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GENERALIZED HARISH-CHANDRA MODULES

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To Yuri Ivanovich Manin with admiration

Abstract. Let $\mathfrak{g}$ be a complex reductive Lie algebra and $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. If $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}$, we call a $\mathfrak{g}$-module M a strict $(\mathfrak{g}, \mathfrak{k})$-module if $\mathfrak{k}$ coincides with the subalgebra of all elements of $\mathfrak{g}$ which act locally finitely on M. For an intermediate $\mathfrak{k}$, i. e., such that $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$, we construct irreducible strict $(\mathfrak{g}, \mathfrak{k})$-modules. The method of construction is based on the $\mathcal{D}$-module localization theorem of Beilinson and Bernstein. The existence of irreducible strict $(\mathfrak{g}, \mathfrak{k})$-modules has been known previously only for very special subalgebras $\mathfrak{k}$, for instance when $\mathfrak{k}$ is the (reductive) subalgebra of fixed points of an involution of $\mathfrak{g}$. In this latter case strict irreducible $(\mathfrak{g}, \mathfrak{k})$-modules are Harish-Chandra modules.

We also give separate necessary and sufficient conditions on $\mathfrak{k}$ for the existence of an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module of finite type, i. e., an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module with finite $\mathfrak{k}$-multiplicities. In particular, under the assumptions that the intermediate subalgebra $\mathfrak{k}$ is reductive and $\mathfrak{g}$ has no simple components of types B_n for $n > 2$ or F_4, we prove a simple explicit criterion on $\mathfrak{k}$ for the existence of an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module of finite type. It implies that, if $\mathfrak{g}$ is simple of type A or C, for every reductive intermediate $\mathfrak{k}$ there is an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module of finite type.

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Notational conventions

The ground field is $\mathbb{C}$, however all our results can be easily carried out over an algebraically closed field of characteristic zero. $\mathbb{R}_+$ (respectively, $\mathbb{Z}_+$) denotes the set of nonnegative real numbers (respectively, integers), and $\langle\cdot\rangle_{\mathbb{C}}$, $\langle\cdot\rangle_{\mathbb{R}_+}$, $\langle\cdot\rangle_{\mathbb{Z}}$, or $\langle\cdot\rangle_{\mathbb{Z}_+}$ stands for linear span, respectively with coefficients in $\mathbb{C}$, $\mathbb{R}_+$, $\mathbb{Z}$, or $\mathbb{Z}_+$. If X

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is a topological space and $\mathcal{F}$ is a sheaf of abelian groups on X, then $\Gamma(\mathcal{F})$ denotes the global sections of $\mathcal{F}$ on X. If $U \subset X$ is an open subset, $\mathcal{F}|_U$ denotes the restriction of $\mathcal{F}$ onto U, and $\mathcal{F}(U) := \Gamma(\mathcal{F}|_U)$. For $v \in \Gamma(\mathcal{F})$, $v|_U$ is the restriction of v to U. If X is an algebraic variety, $\mathcal{O}_X$ stands for the structure sheaf of X, and if $f\colon X \to Y$ is a morphism of algebraic varieties, f^* denotes the inverse image functor of $\mathcal{O}$-modules. A *multiset* is defined as a map from a set Y into $\mathbb{Z}_+\backslash\{0\}$, or, more informally, as a set whose elements have finite multiplicities.

1. Origin of the problem

Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and M be a $\mathfrak{g}$-module. By definition an element $g \in \mathfrak{g}$ acts locally finitely on M if the subspace $\langle m,\, g\cdot m,\, g^2\cdot m,\, \ldots\rangle_{\mathbb{C}} \subset M$ is finite-dimensional for any $m \in M$. It is a result of S. Fernando [F] and of V. Kac [K] (established independently and by different methods), that all elements of $\mathfrak{g}$ which act locally finitely on M form a Lie subalgebra $\mathfrak{g}[M]$ of $\mathfrak{g}$. Given a subalgebra $\mathfrak{k} \subset \mathfrak{g}$, we call M a $(\mathfrak{g}, \mathfrak{k})$*-module*, or a *generalized Harish-Chandra module for the pair* $(\mathfrak{g}, \mathfrak{k})$, if $\mathfrak{k} \subset \mathfrak{g}[M]$. *A strict* $(\mathfrak{g}, \mathfrak{k})$*-module*, or a *strict generalized Harish-Chandra module for the pair* $(\mathfrak{g}, \mathfrak{k})$, is by definition a $(\mathfrak{g}, \mathfrak{k})$-module M for which $\mathfrak{g}[M] = \mathfrak{k}$. Furthermore, we call $\mathfrak{k}$ a *Fernando subalgebra* of $\mathfrak{g}$ if there exists an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module M. (Clearly, for every $\mathfrak{k}$ there exist induced $\mathfrak{g}$-modules which are strict $(\mathfrak{g}, \mathfrak{k})$-modules but are not necessarily irreducible.)

As we show at the end of this section, not all subalgebras of $\mathfrak{sl}(n)$ are Fernando subalgebras. This makes it natural to pose the problem of describing all Fernando subalgebras of a given finite-dimensional Lie algebra $\mathfrak{g}$. Not much is known for nonreductive Lie algebras $\mathfrak{g}$ and the problem is still open in the reductive case. One of our objectives is to desribe all Fernando subalgebras containing a Cartan subalgebra of a reductive Lie algebra $\mathfrak{g}$. In what follows we will automatically assume that $\mathfrak{g}$ is reductive and that $\mathfrak{h}$ is a fixed Cartan subalgebra of $\mathfrak{g}$. In Theorem 1 below we prove that any intermediate subalgebra $\mathfrak{k}$, i. e., such that $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$, is a Fernando subalgebra. The corresponding irreducible strict $(\mathfrak{g}, \mathfrak{k})$-modules M are necessarily weight modules and, as it turns out, $\mathfrak{k}$ determines certain essential invariants of M. Here is a brief description of relevant known results about weight modules.

Recall that a $\mathfrak{g}$-module M is a *weight module* if

$$M = \bigoplus_{\nu \in \mathfrak{h}^*} M^\nu, \tag{1}$$

where $\mathfrak{h}^*$ stands for the dual space of $\mathfrak{h}$ and $M^\nu := \{m \in M\colon h\cdot m = \nu(h)m \;\forall h \in \mathfrak{h}\}$. It is well known that, if M is irreducible, (1) is equivalent to $\mathfrak{h}$ being contained in $\mathfrak{g}[M]$, see for instance Proposition 1 in [PS]. The spaces M^ν are the *weight spaces* of M, and the set of all weights of M, i. e., linear functions $\nu \in \mathfrak{h}^*$ with $M^\nu \neq 0$, is by definition the *support* $\operatorname{supp} M$ of M. The *weight multiplicities* are the dimensions $\dim M^\nu$ that can be finite or infinite. The *formal character* of M is defined as the formal sum

$$\sum_{\nu \in \operatorname{supp} M} (\dim M^\nu) e^\nu.$$

This invariant of M has been studied now for about 80 years: the classical Weyl character formula established in 1924 computes the formal character of any irreducible finite-dimensional $\mathfrak{g}$-module M as a function of its highest (or extremal) weight(s), [W]. In general, the support of an irreducible weight module M carries less information than its formal character, however, at the two extremes, when $\dim M < \infty$ or when $\dim M^\nu = \infty$ for all $\nu \in \operatorname{supp} M$, the support determines the formal character. Furthermore, it is known that if M is irreducible, all weight multiplicities are simultaneously finite or infinite. Not long ago O. Mathieu [M] completed the classification of irreducible weight modules with finite weight multiplicities (see also [BL]), and in particular Mathieu's results lead to a formula for the formal character of any such module. The case of infinite multiplicities is also important and interesting, since most Harish-Chandra weight modules have infinite weight multiplicities. This is the case for which Theorem 1 below is of interest.

