A CATEGORIFICATION OF THE BOSON-FERMION CORRESPONDENCE VIA REPRESENTATION THEORY OF $sl(\infty)$

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ABSTRACT. In recent years different aspects of categorification of the boson-fermion correspondence have been studied. In this paper we propose a categorification of the boson-fermion correspondence based on the category of tensor modules of the Lie algebra $sl(\infty)$ of finitary infinite matrices. By $\mathbb{T}^+$ we denote the category of "polynomial" tensor $sl(\infty)$-modules. There is a natural "creation" functor $\mathcal{T}_N : \mathbb{T}^+ \to \mathbb{T}^+, M \mapsto N \otimes M, \quad M, N \in \mathbb{T}^+$. The key idea of the paper is to employ the entire category $\mathbb{T}$ of tensor $sl(\infty)$-modules in order to define the "annihilation" functor $\mathcal{D}_N : \mathbb{T}^+ \to \mathbb{T}^+$ corresponding to $\mathcal{T}_N$. We show that the relations allowing to express fermions via bosons arise from relations in the cohomology of complexes of linear endofunctors on $\mathbb{T}^+$.

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1. Introduction

The origin of the boson-fermion correspondence can be traced back to the celebrated Jacobi triple product identity

$$\frac{\sum_{n\in\mathbb{Z}} t^n q^{\frac{n^2}{2}}}{\prod_{n\geq 1}(1-q^n)} = \prod_{n\geq 1}(1+tq^{n-\frac{1}{2}})(1+t^{-1}q^{n-\frac{1}{2}}) \tag{1}$$

which one should consider as an equality of two generating functions

$$\sum_{n\in\mathbb{Z},2m\in\mathbb{Z}_{\geq 0}} b_{n,m}t^n q^m = \sum_{n\in\mathbb{Z},2m\in\mathbb{Z}_{\geq 0}} f_{n,m}t^n q^m$$

with nonnegative integral coefficients.

The boson-fermion correspondence can be viewed as a "categorification" of this identity. Namely, it is an isomorphism of doubly graded vector spaces, called bosonic and fermionic Fock spaces,

$$\mathbf{B} = \bigoplus_{n\in\mathbb{Z},2m\geq\mathbb{Z}_{\geq 0}} \mathbf{B}_{n,m} \cong \bigoplus_{n\in\mathbb{Z},2m\geq\mathbb{Z}_{\geq 0}} \mathbf{F}_{n,m} = \mathbf{F} \tag{2}$$

such that

$$\dim \mathbf{B}_{n,m} = b_{n,m}, \quad \dim \mathbf{F}_{n,m} = f_{n,m},$$

and their structure yields respectively the left-hand and right-hand sides of (1). The identity (1) itself follows then from the isomorphism (2). The bosonic Fock space is naturally a representation of an infinite-dimensional Heisenberg Lie algebra, while the fermionic Fock space is naturally a representation of an infinite-dimensional Clifford algebra, see Section 2 for details.

The fact that the Heisenberg Lie algebra can be constructed from the Clifford algebra and vice versa was first noticed in the physics literature [Mn], and was given the name boson-fermion correspondence. It was understood in [F] that the bosonic and fermionic Fock spaces are just two realizations of an affine

Lie algebra representation. In particular, one can see the bosonic and fermionic Fock spaces (2) as a representation of a central extension $\widehat{sl}(\infty)$ of the Lie algebra of infinite matrices with finitely many non-zero diagonals, [DJKM].

If we consider the subspaces $\mathbf{B}_0 \subset \mathbf{B}$, $\mathbf{F}_0 \subset \mathbf{F}$,

$$\mathbf{B}_0 := \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathbf{B}_{0,m} \cong \bigoplus_{m\in\mathbb{Z}_{\geq 0}} \mathbf{F}_{0,m} =: \mathbf{F}_0, \tag{3}$$

the left-hand side of (1) implies

$$\dim \mathbf{B}_{0,m} = \dim \mathbf{F}_{0,m} = p(m)$$

where $p : \mathbb{N} \to \mathbb{N}$ is the partition function. The value $p(m)$ equals the size of the character table of the symmetric group S_m, and the transition matrix between the bases in $\mathbf{B}_{0,m}$ and $\mathbf{F}_{0,m}$ is precisely the character table of S_m. This suggests a second categorification of the boson-fermion correspondence, more specifically of $\mathbf{B}_0 \cong \mathbf{F}_0$, via representation theory of all symmetric groups $\{S_n\}_{n\geq 0}$ or, by Schur-Weyl duality, via representation theory of the Lie algebra $sl(\infty) := sl(\infty, \mathbb{C})$ of traceless finitary infinite matrices.

In this second categorification $sl(\infty)$ appears without a central extension and one is led to consider the category of tensor modules $\mathbb{T}^+$ whose Grothendieck ring is given by (3). The passage to the full boson-fermion correspondence $\mathbf{B} \cong \mathbf{F}$ is achieved by considering the derived category $\mathbb{DT}^+$, or more precisely, $\mathbb{Z}$ copies of the derived category $\mathbb{DT}^+$ which are related by powers of the autoequivalence $\mathcal{S} : \mathbb{DT}^+ \to \mathbb{DT}^+$ arising from shifting the grading, see Section 6.

Initially the idea of categorification was motivated by an attempt to lift three-dimensional invariants to four-dimensional couterparts replacing nonnegative integers by vector spaces of corresponding dimensions, vector spaces with bases by categories, categories by 2-categories, etc. [CF], [K1]. It was quickly realized that the semisimple categories that give rise to three-dimensional invariants should be generalized to more interesting abelian or even triangulated categories, to get a nontrivial categorification at the next level. Besides in topology, these ideas have been successfully used in algebraic geometry and representation theory and have led to many interesting examples of categorification in the last 15 years.

In our example, both $\mathbb{T}^+$ and $\mathbb{DT}^+$ are semisimple categories and, in order to obtain a nontrivial categorification, one should look for non-semisimple generalizations. Luckily, there is a natural category of tensor modules, denoted by $\mathbb{T}$ [DPS], based on the vector representation V of $sl(\infty)$ and its restricted dual V_*. Unlike its finite-dimensional analogue, $\mathbb{T}$ is not a semisimple category, see Section 3 for details. The starting point of our categorification of the boson-fermion correspondence is the projection functor

$$(\,\cdot\,)^+ : \mathbb{T} \to \mathbb{T}^+.$$

This allows us to define, besides the obvious "creation functor"

$$\mathcal{T}_N : \mathbb{T}^+ \to \mathbb{T}^+, \quad M \to N \otimes M, \quad N, M \in \mathbb{T}^+,$$

its important counterpart, the "annihilation functor"

$$\mathcal{D}_N : \mathbb{T}^+ \to \mathbb{T}^+, \quad M \to (N_* \otimes M)^+, \quad N, M \in \mathbb{T}^+.$$

We show in Section 4 that $\mathcal{D}_N$ is left adjoint to $\mathcal{T}_N$.

Together, the two functors $\mathcal{T}_N$ and $\mathcal{D}_N$ generate an abelian subcategory $\widehat{\mathbb{T}}$ of the category of linear endofunctors on $\mathbb{T}^+$, and the category $\widehat{\mathbb{T}}$ can be viewed as a realization of $\mathbb{T}$. In particular, the simplest nontrivial exact sequence in $\mathbb{T}$

$$0 \to sl(V) \to V_* \otimes V \to \mathbb{C} \to 0$$

gives rise to a categorification of the simplest nontrivial relation in the Heisenberg algebra acting on $\mathbf{B}$:

$$0 \to \mathcal{T}_V \circ \mathcal{D}_V \to \mathcal{D}_V \circ \mathcal{T}_V \to Id \to 0. \tag{4}$$

To categorify the boson-fermion correspondence one needs to consider the creation and annihilation functors corresponding to the extreme partitions of n: n itself and $1_n = \underbrace{(1, ..., 1)}_{\text{n}}$. We denote

$$\mathcal{H}_n := \mathcal{T}_{S^n(V)},\ \ \mathcal{E}_n := \mathcal{T}_{\wedge^n(V)},\quad \mathcal{H}_n^* := \mathcal{D}_{S^n(V)},\ \ \mathcal{E}_n^* := \mathcal{D}_{\wedge^n(V)}. \tag{5}$$

The exact sequences between these functors yield a categorification of certain identities that appear in the boson-fermion correspondence, see Section 5.

The creation and annihilation functors (5) and the relations between them provide the building blocks for the categorifications of the operators in the Heisenberg and Clifford algebras. However, to define the corresponding functors we need to work with the category of complexes in $\widehat{\mathbb{T}}$. In Section 6 we introduce complexes of functors

$$\cdots \to \mathcal{H}_{p+1} \circ \mathcal{E}_{q+1}^* \to \mathcal{H}_p \circ \mathcal{E}_q^* \to \mathcal{H}_{p-1} \circ \mathcal{E}_{q-1}^* \ldots,$$

$$\cdots \to \mathcal{E}_{p+1} \circ \mathcal{H}_{q+1}^* \to \mathcal{E}_p \circ \mathcal{H}_q^* \to \mathcal{E}_{p-1} \circ \mathcal{H}_{q-1}^* \ldots$$

for $p-q = a$, which we denote by $\mathcal{X}_a$, and $\mathcal{X}_a^*$, respectively. Then in Section 7 we verify that the commutation relations of these complexes of functors yield, at the Grothendieck ring level, the usual relations between the generators of the Clifford algebra.

