrak{g}[M]$, and moreover $\mathfrak{g}_i \subset \mathfrak{g}[M]$ as $\mathfrak{k} \subset \mathfrak{g}[M]$.

(b) Note that the subalgebra $\tilde{\mathfrak{g}}$ generated by all non-trivial $\mathfrak{k}$-submodules $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}$. On the other hand, by (a), $\tilde{\mathfrak{g}} \subset \mathfrak{g}[M]$. The simplicity

of $\mathfrak{g}$ yields now $\mathfrak{g} = \mathfrak{g}[M]$. Hence M is finite-dimensional as it is finitely generated. □

4. First results on bounded modules and bounded subalgebras

Recall that a $(\mathfrak{g}, \mathfrak{k})$-module M has *finite type* if M^r is finite-dimensional for all $r \in R_{\mathfrak{k}}$, and that a $(\mathfrak{g}, \mathfrak{k})$-module of finite type is a *generalized Harish-Chandra module* according to the definition in [PZ1] and [PSZ]. Any $(\mathfrak{g}, \mathfrak{k})$-module M of finite type is also automatically a $(\mathfrak{g}, \mathfrak{k}')$-module of finite type for any intermediate subalgebra $\mathfrak{k}'$, $\mathfrak{k} \subset \mathfrak{k}' \subset \mathfrak{g}[M]$. Moreover, the condition that M is of finite type implies

$$\mathfrak{k} + \mathfrak{g}^{\mathfrak{k}} \subset \mathfrak{g}[M]. \tag{1}$$

If $\mathfrak{g}$ is reductive, then for any proper reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k}$, there exist infinite-dimensional simple $(\mathfrak{g}, \mathfrak{k})$-modules of finite type over $\mathfrak{k}$. A stronger statement is proved in [PZ2]. A $(\mathfrak{g}, \mathfrak{k})$-module is *bounded* if, for some positive integer C_M, $\dim M^r < C_M$ for all $r \in R_{\mathfrak{k}}$, and is *multiplicity-free* if $\dim M^r \leq 1$ for all $r \in R_{\mathfrak{k}}$.

Theorem 4.1. *Let $\mathfrak{g} = \bigoplus \mathfrak{g}_i$, where $\mathfrak{g}_i$ are simple Lie algebras, let $\mathfrak{k} \subset \mathfrak{g}$ be a reductive in $\mathfrak{g}$ subalgebra, and let M be a simple bounded $(\mathfrak{g}, \mathfrak{k})$-module. Then $\mathfrak{g}^{\mathfrak{k}} = \bigoplus_i \mathfrak{g}_i^{\mathfrak{k}}$, and $\mathfrak{g}_i \subset \mathfrak{g}[M]$ whenever $\mathfrak{g}_i^{\mathfrak{k}}$ is not abelian. Furthermore, $M \simeq M' \otimes M''$ for some simple finite-dimensional $\mathfrak{g}' := \bigoplus_{\mathfrak{g}_i \subset \mathfrak{g}[M]} \mathfrak{g}_i$-module M' and some simple bounded $(\mathfrak{g}'', \mathfrak{k}'')$-module M'', where $\mathfrak{g}'' := \bigoplus_{\mathfrak{g}_i \not\subseteq \mathfrak{g}[M]} \mathfrak{g}_i$ and $\mathfrak{k}'' := \mathfrak{k} \cap \mathfrak{g}''$.*

Proof. The equality $\mathfrak{g}^{\mathfrak{k}} = \bigoplus_i \mathfrak{g}_i^{\mathfrak{k}}$ follows directly from the definition of $\mathfrak{g}^{\mathfrak{k}}$. In addition, each subalgebra $\mathfrak{g}_i^{\mathfrak{k}}$ is reductive in $\mathfrak{g}_i$, hence $\mathfrak{s}_i := [\mathfrak{g}_i^{\mathfrak{k}}, \mathfrak{g}_i^{\mathfrak{k}}]$ is semisimple. Set $\mathfrak{s} := \bigoplus_i \mathfrak{s}_i$. Consider the decomposition

$$M = \bigoplus_{r \in \operatorname{supp}_{\mathfrak{k}} M} V^r \otimes M^r.$$

Since the dimensions of M^r are bounded, Lemmas 3.2 and 3.3 imply that at most finitely many simple $\mathfrak{s}$-modules M^r are non-isomorphic. Hence, M considered as a $(\mathfrak{g}, \mathfrak{s})$-module satisfies the condition of Lemma 3.4. Thus $\mathfrak{g}[M] + \mathfrak{g}^{\mathfrak{s}} = \mathfrak{g}$. Note that the trivial $\mathfrak{s}$-submodule $\mathfrak{g}^{\mathfrak{s}}$ of $\mathfrak{g}$ has a unique $\mathfrak{s}$-submodule complement $\mathfrak{a}$. Moreover, $\mathfrak{a} \subset \mathfrak{g}[M]$ by Lemma 3.4. In addition, as we already noted in the proof of Lemma 3.4 (b), the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{a}$ is an ideal in $\mathfrak{g}$. Since $\mathfrak{s} \subset \mathfrak{a}$, we

have $\bigoplus_{s_i \neq 0} \mathfrak{g}_i \subset \mathfrak{g}[M]$, i.e. we have proved that $\mathfrak{g}_i \subset \mathfrak{g}[M]$ whenever $\mathfrak{g}_i^{\mathfrak{k}}$ is not abelian.

We prove next that $M = M' \otimes M''$. Since $\mathfrak{g}' \subset \mathfrak{g}[M]$, there is a simple finite-dimensional $\mathfrak{g}'$-submodule M' of M. Set $M'' := \mathrm{Hom}_{\mathfrak{g}'}(M', M)$. Clearly M'' is a $\mathfrak{g}''$-module, and there is a non-zero homomorphism of $\mathfrak{g}$-modules

$$\Phi : M' \otimes M'' \to M,$$

$$\Phi(m' \otimes \varphi) := \varphi(m'), \quad m' \in M'.$$

Since M is simple, Φ is surjective. To prove that Φ is injective, note that M is semisimple as a $\mathfrak{g}'$-module. Hence, by the density theorem, every non-zero submodule of M contains a non-zero vector $m' \otimes \varphi$ for some $m' \in M'$ and $\varphi \in M''$. This implies $\ker \Phi = 0$.