Let now $\mathfrak{h} \subset \mathfrak{k}$ and M be an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module. To explain the relationship between $\mathfrak{k}$ and $\operatorname{supp} M$ we need to introduce one more invariant of M, its shadow decomposition. Let $\Delta \subset \mathfrak{h}^*$ denote the root system of $\mathfrak{g}$ and $\Delta_{\mathfrak{k}} \subset \Delta$ be the set of roots of $\mathfrak{k}$. Define $\Gamma_{\mathfrak{k}}$ as the submonoid of $\langle\Delta\rangle_{\mathbb{Z}}$ generated by $\Delta \backslash \Delta_{\mathfrak{k}}$. The *$M$-decomposition* of Δ, or the *shadow decomposition of Δ corresponding to M*,

$$\Delta = \Delta_M^- \sqcup \Delta_M^I \sqcup \Delta_M^F \sqcup \Delta_M^+, \tag{2}$$

is defined by setting

$$\begin{aligned}
\Delta_M^I &:= \{\alpha \in \Delta\colon \alpha \in \langle\Gamma_{\mathfrak{k}}\rangle_{\mathbb{R}_+},\ -\alpha \in \langle\Gamma_{\mathfrak{k}}\rangle_{\mathbb{R}_+}\},\\
\Delta_M^F &:= \{\alpha \in \Delta\colon \alpha \notin \langle\Gamma_{\mathfrak{k}}\rangle_{\mathbb{R}_+},\ -\alpha \notin \langle\Gamma_{\mathfrak{k}}\rangle_{\mathbb{R}_+}\},\\
\Delta_M^+ &:= \{\alpha \in \Delta\colon \alpha \notin \langle\Gamma_{\mathfrak{k}}\rangle_{\mathbb{R}_+},\ -\alpha \in \langle\Gamma_{\mathfrak{k}}\rangle_{\mathbb{R}_+}\},\\
\Delta_M^- &:= \{\alpha \in \Delta\colon \alpha \in \langle\Gamma_{\mathfrak{k}}\rangle_{\mathbb{R}_+},\ -\alpha \notin \langle\Gamma_{\mathfrak{k}}\rangle_{\mathbb{R}_+}\}.
\end{aligned}$$

In particular, the M-decomposition of Δ is determined by $\mathfrak{k}$. The decomposition (2) induces a decomposition of $\mathfrak{g}$,

$$\mathfrak{g} = \left(\mathfrak{g}_M^I + \mathfrak{g}_M^F\right) \oplus \mathfrak{g}_M^+ \oplus \mathfrak{g}_M^-, \tag{3}$$

where

$$\mathfrak{g}_M^F := \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_M^F} \mathfrak{g}^\alpha\right), \quad \mathfrak{g}_M^I := \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Delta_M^I} \mathfrak{g}^\alpha\right), \quad \mathfrak{g}_M^\pm := \bigoplus_{\alpha \in \Delta_M^\pm} \mathfrak{g}^\alpha.$$

It follows from the main results of [DMP] that $\mathfrak{p}_M := (\mathfrak{g}_M^F + \mathfrak{g}_M^I) \oplus \mathfrak{g}_M^+$ is a parabolic subalgebra whose semisimple part is nothing but the direct sum of Lie algebras $[\mathfrak{g}_M^F, \mathfrak{g}_M^F] \oplus [\mathfrak{g}_M^I, \mathfrak{g}_M^I]$. In particular, $[\mathfrak{g}_M^F, \mathfrak{g}_M^F]$ and $[\mathfrak{g}_M^I, \mathfrak{g}_M^I]$ are two commuting semisimple subalgebras of $\mathfrak{p}_M$. Furthermore, if $\mathfrak{k} = \mathfrak{g}[M]$, then

$$\mathfrak{k} = (\mathfrak{g}_M^F + (\mathfrak{k} \cap \mathfrak{g}_M^I)) \oplus \mathfrak{g}_M^+.$$

This is a consequence of the inclusion $\mathfrak{g}_M^F \oplus \mathfrak{g}_M^+ \subset \mathfrak{k}$ and of the fact that $\mathfrak{g}_M^- \cap \mathfrak{k} = 0$. The latter follows easily from some basic representation theory of $\mathfrak{sl}(2)$.

Note now that the very definition of $\mathfrak{g}_{\mathfrak{k}}$ implies

$$\operatorname{supp} M = \operatorname{supp} M^{\mathfrak{k}} + \Gamma_{\mathfrak{k}}, \tag{4}$$

where $M^{\mathfrak{k}}$ is any irreducible finite-dimensional $\mathfrak{k}$-submodule of M; see also Proposition 2 in [PS]. Moreover, as $\Delta_M^I \subset \Gamma_{\mathfrak{k}}$ and $(\Delta_M^F + \Delta_M^I) \cap \Delta = \varnothing$, if M^F is any irreducible $\mathfrak{g}_M^F \oplus \mathfrak{g}_M^+$-submodule of $M^{\mathfrak{k}}$, we also have

$$\operatorname{supp} M = \operatorname{supp} M^F + \Gamma_{\mathfrak{k}}. \tag{5}$$

M^F always exists, is necessarily finite-dimensional, and is irreducible as a $\mathfrak{g}_M^F$-module. Since $\mathfrak{g}_M^F$ is reductive, $\operatorname{supp} M^F$ is nothing but the intersection of the convex hull of all extremal weights of M^F with the weight lattice of $\mathfrak{g}_M^F$ shifted by any element of $\operatorname{supp} M^F$. Furthermore, as $\mathfrak{g}_M^+$ acts by zero on M^F, the right-hand side of (5) is just $\operatorname{supp} M^F + \langle \Delta_M^I \rangle_{\mathbb{Z}} + \langle \Delta_M^- \rangle_{\mathbb{Z}_+}$. Therefore,

$$\operatorname{supp} M = \operatorname{supp} M^F + \langle \Delta_M^I \rangle_{\mathbb{Z}_+} + \langle \Delta_M^- \rangle_{\mathbb{Z}_+}, \tag{6}$$

and $\operatorname{supp} M^F$ is not determined by $\mathfrak{k}$ (and could be arbitrary for a fixed $\mathfrak{k}$) while $\langle \Delta_M^I \rangle_{\mathbb{Z}_+} + \langle \Delta_M^- \rangle_{\mathbb{Z}_+}$ is determined by $\mathfrak{k}$. In other words, the support of an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module is determined by $\mathfrak{k}$ up to adding the support of an arbitrary irreducible finite-dimensional $\mathfrak{g}_M^F$-module. (A more general version of this statement for a not necessarily reductive Lie algebra see in [PS].)