Finally, in Section 8 we introduce the complexes of functors $\mathcal{P}_k$ and $\mathcal{P}_k^*$ and verify that their commutation relations categorify the familiar relations between the generators of the Heisenberg algebra.

Different aspects of categorification of the boson-fermion correspondence have been studied by several authors. A combinatorial version of the categorification of the Heisenberg algebra using a counterpart of the functors (5) is obtained in [K2]. It has been further extended in [LS1] and a survey of related works can be found in [LS2]. Another categorification of the bosonic Fock space and the action of the Lie algebra $\widehat{sl}(\infty)$ has been obtained in a series of papers [HY1], [HTY], [HY2]. This approach is closest to ours, though our tensor representations of $sl(\infty)$ are "rational" (not "polynomial") which leads to a non-semisimple category. In a more general setting, the authors of [CL1], [CL2] have constructed a categorification of the basic representation of the affine Lie algebras using the derived categories of coherent sheaves on Hilbert schemes of points on ALE spaces. For the simplest ALE space $\mathbb{C}^2$ their construction should yield a categorification of the boson-fermion correspondence.[1] It is a very interesting problem to obtain these more general geometric categories as representation categories for appropriate generalizations of $sl(\infty)$. Extending the existing terminology of categorification it would be natural to call a solution to this problem — representification — namely, a realization of a geometric category as a certain representation category associated with a particular Lie algebra. Strictly speaking, one still has to verify that our representation-theoretic categorification of boson-fermion correspondence is equivalent to the earlier geometric categorification. If so, one can view our present paper as a first example of representification of the categories of sheaves on Hilbert schemes of points on $\mathbb{C}^2$.

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[1]Note added in proof. In fact, the work of [CL1], [CL2] has recently been extended, independently of our work, to a categorification of the boson-fermion correspondence in [CS].

2. Recollection of the boson-fermion correspondence

Recall that the boson-fermion correspondence relates the actions of the infinite Heisenberg and Clifford algebras on the Fock space, see for instance [F] or [DJKM].

The ground field is $\mathbb{C}$. Let **Cf** be the infinite-dimensional Clifford algebra over with generators $\{\psi_i, \psi_i^*\}_{i\in\mathbb{Z}}$ and relations

$$\psi_i\psi_j + \psi_j\psi_i = \psi_i^*\psi_j^* + \psi_j^*\psi_i^* = 0,\ \ \psi_i\psi_j^* + \psi_j^*\psi_i = \delta_{ij}. \tag{6}$$

The *Fock space* **F** can be defined as the induced module $\mathbf{Cf}\otimes_{\mathbf{Cf}^+}\mathbb{C}$, where $\mathbf{Cf}^+$ is the subalgebra generated by ψ_i for $i \geq 0$ and ψ_i^* for $i < 0$. The Fock space is an irreducible **Cf**-module and is equipped with the grading $\mathbf{F} = \bigoplus_{m\in\mathbb{Z}}\mathbf{F}(m)$ induced by the grading on **Cf** given by $\deg\psi_i = 1$, $\deg\psi_i^* = -1$.

Let **H** denote the (infinitely generated) Weyl algebra with generators $\{p_n, p_n^*\}_{n\geq 1}$ and relations

$$[p_n, p_m] = [p_n^*, p_m^*] = 0,\ \ [p_n, p_m^*] = n\delta_{mn}. \tag{7}$$

The notation **H** reflects the fact that the Lie algebra defined by the relations (7) is the infinite-dimensional Heisenberg algebra. One can define an action of **H** on **F** by setting

$$p_n = \sum_{i\in\mathbb{Z}}\psi_i\psi_{i+n}^*,\ \ p_n^* = \sum_{i\in\mathbb{Z}}\psi_i\psi_{i-n}^*.$$

Note that **H** has two commutative subalgebras: $\mathbf{H}^+$, generated by p_n for $n \geq 1$, and $\mathbf{H}_*^+$, generated by p_n^* for $n \geq 1$. We identify $\mathbf{H}^+$ with the algebra of symmetric functions of infinitely many variables,

$$\mathbf{H}^+ = \bigoplus_{\lambda\in\mathbf{Part}}\mathbb{C}s_\lambda \cong \mathbb{C}[p_1, p_2, ...] = \mathbb{C}[h_1, h_2, ...] = \mathbb{C}[e_1, e_2, ...].$$

Here **Part** is the set of all partitions, s_λ are the Schur functions, p_i-s are the sums of powers, e_i-s are the elementary symmetric functions, h_i-s are the sums of all monomials of degree i. Recall [M] that $\{h_i\}_{i\geq 1}$, $\{e_i\}_{i\geq 1}$ are expressed in terms of $\{p_i\}_{i\geq 1}$ as follows

$$H(z) \stackrel{def}{=} \sum_{n\geq 0} h_n z^n = \exp\left(\sum_{n\geq 1}\frac{p_n z^n}{n}\right),\ E(z) \stackrel{def}{=} \sum_{n\geq 0} e_n z^n = \exp\left(\sum_{n\geq 1}(-1)^{n-1}\frac{p_n z^n}{n}\right), \tag{8}$$

where we set $h_0 := 1$ and $e_0 := 1$.

In a similar way

$$\mathbf{H}_*^+ \cong \mathbb{C}[p_1^*, p_2^*, ...] \cong \mathbb{C}[h_1^*, h_2^*, ...] \cong \mathbb{C}[e_1^*, e_2^*, ...],$$

and

$$H^*(z) \stackrel{def}{=} \sum_{n\geq 0} h_n^* z^{-n} = \exp\left(\sum_{n\geq 1}\frac{p_n^* z^{-n}}{n}\right),\ E^*(z) \stackrel{def}{=} \sum_{n\geq 0} e_n^* z^{-n} = \exp\left(\sum_{n\geq 1}(-1)^{n-1}\frac{p_n^* z^{-n}}{n}\right); \tag{9}$$

here again $h_0^* := 1$ and $e_0^* := 1$. Note that $H^*(z), E^*(z)$ contain only non-positive powers of z.

It is possible to show [DJKM] that for each $m \in \mathbb{Z}$ there is an isomorphism of **H**-modules

$$\mathbf{F}(m) \simeq \mathbf{H}\otimes_{\mathbf{H}_*^+}\mathbb{C}.$$

In particular, each $\mathbf{F}(m)$ is an irreducible **H**-module. Moreover, as an $\mathbf{H}^+$-module $\mathbf{F}(m)$ can be identified with $\mathbf{H}^+$. In what follows we consider $h_n, e_n, p_n, h_n^*, e_n^*$ and p_n^* as linear operators on $\mathbf{H}^+$ and the identities (8) and (9) as identities relating these linear operators. Furthermore, it is a known fact that h_n^* (respectively, e_n^*) are operators dual to h_n (respectively, e_n) with respect to the natural inner product on $\mathbf{H}^+$ for which $\{s_\lambda\}_{\lambda\in\mathbf{Part}}$ forms an orthonormal basis.

Set

$$X(z) = H(z)E^*(-z),\ \ X^*(z) = E(-z)H^*(z). \tag{10}$$

Roughly speaking, up to a shift, the coefficients of $X(z)$ and $X^*(z)$ are the fermions ψ_i and ψ_i^*. More precisely, there exists an automorphism s of the $\mathbf{H}$-module $\mathbf{F}$ such that $s(\mathbf{F}(m)) = \mathbf{F}(m+1)$ and the restrictions of ψ_i and ψ_i^* on $\mathbf{F}(m)$ are recovered by the formulas

$$\sum_{i\in\mathbb{Z}} \psi_i z^i = sz^m X(z), \ \sum_{i\in\mathbb{Z}} \psi_i^* z^i = s^{-1} z^{-m} X^*(z). \tag{11}$$

To prove (11) one has to show that the Fourier coefficients of the vertex operators $X(z)$ and $X^*(z)$ satisfy a version of the Clifford relations (6). It is known [F] that (6) is equivalent to the following vertex operator identities

$$X(z)X(w) + \frac{w}{z} X(w)X(z) = 0, \tag{12}$$

$$X^*(z)X^*(w) + \frac{w}{z} X^*(w)X^*(z) = 0, \tag{13}$$

$$X(z)X^*(w) + \frac{z}{w} X^*(w)X(z) = \sum_{n\in\mathbb{Z}} \frac{z^n}{w^n}. \tag{14}$$

In Section 7 we provide a categorical proof of these identities by showing that they are corollaries of certain relations between endofunctors on a category of $sl(\infty)$-modules.

3. The category $\mathbb{T}$ and the projection functor

Denote by V and V_* a pair of countable dimensional vector spaces in perfect duality, i.e. with a fixed non-degenerate linear map $\langle\cdot,\cdot\rangle : V \times V_* \to \mathbb{C}$. As proved by G. Mackey [Mk], all such triples $(V, V_*, \langle\cdot,\cdot\rangle)$ are isomorphic. The vector space $V \otimes V_*$ is naturally endowed with the structure of an associative algebra such that

$$(v \otimes w) \cdot (v' \otimes w') = \langle v' \otimes w \rangle v \otimes w',$$

and hence also with a Lie algebra structure. It is easy to check that this Lie algebra is isomorphic to the Lie algebra $gl(\infty)$ which by definition is the Lie algebra of infinite matrices $(a_{ij})_{i,j\in\mathbb{Z}}$ with finitely many non-zero entries. The subalgebra $\mathfrak{g} = \mathrm{Ker}\langle\cdot,\cdot\rangle$ is isomorphic to $sl(\infty)$, i.e. to the subalgebra of traceless matrices in $gl(\infty)$.