The irreducibility of M now yields the irreducibility of M''. To see that M'' is a bounded $(\mathfrak{g}'', \mathfrak{k}'')$-module it suffices to notice that M is a bounded $(\mathfrak{g}, \mathfrak{g}' \oplus \mathfrak{k}'')$-module as $\mathfrak{k} \subset \mathfrak{g}' \oplus \mathfrak{k}''$ and that the multiplicity of $M' \otimes V^{r''}$ in M equals the multiplicity of $V^{r''}$ in M'' for any $r'' \in R_{\mathfrak{k}''}$. □

In the rest of this section and in Sections 5 and 6 below, $\mathfrak{g}$ is a reductive Lie algebra unless further restrictions are explicitly stated. We call $\mathfrak{k}$ a *bounded subalgebra* of $\mathfrak{g}$ if there exists an infinite-dimensional bounded simple $(\mathfrak{g}, \mathfrak{k})$-module. Theorem 4.1 suggests also the following stronger notion: a bounded subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is *strictly bounded*, if there exists an infinite-dimensional bounded simple $(\mathfrak{g}, \mathfrak{k})$-module M such that $\mathfrak{g}[M]$ contains no simple ideal of $\mathfrak{g}$. Clearly, if $\mathfrak{g}$ is simple, a subalgebra $\mathfrak{k}$ is bounded if and only if it is strictly bounded.

Corollary 4.2. *If $\mathfrak{k}$ is a strictly bounded subalgebra of a reductive Lie algebra $\mathfrak{g}$, then $\mathfrak{g}^{\mathfrak{k}} \subset \mathfrak{g}$ is an abelian subalgebra.*

Theorem 4.3. *Let C be a positive integer and M be a simple bounded $(\mathfrak{g}, \mathfrak{k})$-module with $\dim M^r < C$ for all $r \in R_{\mathfrak{k}}$. Let N be a simple $(\mathfrak{g}, \mathfrak{k})$-module with $\mathrm{Ann}N = \mathrm{Ann}M$. Then N is also bounded and $\dim N^r < C$ for all $r \in R_{\mathfrak{k}}$.*

Proof. Set $U_M := U(\mathfrak{g})/\mathrm{Ann}M$ and $Z_M := (U_M)^{\mathfrak{k}}$. The $(\mathfrak{g}, \mathfrak{k})$-module M determines an injective algebra homomorphism

$$Z_M \to \prod_{r \in R_{\mathfrak{k}}} \mathrm{End}(M^r),$$

and $\dim M^r < C$ for all r. By Lemma 3.3, N^r is a simple Z_M-module for any r. Therefore, $\dim N^r < C$ by Lemma 3.1. □

Recall that, for any simple $\mathfrak{g}$-module M, its *Gelfand-Kirillov dimension* $\mathrm{GKdim}M \in \mathbb{Z}_{\geq 0}$ is defined by the formula

$$\mathrm{GKdim}M = \overline{\lim_{n\to\infty}} \frac{\log \dim (U(\mathfrak{g})_n \cdot v)}{\log n}$$

for any non-zero $v \in M$, [KL] [p. 91]. Recall also that the *associated variety* X_M of $\mathrm{Ann}M$ is the nil-variety in $\mathfrak{g}^*$ of the associated graded ideal in $S^{\cdot}(\mathfrak{g})$ of $\mathrm{Ann}M$. We next prove an explicit bound for $\dim X_M$ by $\dim \mathfrak{k} + \mathrm{rk}\mathfrak{k}$ for any simple bounded $(\mathfrak{g}, \mathfrak{k})$-module M. For this purpose we will use the inequality

$$\mathrm{GKdim}M \geq \frac{\dim X_M}{2},$$

due to O. Gabber and A. Joseph, see [KL] [p. 135].

Theorem 4.4. *Let M be a simple bounded $(\mathfrak{g}, \mathfrak{k})$-module. Then*

$$\mathrm{GKdim}M \leq b_{\mathfrak{k}}, \tag{2}$$

where $b_{\mathfrak{k}} := \dfrac{\dim \mathfrak{k} + \mathrm{rk}\mathfrak{k}}{2}$.

Proof. Fix a Cartan subalgebra $\mathfrak{h}_{\mathfrak{k}} \subset \mathfrak{k}$ and a Borel subalgebra $\mathfrak{b}_{\mathfrak{k}} \subset \mathfrak{k}$ with $\mathfrak{h}_{\mathfrak{k}} \subset \mathfrak{b}_{\mathfrak{k}}$. Note that $b_{\mathfrak{k}} = \dim \mathfrak{b}_{\mathfrak{k}}$. Fix also $r \in R_{\mathfrak{k}}$ with $M^r \neq 0$ and let $\mu_0 \in \mathfrak{h}_{\mathfrak{k}}^*$ be the $\mathfrak{b}_{\mathfrak{k}}$ -highest weight of V^r. Set

$$M_n := U(\mathfrak{g})_n \cdot V^r$$

for $n \in \mathbb{Z}_{\geq 0}$. It suffices to prove that there exists a polynomial $f(n)$ of degree $b_{\mathfrak{k}}$ such that $\dim M_n \leq f(n)$.

Let $\nu_1, \ldots, \nu_s$ be the $\mathfrak{b}_{\mathfrak{k}}$-highest weights of all simple $\mathfrak{k}$-submodules of $\mathfrak{g}$. Put $\nu := \sum_i \nu_i$. Then, if V_μ is the simple finite-dimensional $\mathfrak{k}$-module with $\mathfrak{b}_{\mathfrak{k}}$- highest weight μ, $\mathrm{Hom}_{\mathfrak{k}}(V_\mu, M_n) \neq 0$ implies

$$\mu \leq n\nu + \mu_0 \tag{3}$$

where $\leq$ is the partial order on $\mathfrak{h}_{\mathfrak{k}}^*$ determined by $\mathfrak{b}_{\mathfrak{k}}$. The cardinality of the set of all integral $\mathfrak{b}_{\mathfrak{k}}$-dominant weights μ satisfying (3) is bounded by some polynomial $g(n)$ of degree $\mathrm{rk}\mathfrak{k}$. Weyl's dimension formula implies that the dimension of V_μ is bounded by a polynomial $h(n)$ of degree equal to the number of roots of $\mathfrak{b}_{\mathfrak{k}}$. If $\dim M^r < C$, then

$$\dim M_n \leq Ch(n)g(n).$$

□

In the particular case when $\mathfrak{k} = \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and M is a simple bounded $(\mathfrak{g}, \mathfrak{h})$-module, A. Joseph [J][4.8] proved that (2) is an equality, i.e. $2\mathrm{rk}\mathfrak{h} = \dim X_M$.