The starting point of this paper is the observation that, when M has infinite weight multiplicities, its formal character (or equivalently its support) does not fully determine the Lie algebra $\mathfrak{g}[M]$, nor, more precisely, its subalgebra $\mathfrak{g}[M] \cap \mathfrak{g}_M^I$. In particular, the existing theory of weight modules provides no answer to the following question: Let $\mathfrak{k} \subset \mathfrak{g}$ be a subalgebra with $\mathfrak{k} \supset \mathfrak{h}$. Is it true that there exists an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module? Theorem 1 below gives an affirmative answer.

Once the existence problem for irreducible strict $(\mathfrak{g}, \mathfrak{k})$-modules is resolved, a further natural question arises. For which $\mathfrak{k}$ does there exist an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module M of finite type, i. e., such that, considered as a module over the reductive part $\mathfrak{k}_{\mathrm{red}}$ of $\mathfrak{k}$, M has finite-dimensional isotypic components? Our second objective is to give a partial answer to this latter question. Propositions 4 and 5 give, respectively, a necessary and a sufficient condition on $\mathfrak{k}$ for the existence of an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module of finite type. For a reasonably large class of subalgebras $\mathfrak{k}$ these conditions are directly verifiable and provide a definitive result, see Corollaries 1 and 2. In particular, if $\mathfrak{k}$ is reductive and $\mathfrak{g}$ has no simple components of type B_n for $n > 2$ or F_4, then $\mathfrak{g}$ admits an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module of finite type if and only if the centralizer in $\mathfrak{g}$ of the semisimple part of $\mathfrak{k}$ has simple components only of types A and C.

The $\mathfrak{k}$-finiteness problem has been studied previously in two particular cases: when $\mathfrak{k}$ coincides with the fixed points of an involution on $\mathfrak{g}$, and when $\mathfrak{k}$ is replaced by $\mathfrak{h}$. It is a classical result that in the first case any irreducible $(\mathfrak{g}, \mathfrak{k})$-module has finite type. In the second case finite type means nothing but finite weight multiplicities, and the following Proposition summarizes known results.

Proposition 1. *Let $\mathfrak{k} \subset \mathfrak{g}$ be a subalgebra with $\mathfrak{k} \supset \mathfrak{h}$, and let M be a strict irreducible $(\mathfrak{g}, \mathfrak{k})$-module.*

(a) *If $\mathfrak{k} \cap \mathfrak{g}_M^I \neq \mathfrak{h}$, then M necessarily has infinite weight multiplicities.*

(b) *If* $\mathfrak{k} \cap \mathfrak{g}_M^I = \mathfrak{h}$ *and* M *has finite weight multiplicities, then* $[\mathfrak{g}_M^I, \mathfrak{g}_M^I]$ *is isomorphic to a direct sum of simple Lie algebras of types* A *and* C.

(c) *If* $\mathfrak{g}_M^I = \mathfrak{h}$*, then* M *necessarily has finite weight multiplicities.*

Proof. Fernando has proved in [F] that $\mathfrak{k} \cap \mathfrak{g}_M^I = \mathfrak{h}$ whenever M has finite weight multiplicities. This implies (a). To prove (b), consider a nonzero vector $m \in M$ with $\mathfrak{g}_M^+ \cdot m = 0$. Recall that $\mathfrak{p}_M = (\mathfrak{g}_M^F + \mathfrak{g}_M^I) \oplus \mathfrak{g}_M^+$ is a parabolic subalgebra of $\mathfrak{g}$ and set $M^{FI} := U(\mathfrak{p}_M) \cdot m$. It is straightforward to verify that M^{FI} is irreducible as a $[\mathfrak{g}_M^F, \mathfrak{g}_M^F] \oplus [\mathfrak{g}_M^I, \mathfrak{g}_M^I]$-module. Therefore $M^{FI} \simeq M^F \boxtimes M^I$ for some irreducible finite-dimensional $[\mathfrak{g}_M^F, \mathfrak{g}_M^F]$-module M^F and some irreducible $[\mathfrak{g}_M^I, \mathfrak{g}_M^I]$-module M^I with $[\mathfrak{g}_M^I, \mathfrak{g}_M^I][M^I] = \mathfrak{h} \cap [\mathfrak{g}_M^I, \mathfrak{g}_M^I]$. Furthermore, one notes that, as M has finite weight multiplicities, M^I has also finite $\mathfrak{h} \cap [\mathfrak{g}_M^I, \mathfrak{g}_M^I]$-weight multiplicities. Then the main result of Fernando [F] implies that $[\mathfrak{g}_M^I, \mathfrak{g}_M^I]$ is isomorphic to a direct sum of simple Lie algebras of types A and C only. (b) is proved. Claim (c) follows from the fact that M^{FI} is a finite-dimensional $\mathfrak{p}_M$-module when $\mathfrak{g}_M^I = \mathfrak{h}$. Indeed, then the surjection $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_M)} M^{FI} \to M$ gives that M is a highest weight $\mathfrak{g}$-module with respect to any Borel subalgebra $\mathfrak{b} \subset \mathfrak{g}$ with $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{p}_M$. Hence M has finite weight multiplicities. □

We will need the following slightly stronger version of claim (b). If M is a strict $(\mathfrak{g}, \mathfrak{k})$-module which is not necessarily irreducible, we call M *isotropic* if, for every nonzero $m \in M$, $\mathfrak{k}$ coincides with the set of elements $g \in \mathfrak{g}$ which act locally finitely on m. Any irreducible $\mathfrak{g}$-module M is automatically isotropic as a strict $(\mathfrak{g}, \mathfrak{g}[M])$-module.

Lemma 1. *Let* $\mathfrak{k}$ *be solvable,* $\mathfrak{k} \supset \mathfrak{h}$*, and let* M *be an isotropic strict* $(\mathfrak{g}, \mathfrak{k})$*-module with finite weight multiplicities. Then there exists a parabolic subalgebra* $\mathfrak{p} \subset \mathfrak{g}$ *with* $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$*,* $\mathfrak{p} \cap \mathfrak{k} = \mathfrak{h}$*, and such that its semisimple part* $\mathfrak{p}_{\mathrm{ss}}$ *is a direct sum of simple Lie algebras of types* A *and* C.

Proof. Without loss of generality we can assume that M is generated by some 1-dimensional $\mathfrak{k}$-submodule $E^\lambda \subset M^\lambda$. Then (similarly to (4)) $\operatorname{supp} M = \lambda + \Gamma_{\mathfrak{k}}$. We claim that $\Delta_{\mathfrak{k}} \cap \Gamma_{\mathfrak{k}} = \varnothing$. Indeed, if $\alpha \in \Delta_{\mathfrak{k}} \cap \Gamma_{\mathfrak{k}}$, the solvability of $\mathfrak{k}$ implies that $\mathfrak{g}^\alpha$ acts locally finitely and $\mathfrak{g}^{-\alpha}$ acts freely on M. Furthermore, $\{\lambda + \mathbb{Z}\alpha\} \subset \operatorname{supp} M$, and therefore there are infinitely many nonzero vectors $m^\nu \in M^\nu$ for $\nu \in \{\lambda + \mathbb{Z}\alpha\}$ with $\mathfrak{g}^\alpha \cdot m^\nu = 0$. Hence M has infinitely many Verma submodules of the rank one subalgebra $\mathfrak{g}^{-\alpha} \oplus \mathfrak{h} \oplus \mathfrak{g}^\alpha$, and consequently M has infinite weight multiplicities. Contradiction. Therefore $\mathfrak{p} := \mathfrak{h} \oplus (\bigoplus_{\alpha \in \Delta \cap \Gamma_{\mathfrak{k}}} \mathfrak{g}^\alpha)$ is a parabolic subalgebra with $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and $\mathfrak{p} \cap \mathfrak{k} = \mathfrak{h}$.