In [DPS] a category $\mathbb{T}$ of tensor $\mathfrak{g}$-modules has been introduced. More precisely, $\mathbb{T}$ is the category of finite-length submodules (equivalently, finite-length subquotients) of direct sums of copies of the tensor algebra $T^{\cdot}(V \oplus V_*)$ considered as a $\mathfrak{g}$-module. It is also possible to define this category intrinsically, and this is done in three different ways in [DPS]. Note that $\mathbb{T}$ is a monoidal category with respect to tensor product of $\mathfrak{g}$-modules.

Consider the tensor algebra $T^{\cdot}(V)$. By Schur–Weyl duality,

$$T^{\cdot}(V) = \bigoplus_{\lambda\in\mathbf{Part}} V_\lambda \otimes A^\lambda$$

where V_λ is the image of a Young projector corresponding to λ and A^λ stands for the irreducible representation of the symmetric group $\mathbb{S}_{|\lambda|}$ corresponding to the partition λ; here $|\lambda|$ is the degree of λ, i.e. $|\lambda| = \sum_i \lambda_i$, where $\lambda = \{\lambda_i\}$. Note that each V_λ is an irreducible $\mathfrak{g}$-module. Similarly,

$$T^{\cdot}(V_*) = \bigoplus_{\lambda\in\mathbf{Part}} (V_*)_\lambda \otimes A^\lambda.$$

The tensor algebra $T^{\cdot}(V \oplus V_*)$ is not a semisimple $\mathfrak{g}$-module. More precisely, the $\mathfrak{g}$-module $T^{m,n} = V^{\otimes m} \otimes V_*^{\otimes n}$ is $\mathfrak{g}$-semisimple if and only if $mn = 0$. In [PStyr] the socle filtration of $T^{m,n}$ is described as follows. Let $\Phi_{i_1 \dots i_k | j_1 \dots j_k} : T^{m,n} \to T^{m-k,n-k}$ be the contraction map given by

$$\Phi_{i_1 \dots i_k | j_1 \dots j_k}(v_1 \otimes \dots \otimes v_m \otimes w_1 \otimes \dots \otimes w_n) = \langle v_{i_1}, w_{j_1} \rangle \dots \langle v_{i_k}, w_{j_k} \rangle v_1 \otimes \dots \otimes \hat{v}_{i_1} \otimes \dots \otimes \hat{v}_{i_k} \otimes \dots \otimes v_m \otimes w_1 \otimes \dots \otimes \hat{w}_{j_1} \otimes \dots \otimes \hat{w}_{j_k} \otimes \dots \otimes w_n.$$

Then the socle filtration of $T^{m,n}$ is given by

$$\mathrm{soc}^k(T^{m,n}) = \bigcap_{i_1 \dots i_k | j_1 \dots j_k} \mathrm{Ker}(\Phi_{i_1 \dots i_k | j_1 \dots j_k}). \tag{15}$$

Here $k = 1, \dots, \min(m, n) + 1$ and we use the convention $\mathrm{soc} = \mathrm{soc}^1$.

The modules $V_\lambda \otimes (V_*)_\mu$ are indecomposable and represent the isomorphism classes of indecomposable direct summands in $T^{\cdot}(V \oplus V_*)$. Moreover, using (15) it is shown in [PStyr] that $V_\lambda \otimes (V_*)_\mu$ has a simple socle. Denote this simple module by $V_{\lambda,\mu}$. Then, for variable λ and μ, $V_{\lambda,\mu}$ exhaust all (up to isomorphism) simple modules of $\mathbb{T}$. In this way, the simple objects of $\mathbb{T}$ are labeled by pairs of partitions λ, μ. Note that $V_\lambda = V_{\lambda,\emptyset}$ and $(V_*)_\lambda = V_{\emptyset,\lambda}$.

Furthermore, $V_\lambda \otimes (V_*)_\mu$ is an injective hull of $V_{\lambda,\mu}$. It is shown in [DPS] that any indecomposable injective object of $\mathbb{T}$ is isomorphic to $V_\lambda \otimes (V_*)_\mu$ for some λ, μ. For the layers of the socle filtration of $V_\lambda \otimes (V_*)_\mu$ we have

$$[\mathrm{soc}^{k+1}(V_\lambda \otimes (V_*)_\mu)/\mathrm{soc}^k(V_\lambda \otimes (V_*)_\mu) : V_{\lambda',\mu'}] = \sum_{\gamma \in \mathbf{Part},\, |\gamma| = k} N^\lambda_{\lambda',\gamma} N^\mu_{\mu',\gamma} \tag{16}$$

where $N^\nu_{\nu',\gamma}$ denote the Littlewood–Richardson coefficients [PStyr].

Lemma 1. *Let $|\lambda| - |\lambda'| = |\mu| - |\mu'| = 1$. Then*

$$\dim \mathrm{Hom}(V_\lambda \otimes (V_*)_\mu, V_{\lambda'} \otimes (V_*)_{\mu'}) \leq 1,$$

and

$$\mathrm{Hom}(V_\lambda \otimes (V_*)_\mu, V_{\lambda'} \otimes (V_*)_{\mu'}) \simeq \mathbb{C}$$

if and only if λ' and μ' are obtained from λ and μ respectively by removing one box.

Proof The statement follows from (16) and the injectivity of $V_{\lambda'} \otimes (V_*)_{\mu'}$. □

Twisting by an outer involution of $\mathfrak{g}$ yields an involution (equivalence of monoidal categories whose square is the identity functor)

$$(\,\cdot\,)_* : \mathbb{T} \to \mathbb{T}$$

which maps $V_{\lambda,\mu}$ to $V_{\mu,\lambda}$.

By $\mathbb{T}^+$ we denote the full semisimple subcategory of $\mathbb{T}$ consisting of $\mathfrak{g}$-modules whose simple constituents are isomorphic to V_λ for $\lambda \in \mathbf{Part}$. Note that V_λ is injective as an object of $\mathbb{T}$, hence any object of $\mathbb{T}^+$ is injective in $\mathbb{T}$.

By $\mathbb{DT}^+$ we denote the derived category of $\mathbb{T}^+$. As $\mathbb{T}^+$ is semisimple, $\mathbb{DT}^+$ is semisimple, and its simple objects $V_\lambda[n]$ are labeled by pairs $\lambda \in \mathbf{Part}, n \in \mathbb{Z}$.

If M is an object of $\mathbb{T}$, $\mathbb{T}^+$ or $\mathbb{DT}^+$, then $[M]$ denotes the class of M in the corresponding Grothendieck ring; furthermore, $\mathcal{K}(\cdot)$ stands for complexified Grothendieck ring.

Lemma 2. *The map $[V_\lambda] \mapsto s_\lambda$ extends to an isomorphsim* $\mathrm{ch} : \mathcal{K}(\mathbb{T}^+) \to \mathbf{H}^+$.

Proof The character of any representation in $\mathbb{T}^+$ is a symmetric function on the diagonal subalgebra of $gl(\infty)$. By definition, the homomorphism ch assigns to any element of the complexified Grothendieck ring the corresponding linear combination of characters of modules. It is well known that $\mathrm{ch}([V_\lambda]) = s_\lambda$. Since $\{s_\lambda\}_{\lambda \in \mathbf{Part}}$ is a basis in $\mathbf{H}^+$, ch is an isomorphism. □

We denote by Sch the map from $\mathbb{T}^+ \to \mathbf{H}^+$ defined by $\mathrm{Sch}(\cdot) := \mathrm{ch}([\cdot])$. For a given exact functor $\mathcal{F} : \mathbb{T}^+ \to \mathbb{T}^+$ there exists a unique linear operator $[\mathcal{F}] : \mathbf{H}^+ \to \mathbf{H}^+$ such that $[\mathcal{F}] \circ \mathrm{Sch} = \mathrm{Sch} \circ \mathcal{F}$.

Define a functor $(\ )^+ : \mathbb{T} \to \mathbb{T}^+$ by setting

$$M^+ := M/(\bigcap_{\varphi \in \mathrm{Hom}_{\mathfrak{g}}(M, T^{\cdot}(V))} \mathrm{Ker}\varphi)$$

for $M \in \mathbb{T}$.