Corollary 4.5. *Let M be a bounded simple $(\mathfrak{g},\mathfrak{k})$-module. Then*

$$\frac{\dim X_M}{2} \leq b_{\mathfrak{k}}.$$

In the remainder of the paper G will be a fixed reductive algebraic group with Lie algebra $\mathfrak{g}$. Denote by $r_{\mathfrak{g}}$ the half-dimension of a nilpotent orbit of minimal positive dimension in $\mathfrak{g}$. If $\mathfrak{g}$ is simple, such an orbit is unique. It coincides with the orbit of a highest vector in the adjoint representation, and

$$r_{\mathfrak{g}} = \begin{cases} \mathrm{rk}\mathfrak{g} = n & \text{for } \mathfrak{g} = \mathrm{sl}(n+1), \mathrm{sp}(2n) \\ 2n-2 & \text{for } \mathfrak{g} = \mathrm{so}(2n+1) \\ 2n-3 & \text{for } \mathfrak{g} = \mathrm{so}(2n) \\ 3 & \text{for } \mathfrak{g} = G_2 \\ 8 & \text{for } \mathfrak{g} = F_4 \\ 11 & \text{for } \mathfrak{g} = E_6 \\ 17 & \text{for } \mathfrak{g} = E_7 \\ 29 & \text{for } \mathfrak{g} = E_8. \end{cases}$$

Corollary 4.6. *If $\mathfrak{k}$ is a bounded subalgebra, then*

$$r_{\mathfrak{g}} \leq b_{\mathfrak{k}}. \tag{4}$$

If $\mathfrak{g} = \mathfrak{g}_1 \oplus ... \oplus \mathfrak{g}_s$ is a sum of simple ideals and $\mathfrak{k} \subset \mathfrak{g}$ is strictly bounded, then

$$r_{\mathfrak{g}_1} + ... + r_{\mathfrak{g}_s} \leq b_{\mathfrak{k}}. \tag{5}$$

Proof. Let M be a infinite-dimensional simple bounded $(\mathfrak{g},\mathfrak{k})$-module. Then X_M is the closure of a nilpotent G-orbit in $\mathfrak{g}$ [J1]. Since M is infinite-dimensional, the dimension of X_M is positive. Hence $\frac{\dim X_M}{2} \geq r_{\mathfrak{g}}$, and (4) follows from Corollary 4.5. If $\mathfrak{k}$ is strictly bounded, then there exists a simple bounded $(\mathfrak{g},\mathfrak{k})$-module M such that $\mathfrak{g}[M]$ does not contain $\mathfrak{g}_i$ for all $i = 1, ..., s$. This implies that $X_M \cap \mathfrak{g}_i \neq 0$ for all $i = 1, ..., s$ as $X_M \cap \mathfrak{g}_i = 0$ forces $\mathfrak{g}_i \subset \mathfrak{g}[M]$. Hence $\frac{\dim X_M}{2} \geq r_{\mathfrak{g}_1} + ... + r_{\mathfrak{g}_s}$. □

Corollary 4.6 implies that if $\mathfrak{k} \simeq \mathrm{sl}(2)$ is a strictly bounded subalgebra of a semisimple Lie algebra $\mathfrak{g}$, then there are only the following three choices for $\mathfrak{g}$:

$$\mathfrak{g} \simeq \mathrm{sl}(2) \oplus \mathrm{sl}(2), \quad \mathfrak{g} \simeq \mathrm{sl}(3), \quad \mathfrak{g} \simeq \mathrm{sp}(4). \tag{6}$$

In the forthcoming paper [PS2] we show that, up to conjugation, there are five possible embeddings $\mathrm{sl}(2) \hookrightarrow \mathfrak{g}$ (with $\mathfrak{g}$ in (6)) whose image is a bounded subalgebra; in [PS2] we describe in detail all simple bounded $(\mathfrak{g}, \mathrm{sl}(2))$-modules. This example shows that the inequality $r_{\mathfrak{g}} \leq b_{\mathfrak{k}}$

together with the requirement that $\mathfrak{g}^{\mathfrak{k}}$ is abelian are not sufficient for a reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k}$ to be bounded.

Consider a chain of subalgebras $\mathfrak{k} \subset \mathfrak{g}' \subset \mathfrak{g}$ where $\mathfrak{k} \simeq \mathrm{so}(n)$, $\mathfrak{g}' \simeq \mathrm{sl}(n)$ and $\mathfrak{g} \simeq \mathrm{sl}(n+1)$. Fix Borel subalgebras $\mathfrak{b}_{\mathfrak{k}} \subset \mathfrak{k}$, $\mathfrak{b}' \subset \mathfrak{g}'$ such that $\mathfrak{b}_{\mathfrak{k}} \subset \mathfrak{b}'$. As a $\mathfrak{g}'$-module $\mathfrak{g}$ has the decomposition $\mathfrak{g} = \mathfrak{g}' \oplus V \oplus V^* \oplus \mathbb{C}$, where V and V^* stand for the natural and conatural representation of $\mathfrak{g}'$ respectively. Therefore we can fix $\mathfrak{b}'$- singular vectors $x \in V$, $y \in V^*$. Set $z = [x, y]$. Then it is easy to check that $[x, z] = [y, z] = 0$.

Let M be an infinite-dimensional simple bounded $(\mathfrak{g}, \mathfrak{k})$-module. Then $\mathfrak{g}[M] \neq \mathfrak{g}$, while the trivial $\mathfrak{g}'$-submodule of $\mathfrak{g}$ belongs to $\mathfrak{g}[M]$ by (1). Since $\mathfrak{k} \oplus \mathbb{C}, x, y$ generate $\mathfrak{g}$, either x or y does not lie in $\mathfrak{g}[M]$. Without loss of generality assume that $x \notin \mathfrak{g}[M]$.

We show now that $z \in \mathfrak{g}[M]$. Assume the contrary. Then for any $\mathfrak{b}_{\mathfrak{k}}$-singular vector $m \in M$ the set $\{x^a z^b \cdot m\}_{a,b \in \mathbb{Z}_{\geq 0}}$ is a linearly independent set of $\mathfrak{b}_{\mathfrak{k}}$-singular vectors in M. Since the $\mathfrak{k}$-weights of x and z are ω_1 and $2\omega_1$ respectively (ω_1 being the first fundamental weight), the weight of the vector $x^a z^b \cdot m$ is $\mu + (a+2b)\omega_1$, where μ is the weight of m. Hence the mutiplicity in M of the simple $\mathfrak{k}$-module $V^{\mu + n\omega_1}$ (of highest weight $\mu + n\omega_1$) grows linearly in n. This contradicts the boundness of M.