It remains to show that the simple components of $\mathfrak{p}_{\mathrm{ss}}$ are of types A and C only. Consider $E := U(\mathfrak{p}_{\mathrm{ss}}) \cdot E^\lambda$. Since for every $\alpha \in \Delta_{\mathfrak{p}_{\mathrm{ss}}}$ both $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$ act freely on M, $\dim E^\mu = \dim E^{\mu \pm \alpha}$ for any $\mu \in \operatorname{supp} M$, i.e., all weight multiplicities of E are equal. Then E is admissible as defined in [M], and thus has finite length [M, Lemma 3.3]. As E is obviously a strict $(\mathfrak{p}_{\mathrm{ss}}, \mathfrak{h} \cap \mathfrak{p}_{\mathrm{ss}})$-module, any irreducible submodule of E is also a strict $(\mathfrak{p}_{\mathrm{ss}}, \mathfrak{h} \cap \mathfrak{p}_{\mathrm{ss}})$-module, and by Proposition 1(b) $\mathfrak{p}_{\mathrm{ss}}$ is isomorphic to a direct sum of simple Lie algebras of types A and C. □

We conclude this introductory section with an example of a subalgebra $\mathfrak{k}$ which is not a Fernando subalgebra. Let $\mathfrak{g} = \mathfrak{sl}(n)$ for $n > 2$, let $\mathfrak{b}$ denote the subalgebra of upper triangular matrices, $\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$, and let $\mathfrak{k} \subset \mathfrak{b}$ be the subalgebra of all upper triangular matrices with zero first column. Note that $\mathfrak{k}$ contains no Cartan subalgebra. We claim that there is no irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module. Indeed, assume on the contrary that M is such a $\mathfrak{g}$-module. Then, since $\mathfrak{n} = [\mathfrak{k}, \mathfrak{k}]$, one can find a nonzero $m \in M$ such that $\mathfrak{k} \cdot m \subset \langle m \rangle_{\mathbb{C}}$ and $\mathfrak{n} \cdot m = 0$. Choose a basis $h_1, \ldots, h_{n-2}$ in $\mathfrak{h} \cap \mathfrak{k}$. The Casimir operator Ω can be written as

$$\Omega = u^2 + up(h_1, \ldots, h_{n-2}) + q(h_1, \ldots, h_{n-2}) + \sum_i x_i y_i$$

for some nonzero u in the 1-dimensional orthogonal complement of $\mathfrak{h} \cap \mathfrak{k}$ (with respect to the Killing form) in $\mathfrak{h}$, for certain polynomials p and q, $\deg p = 1$, $\deg q = 2$, and for certain strictly upper (respectively, strictly lower) triangular matrices y_i (respectively, x_i). As $\Omega \cdot m = \mu m$ for some $\mu \in \mathbb{C}$, and $h_i \cdot m = \lambda_i m$ for some $\lambda_i \in \mathbb{C}$, we obtain that $u^2 \cdot m \in \langle m \rangle_{\mathbb{C}} + \langle u \cdot m \rangle_{\mathbb{C}}$. Therefore u acts locally finitely on M, which is a contradiction since $u \notin \mathfrak{k}$.

2. Existence theorem

Theorem 1. *Any subalgebra $\mathfrak{k} \subset \mathfrak{g}$ such that $\mathfrak{h} \subset \mathfrak{k}$ is a Fernando subalgebra.*

The proof splits naturally into two cases: that of a solvable $\mathfrak{k}$ and that of a general $\mathfrak{k}$. We start with some preliminaries, most of which amount to fixing notation.

2.1. Preliminaries needed in the proof. Let G denote a connected reductive algebraic group with Lie algebra $\mathfrak{g}$. All considered subgroups of G are automatically assumed to be connected and are denoted by capital letters B, P, etc. The lower case German letters $\mathfrak{b}$, $\mathfrak{p}$, etc. stand for the corresponding Lie subalgebras of $\mathfrak{g}$, and determine the subgroups B, P, etc.

B will always denote a fixed Borel subgroup of G whose Borel subalgebra $\mathfrak{b}$ contains the fixed Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Sometimes $\mathfrak{b}$ will satisfy additional requirements which will be stated explicitly. The choice of $\mathfrak{b}$ fixes a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ where $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. By Δ^+ and Δ^- we denote respectively the roots of $\mathfrak{b}$ and $\mathfrak{b}^- := \mathfrak{h} \oplus \mathfrak{n}^-$. N^- is the subgroup of G with Lie algebra $\mathfrak{n}^-$. The big cell U of the flag variety G/B is defined as the (open) orbit $N^- \cdot B$ of the point $B \in G/B$. N^- acts freely on U. This enables us to identify N^- with $\mathfrak{n}^-$, and therefore obtain a set of coordinates $\{x_\alpha\}_{\alpha \in \Delta^-}$ on U which correspond to coordinates on $\mathfrak{n}^-$ arising from a basis of root vectors in $\mathfrak{n}^-$. The algebra $\mathcal{O}_{G/B}(U)$ is then identified with the polynomial algebra $\mathbb{C}[x_\alpha]_{\alpha \in \Delta^-}$. Note also that $\mathfrak{g}$ acts by derivations on $\mathcal{O}_{G/B}$ via its canonical homomorphism into the tangent bundle $\mathcal{T}_{G/B}$.

If $i \colon Q \hookrightarrow G/B$ is the embedding of a (possibly singular) subvariety of G/B, $\bar{Q}$ will denote the closure of Q in G/B, and $\operatorname{Stab}_{\mathfrak{g}} Q \subset \mathfrak{g}$ is defined as the Lie algebra of the subgroup of G which preserves Q.

Throughout the rest of the paper μ will denote a fixed regular $\mathfrak{b}$-dominant weight, sometimes satisfying explicit additional conditions. Following Beilinson and Bernstein, we denote by $\mathcal{D}^\mu$ the sheaf of twisted differential operators (or "tdo" for short) on G/B corresponding to μ, see [BB1]. For a nonsingular locally closed subvariety Q of G/B we define the tdo $\mathcal{D}^\mu_Q$ as the sheaf of left differential endomorphisms of the inverse image $i^*\mathcal{D}^\mu$. (If Q is an open subvariety, $\mathcal{D}^\mu_Q$ is simply the restriction $\mathcal{D}^\mu|_Q$.) If $\mu = \rho$ (where $\rho := 1/2\sum_{\alpha\in\Delta^+}\alpha$), then $\mathcal{D} := \mathcal{D}^\rho$ is nothing but the sheaf of differential operators on G/B. Furthermore, $\mathcal{D}^\mu(U)$ can be identified with the Weyl algebra $\mathbb{C}\big[x_\alpha, \frac{\partial}{\partial x_\alpha}\big]_{\alpha\in\Delta^-} = \mathcal{D}(U)$.