Lemma 3. *Let M_{gr} denote the semisimplification of $M \in \mathbb{T}$. Then $(M_{gr})^+ \simeq M^+$.*

Proof Since $T^{\cdot}(V)$ is semisimple, $\mathrm{rad}M \subset \bigcap_{\varphi \in \mathrm{Hom}_{\mathfrak{g}}(M, T(V))} \mathrm{Ker}\varphi$, where $\mathrm{rad}M$ stands for the radical of M considered as a $\mathfrak{g}$-module. Therefore $M^+ \simeq (M/\mathrm{rad}M)^+$. On the other hand, the Jordan-Hölder multiplicity of V_λ in $\mathrm{rad}M$ is zero for any partition λ, as V_λ is injective and simple. Therefore $(\mathrm{rad}M)^+ = ((\mathrm{rad}M)_{gr})^+ = 0$. This shows that

$$(M_{gr})^+ \simeq ((M/\mathrm{rad}M) \oplus (\mathrm{rad}M)_{gr})^+ \simeq (M/\mathrm{rad}M)^+ \oplus ((\mathrm{rad}M)_{gr})^+ = (M/\mathrm{rad}M)^+ \simeq M^+.$$

□

Corollary 4. $(\ )^+ : \mathbb{T} \to \mathbb{T}^+$ *is an exact functor.*

4. The functors $\mathcal{D}_N$ and $\mathcal{T}_N$

Let $N \in \mathbb{T}$, then $\mathcal{T}_N(\cdot) := (N \otimes \cdot)^+$ is an exact functor $\mathbb{T}^+ \to \mathbb{T}^+$. If $\mathcal{E}nd_l(\mathbb{T}^+)$ denotes the category of all linear endofunctors of $\mathbb{T}^+$ (i.e. all functors from $\mathbb{T}^+$ to $\mathbb{T}^+$ which induce linear operators on Homs), then $\mathcal{T} : \mathbb{T} \to \mathcal{E}nd_l(\mathbb{T}^+)$ is a faithful functor. In particular, for any $M, N \in \mathbb{T}$, a morphism $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(M, N)$ induces a morphism of functors

$$\mathcal{T}_\varphi : \mathcal{T}_M \to \mathcal{T}_N,$$
$$\mathcal{T}_\varphi(X) := (\varphi \otimes \mathrm{id})^+ : (M \otimes X)^+ \to (N \otimes X)^+$$

for $X \in \mathbb{T}^+$. In particular, any exact sequence in $\mathbb{T}$

$$0 \to N \to M \to L \to 0$$

induces an exact sequence of linear endofunctors on $\mathbb{T}^+$

$$0 \to \mathcal{T}_N \to \mathcal{T}_M \to \mathcal{T}_L \to 0.$$

In what follows we use the notation $\mathcal{D}_N := \mathcal{T}_{N_*}$.

Lemma 5. *If $N \in \mathbb{T}^+$, then $\mathcal{D}_N$ is left adjoint to $\mathcal{T}_N$.*

Proof We have to construct a canonical isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}((N_* \otimes X)^+, Y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}(X, N \otimes Y) \tag{17}$$

for any $X, Y \in \mathbb{T}^+$. First, from the definition of $\cdot^+$ we have a canonical isomorphsim

$$\mathrm{Hom}_{\mathfrak{g}}(N_* \otimes X, Y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}((N_* \otimes X)^+, Y). \tag{18}$$

Next, define a morphsim

$$\gamma : \mathrm{Hom}_{\mathfrak{g}}(X, N \otimes Y) \to \mathrm{Hom}_{\mathfrak{g}}(N_* \otimes X, Y)$$

by setting $\gamma(\varphi)(m \otimes x) = \sum_i \langle n_i, m \rangle y_i$ if $\varphi(x) = \sum_i n_i \otimes y_i$. By construction γ is injective. It remains to show that

$$\dim \mathrm{Hom}_{\mathfrak{g}}(X, N \otimes Y) = \dim \mathrm{Hom}_{\mathfrak{g}}(N_* \otimes X, Y). \tag{19}$$

Since $\mathbb{T}^+$ is semisimple, it suffices to check (19) for simple $N = V_\mu$, $X = V_\lambda$ and $Y = V_\nu$. By the very definition of the Littlewood-Richardson coefficients, the left-hand side equals $N^\lambda_{\mu,\nu}$. The equality (16) implies that the right-hand side is given by the same Littlewood-Richardson coefficient. Therefore γ is an isomorphism and the desired isomorphism (17) is the composition of (18) with γ^{-1}. $\square$

Corollary 6. *If $N \in \mathbb{T}^+$, then $[\mathcal{D}_N]$ and $[\mathcal{T}_N]$ are mutually dual operators on $\mathbf{H}^+$.*

It is clear that for all $L, N \in \mathbb{T}^+$ we have isomorphisms of functors

$$\mathcal{T}_N \circ \mathcal{T}_L \simeq \mathcal{T}_{N\otimes L}$$

and

$$\mathcal{D}_L \circ \mathcal{T}_N \simeq \mathcal{T}_{L_*\otimes N}.$$

Lemma 7. *For $L, N \in \mathbb{T}^+$ there is an isomorphism of functors $\mathcal{D}_N \circ \mathcal{D}_L \simeq \mathcal{D}_{N\otimes L}$.*

Proof Consider the natural exact sequence

$$0 \to K \to L_* \otimes M \to (L_* \otimes M)^+ \to 0,$$

where $M \in \mathbb{T}^+$. It implies the exact sequence

$$0 \to N_* \otimes K \to N_* \otimes L_* \otimes M \to N_* \otimes (L_* \otimes M)^+ \to 0.$$

By Lemma 3 $K^+ = 0$ and therefore, again by Lemma 3, $(N_* \otimes K)^+ = 0$. Since $(\cdot)^+$ is an exact functor, we obtain a natural isomorphism

$$\mathcal{D}_{N\otimes L}(M) = (N_* \otimes L_* \otimes M)^+ \simeq (N_* \otimes (L_* \otimes M)^+)^+ = \mathcal{D}_N \circ \mathcal{D}_L(M).$$

$\square$

Under a direct sum of linear operators $L_i : A \to B_i$ we understand the operator $\oplus L_i : A \to \bigoplus_i B_i$.

Lemma 8. *(a) $(T^{m,n})^+ = 0$ if $m < n$.*
(b) If $m \geq n$, then $(T^{m,n})^+ \simeq \mathrm{Im}\left(\bigoplus_{i_1\dots i_n} \Phi_{i_1\dots i_n|1\dots n}\right)$.
(c) If M is a submodule of $T^{m,n}$, then $M^+ \simeq \left(\bigoplus_{i_1\dots i_n} \Phi_{i_1\dots i_n|1\dots n}\right)(M)$.

Proof By (16) all simple constituents of $\mathrm{soc}^{k+1}(T^{m,n})/(\mathrm{soc}^k(T^{m,n}))$ are isomorphic to $V_{\lambda,\mu}$ with $|\lambda| = n-k$, $|\mu| = m-k$. Therefore (a) follows from Lemma 3.

In the case $n \geq m$ we have $(T^{m,n})^+ \simeq T^{m,n}/(\mathrm{soc}^{n+1}(T^{m,n}))$. Since $\mathrm{soc}^{n+1}(T^{m,n}) = \bigcap_{i_1\dots i_n} \mathrm{Ker}(\Phi_{i_1\dots i_n|1\dots n})$, statement (b) follows.

To prove (c) note that $T^\cdot(V)$ is an injective object of $\mathbb{T}$. Therefore any $\varphi \in \mathrm{Hom}_{\mathfrak{g}}(M, T^\cdot(V))$ extends to $\tilde{\varphi} \in \mathrm{Hom}_{\mathfrak{g}}(T^{m,n}, T^\cdot(V))$, $\tilde{\varphi}|_M = \varphi$. This shows that $M^+ \simeq M/(M \cap (\bigcap_{i_1\dots i_n} \mathrm{Ker}\Phi_{i_1\dots i_n|1\dots n}))$, and (c) follows. $\square$

Proposition 1. *For any $L, N \in \mathbb{T}^+$ there is an isomorphism of functors*

$$\mathcal{T}_{\mathrm{soc}(L_*\otimes N)} \simeq \mathcal{T}_N \circ \mathcal{D}_L.$$

Proof One has to prove that for any $M \in \mathbb{T}^+$ there exists a canonical isomorphsim

$$(\mathrm{soc}(L_* \otimes N) \otimes M)^+ \simeq \mathcal{T}_N \circ \mathcal{D}_L(M).$$

Without loss of generality we may assume that $L \subset V^{\otimes l}$, $M \subset V^{\otimes m}$, $N \subset V^{\otimes n}$. By Lemma 8,

$$(\mathrm{soc}(L_* \otimes N) \otimes M)^+ = \left(\bigoplus_{1\leq j_1,\dots,j_l\leq m+n} \Phi_{1\dots l|j_1\dots j_l}\right)(\mathrm{soc}(L_* \otimes N) \otimes M).$$

If $j_i \leq n$ at least for one j_i, then $\Phi_{1\dots l|j_1\dots j_l}(\mathrm{soc}(L_* \otimes N) \otimes M) = 0$. Therefore

$$(\mathrm{soc}(L_* \otimes N) \otimes M)^+ = \left(\bigoplus_{n<j_1,\dots,j_l\leq m+n} \Phi_{1\dots l|j_1\dots j_l}\right)(\mathrm{soc}(L_* \otimes N) \otimes M).$$

We will prove that

$$\left(\bigoplus_{n<j_1,\dots,j_l\leq m+n}\Phi_{1\dots l|j_1\dots j_l}\right)((\mathrm{soc}(L_*\otimes N)\otimes M)=\left(\bigoplus_{n<j_1,\dots,j_l\leq m+n}\Phi_{1\dots l|j_1\dots j_l}\right)(L_*\otimes N\otimes M). \tag{20}$$

Note that the left-hand side of (20) is a submodule of the right-hand side.