Since $\mathfrak{k}$ and z generate $\mathfrak{g}'$ we obtain that $\mathfrak{g}' \subset \mathfrak{g}[M]$. Consider now a $\mathfrak{b}'$-singular vector $v \in M$ of weight ν. Then $x^n \cdot v$ is a $\mathfrak{b}'$-singular vector for any $n \in \mathbb{Z}_{\geq 0}$. Hence $\mathrm{Hom}_{\mathfrak{g}'}((V')^{\nu + n\omega_1'}, M) \neq 0$, where $(V')^{\nu+n\omega_1'}$ is the simple $\mathfrak{g}'$-module of highest weight $\nu + n\omega_1'$ (ω_1' now denoting the first fundamental weight of $\mathfrak{g}'$).

We claim that this implies that M is a $(\mathfrak{g}, \mathfrak{k})$-module of infinite type. Indeed, for any positive n $S^n(V)$ is a simple $\mathfrak{g}'$-module and

$$0 \neq \mathrm{Hom}_{\mathfrak{g}'}(S^n(V) \otimes (V')^{\nu}, (V')^{\nu + n\omega_1'}) = \mathrm{Hom}_{\mathfrak{g}'}(S^n(V), ((V')^{\nu})^* \otimes (V')^{\nu + n\omega_1'}).$$

However, for any even n $S^n(V)$ contains a trivial $\mathfrak{k}$-constituent. Therefore

$$0 \neq ((V')^{\nu})^* \otimes (V')^{\nu + n\omega_1'})^{\mathfrak{k}} = \mathrm{Hom}_{\mathfrak{k}}((V')^{\nu}, (V')^{\nu + n\omega_1'})$$

for all even n. Since $(V')^{\nu}$ has finitely many simple $\mathfrak{k}$-constituents, there is a simple $\mathfrak{k}$-constituent V^r of $(V')^{\nu}$ such that $\mathrm{Hom}_{\mathfrak{k}}(V^r, (V')^{\nu + n\omega_1'}) \neq 0$ for infinitely many n. This yields $\dim M^r = \infty$. Contradiction.

We conclude this section by a brief discussion of the action of the translation functor on bounded $(\mathfrak{g}, \mathfrak{k})$-modules. For any $\xi \in \mathfrak{h}^*$, denote by $U^{\chi(\xi)}$ the quotient of $U(\mathfrak{g})$ by the two sided ideal generated by the kernel of the character $\chi(\xi) : Z_U \to \mathbb{C}$ via which Z_U acts on the Verma module with $\mathfrak{b}$-highest weight $\xi - \rho$. Let now $\xi, \eta \in \mathfrak{h}^*$ be two weights whose difference $\eta - \xi$ is a $\mathfrak{g}$-integral weight. There is a unique simple finite-dimensional $\mathfrak{g}$-module E such that $\eta - \xi$ is its extremal weight.

The following functor is known as translation functor [BJ],[Z]

$$\begin{aligned} T_\xi^\eta : U^{\chi(\xi)} - \mathrm{mod} &\to U^{\chi(\eta)} - \mathrm{mod}\ , \\ M &\mapsto U^{\chi(\eta)} \otimes_{U(\mathfrak{g})} (M \otimes E). \end{aligned}$$

It is clear that the image of a bounded $(\mathfrak{g}, \mathfrak{k})$-module under any translation functor is a bounded $(\mathfrak{g}, \mathfrak{k})$-module.

Under the additional condition ξ and η have the same stabilizer in the Weyl group $W_\mathfrak{g}$ and $(\xi, \check{\alpha}) \in \mathbb{Z}_{\geq 0} \iff (\eta, \check{\alpha}) \in \mathbb{Z}_{\geq 0}$ and $(\xi, \check{\alpha}) \in \mathbb{Z}_{\leq 0} \iff (\eta, \check{\alpha}) \in \mathbb{Z}_{\leq 0}$ for any root α of $\mathfrak{b}$ (as usual, $\check{\alpha} = \frac{2\alpha}{(\alpha,\alpha)}$), the functors T_ξ^η and T_η^ξ are known to be mutually inverse equivalences of categories [BG]. This implies in particular that if $\mathfrak{B}_\mathfrak{k}^{\chi(\xi)}$ (respectively, $\mathfrak{B}_\mathfrak{k}^{\chi(\eta)}$) is the full subcategory of $U^{\chi(\xi)} - \mathrm{mod}$ (resp., of $U^{\chi(\eta)} - \mathrm{mod}$) whose objects are bounded generalized $(\mathfrak{g}, \mathfrak{k})$-modules, T_ξ^η and T_η^ξ induce mutually inverse equivalences of the categories $\mathfrak{B}_\mathfrak{k}^{\chi(\xi)}$ and $\mathfrak{B}_\mathfrak{k}^{\chi(\eta)}$.

5. A construction of bounded $(\mathfrak{g}, \mathfrak{k})$-modules

Let $\mathscr{D}^\xi$ be the sheaf of twisted differential operators on G/B as introduced in [BB]. Recall that if $(\xi, \check{\alpha}) \neq 0$ for any $\alpha \in \Delta$, then $\Gamma(G/B, \mathscr{D}^\xi) = U^{\chi(\xi)}$. Furthermore, if $(\xi, \check{\alpha}) \notin \mathbb{Z}_{\leq 0}$ for any root α of $\mathfrak{b} = \mathrm{Lie}B$, then the functors

$$\Gamma : \quad \mathscr{D}^\xi - \mathrm{mod} \rightsquigarrow U^{\chi(\xi)} - \mathrm{mod}\ ,$$

$$\mathscr{D}^\xi \otimes_{U^\chi} \cdot : \quad U^{\chi(\xi)} - \mathrm{mod} \rightsquigarrow \mathscr{D}^\chi - \mathrm{mod}$$

are mutually inverse equivalences of categories. Here $\mathscr{D}^\xi - \mathrm{mod}$ denotes the category of sheaves of left $\mathscr{D}^\xi$-modules on G/B which are quasicoherent as sheaves of $\mathscr{O} = \mathscr{O}_{G/B}$-modules, [BB].

Note that if $\xi, \eta \in \mathfrak{h}^*$ satisfy $(\xi, \check{\alpha}) \notin \mathbb{Z}_{\leq 0}$, $(\eta, \check{\alpha}) \notin \mathbb{Z}_{\leq 0}$ for any root α of $\mathfrak{b}$, and $\xi - \eta$ is a $\mathfrak{g}$-integral weight, then the translation functor

$$T_\xi^\eta : U^{\chi(\eta)} - \mathrm{mod} \rightsquigarrow U^{\chi(\xi)} - \mathrm{mod}$$

coincides with the composition $\Gamma \circ (\mathscr{O}(\xi - \eta) \otimes_\mathscr{O} \cdot) \circ (\mathscr{D}^\eta \otimes_{U^\eta} \cdot)$, where $\mathscr{O}(\xi - \eta)$ stands for the invertible sheaf on G/B on whose geometric fibre at the point $B \in G/B$ the Lie algebra $\mathfrak{b}$ acts via the weight $w_m(\xi - \eta)$, w_m being the element of maximal length in the Weyl group $W_\mathfrak{g}$. This yields a geometric description of the translation functor T_ξ^η.