A $\mathcal{D}^\mu_Q$-module $\mathcal{F}$ is by definition a sheaf of $\mathcal{D}^\mu_Q$-modules on Q which is quasi-coherent as a sheaf of $\mathcal{O}_Q$-modules. The support of $\mathcal{F}$ is the closure of the subvariety of all closed points for which the sheaf-theoretic fiber of $\mathcal{F}$ is nonzero. We will make extensive use of the $\mathcal{D}^\mu$-affinity theorem of A. Beilinson and J. Bernstein [BB1]. This theorem states that G/B is $\mathcal{D}^\mu$-affine, i. e., that every $\mathcal{D}^\mu$-module $\mathcal{F}$ is generated over $\mathcal{D}^\mu$ by its global sections $\mathfrak{g}(\mathcal{F})$ and that all higher cohomology groups of the sheaf $\mathcal{F}$ vanish. Consequently $\mathfrak{g}$ is an equivalence between the category of $\mathcal{D}^\mu$-modules and the category of modules over the associative algebra $\mathfrak{g}(\mathcal{D}^\mu)$. Moreover, a further result of Beilinson and Bernstein [BB1] claims that $\mathfrak{g}(\mathcal{D}^\mu)$ is canonically isomorphic to the quotient $U(\mathfrak{g})/(\operatorname{Ker}\theta^\mu)$, where $U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ and $(\operatorname{Ker}\theta^\mu)$ is the two-sided ideal generated by the kernel of the central character θ^μ of a module with $\mathfrak{b}$-highest weight $\mu - \rho$.

The following lemma relates the Fernando subalgebra of the global sections of a $\mathcal{D}^\mu$-module with its support.

Lemma 2. *If Q is the support of a $\mathcal{D}^\mu$-module $\mathcal{F}$, then $\mathfrak{g}[\Gamma(\mathcal{F})] \subset \operatorname{Stab}_\mathfrak{g} Q$.*

Proof. Set $R := (G/B)\backslash Q$. For any $x \in \mathfrak{g}[\Gamma(\mathcal{F})]$, $g_x := \exp x$ is an automorphism of G/B as well as of $\Gamma(\mathcal{F})$. Furthermore, for any $v \in \Gamma(\mathcal{F})$,

$$v|_{g_x(R)} = g_x^{-1}(v)|_R = 0. \tag{7}$$

Since $\mathcal{F}$ is generated by $\Gamma(\mathcal{F})$ over $\mathcal{D}^\mu$, (7) implies $\mathcal{F}(g_x(R)) = 0$. Then $g_x(R) = R$ and therefore $x \in \operatorname{Stab}_\mathfrak{g} Q$. □

Assume that $i\colon Q \hookrightarrow G/B$ is a locally closed embedding of a nonsingular subvariety Q. We denote by i_* the $\mathcal{D}^\mu_Q$-module direct image functor. By definition, $i_*\mathcal{F} := \mathcal{D}^\mu_{\leftarrow} \otimes_{\mathcal{D}^\mu_Q} \mathcal{F}$, where $\mathcal{D}^\mu_{\leftarrow} := i^*(\mathcal{D}^\mu \otimes_{\mathcal{O}_{G/B}} \Omega^*_{G/B}) \otimes_{\mathcal{O}_Q} \Omega_Q$ and Ω stands for volume forms. Furthermore the $\mathcal{D}^\mu$-module $i_*\mathcal{F}$ has a natural $\mathcal{O}_{G/B}$-module filtration with successive quotients

$$\Lambda^{\max}(\mathcal{N}_Q) \otimes_{\mathcal{O}_Q} S^i(\mathcal{N}_Q) \otimes_{\mathcal{O}_Q} \mathcal{F}$$

for $i \in \mathbb{Z}_+$, where $\mathcal{N}_Q$ denotes the normal bundle of Q in G/B, S^i stands for i^{th} symmetric power, and $\Lambda^{\max}$ stands for maximal exterior power. In the case when i is a closed embedding, Kashiwara's theorem [Ka] claims that i_* is an equivalence between the category of $\mathcal{D}^\mu_Q$-modules and the category of $\mathcal{D}^\mu$-modules with support in Q.

A locally closed embedding i is the composition of a closed embedding $j\colon Q \to V$ and an open embedding $l\colon V \to X$. Then $i_* = l_* j_*$, where l_* coincides with the sheaf-theoretic direct image. Under the additional assumptions that $\mathcal{F}$ is irreducible and locally free (of finite rank) as $\mathcal{O}_Q$-module, $i_*\mathcal{F}$ contains a unique irreducible

$\mathcal{D}^\mu$-submodule $i_{*!}\mathcal{F}$, see [B], [BB1]. Furthermore, $i_{*!}\mathcal{F}(V) = i_*\mathcal{F}(V)$. In particular the support of $i_{*!}\mathcal{F}$ is $\bar{Q}$. If $i = j$, i. e., if i is closed, then $i_{*!}\mathcal{F}$ simply equals $i_*\mathcal{F}$.

We are now ready to start the proof of Theorem 1. Throughout the proof $\mathfrak{k}$ is a fixed subalgebra of $\mathfrak{g}$ with $\mathfrak{h} \subset \mathfrak{k}$. Our goal will be to construct an irreducible holonomic $\mathcal{D}^\mu$-module such that its global sections form a strict $(\mathfrak{g}, \mathfrak{k})$-module. We will do this in a very explicit way, without referring to the structure theory of holonomic modules, and only using the minimal preliminaries stated above.

2.2. The case of a solvable $\mathfrak{k}$. Assume first that $\mathfrak{k}$ is solvable and let $\mathfrak{b}^- \supset \mathfrak{k}$. Set $\tilde{\Delta}_{\mathfrak{k}} := \Delta^-\backslash\Delta_{\mathfrak{k}}$.

Lemma 3. *There are global coordinates $\{u_\alpha\}_{\alpha\in\Delta_-}$ on U such that $h(u_\alpha) = \alpha(h)\, u_\alpha$ for any $h \in \mathfrak{h}$ and any $\alpha \in \Delta_-$, and $k(u_\alpha) = 0$ for all $k \in \mathfrak{k}\cap\mathfrak{n}^-$ and all $\alpha \in \tilde{\Delta}_{\mathfrak{k}}$.*

Proof. Consider the K-orbit $K\cdot B \subset U$. The polynomial algebra $\mathcal{O}(U) = \mathbb{C}[x_\alpha]_{\alpha\in\Delta^-}$, considered as a $\mathfrak{k}$-module, is a weight module (with respect to $\mathfrak{h}$). The ideal $I_{K\cdot B} := \Gamma(\mathcal{I}_{K\cdot B})$ is a $\mathfrak{k}$-submodule of $\mathcal{O}(U)$, and splits as a direct summand as an $\mathfrak{h}$-submodule. The quotient $\mathcal{O}(U)/I_{K\cdot B}$ is nothing but $\Gamma(\mathcal{O}_{K\cdot B}) = \mathbb{C}[x_\alpha]_{\alpha\in\Delta_{\mathfrak{k}}}$, and therefore we have an $\mathfrak{h}$-module decomposition

$$\mathcal{O}(U) = \mathbb{C}[x_\alpha]_{\alpha\in\Delta_{\mathfrak{k}}} \oplus I_{K\cdot B}.$$