We use the fact that

$$U:=\{w_1\otimes\dots\otimes w_n\otimes x_1\otimes\dots\otimes x_m|w_1,\dots,w_n,x_1,\dots,x_m\in V,\mathrm{span}(w_1,\dots,w_n)\cap\mathrm{span}(x_1,\dots,x_m)=\{0\}\}$$

spans $T^{0,m+n}$. Let $u=v_1\otimes\dots\otimes v_l\otimes w_1\otimes\dots\otimes w_n\otimes x_1\otimes\dots\otimes x_m\in T^{l,m+n}$. Set

$$\pi(u)=\pi_{L_*}(v_1\otimes\dots\otimes v_l)\otimes\pi_N(w_1\otimes\dots\otimes w_n)\otimes\pi_M(x_1\otimes\dots\otimes x_m),$$

where $\pi_{L_*}:V_*^{\otimes l}\to L_*$, $\pi_N:V^{\otimes n}\to N$ and $\pi_M:V^{\otimes m}\to M$ are respective projectors. To prove (20) it is sufficient to show that for any u such that $\mathrm{span}(w_1,\dots,w_n)\cap\mathrm{span}(x_1,\dots,x_m)=\{0\}$ there exists $\tilde{u}\in\mathrm{soc}(T^{l,n})\otimes V^{\otimes m}$ such that

$$\Phi_{1\dots l|j_1\dots j_l}(\pi(u))=\Phi_{1\dots l|j_1\dots j_l}(\pi(\tilde{u})) \tag{21}$$

for any choice of $j_1\dots j_l$, $n<j_1,\dots,j_l\leq m+n$.

For this purpose consider $\tilde{u}=\tilde{v}_1\otimes\cdots\otimes\tilde{v}_l\otimes w_1\otimes\dots\otimes w_n\otimes x_1\otimes\dots\otimes x_m$, where $\tilde{v}_i\in V_*$ satisfy the conditions $\langle\tilde{v}_i,w_j\rangle=0$ for $1\leq i\leq l$, $1\leq j\leq n$, and $\langle\tilde{v}_i,x_j\rangle=\langle v_i,x_j\rangle$ for $1\leq i\leq l$, $1\leq j\leq m$. Such a choice of $\tilde{v}_i$ is possible since $\mathrm{span}(w_1,\dots,w_n)\cap\mathrm{span}(x_1,\dots,x_m)=\{0\}$. It is a direct computation to verify that $\tilde{u}$ satisfies (21).

To finish the proof, note that Lemma 8 (c) implies

$$\left(\bigoplus_{n<j_1,\dots,j_l\leq m+n}\Phi_{1\dots l|j_1\dots j_l}\right)(L_*\otimes N\otimes M)\simeq N\otimes\left(\left(\bigoplus_{1\leq j_1,\dots,j_l\leq m}\Phi_{1\dots l|j_1\dots j_l}\right)(L_*\otimes M)\right)=\mathcal{T}_N\circ\mathcal{D}_L(M).$$

□

It is an interesting problem to characterize the image $\hat{\mathbb{T}}$ of $\mathbb{T}$ inside the category of linear endofunctors $\mathcal{E}nd_l(\mathbb{T}^+)$.

5. Categorifying identities for halfs of vertex operators

As a preliminary step to categorifying the identites (12)–(14), in this section we categorify the following identities:

$$H(z)E(-z)=E(-z)H(z)=1, \tag{22}$$

$$H^*(z)E^*(-z)=E^*(-z)H^*(z)=1, \tag{23}$$

$$H(z)H^*(w)=(1-z/w)H^*(w)H(z), \tag{24}$$

$$E(-z)E^*(-w)=(1-z/w)E^*(-w)E(-z), \tag{25}$$

$$(1-z/w)H(z)E^*(-w)=E^*(-w)H(z), \tag{26}$$

$$(1-z/w)E(-z)H^*(w)=H^*(w)E(-z). \tag{27}$$

For $n\in\mathbb{Z}_{\geq0}$, there are two "extreme" partitions of n: n itself and $1_n=\underbrace{(1,\dots,1)}_{n}$. Set $S^n:=S^n(V)=V_n$, $\Lambda^n:=\Lambda^n(V)=V_{1_n}$. Set furthermore

$$\mathcal{H}_n:=\mathcal{T}_{S^n},\ \mathcal{E}_n:=\mathcal{T}_{\Lambda^n},\ \mathcal{H}_n^*:=\mathcal{D}_{S^n},\ \mathcal{E}_n^*:=\mathcal{D}_{\Lambda^n}$$

(cf (5)).

Proposition 2. *For $m, n \in \mathbb{Z}_{>0}$ there are exact sequences of functors*

$$0 \to \mathcal{H}_n \circ \mathcal{H}_m^* \to \mathcal{H}_m^* \circ \mathcal{H}_n \to \mathcal{H}_{m-1}^* \circ \mathcal{H}_{n-1} \to 0, \tag{28}$$

$$0 \to \mathcal{E}_n \circ \mathcal{E}_m^* \to \mathcal{E}_m^* \circ \mathcal{E}_n \to \mathcal{E}_{m-1}^* \circ \mathcal{E}_{n-1} \to 0, \tag{29}$$

$$0 \to \mathcal{H}_n \circ \mathcal{E}_m^* \to \mathcal{E}_m^* \circ \mathcal{H}_n \to \mathcal{H}_{n-1} \circ \mathcal{E}_{m-1}^* \to 0, \tag{30}$$

$$0 \to \mathcal{E}_n \circ \mathcal{H}_m^* \to \mathcal{H}_m^* \circ \mathcal{E}_n \to \mathcal{E}_{n-1} \circ \mathcal{H}_{m-1}^* \to 0. \tag{31}$$

Proof We claim that there are the following exact sequences:

$$0 \to V_{m,n} \to S^m \otimes (S^n)_* \to S^{m-1} \otimes (S^{n-1})_* \to 0, \tag{32}$$

$$0 \to V_{1_m,1_n} \to \Lambda^m \otimes (\Lambda^n)_* \to \Lambda^{m-1} \otimes (\Lambda^{n-1})_* \to 0, \tag{33}$$

$$0 \to V_{1_m,n} \to \Lambda^m \otimes (S^n)_* \to V_{1_{m-1},n-1} \to 0, \tag{34}$$

$$0 \to V_{m,1_n} \to S^m \otimes (\Lambda^n)_* \to V_{m-1,1_{n-1}} \to 0. \tag{35}$$

Let us for instance construct (32). By Lemma 1 there is a non-zero homomorphism $\varphi : S^m \otimes (S^n)_* \to S^{m-1} \otimes (S^{n-1})_*$. Clearly, $V_{m,n} = \mathrm{soc}(S^m \otimes (S^n)_*) \subset \mathrm{Ker}\varphi$ since $V_{m,n}$ is not a constituent of $S^{m-1} \otimes (S^{n-1})_*$ by (16). Again by (16),

$$[S^m \otimes (S^n)_*] = [S^{m-1} \otimes (S^{n-1})_*] + [V_{m,n}]$$

in the Grothendieck ring of $\mathbb{T}$, and the socle of $(S^m \otimes (S^n)_*)/V_{m,n}$ is isomorphic to the socle of $S^{m-1} \otimes (S^{n-1})_*$. Hence φ is surjective and $V_{m,n} = \mathrm{Ker}\varphi$. The sequences (33)-(35) are constructed by similar considerations.

We will now show that the sequence (32) implies the existence of (28). Indeed, by the remark at the beginning of Section 4, the exact sequence (32) induces an exact sequence

$$0 \to \mathcal{T}_{V_{m,n}} \to \mathcal{D}_{S^m} \circ \mathcal{T}_{S^n} \to \mathcal{D}_{S^{m-1}} \circ \mathcal{T}_{S^{n-1}} \to 0.$$

Since $\mathcal{T}_{V_{m,n}} \simeq \mathcal{T}_{S^n} \circ \mathcal{D}_{S^m}$ by Proposition 1, the existence of (28) follows.

The existence of sequences (29)-(31) is proved in a similar way by using the sequences (33)-(35). □

Lemma 9. *For any $n \geq 0$*

$$[\mathcal{H}_n] = h_n, \; [\mathcal{E}_n] = e_n, \; [\mathcal{H}_n^*] = h_n^*, \; [\mathcal{E}_n^*] = e_n^*,$$

where h_n, e_n, h_n^*, e_n^* are considered as operators on $\mathbf{H}^+$ as explained in Section 1.

Proof First, observe that $[\mathcal{T}_N] = \mathrm{Sch}(N)$ for $N \in \mathbb{T}^*$. This implies the equalities $[\mathcal{H}_n] = h_n, [\mathcal{E}_n] = e_n$. The two other equalities follow via Corollary 6. □

We now see that the exact sequences (28)-(31) categorify the identities (24)-(27). More precisely we have the following.

Corollary 10. *Proposition 2 implies (24)-(27).*

Proof Consider for instance the identity (24)

$$H(z)H^*(w) = (1 - z/w)H^*(w)H(z).$$

It is equivalent to the equality

$$h_m^* h_n - h_n h_m^* = h_{m-1}^* h_{n-1}$$

which follows immediately from (28) and Lemma 9.

The arguments in the remaining three cases are similar. □

Now we proceed to the categorification of the identities (22)-(23). Clearly, $H(z)$ and $E(-z)$ commute, therefore (22) is equivalent to

$$H(z)E(-z) = 1 \tag{36}$$

The equality (36) can be rewritten as

$$\sum_{m+n=k} (-1)^m h_n e_m = 0 \text{ for } k > 0. \tag{37}$$

Similarly, (23) can be rewritten as

$$\sum_{m+n=k} (-1)^m h_n^* e_m^* = 0 \text{ for } k > 0. \tag{38}$$

Lemma 9 together with Lemma 11 below give a categorical proof of (37) and (38).