We need one more basic $\mathscr{D}$-module construction. For any parabolic subgroup $P \subset G$ there is a well-known ring homomorphism

$U(\mathfrak{g}) \to \Gamma(G/P, \mathscr{D}_{G/P})$ which extends the obvious homomorphism $\mathfrak{g} \to \Gamma(G/P, \mathcal{T}_{G/P})$. Therefore the functor

$$\Gamma : \mathscr{D}_{G/P} - \text{mod} \ \to \Gamma(G/P, \mathscr{D}_{G/P}) - \text{mod}$$

can be considered as a functor into $U(\mathfrak{g})$-mod.

Let Z be a smooth closed subvariety of G/P, and let $(\mathscr{D}_{G/P} - \text{mod}\ )^Z$ be the full subcategory of $\mathscr{D}_{G/P}$-mod with objects $\mathscr{D}_{G/P}$-modules supported on Z as sheaves. Furthermore, denote by $\mathscr{D}_{X\leftarrow Z}$ the $(\mathscr{D}_{G/P}, \mathscr{D}_Z)$-bimodule $((\mathscr{D}_{G/P} \otimes_{\mathscr{O}_{G/P}} \Omega^*_{G/P})_{|Z}) \otimes_{\mathscr{O}_Z} \Omega_Z$. A well-known theorem of Kashiwara [K] claims that the functor

$$i_\bigstar : \mathscr{D}_Z - \text{mod} \ \rightsquigarrow (\mathscr{D}_{G/P} - \text{mod}\ )^Z$$
$$\mathcal{F} \mapsto \mathscr{D}_{X\leftarrow Z} \otimes_{\mathscr{D}_Z} \mathcal{F}$$

is an equivalence of categories. In addition, it is easy to see that $i^{-1}i_\bigstar \mathscr{O}_Z$ has a natural $\mathscr{O}_Z$-module filtration with successive quotients

$$\Lambda^{max}(\mathcal{N}) \otimes_{\mathscr{O}_Z} S^i(\mathcal{N}), \tag{7}$$

where $\mathcal{N}$ denotes the normal bundle of Z in G/P and Λ^{max} stands for maximal exterior power.

Let K be a reductive algebraic group and B_K be a Borel subgroup of K. A K-module V is called *spherical* when B_K has an open orbit in V. If V is spherical then any rational B_K-invariant on V is constant and therefore any two $\mathfrak{b}_\mathfrak{k}$-singular vectors in $S^\cdot(V)^*$ have different weights. Thus, the symmetric algebra $S^\cdot(V)$ is a multiplicity-free K-module [VK] [Thm.2].

In the sequel we assume that K is a reductive proper subgroup of our fixed reductive algebraic group G, $\mathfrak{k} = \text{Lie}K$, and let $P \subset G$ be a proper parabolic subgroup such that $Q := K \cap P$ is a parabolic subgroup in K. There is a closed immersion

$$i : K \cdot P = K/Q \hookrightarrow G/P.$$

Since P is Q-stable, Q acts in the fiber $\mathcal{N}_P \simeq \mathfrak{g}/(\mathfrak{k} + \mathfrak{p})$ at the point P of the normal bundle $\mathcal{N}$ of K/Q in G/P. Let Q_0 denote a reductive part of Q.

The following result is one of the key observations in this paper.

Theorem 5.1. *Let G, P, K, Q be as above. If $\mathcal{N}_P$ is a non-zero spherical Q_0-module, then $\Gamma(G/P, i_\bigstar \mathscr{O}_{K/Q})$ is an infinite-dimensional multiplicity-free $(\mathfrak{g}, \mathfrak{k})$-module.*

Proof. By (7) $i^{-1}i_\bigstar \mathscr{O}_{K/Q}$ has a natural $\mathscr{O}_{K/Q}$-module filtration with successive quotients

$$\Lambda^{max}(\mathcal{N}) \otimes_{\mathscr{O}_{K/Q}} S^i(\mathcal{N}).$$

Moreover, $i^{-1}i_\star\mathscr{O}_{K/Q}$ is K-equivariant, and at the point P, the above filtration induces a Q-module filtration and thus also a Q_0-module filtration of the fiber $(i^{-1}i_\star\mathscr{O}_{K/Q})_P$ with successive quotients

$$\Lambda^{max}(\mathcal{N}_P)\otimes_{\mathbb{C}} S^i(\mathcal{N}_P). \tag{8}$$

Theorem 5.1 implies that the direct sum of all modules (8) for $i \geq 0$ is a multiplicity-free Q_0-module.

According to the Bott-Borel-Weil Theorem the global sections of an irreducible K-bundle induced from a simple Q_0-module E, whenever non-zero, form a simple K-module with the same highest weight as E. Therefore the K-module $\Gamma(K/Q, \bigoplus_{i\geq 0}(\Lambda^{max}(\mathcal{N})\otimes_{\mathscr{O}_{K/Q}} S^i(\mathcal{N})))$ is a multiplicity-free K-module. Since K is reductive, $\Gamma(G/P, i_\star\mathscr{O}_{K/Q})$ is a semisimple K-module. Moreover, $\Gamma(G/P, i_\star\mathscr{O}_{K/Q})$ has an obvious K-module filtration induced by the $\mathscr{O}_{K/Q}$-module filtration on $i^{-1}i_\star\mathscr{O}_{K/Q}$. The associated graded of this filtaration is clearly a submodule of $\Gamma(K/Q, \bigoplus_{i\geq 0}(\Lambda^{max}(\mathcal{N})\otimes_{\mathscr{O}_{K/Q}} S^i(\mathcal{N})))$. Hence $\Gamma(G/Q, i_\star\mathscr{O}_{K/Q})$, being isomorphic as a K-module to this associated graded, is itself K-multiplicity-free. The fact that $\Gamma(G/Q, i_\star\mathscr{O}_{K/Q})$ is infinite-dimensional follows from our assumption that $\mathcal{N}_P$ is not zero. □

We would like to point out that it is relatively straightforward to generalize Theorem 5.1 to the case when $\mathscr{O}_{K/Q}$ is replaced by a K-equivariant line bundle on K/Q. This more general theorem should play an important role in a future study of bounded $(\mathfrak{g}, \mathfrak{k})$-modules with central characters different from that of a trivial $\mathfrak{g}$-module. In the subsequent paper [PS2] we will discuss this construction in a special case.