Set then $u_\alpha := x_\alpha$ for $\alpha \in \Delta_{\mathfrak{k}}$, and define u_α for $\alpha \in \tilde{\Delta}_{\mathfrak{k}}$ to be the projection of x_α onto $I_{K\cdot B}$. It is straightforward to check that u_α are as required. □

Lemma 4. *Set $Z := \bigcup_{\alpha\in\tilde{\Delta}_{\mathfrak{k}}} Z_\alpha$, where Z_α is the set of zeroes of u_α in U, and $V := U\backslash Z$. Then $\operatorname{Stab}_{\mathfrak{g}} V = \mathfrak{k}$.*

Proof. Put $Y := (G/B)\backslash V$. Then Y is the union of irreducible components

$$Y = \left(\bigcup_{\alpha\in\tilde{\Delta}_{\mathfrak{k}}} \bar{Z}_\alpha\right) \cup Z_1 \cup \cdots \cup Z_n,$$

where $Z_1, \ldots, Z_n$ are all Schubert varieties of codimension 1 in G/B. Furthermore, $\operatorname{Stab}_{\mathfrak{g}} V = \operatorname{Stab}_{\mathfrak{g}} Y$ preserves each irreducible component, in particular

$$\operatorname{Stab}_{\mathfrak{g}} V \subset (\operatorname{Stab}_{\mathfrak{g}} \bar{Z}) \cap \mathfrak{n}^- = \mathfrak{k}.$$

On the other hand, the definition of u_α implies that Z_α is $\mathfrak{k}$-invariant. Therefore V is $\mathfrak{k}$-invariant and

$$\operatorname{Stab}_{\mathfrak{g}} V = (\operatorname{Stab}_{\mathfrak{g}} \bar{Z}) \cap \mathfrak{n}^- = \mathfrak{k}.$$ □

Fix now a map $\lambda\colon \tilde{\Delta}_{\mathfrak{k}} \to \mathbb{C}$ and consider the vector space

$$F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu} := \mathcal{O}(V) \otimes_{\mathbb{C}} \left\langle \prod_{\alpha\in\tilde{\Delta}_{\mathfrak{k}}} u_\alpha^{\lambda(\alpha)} \right\rangle_{\mathbb{C}}.$$

It has a natural structure of a $\mathcal{D}^\mu(V)$-module and therefore of a $\Gamma(\mathcal{D}^\mu)$-module (as $\Gamma(\mathcal{D}^\mu)$ is a subalgebra of $\mathcal{D}^\mu(V)$). Define $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$ as the localization of $F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$ to a $\mathcal{D}^\mu$-module, i. e., set

$$\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu} := \mathcal{D}^\mu \otimes_{\Gamma(\mathcal{D}^\mu)} F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}.$$

Then $\Gamma(\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}) = F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$.

Proposition 2. *For almost all pairs* λ, μ, $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$ *is an irreducible* $\mathcal{D}^{\mu}$*-module and* $\mathfrak{g}[F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}] = \mathfrak{k}$.

Proof. First we show that $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$ is an irreducible $\mathcal{D}^{\mu}$-module whenever μ is generic and $(\operatorname{im}\lambda) \cap \mathbb{Z} = \varnothing$. Note that, as I_Z is generated by u_α for $\alpha \in \tilde{\Delta}_{\mathfrak{k}}$,

$$F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu} = \mathbb{C}[u_\beta]_{\beta\in\Delta_{\mathfrak{k}}} \otimes_{\mathbb{C}} \mathbb{C}[u_\alpha^{\pm 1}]_{\alpha\in\tilde{\Delta}_{\mathfrak{k}}} \otimes_{\mathbb{C}} \Big\langle \prod_{\alpha\in\tilde{\Delta}_{\mathfrak{k}}} u_\alpha^{\lambda(\alpha)} \Big\rangle_{\mathbb{C}}.$$

Therefore an immediate verification, using the condition $(\operatorname{im}\lambda)\cap\mathbb{Z} = \varnothing$, shows that as a $\mathcal{D}(U)$-module $F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$ is generated by each monomial of the form $\prod_{\alpha\in\Delta^-} u_\alpha^{m_\alpha}$, where $m_\alpha \in \mathbb{Z}_+$ for $\alpha \in \Delta_{\mathfrak{k}}$ and $m_\alpha \in \lambda_\alpha + \mathbb{Z}$ for $\alpha \in \tilde{\Delta}_{\mathfrak{k}}$. On the other hand, $F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$ is a semisimple module over the commutative subalgebra generated by $u_\alpha \frac{\partial}{\partial u_\alpha}$ for $\alpha \in \Delta^-$, and the corresponding weight spaces of $F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$ are 1-dimensional and are spanned precisely by the above monomials. As any submodule of a weight module is also a weight module, every nonzero $\mathcal{D}(U)$-submodule of $F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$ contains at least one monomial, i. e., $F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$ is necessarily an irreducible $\mathcal{D}(U)$-module. Thus $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}|_U$ is an irreducible $\mathcal{D}^{\mu}|_U$-module (as U is affine and thus $\mathcal{D}^{\mu}|_U$-affine).

To prove the irreducibility of $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$, consider the atlas $\{U_w\}_{w\in W}$ where $U_w := w(U)$ ($U = U_{\mathrm{id}}$). We will show first that $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}|_{U_w}$ is an irreducible $\mathcal{D}|_{U_w}$-module for each w and that it is free as $\mathcal{O}_{U_w}$-module. The crucial point is that, if w is the reflection corresponding to a simple root α of $\mathfrak{b}$, then

$$\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}(U_w) = F^{w(\mathfrak{k})\cap\mathfrak{k},w(\mathfrak{b})}_{\lambda',\mu}, \tag{8}$$

where

$$\lambda'(\beta) := \begin{cases} \lambda(\beta) & \text{if } \beta \neq \alpha, \\ -\lambda(\beta) + 2\dfrac{(\mu,\beta)}{(\beta,\beta)} & \text{if } \beta = \alpha \text{ and } -\alpha \in \tilde{\Delta}_{\mathfrak{k}}, \\ -2\dfrac{(\mu,\beta)}{(\beta,\beta)} & \text{if } \beta = \alpha \text{ and } -\alpha \in \Delta_{\mathfrak{k}}. \end{cases}$$

The fact that μ is generic ensures that λ' also satisfies the condition $(\operatorname{im}\lambda')\cap\mathbb{Z} = \varnothing$, and thus $F^{w(\mathfrak{k})\cap\mathfrak{k},w(\mathfrak{b})}_{\lambda',\mu}$ is an irreducible $\mathcal{D}(U_w)$-module, free as $\mathcal{O}(U_w)$-module. Induction by the length of w yields the same statement for an arbitrary $w \in W$. As U_w is affine, this is equivalent to the fact that $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}|_{U_w}$ is an irreducible $\mathcal{D}|_{U_w}$-module, which is free as $\mathcal{O}_{U_w}$-module. The irreducibility of $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$ follows now immediately. For a proper submodule $\mathcal{F}'$ of $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$ would have to equal zero after restriction on some U_w and equal $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}|_{U_w}$ for another U_w, which is impossible.