Lemma 11. *The complexes C_k,*

$$0 \to S^k \to \dots \to S^m \otimes \Lambda^{k-m} \to S^{m-1} \otimes \Lambda^{k-m+1} \to \dots \to \Lambda^k \to 0,$$

and $(C_k)_*$,

$$0 \to (S_*)^k \to \cdots \to (S_*)^m \otimes (\Lambda_*)^{k-m} \to \dots \to (\Lambda_*)^k \to 0,$$

are exact except for $k = 0$. They induce complexes of functors

$$0 \to \mathcal{H}_k \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathcal{H}_m \circ \mathcal{E}_{k-m} \xrightarrow{\delta} \dots \xrightarrow{\delta} \mathcal{E}_k \to 0, \tag{39}$$

$$0 \to \mathcal{H}_k^* \xrightarrow{\delta^*} \dots \xrightarrow{\delta^*} \mathcal{H}_m^* \circ \mathcal{E}_{k-m}^* \xrightarrow{\delta^*} \dots \xrightarrow{\delta^*} \mathcal{E}_k^* \to 0, \tag{40}$$

which are exact for $k \geq 1$.

Proof Note that $\bigoplus_{k\geq 0} C_k$ and $\bigoplus_{k\geq 0}(C_k)_*$ are Koszul complexes. Hence they are exact except for $k = 0$. In particular, C_k and $(C_k)_*$ are exact for $k \geq 1$. After application of the functor $\mathcal{T}$, we obtain that the complexes (39) and (40) are also exact for $k \geq 1$. □

6. Categorification of Clifford algebra

Write the vertex operators $X(z)$ and $X^*(z)$ as

$$X(z) = \sum_{a\in\mathbb{Z}} X_a z^a, \;\; X^*(z) = \sum_{a\in\mathbb{Z}} X_a^* z^a.$$

Our next goal is to categorify the coefficients X_a and X_a^*. Let

$$R_a(q) := S^{a+q} \otimes \Lambda_*^q \otimes T^{\cdot}(V), \;\; R_a := \bigoplus_{q\geq\max(-a,0)} R_a(q),$$

and

$$R_a^*(p) := \Lambda^{a+p} \otimes S_*^p \otimes T^{\cdot}(V), \;\; R_a^* := \bigoplus_{p\geq\max(-a,0)} R_a^*(p).$$

Define $\theta_a : R_a \to R_a$ and $\theta_a^* : R_a^* \to R_a^*$ by the formulas

$$\theta_a(x_1 \dots x_p \otimes y_1 \wedge \cdots \wedge y_q \otimes v_1 \otimes \cdots \otimes v_k) :=$$

$$\sum_{i,j,s} (-1)^i \langle y_i, v_j\rangle x_1 \dots \hat{x}_s \dots x_p \otimes y_1 \wedge \cdots \wedge \hat{y}_i \wedge \cdots \wedge y_q \otimes v_1 \otimes \cdots \otimes v_{j-1} \otimes x_s \otimes v_{j+1} \otimes \cdots \otimes v_k,$$

$$\theta_a^*(x_1 \wedge \cdots \wedge x_p \otimes y_1 \dots y_q \otimes v_1 \otimes \cdots \otimes v_k) :=$$

$$\sum_{i,j,s}(-1)^i\langle y_i, v_j\rangle x_1\wedge\cdots\wedge\hat{x}_s\wedge\cdots\wedge x_p\otimes y_1\ldots\hat{y}_i\ldots y_q\otimes v_1\otimes\cdots\otimes v_{j-1}\otimes x_s\otimes v_{j+1}\otimes\cdots\otimes v_k.$$

Define also operators $d : R_a \to R_a$ and $d^* : R^*_a \to R^*_a$:

$$d(x_1\ldots x_p\otimes y_1\wedge\cdots\wedge y_q\otimes v) := \sum_{i,j}(-1)^i\langle y_i, x_j\rangle x_1\ldots\hat{x}_j\ldots x_p\otimes y_1\wedge\cdots\wedge\hat{y}_i\wedge\cdots\wedge y_q\otimes v,$$

$$d^*(x_1\wedge\cdots\wedge x_p\otimes y_1\ldots y_q\otimes v) := \sum_{i,j}(-1)^j\langle y_i, x_j\rangle x_1\wedge\cdots\wedge\hat{x}_j\wedge\cdots\wedge x_p\otimes y_1\ldots\hat{y}_i\ldots y_q\otimes v$$

for $v = v_1\otimes\cdots\otimes v_k$.

Lemma 12. *(a) θ_a, θ^*_a, d and d^* commute with the action of the symmetric group $\mathbb{S}_k$ on $V^{\otimes k}$;*
(b) $d^2 = 0$, $(d^*)^2 = 0$ and the complexes (R_a, d), (R^*_a, d^*) are acyclic for all $a\in\mathbb{Z}$;
(c) $\theta_a^2 = d\theta_a = -\theta_a d$, $(\theta^*_a)^2 = d^*\theta^*_a = -\theta^*_a d^*$.

Proof (a) is straightforward, (b) is well known and also follows from (34)-(35). We will prove (c) by a direct calculation. Indeed, set

$$z := x_1\ldots x_p\otimes y_1\wedge\cdots\wedge y_q\otimes v_1\otimes\cdots\otimes v_k.$$

Furthermore, let $X_{s,t}$ stand for $x_1\ldots x_p$ with x_s and x_t removed, and $Y_{i,c}$ stand for $y_1\wedge\cdots\wedge y_q$ with y_c and y_i removed and multiplied by $(-1)^{i+c}$ if $i < c$ and $(-1)^{i+c-1}$ if $i > c$. Then

$$\theta_a(\theta_a(z)) = \sum_{s\neq t, j, i\neq c}\langle y_i, v_j\rangle\langle y_c, x_s\rangle X_{s,t}\otimes Y_{c,i}\otimes v_1\otimes\cdots\otimes v_{j-1}\otimes x_s\otimes v_{j+1}\otimes\cdots\otimes v_k =$$

$$\theta_a\left(\sum_{c,s}(-1)^c\langle y_c, x_s\rangle x_1\ldots\hat{x}_s\ldots x_p\otimes y_1\wedge\cdots\wedge\hat{y}_c\wedge\cdots\wedge y_q\otimes v_1\otimes\cdots\otimes v_k\right) = \theta_a(d(z)) =$$

$$-d\left(\sum_{i,t,j}(-1)^i\langle y_i, v_j\rangle x_1\ldots\hat{x}_t\ldots x_p\otimes y_1\wedge\cdots\wedge\hat{y}_i\wedge\cdots\wedge y_q\otimes v_1\otimes\ldots v_{j-1}\otimes x_t\otimes\cdots\otimes v_k\right) = d(\theta_a(z)).$$

Checking (c) for θ^*_a and d^* is similar. □

Let M be a submodule of $T^\cdot(V)$, and $\pi_M\in\mathrm{End}_{\mathfrak{g}}(T^\cdot(V))$ denote a projector onto M. Since p_M is an element of the direct sum $\bigoplus_{n\geq o}\mathbb{C}[\mathbb{S}_n]$, Lemma 12(a) implies that d, d^* and θ_a, θ^*_a commute with π_M. Therefore d, d^* and θ_a, θ^*_a are well defined linear operators in $R_a(M) = \pi_M R_a$ and in $R^*_a(M) = \pi_M R^*_a$. Moreover, since any $M\in\mathbb{T}^+$ is a submodule of a direct sum of finitely many copies of $T^\cdot(V)$, Schur–Weyl dualty implies that $R_a(M)$, $R^*_a(M)$ and $d, \theta_a\in\mathrm{End}_{\mathfrak{g}}(R_a(M))$, $d^*, \theta^*_a\in\mathrm{End}_{\mathfrak{g}}(R^*_a(M))$ are well-defined for any $M\in\mathbb{T}^+$.

Lemma 12(c) shows that Ker$d$ is θ_a-stable and Kerd^* is θ^*_a-stable. If we define

$$S_a(M) := \mathrm{Ker}d\cap R_a(M),\quad S^*_a(M) := \mathrm{Ker}d^*\cap R^*_a(M),$$

and denote by $\theta_a(M)$ (respectively, $\theta^*_a(M)$) the restriction of θ_a (respectively, θ^*_a) on $S_a(M)$ (respectively, $S^*_a(M)$) then, again by Lemma 12(c), $(S_a(M), \theta_a(M))$ and $(S^*_a(M), \theta^*_a(M))$ become complexes. Next we set

$$\mathcal{X}_a(M) := (S_a(M))^+,\quad \mathcal{X}^*_a(M) := (S^*_a(M))^+.$$

The exact sequences (30) and (31) imply Ker$d\cap R_a(q)(M) = \mathcal{T}_{S^{a+q}}\circ\mathcal{D}_{\Lambda^q}(M)$ and Ker$d\cap R^*_a(p)(M) = \mathcal{T}_{\Lambda^{a+p}}\circ\mathcal{D}_{S^p}(M)$. Therefore we have

$$\mathcal{X}_a(M) = \bigoplus_{p-q=a}\mathcal{H}_p\circ\mathcal{E}^*_q(M),\ \ \mathcal{X}^*_a(M) = \bigoplus_{p-q=a}\mathcal{E}_p\circ\mathcal{H}^*_q(M), \tag{41}$$

or simply

$$\mathcal{X}_a = \bigoplus_{p-q=a} \mathcal{H}_p \circ \mathcal{E}_q^*,\ \mathcal{X}_a^* = \bigoplus_{p-q=a} \mathcal{E}_p \circ \mathcal{H}_q^*.$$

Moreover, $\mathcal{X}_a$ and $\mathcal{X}_a^*$ are well-defined complexes of linear endofunctors on $\mathbb{T}^+$:

$$\mathcal{X}_a(q) = \mathcal{H}_{q+a} \circ \mathcal{E}_q^*,\ \mathcal{X}_a^*(q) = \mathcal{E}_{q+a} \circ \mathcal{H}_q^*,$$

and the differentials $\mathcal{X}_a(q) \to \mathcal{X}_a(q-1)$, $\mathcal{X}_a^*(q) \to \mathcal{X}_a^*(q-1)$ are simply the restrictions of θ_a and θ_a^* respectively. By abuse of notation we denote these differentials by the same letters θ_a and θ_a^*. These complexes can be considered as functors from $\mathbb{T}^+$ to $\mathbb{DT}^+$. However, we will consider them as functors from $\mathbb{DT}^+$ to $\mathbb{DT}^+$. Indeed, an object of $\mathbb{DT}^+$ is isomorphic to a direct sum of simple objects $V_\lambda[n]$ for $\lambda \in \mathbf{Part}, n \in \mathbb{Z}$. Then applying $\mathcal{X}_a$ (or $\mathcal{X}_a^*$) yields a direct sum of appropriately shifted complexes.