6. On Bounded Subalgebras

Theorem 5.1 leads to the following results about bounded subalgebras.

Corollary 6.1. *Let $K \subset G \subset GL(V)$ be a chain of reductive algebraic groups, $\mathfrak{k} = \mathrm{Lie}K$, and let $V' \subset V$ be a 1-dimensional space whose stabilizers in G and K are parabolic subgroups $P \subset G$ and $Q \subset K$. Then, if $(V')^* \otimes (\mathfrak{g}\cdot V'/\mathfrak{k}\cdot V')$ is a non-zero spherical Q_0-module, $\mathfrak{k}$ is a bounded subalgebra of $\mathfrak{g}$.*

Proof. We identify G/P with the G-orbit of V' in $\mathbb{P}(V)$. Then K/Q is identified with the K-orbit of V' in $\mathbb{P}(V)$. Moreover $(\mathcal{T}_{G/P})_{V'} = (V')^*\otimes\mathfrak{g}\cdot V', (\mathcal{T}_{K/Q})_{V'} = (V')^*\otimes\mathfrak{k}\cdot V'$, and hence $\mathcal{N}_P$ is identified with $((\mathcal{T}_{G/P})_{V'}/(\mathcal{T}_{K/Q})_{V'}) = (V')^*\otimes(\mathfrak{g}\cdot V'/\mathfrak{k}\cdot V')$. Therefore the claim follows

from Theorem 5.1 (which in addition yields an infinite-dimensional multiplicity-free $(\mathfrak{g},\mathfrak{k})$-module). □

Corollary 6.2. *Let K be a reductive subgroup in $GL(\tilde{V})$ such that $\tilde{V}$ is a spherical K-module. Then $\mathfrak{k} = \mathrm{Lie}K$ is a bounded subalgebra of $\mathrm{gl}(\tilde{V}\oplus\mathbb{C})$, where $\mathfrak{k}$ is embedded in $\mathrm{gl}(\tilde{V}\oplus\mathbb{C})$ via the composition $\mathrm{Lie}K \subset \mathrm{gl}(\tilde{V}) \subset \mathrm{gl}(\tilde{V}\oplus\mathbb{C})$.*

Proof. One sets $V := \tilde{V}\oplus\mathbb{C}$ and applies Corollary 6.1 to the chain $K \subset G := GL(V)$ with the choice of V' as the fixed one dimensional subspace $\mathbb{C} \subset V$. Then $(V')^* \otimes (\mathfrak{g}\cdot V'/\mathfrak{k}\cdot V') = \tilde{V}$ as $\mathfrak{g}\cdot V' = V$, $\mathfrak{k}\cdot V' = V'$. □

All faithful simple spherical modules of reductive Lie groups are listed in [K1] [Thm. 3]. This list provides via Corollary 6.2 a lot of examples of bounded subalgebras of $\mathrm{gl}(n)$.

Before we proceed to applications of Corollary 6.1, let us briefly discuss what is known in the cases when $\mathfrak{k}$ is a symmetric or a Cartan subalgebra of $\mathfrak{g}$. In the first case, there is the celebrated classification of Harish-Chandra modules, pioneered by R. Langlands [L], see also [V1], [KV] and the references therein. In addition, bounded Harish-Chandra modules have been studied in detail in many cases, and the corresponding very interesting results are somewhat scattered throughout the literature. It is an important fact that every symmetric subalgebra of a semisimple Lie algebra is bounded, and this follows from a combination of published and unpublished results, communicated to us by D. Vogan, Jr. and G. Zuckerman.

More precisely, if the pair $(\mathfrak{g},\mathfrak{k})$ is Hermitian, i. e. if $\mathfrak{k}$ is contained in a proper maximal parabolic subalgebra, any simple highest weight Harish-Chandra module is bounded. This follows from results of W. Schmid, [Sch]. If $\mathfrak{g}$ is simply laced, then (published and unpublished) results of D. Vogan, Jr. imply that any symmetric subalgebra $\mathfrak{k}\subset\mathfrak{g}$ is bounded. In all remaining cases, the boundedness of a symmetric subalgebra follows from the existence of a simple ladder module (this is a special type of multiplicity-free $(\mathfrak{g},\mathfrak{k})$-module), or a bounded degenerate principal series module, or a bounded Zuckerman derived functor module. The corresponding results can be found in [V1], [V3], [BS], [GW], [Str], and [EPWW]. A systematic study of bounded Harish-Chandra modules would be very desirable but is not part of this paper.

In the case when $\mathfrak{k} = \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ the simple bounded $(\mathfrak{g},\mathfrak{k})$-modules have played a quite visible role in the literature on weight modules. Here it is easy to check that, if $\mathfrak{g}$ is simple, (4) is satisfied only for $\mathfrak{g}\simeq \mathrm{sl}(m), \mathrm{sp}(n)$. This observation, due

to A. Joseph [J][5.6], easily implies S. Fernando's result that a Cartan subalgebra is a bounded subalgebra of a simple Lie algebra $\mathfrak{g}$ if and only if $\mathfrak{g} \simeq \mathrm{sl}(m), \mathrm{sp}(n)$. Furthermore, the works of S. Fernando, O. Mathieu and others, see [M], [F] and the references therein, have lead to an explicit description of all simple bounded $(\mathfrak{g}, \mathfrak{h})$-modules for $\mathfrak{g} = \mathrm{sl}(m), \mathrm{sp}(n)$, see [M] for comprehensive results.

We now proceed to direct applications of Corollary 6.1: we classify all bounded reductive subalgebras $\mathfrak{k} \subset \mathrm{sl}(n)$ which are maximal as subalgebras, and give examples of bounded non-maximal subalgebras of $\mathrm{sl}(n)$.

Theorem 6.3. *Let $\mathfrak{g} = \mathrm{sl}(n)$. A proper reductive in $\mathfrak{g}$ subalgebra $\mathfrak{k}$ which is maximal as a subalgebra of $\mathfrak{g}$ is bounded if and only if it satisfies the inequality (4), i.e. iff $b_{\mathfrak{k}} \geq n-1$.*

We need the following preparatory statements. For a simple Lie algebra $\mathfrak{k}$ we denote by $\omega_1, ..., \omega_{\mathrm{rk}\mathfrak{k}}$ the fundamental weights of $\mathfrak{k}$, where for the enumeration of simple roots we follow the convention of [OV]. Furthermore, in what follows we denote by V_λ the simple finite-dimensional $\mathfrak{k}$-module with highest weight λ.