Finally, it remains to check that $\mathfrak{g}[F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}] = \mathfrak{k}$. Note that the restriction homomorphism $\Gamma(\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}) \to \mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}(V)$ is an isomorphism. If $x \in \mathfrak{g}[F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}]$ and $V_x := g_x \cdot V$, the restriction homomorphism $\Gamma(\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}) \to \mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}(V_x)$ is also an isomorphism. Thus $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}(V)$ and $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}(V_x)$ are canonically identified. Furthermore, $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}(V_x) = \mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}(V \cap V_x)$ by the very definition of $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$. This is sufficient to conclude that $V = V_x$, as otherwise the codimension of $V\backslash(V \cap V_x)$ in V would necessarily be 1 and the restriction map $\mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}(V) \to \mathcal{F}^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}(V \cap V_x)$ would not be surjective. Therefore $x \in \operatorname{Stab}_{\mathfrak{g}} V$, and since $\operatorname{Stab}_{\mathfrak{g}} V = \mathfrak{k}$ (Lemma 4), we obtain

$$\mathfrak{g}[F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}] \subset \mathfrak{k}.$$

The opposite inclusion $\mathfrak{k} \subset \mathfrak{g}[F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}]$ is verified directly from the definition of $F^{\mathfrak{k},\mathfrak{b}}_{\lambda,\mu}$. □

2.3. The case of a general $\mathfrak{k}$. Assume now that $\mathfrak{k}$ is arbitrary. The subalgebra $\mathfrak{k}_{\mathrm{ss}}$ of $\mathfrak{g}$ generated by all $\mathfrak{g}^\alpha$ for $\alpha \in \Delta_{\mathfrak{k}} \cap -\Delta_{\mathfrak{k}}$ is a Levi subalgebra of $\mathfrak{k}$. Let $\mathfrak{c}$ denote the centralizer of $\mathfrak{k}_{\mathrm{ss}}$ in $\mathfrak{g}$. Furthermore, for a basis $h_1, \ldots, h_r$ of $\mathfrak{k}_{\mathrm{ss}} \cap \mathfrak{h}$ arising from a Chevalley basis of $\mathfrak{k}_{\mathrm{ss}}$, set $h := h_1 + \cdots + h_r$. Then

$$\mathfrak{p}^h := \mathfrak{h} \oplus \left(\bigoplus_{\alpha(h) \geq 0} \mathfrak{g}^\alpha \right)$$

is a parabolic subalgebra. We call any Borel subalgebra $\mathfrak{b}$ $\mathfrak{k}$-*preferable* if $\mathfrak{h} \subset \mathfrak{b} \subset \mathfrak{p}^h$ for a suitable choice of a Chevalley basis.

Fix a $\mathfrak{k}$-preferable Borel subalgebra $\mathfrak{b}$ and set $S := C \cdot B$. Clearly, S is a closed subvariety of G/B isomorphic to the flag variety $C/(C \cap B)$. As $\mathfrak{c} \cap \mathfrak{k}$ is a solvable subalgebra we can consider U_S, Z_S, $V_S \subset S$ as in Section 2.2 for the pair $(\mathfrak{c}, \mathfrak{c} \cap \mathfrak{k})$ instead of $(\mathfrak{g}, \mathfrak{k})$. Furthermore, we define Y_S to be the set of all points $s \in V_S$ for which $\dim((\mathfrak{k} + \mathfrak{c}) \cap \mathfrak{b}_s)$ is minimal possible. Obviously, Y_S is an affine open $K \cap C$-invariant subvariety of V_S.

Lemma 5. *$Q := K \cdot Y_S$ is a nonsingular subvariety in G/B and* $\mathfrak{k} \subset \operatorname{Stab}_{\mathfrak{g}}(\bar{Q}) \subset \mathfrak{k} + \mathfrak{c}$.

Proof. To prove that Q is nonsingular it is sufficient to show that every $s \in Y_S$ is a nonsingular point in $K \cdot S$, i. e., that the dimension of the tangent space $\mathcal{T}_{K \cdot S}(s)$ is constant for all $s \in Y_S$. The latter follows immediately from the equality

$$\dim T_{K \cdot S}(s) = \dim(\mathfrak{k} + \mathfrak{c}) - \dim((\mathfrak{k} + \mathfrak{c}) \cap \mathfrak{b}_s),$$

as by the definition Y_S consists of s for which the right-hand side is maximal possible.

It is obvious that $\mathfrak{k} \subset \operatorname{Stab}_{\mathfrak{g}} \bar{Q}$. Furthermore $S = \bar{V}_S$, and thus $K \cdot S \subset \bar{Q}$. If $x \in \operatorname{Stab}_{\mathfrak{g}} \bar{Q}$, then $g_{tx} \cdot B \in K \cdot S$ for sufficiently small $t > 0$ as $K \cdot S$ is locally closed in G/B. This implies that $x \in \mathfrak{k} + \mathfrak{c} + \mathfrak{b}$, i. e.,

$$\mathfrak{k} \subset \operatorname{Stab}_{\mathfrak{g}} \bar{Q} \subset \mathfrak{k} + \mathfrak{c} + \mathfrak{b}.$$

To show that actually $\operatorname{Stab}_{\mathfrak{g}} \bar{Q} \subset \mathfrak{k} + \mathfrak{c}$, consider $\operatorname{Stab}_{\mathfrak{g}} \bar{Q}$ as a $\mathfrak{k}_{\mathrm{ss}}$-module. $\operatorname{Stab}_{\mathfrak{g}} \bar{Q}$ is isomorphic to $\mathfrak{k} \oplus \mathfrak{m}$ for some $\mathfrak{k}_{\mathrm{ss}}$-submodule $\mathfrak{m}$ of $\mathfrak{c} + \mathfrak{b}$. Let α be a $\mathfrak{b} \cap \mathfrak{k}_{\mathrm{ss}}$-minimal weight of $\mathfrak{m}$. Then $\alpha(h_i) \leq 0$ for $i = 1, \ldots, r$ and, as $\mathfrak{m} \subset \mathfrak{c} + \mathfrak{b}$, we have $\mathfrak{g}^\alpha \subset \mathfrak{b}$. Thus, $\alpha(h) \geq 0$. This is possible only when $\alpha(h_i) = 0$ for each $i = 1, \ldots, r$, i. e., when $\mathfrak{g}^\alpha \in \mathfrak{c}$. Since $\mathfrak{m}$ is generated by $\mathfrak{g}^\alpha$ for all minimal α, we have finally $\mathfrak{m} \subset \mathfrak{c}$. Therefore $\operatorname{Stab}_{\mathfrak{g}} \bar{Q} \subset \mathfrak{k} + \mathfrak{c}$. □

Let $\mathfrak{k}_r$ denote the radical of $\mathfrak{k}$. As an $\mathfrak{h}$-module $\mathfrak{k}_r$ equals $(\mathfrak{k} \cap \mathfrak{c}) \oplus \mathfrak{t}$ for a unique $\mathfrak{h}$-invariant subspace $\mathfrak{t}$ of $\mathfrak{k}_r$. Set $T := \exp \mathfrak{t}$. Since $\mathfrak{k}_r$ is solvable and $\mathfrak{t}$ is contained in the nilpotent radical of $\mathfrak{k}_r$, one can easily show that the multiplication map $m \colon T \times (K \cap C) \to K_r$ is an isomorphism of algebraic varieties. Therefore $K_{\mathrm{ss}} \times$

$T \times (K \cap C) \simeq K$, and furthermore $K_{\mathrm{ss}} \times T \times C \simeq K \cdot C$. Hence one can define a projection $K \times C \to C$, and this projection induces the morphism $p\colon K \cdot C \cdot B \to S$. Obviously $p(Q) = Y_S$.