Lemma 13. *Let $\chi : \mathbb{DT}^+ \to \mathbf{H}^+$ denote the Euler characteristic map. Then*

$$\chi \circ \mathcal{X}_a = X_a \circ \chi,\ \ \chi \circ \mathcal{X}_a^* = X_a^* \circ \chi.$$

Proof By (41) for any $M \in \mathbb{T}^+$ we have

$$\chi(\mathcal{X}_a(M)) = \sum_{p-q=a} (-1)^q [\mathcal{H}_p \circ \mathcal{E}_q^*(M)],\ \chi(\mathcal{X}_a^*(M)) = \sum_{p-q=a} (-1)^p [\mathcal{E}_p \circ \mathcal{S}_q^*(M)].$$

Therefore the statement follows from Lemma 9 and (10). □

Recall that by (11) the operators X_a and X_a^* up to a shift coincide with the generators ψ_i, ψ_i^* of $\mathbf{Cf}$. Therefore for an irreducible $M \in \mathbb{T}^+$, we have $X_a[M]$ (or $X_a^*[M]$) is either zero or equals the class of an irreducible object in $\mathbb{T}^+$. This suggests the following.

Conjecture 14. *If $M \in \mathbb{T}^+$ is irreducible, then the cohomology of each complex $\mathcal{X}_a(M)$ and $\mathcal{X}_a^*(M)$ is nonzero in at most one degree and is an irreducible $\mathfrak{g}$-module.*

To obtain a categorification of the Clifford generators ψ_i, ψ_i^* we use the equation (11). First, we identify $\mathbf{F}$ with the Grothendieck group of $\mathbb{DT}^+$ by identifying $\mathcal{K}(\mathbb{T}^+[m])$ with $\mathbf{F}(m)$ for $m \in \mathbb{Z}$. Denote by $\mathcal{S}$ the autoequivalence $\mathbb{DT}^+ \to \mathbb{DT}^+$ arising from shifting the grading: $\mathcal{S}(M) := M[1]$. Define functors $\Psi_a : \mathbb{T}[m] \to \mathbb{T}[m+1]$ and $\Psi_a^* : \mathbb{T}[m] \to \mathbb{T}[m-1]$:

$$\Psi_a := \mathcal{S} \circ \mathcal{X}_{a+m},\ \Psi_a^* := \mathcal{S}^{-1} \circ \mathcal{X}_{a-m}.$$

Using the semisimplicity of $\mathbb{DT}$ we extend Ψ_a and Ψ_a^* by additivity to functors $\mathbb{DT}^+ \to \mathbb{DT}^+$. To check that they satisfy the Clifford relations it suffices to categorify the identities (12)–(14).

7. Categorifying vertex operator identities

In this section we categorify the identities (12)–(14) by using the complexes of functors $\mathcal{X}_a$ and $\mathcal{X}_a^*$.

Let us start with (12). From Proposition 2 we have the following exact sequences of linear endofunctors on $\mathbb{T}^+$

$$0 \to \mathcal{H}_m \circ \mathcal{H}_p \circ \mathcal{E}_n^* \circ \mathcal{E}_q^* \to \mathcal{H}_m \circ \mathcal{E}_n^* \circ \mathcal{H}_p \circ \mathcal{E}_q^* \to \mathcal{H}_m \circ \mathcal{H}_{p-1} \circ \mathcal{E}_{n-1}^* \circ \mathcal{E}_q^* \to 0,$$

$$0 \to \mathcal{H}_{p-1} \circ \mathcal{H}_m \circ \mathcal{E}_q^* \circ \mathcal{E}_{n-1}^* \to \mathcal{H}_{p-1} \circ \mathcal{E}_q^* \circ \mathcal{H}_m \circ \mathcal{E}_{n-1}^* \to \mathcal{H}_{p-1} \circ \mathcal{H}_{m-1} \circ \mathcal{E}_{q-1}^* \circ \mathcal{E}_{n-1}^* \to 0.$$

Combining these sequences we obtain a long exact sequence

$$\cdots \to \mathcal{H}_m \circ \mathcal{E}_n^* \circ \mathcal{H}_p \circ \mathcal{E}_q^* \to \mathcal{H}_{p-1} \circ \mathcal{E}_q^* \circ \mathcal{H}_m \circ \mathcal{E}_{n-1}^* \to \mathcal{H}_{m-1} \circ \mathcal{E}_{n-1}^* \circ \mathcal{H}_{p-1} \circ \mathcal{E}_{q-1}^* \to \dots. \tag{42}$$

Consider the $\mathbb{Z} \times \mathbb{Z}$-graded functor

$$\mathcal{X}_a \circ \mathcal{X}_b = \bigoplus_{n,q} \mathcal{X}_a(n) \circ \mathcal{X}_b(q),$$

where here and below $\mathcal{K}(n)$ stands for the n-th term of a complex of endofunctors $\mathcal{K}$. Let

$$\chi(\mathcal{X}_a \circ \mathcal{X}_b) = \sum_{n,q} (-1)^{n+q} [\mathcal{X}_a(n)] \circ [\mathcal{X}_b(q)].$$

We rewrite (42) in the form

$$\cdots \to \mathcal{X}_a(n) \circ \mathcal{X}_b(q) \to \mathcal{X}_{b-1}(q) \circ \mathcal{X}_{a+1}(n-1) \to \mathcal{X}_a(n-1) \circ \mathcal{X}_b(q-1) \to \cdots . \tag{43}$$

Then (43) implies

$$\chi(\mathcal{X}_a \circ \mathcal{X}_b) + \chi(\mathcal{X}_{b-1} \circ \mathcal{X}_{a+1}) = 0$$

which yields (12). The proof of (13) is similar.

Note that we can define two morthisms of functors :

$$\theta'_{ab} : \mathcal{X}_a(n) \circ \mathcal{X}_b(q) \to \mathcal{X}_a(n) \circ \mathcal{X}_b(q-1),$$
$$\theta''_{ab} : \mathcal{X}_a(n) \circ \mathcal{X}_b(q) \to \mathcal{X}_a(n-1) \circ \mathcal{X}_b(q)$$

by setting

$$\theta'_{ab}\,(\mathcal{X}_a(n) \circ \mathcal{X}_b(q)(M)) := \mathcal{X}_a(n)(\theta_b(X_b(q)(M))),$$
$$\theta''_{ab}\,(\mathcal{X}_a(n) \circ \mathcal{X}_b(q)(M)) := \theta_a(\mathcal{X}_a(n)(X_b(q)(M)))$$

for every $M \in \mathbb{T}^+$. It is easy to check that $(\theta'_{ab})^2 = (\theta''_{ab})^2 = 0$, and moreover we have the following.

Conjecture 15. *(a) $\mathcal{X}_a \circ \mathcal{X}_b$ is a bicomplex of functors with differentials $\theta'_{ab}, \theta''_{ab}$.*
(b) (43) is the cone of a map of total complexes $\mathcal{X}_a \circ \mathcal{X}_b \to \mathcal{X}_{b-1} \circ \mathcal{X}_{a+1}$.
(c) Analogous statements hold for $\mathcal{X}^*_a \circ \mathcal{X}^*_b$.