Lemma 6.4. *Let $\mathfrak{k}$ be a simple Lie algebra and V be a simple $\mathfrak{k}$ module. Assume that*

$$\dim V - 1 \leq \frac{\dim \mathfrak{k} + \mathrm{rk}\mathfrak{k}}{2}. \tag{9}$$

Then V is trivial, or we have the following possibilities for $\mathfrak{k}$ and V:

1. $\mathfrak{k} = \mathrm{sl}(m)$, $V = V_{\omega_1}$, $V_{\omega_{m-1}}$, V_{ω_2}, $V_{\omega_{m-2}}$, $V_{2\omega_1}$, $V_{2\omega_{m-1}}$,
2. $\mathfrak{k} = \mathrm{so}(m)$ *or* $\mathrm{sp}(m)$, $V = V_{\omega_1}$,
3. $\mathfrak{k} = \mathrm{so}(m)$, $5 \leq m \leq 10$ *or* $m = 11$, $V = V_{\omega_{(m-1)/2}}$ *for odd* m, $V = V_{\omega_{m/2}}$ *and* $V = V_{\omega_{m/2-1}}$ *for even* m,
4. $\mathfrak{k} = G_2$, $V = V_{\omega_1}$,
5. $\mathfrak{k} = F_4$, $V = V_{\omega_1}$,
6. $\mathfrak{k} = E_6$, $V = V_{\omega_1}$ *or* V_{ω_6},
7. $\mathfrak{k} = E_7$, $V = V_{\omega_1}$.

Proof. We start with the observation that $(\lambda, \alpha_i) = k \in \mathbb{Z}_{\geq 0}$ implies $\dim V_\lambda > \dim V_{k\omega_i}$. This follows immediately from Weyl's dimension formula. Therefore it suffices to find all fundamental representations for which the inequality (9) holds.

Let $\mathfrak{k} = \mathrm{sl}(m)$. The dimensions of the fundamental representations are $\binom{m}{k}$ for $k = 1, \ldots, m-1$. The condition

$$\binom{m}{k} \leq \frac{m(m+1)}{2} = \frac{1}{2}(\dim \mathfrak{k} + \mathrm{rk}\mathfrak{k}) + 1$$

is equivalent to (9) and implies $k = 1, 2, m-2, m-1$. Obviously, $\dim V_{2\omega_{m-2}} = \dim V_{2\omega_2}$ is greater than $\frac{m(m+1)}{2}$. On the other hand, $\dim V_{2\omega_1} = \dim V_{2\omega_{m-1}} = \frac{m(m+1)}{2}$. Hence (1).

Let $\mathfrak{k} = \mathrm{so}(m)$, $m = 2p$. We may assume $m \geq 8$. The inequality (9) is equivalent to

$$\dim V \leq p^2 + 1.$$

The dimensions of the fundamental representations are $\binom{m}{k}$ for $k \leq p-2$ and 2^{p-1}. It is not hard to check that for an arbitrary p the inequality holds only for V_{ω_1}; moreover it holds for $V_{\omega_{p-1}}$, V_{ω_p} if $p = 4, 5, 6$.

Let $\mathfrak{k} = \mathrm{so}(m)$, $m = 2p+1$. The inequality (9) is equivalent to

$$\dim V \leq p^2 + p + 1,$$

and holds for V_{ω_1} for any p, and for V_{ω_p} if $p \leq 4$.

Let $\mathfrak{k} = \mathrm{sp}(m), m = 2p$. Assume $p \geq 3$. The inequality is the same as in the previous case, but

$$\dim V_{\omega_k} = \binom{2p}{k} - \binom{2p}{k-2}.$$

One can check that here the inequality holds only for $k = 1$. This proves (2) and (3).

The cases (4)-(7) can be checked using the tables in [OV]. □

Lemma 6.5. *Let $\mathfrak{k}$ and V be as in Lemma 6.4. The following is a complete list of pairs $\mathfrak{k}, V$ such that V has no non-degenerate $\mathfrak{k}$-invariant bilinear form:*

(1) $\mathfrak{k} = \mathrm{sl}(m)$, $V = V_{\omega_1}$, $V_{\omega_{m-1}}$, V_{ω_2} $(m \geq 5)$, $V_{\omega_{m-2}}$, $(m \geq 5)$, $V_{2\omega_1}$, $V_{2\omega_{m-1}}$;
(2) $\mathfrak{k} = \mathrm{so}(10)$, $V = V_{\omega_4}$ *or* V_{ω_5};
(3) $\mathfrak{k} = E_6$, $V = V_{\omega_1}$ *or* V_{ω_6}.

Proof. If V is not self-dual, the Dynkin diagram of $\mathfrak{k}$ admits an involutive automorphism which does not preserve the highest weight. Moreover, in the case of $\mathrm{so}(2p)$, p must be odd. These conditions reduce the list of representations in Lemma 6.4 to the list in the Lemma. □

Proof of Theorem 6.3 According to E. Dynkin's classification [D] [Ch.1.], if $\mathfrak{k} \subset \mathfrak{g} = \mathrm{sl}(n)$ is a reductive in $\mathfrak{g}$ subalgebra which is maximal as a subalgebra of $\mathfrak{g}$, one of the following alternatives holds:

(i) $\mathfrak{k}$ is simple, the natural $\mathrm{sl}(n)$-module V is a simple $\mathfrak{k}$-module with no non-degenerate invariant bilinear form, or $\mathfrak{k} = \mathrm{so}(n)$ and $\mathrm{sp}(n)$.
(ii) $\mathfrak{k} \simeq \mathrm{sl}(r) \oplus \mathrm{sl}(s)$ with $rs = n$, and $V \simeq S_r \otimes S_s$, where S_r and S_s are respectively the natural modules of $\mathrm{sl}(r)$ and $\mathrm{sl}(s)$.

If (i) holds, then $\mathfrak{k} \simeq \mathrm{so}(n), \mathrm{sp}(n)$ or $\mathfrak{k}$ is among the Lie algebras listed in Lemma 6.5, where $\mathfrak{g}$ is identified with $\mathrm{sl}(V)$. Consider first the case $\mathfrak{k} \simeq \mathrm{sp}(n)$, $n = 2p$. To show that $\mathfrak{k}$ is bounded in $\mathfrak{g}$, we apply Theorem 5.1 with G/P being the Grassmannian of p-dimensional subspaces in $\mathbb{C}^n$ and K/Q being the Grassmannian of Lagrangian subspaces in $\mathbb{C}^n$. Then $Q_0 \simeq GL(p)$ and $\mathcal{N}_P$ is the exterior square of the natural representation. The Q_0-module $\mathcal{N}_P$ is spherical, [K1].