It remains to prove (14). Define the functors $\mathcal{Z}_{a,b}(n,q) : \mathbb{T}^+ \to \mathbb{T}^+$ by setting

$$\mathcal{Z}_{a,b}(n,q) := \mathcal{H}_{a+n} \circ \mathcal{E}_{b+q} \circ \mathcal{E}^*_n \circ \mathcal{H}^*_q.$$

We rewrite (28) and (29) in the form

$$0 \to \mathcal{Z}_{a,b}(n,q) \to \mathcal{X}_a(n) \circ \mathcal{X}^*_b(q) \to \mathcal{X}_{a+1}(n-1) \circ \mathcal{X}^*_{b-1}(q) \to 0, \tag{44}$$

$$0 \to \mathcal{Z}_{a,b}(n,q) \to \mathcal{X}^*_b(n) \circ \mathcal{X}_a(q) \to \mathcal{X}^*_{b+1}(q-1) \circ \mathcal{X}_{a-1}(n) \to 0. \tag{45}$$

Conjecture 16. *$\mathcal{Z}_{a,b} = \bigoplus_{n,q} \mathcal{Z}_{a,b}(n,q)$ has the structure of a bicomplex, and the morphisms in (44) and (45) commute with the differentials.*

Set

$$\mathcal{X}_a \circ \mathcal{X}^*_b = \bigoplus_{n,q} \mathcal{X}_a(n) \circ \mathcal{X}^*_b(q), \;\; \mathcal{X}^*_b \circ \mathcal{X}_a = \bigoplus_{n,q} \mathcal{X}^*_b(q) \circ \mathcal{X}_a(n).$$

Observe that for any fixed $M \in \mathbb{T}^+$ we have $\mathcal{Z}_{a,b}(M) = 0$ for sufficiently large negative b. Therefore (44) implies

$$\mathcal{X}_a \circ \mathcal{X}^*_b \simeq \bigoplus_{i>0} \mathcal{Z}_{a+i,b-i}.$$

Similarly by (45) we have

$$\mathcal{X}^*_b \circ \mathcal{X}_a \simeq \bigoplus_{i<0} \mathcal{Z}_{a+i,b-i}.$$

Hence

$$\mathcal{X}_a \circ \mathcal{X}^*_b \oplus \mathcal{X}^*_{b+1} \circ \mathcal{X}_{a-1} = \bigoplus_{i\in\mathbb{Z}} \mathcal{Z}_{a+i,b-i}.$$

Set

$$\chi(\mathcal{Z}_{a,b}) = \sum (-1)^{n+q} [\mathcal{Z}_{a,b}(n,q)].$$

Lemma 17.
$$\sum_{i\in\mathbb{Z}} \chi(\mathcal{Z}_{a+i,b-i}) = \delta_{a+b,0}\,\mathrm{Id}\,.$$

Proof Using the complex C_k (see Lemma 11) we obtain
$$\sum_{m+p=k} (-1)^p[\mathcal{H}_m \circ \mathcal{E}_p] = \delta_{k,0}\,\mathrm{Id},$$
and from the complex $(C_l)_*$ we obtain
$$\sum_{n+q=l} (-1)^q[\mathcal{E}^*_n \circ \mathcal{H}^*_q] = \delta_{l,0}\,\mathrm{Id}\,.$$
Combining the both above identities we have
$$\sum_{m+p=k,n+q=l} (-1)^{p+q}[\mathcal{H}_m \circ \mathcal{E}_p \circ \mathcal{E}^*_n \circ \mathcal{H}^*_q] = \delta_{k,0}\delta_{l,0}\,\mathrm{Id}\,.$$
After summation over all $k-l=s$ we get
$$\sum_{m+p-n-q=s} (-1)^{p+q}[\mathcal{H}_m \circ \mathcal{E}_p \circ \mathcal{E}^*_n \circ \mathcal{H}^*_q] = \delta_{s,0}\,\mathrm{Id}\,.$$
The left-hand side of the last identity is equal to $\sum_{i\in\mathbb{Z}} \chi(\mathcal{Z}_{s+i,-i})$. Hence the statement. □

Corollary 18. *We have*
$$\chi(\mathcal{X}_a \circ \mathcal{X}^*_b) + \chi(\mathcal{X}^*_{b+1} \circ \mathcal{X}_{a-1}) = \delta_{a+b,0}\,\mathrm{Id}\,.$$
Hence (14) holds.

8. Categorification of Weyl algebra

We will now construct certain graded endofunctors $\mathcal{P}_n, \mathcal{P}^*_n$ categorifying p_n, p^*_n, and will use $\mathcal{P}_n, \mathcal{P}^*_n$ to show that p_n and p^*_n satisfy the relations (7). Recall that the partitions (or equivalently, diagrams) of the form $(p, 1_q)$ are called *hooks*. Define $\mathcal{P}_k(i) := \mathcal{T}_{V_{k-i,1_i}}, \mathcal{P}_k(i) := \mathcal{D}_{V_{k-i,1_i}}$ and set
$$\mathcal{P}_k := \bigoplus_{i=0}^{k-1} \mathcal{P}_k(i),\ \ \mathcal{P}^*_k := \bigoplus_{i=0}^{k-1} \mathcal{P}^*_k(i).$$

Consider again the complexes of functors (39) and (40) with differentials δ and δ^*.

Lemma 19. *We have the following exact sequence of functors*
$$0 \to \mathcal{P}_k(i) \to \mathcal{H}_{k-i} \circ \mathcal{E}_i \xrightarrow{\delta} \mathcal{H}_{k-i-1} \circ \mathcal{E}_{i+1},$$
and
$$0 \to \mathcal{P}^*_k(i) \to \mathcal{H}^*_{k-i} \circ \mathcal{E}^*_i \xrightarrow{\delta} \mathcal{H}^*_{k-i-1} \circ \mathcal{E}^*_{i+1}.$$

Proof The first exact sequence follows from the following well-known exact sequence (see, for instance, [FH], Exercise 6.20)
$$0 \to V_{p,1_q} \to S^p \otimes \Lambda^{q-1} \to S^{p-1} \otimes \Lambda^q.$$
The second one follows from the exact sequence
$$0 \to (V_*)_{p,1_q} \to S^p_* \otimes \Lambda^{q-1}_* \to S^{p-1}_* \otimes \Lambda^q_*.$$
□

Lemma 19 motivates us to define $\mathcal{P}_k$ and $\mathcal{P}^*_k$ as complexes of endofunctors in $\mathbb{T}^+$ with zero differentials. As in the fermion case we will consider $\mathcal{P}_k$ and $\mathcal{P}^*_k$ as functors from $\mathbb{D}\mathbb{T}^+$ to $\mathbb{D}\mathbb{T}^+$.

Lemma 20.

$$\chi \circ \mathcal{P}_k = p_k \circ \chi, \quad \chi \circ \mathcal{P}^*_k = p^*_k \circ \chi.$$

Proof The well-know identity [M] in $\mathbf{H}^+$

$$p_k = \sum_{i=0}^{k-1} (-1)^i s_{k-i,1_i}$$

implies

$$\chi \circ \mathcal{P}_k = \sum_{i=0}^{k-1} (-1)^i [\mathcal{P}_k(i)] = \sum_{i=0}^{k-1} (-1)^i [\mathcal{T}_{V_{k-i,1_i}}] = p_k \circ \chi.$$

The second equality is similar. □

Now we are going to use $\mathcal{P}_n$ and $\mathcal{P}^*_n$ in order to show that p_n and p^*_n satisfy the relations (7). The equality $[p_n, p_m] = 0$ follows directly from the commutativity of the tensor product, and the equality $[p^*_n, p^*_m] = 0$ follows from Lemma 7 and the commutativity of the tensor product. So it remains to prove that

$$[p^*_m, p_k] = k\delta_{m,k}.$$

At this time we do not have a truly categorical proof of the last identity. Instead we will pass to the Grothendieck ring of $\mathbb{T}^+$. Note that by Proposition 1 it is sufficient to prove the following identity in $\mathcal{K}(\mathbb{T}^+)$:

$$\sum_{i=0}^{k-1} \sum_{j=0}^{m-1} (-1)^{i+j} [(V_{(k-i,1_i)} \otimes (V_{(m-j,1_j)})_* / V_{(k-i,1_i),(m-j,1_j)}] = k\delta_{m,k}[\mathbb{C}].$$

We are going to use (16). If λ is a hook, then $N^\lambda_{\gamma,\nu} \leq 1$. Moreover, $N^\lambda_{\gamma,\nu} = 1$ implies that γ and ν are also hooks. If $\lambda = (c, 1_d)$, $\nu = (p, 1_q)$ and $\gamma = (s, 1_t)$, then $N^\lambda_{\gamma,\nu} = 1$ if and only if $s + p = c, t + q = d$ or $s + p = c - 1, t + q = d + 1$. If $\gamma = (p, 1_q)$ we set $s(\gamma) = (-1)^q$. The above implies that for any ν such that $|\nu| < m$ and for any hook γ

$$\sum_{j=0}^{m-1} (-1)^j N^{m-j,1_j}_{\nu,\gamma} = (-1)^{s(\gamma)} \delta_{\nu,\emptyset} \delta_{|\gamma|,m}. \tag{46}$$

Now by (16) we have

$$\sum_{i=0}^{k-1} \sum_{j=0}^{m-1} (-1)^{i+j} [(V_{(k-i,1_i)} \otimes (V_{(m-j,1_j)})_*)/V_{(k-i,1_i)),(m-j,1_j)} : V_{\mu,\nu}] = \sum_{i=0}^{k-1} \sum_{j=0}^{m-1} \sum_{\gamma} (-1)^{i+j} N^{(k-i,1_i)}_{\mu,\gamma} N^{(m-j,1_j)}_{\nu,\gamma}.$$

By (46) we obtain

$$\sum_{i=0}^{k-1} \sum_{j=0}^{m-1} \sum_{\gamma} (-1)^{i+j} N^{(k-i,1_i)}_{\mu,\gamma} N^{(m-j,1_j)}_{\nu,\gamma} = \sum_{\gamma} \left(\sum_{i=0}^{k-1} (-1)^i N^{m-i,1_i}_{\mu,\gamma} \right) \left(\sum_{j=0}^{m-1} (-1)^j N^{m-j,1_j}_{\nu,\gamma} \right) = \sum_{\gamma} \delta_{\mu,\emptyset} \delta_{|\gamma|,k} \delta_{\nu,\emptyset} \delta_{|\gamma|,m} = k\delta_{k,m} \delta_{\nu,\emptyset} \delta_{\mu,\emptyset}.$$

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