We now consider the remaining cases of (i), which can all be settled using Corollary 6.1. Note that, if $\mathfrak{k}$ is embedded into $\mathrm{sl}(n)$ via a simple $\mathfrak{k}$-module or via its dual, the corresponding embeddings are conjugate by an automorphism of $\mathrm{sl}(n)$, hence it suffices to consider only one such embedding. The list of Lemma 6.5 reduces therefore to the following cases, in which all Q_0-modules are spherical, [K1]:

-$\mathfrak{k} = \mathrm{sl}(k)$, $V = V_{\omega_2}$, $Q_0 \simeq SL(2) \times GL(k-2)$ and $(V')^* \otimes (V/\mathfrak{k} \cdot V')$ is isomorphic to the tensor product of the exterior square of the natural representation with the determinant representation of $GL(k-2)$, the action of $SL(2)$ being trivial;

-$\mathfrak{k} = \mathrm{sl}(k)$, $V = V_{2\omega_1}$, $Q_0 \simeq GL(k-1)$ and $(V')^* \otimes (V/\mathfrak{k} \cdot V')$ is isomorphic to the tensor product of the symmetric square of the natural representation with the determinant representation of $GL(k-1)$;

-$\mathfrak{k} = \mathrm{so}(10)$, $V = V_{\omega_4}$, $Q_0 \simeq GL(5)$ and $(V')^* \otimes (V/\mathfrak{k} \cdot V')$ is isomorphic to the tensor product of the natural representation of $GL(5)$ with the determinant representation of $GL(5)$; the case $V = V_{\omega_5}$ can be reduced to the case $V = V_{\omega_4}$ by dualization;

-$\mathfrak{k} = E_6$, $V = V_{\omega_1}$, then $Q_0 \simeq SO(10) \times \mathbb{C}^*$ and $(V')^* \otimes (V/\mathfrak{k} \cdot V')$ is isomorphic to the natural 10-dimensional representation of $SO(10)$, and the action of the center of Q_0 is not trivial.

The only remaining case in (i) is when $\mathfrak{k} = \mathrm{so}(n)$, $Q_0 \simeq SO(n-2) \times \mathbb{C}^*$ and $(V')^* \otimes (V/\mathfrak{k} \cdot V')$ is a one-dimensional non-trivial, hence spherical, Q_0-module.

If (ii) holds, then $\mathfrak{k} \simeq \mathrm{sl}(r) \oplus \mathrm{sl}(s)$ for some rs with $rs = n$, and we claim that in this case all pairs r, s with $rs = n$ yield a bounded subalgebra $\mathfrak{k}$. To see this, fix V' of the form $S'_r \otimes S'_s$ for some 1-dimensional spaces $S'_r \subset S_r, S'_s \subset S_s$. Then Q_0 is isomorphic to $GL(S_r/S'_r) \times GL(S_s/S'_s)$ and $\mathfrak{g} \cdot V'/\mathfrak{k} \cdot V' = V/\mathfrak{k} \cdot V' \simeq (S_r/S'_r) \otimes (S_s/S'_s)$. Since the action of $GL(r-1) \times GL(s-1)$ on V' is given by the inverse of the determinant, $(V')^* \otimes (V/\mathfrak{k} \cdot V')$ is isomorphic as a $GL(r-1) \times GL(s-1)$-module to $S_{r-1} \boxtimes S_{s-1}$ twisted by the determinant. This representation is spherical, [K1]. □

We give now three more examples of bounded subalgebras of $\mathrm{sl}(n)$ which are not maximal in the class of reductive subalgebras of $\mathrm{sl}(n)$.

(i) Let $\mathfrak{k} \simeq \mathrm{sl}(k+1)$, $k \geq 2$. The $\mathfrak{k}$-module $V := V_{\omega_1} \oplus V_{\omega_k}$ defines an embedding $\mathfrak{k} \subset \mathfrak{g} = \mathrm{sl}(V)$, and Corollary 6.1 implies that $\mathfrak{k}$ is a bounded subalgebra of $\mathfrak{g}$. Indeed, choose V' to be a 1-dimensional subspace $V' \subset V_{\omega_1}$ and note that the conditions of Corollary 6.1 are satisfied. In this case $Q_0 \simeq GL(k)$ and $(V')^* \otimes (V/\mathfrak{k} \cdot V')$ is isomorphic to $\Lambda^k(S_k) \otimes (\Lambda^k(S_k) \oplus S_k^*)$, S_k being the natural Q_0-module. A straightforward calculation shows that this representation is spherical.

(ii) Consider the embedding $\mathfrak{k} = \mathrm{so}(7) \subset \mathfrak{g} = \mathrm{sl}(8)$, where the natural sl(8)-module restricts to the 8-dimensional spinor representation of so(7). Corollary 6.1 implies that $\mathfrak{k}$ is a bounded subalgebra of $\mathfrak{g}$. Here $V = \mathbb{C}^8$, $G = SL(V)$, $K = Spin(7)$ and V' is a B_K-stable line, where B_K is a fixed Borel subgroup of K. Then $\mathfrak{g} \cdot V' = V$ and $\dim \mathfrak{k} \cdot V' = 7$, hence $\dim(\mathfrak{g} \cdot V'/\mathfrak{k} \cdot V') = 1$. Since Q_0 acts non-trivially on $(V')^* \otimes (V/\mathfrak{k} \cdot V')$, the latter Q_0-module is spherical.

(iii) Let $\mathfrak{k} = G_2 \subset \mathfrak{g} = \mathrm{sl}(7)$. Then again, Corollary 6.1 implies that $\mathfrak{k}$ is a bounded subalgebra. The argument is similar to the argument in (ii) as $\dim \mathfrak{g} \cdot V/\mathfrak{k} \cdot V' = 1$.

We conclude the paper by the following conjecture which is supported by all the empirical evidence available to us. *Let $\mathfrak{k} \subset \mathfrak{g}$ be a reductive in $\mathfrak{g}$ subalgebra. Then $\mathfrak{k}$ is bounded if and only if there exists a simple infinite-dimensional multiplicity-free $(\mathfrak{g}, \mathfrak{k})$-module.* If $\mathfrak{g} = \mathrm{sl}(n)$ and $\mathfrak{k}$ is a maximal proper subalgebra, then the claim of the conjecture follows from the proof of Theorem 6.3 (which is in turn based on Corollary 6.1 and Corollary 6.2).

Note added in proof. While the present paper has been under review, A. Petukhov has posted on the arXiv a proof of the above conjecture for $\mathfrak{g} = \mathrm{sl}(n)$.

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Jacobs University Bremen, School of Engineering and Science, Campus Ring 1, 28759 Bremen, Germany

E-mail address: i.penkov@jacobs-university.de

Department of Mathematics, University of California Berkeley, Berkeley CA 94720, USA

E-mail address: serganov@math.berkeley.edu