Recall our construction for the solvable case and apply it to the pair $(\mathfrak{c}, \mathfrak{c} \cap \mathfrak{k})$. Under the assumptions that the the restriction of μ to $\mathfrak{c} \cap \mathfrak{h}$ is generic (and $\mathfrak{b} \cap \mathfrak{c}$-dominant) and that the restriction of μ to $\mathfrak{k}_{\mathrm{ss}} \cap \mathfrak{h}$ equals the half-sum of $\mathfrak{k}_{\mathrm{ss}} \cap \mathfrak{b}$-positive roots of $\mathfrak{k}_{\mathrm{ss}}$, this construction yields an irreducible $\mathcal{D}^{\mu}_{S}$-module $\mathcal{F}_S$. Put $\mathcal{F} := \mathcal{F}_S|_{Y_S}$.

Proposition 3. *$M := \Gamma(i_{*!} p^* \mathcal{F})$ is an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module.*

Proof. Obviously $p^*\mathcal{F}$ is an irreducible $\mathcal{D}^{\mu}_{Q}$-module which is a locally free $\mathcal{O}_Q$-module of finite rank. Therefore $i_{*!}p^*\mathcal{F}$ is an irreducible $\mathcal{D}^{\mu}$-module, and M is an irreducible $\mathfrak{g}$-module. All it remains to show is that $\mathfrak{g}[M] = \mathfrak{k}$.

We have $\mathfrak{k} = \tilde{\mathfrak{k}} \oplus (\mathfrak{c} \cap \mathfrak{h})$ where $\tilde{\mathfrak{k}} := [\mathfrak{k}, \mathfrak{k}]$. As a $\tilde{\mathfrak{k}}$-sheaf $i_{*!}p^*\mathcal{F}$ is isomorphic to $i_{*!}\mathcal{O}_Q$, and thus the $\tilde{\mathfrak{k}}$-action on $i_{*!}p^*\mathcal{F}$ comes from the action of $\tilde{K} \subset G$ on $\mathcal{O}_Q$. This implies that $\tilde{\mathfrak{k}}$ acts locally finitely on M. As a $\mathfrak{c} \cap \mathfrak{h}$-sheaf $i_{*!}p^*\mathcal{F}$ is isomorphic to $i_{*!}\mathcal{O}_Q \otimes L$ for some one-dimensional $\mathfrak{c} \cap \mathfrak{h}$-module L. Therefore $\mathfrak{c} \cap \mathfrak{h}$ acts also locally finitely on M, and $\mathfrak{k} \subset \mathfrak{g}[M]$.

To verify the opposite inclusion, note that, by Lemma 2, $\mathfrak{g}[M] \subset \mathrm{Stab}_{\mathfrak{g}}\, \bar{Q} \subset \mathfrak{k} + \mathfrak{c}$. Thus we need to check only that $\mathfrak{g}[M] \cap \mathfrak{c} \subset \mathfrak{k} \cap \mathfrak{c}$. By construction, $\Gamma(\mathcal{F})$ is an isotropic strict $(\mathfrak{c}, \mathfrak{k} \cap \mathfrak{c})$-module. Therefore $\Gamma(p^*\mathcal{F}|_Q)$ is an isotropic strict $(\mathfrak{c}, \mathfrak{k} \cap \mathfrak{c})$-module. Furthermore, $i_* p^* \mathcal{F}$ has a natural $\mathfrak{c}$-sheaf filtration whith successive quotients

$$\Lambda^{\max}(\mathcal{N}_Q) \otimes_{\mathcal{O}_Q} S^i(\mathcal{N}_Q) \otimes_{\mathcal{O}_Q} p^*\mathcal{F}.$$

This implies that $\Gamma(i_* p^* \mathcal{F})$ is also an isotropic strict $(\mathfrak{c}, \mathfrak{k} \cap \mathfrak{c})$-module. Since M is a submodule of $\Gamma(i_* p^* \mathcal{F})$, finally $\mathfrak{c}[M] = \mathfrak{g}[M] \cap \mathfrak{c} = \mathfrak{k} \cap \mathfrak{c}$. □

Example 1. If $\mathfrak{c}$ is abelian, Q is a (possibly nonclosed) K-orbit in G/B. In this case our construction is the same as the Beilinson–Bernstein construction of Harish-Chandra modules, see [B], [BB1], [BB2]. Note however that in the classical Harish-Chandra setting (when $\mathfrak{k}$ coincides with the fixed points of an inner involution) $\mathfrak{c}$ is abelian if and only if $\mathfrak{g}$ has no $\mathfrak{sl}(2)$-components.

3. On $(\mathfrak{g}, \mathfrak{k})$-modules of finite type

Let M be a $\mathfrak{g}$-module and $\mathfrak{k}$ be a Lie subalgebra of $\mathfrak{g}[M]$. Given a finite-dimensional irreducible $\mathfrak{k}$-module N, define the multiplicity $[M : N]$ as the supremum of $[M' : N]$ over all finite-dimensional $\mathfrak{k}$-submodules $M' \subset M$. We say that M is of *finite type* over $\mathfrak{k}$ if $[M : N] < \infty$ for any N. Respectively M is of *infinite type* over $\mathfrak{k}$ if $[M : N] \neq 0$ implies $[M : N] = \infty$ for any N. When M is a strict $(\mathfrak{g}, \mathfrak{k})$-module we will say simply that M is of *finite* (or, *infinite*) *type.* A Fernando subalgebra $\mathfrak{k}$ of $\mathfrak{g}$ is, by definition, *of finite type* if there exists an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module of finite type. Otherwise $\mathfrak{k}$ is *of infinite type.*

If $\mathfrak{k}$ equals the fixed points of an involution of $\mathfrak{g}$, it is a classical theorem of Harish-Chandra that any irreducible $(\mathfrak{g}, \mathfrak{k})$-module has finite type over $\mathfrak{k}$. It is also well known that in this case $\mathfrak{k}$ is a Fernando subalgebra, consequently of finite type. The problem of classifying all Fernando subalgebras $\mathfrak{k}$ of finite type appears to be important as it could be a first step in a classification of all irreducible $(\mathfrak{g}, \mathfrak{k})$-modules of finite type over $\mathfrak{k}$. Such a classification would unify the celebrated classification of Harish-Chandra modules (see [KV, Chapter 11]) with Mathieu's classification [M]. Below we give separate necessary and sufficient conditions for a subalgebra $\mathfrak{k}$ with $\mathfrak{k} \supset \mathfrak{h}$ ($\mathfrak{k}$ is a Fernando subalgebra by Theorem 1) to be of finite type.

Our starting point is the following Lemma which is similar to the fact that the weight multiplicities of an irreducible weight $\mathfrak{g}$-module are either all finite or all infinite.

Lemma 6. *Let $\mathfrak{k}$ be any subalgebra reductive in $\mathfrak{g}$. An irreducible $(\mathfrak{g}, \mathfrak{k})$-module is either of finite or infinite type over $\mathfrak{k}$.* (See Erratum on the last page.)

Proof. Fix an irreducible $(\mathfrak{g}, \mathfrak{k})$-module M. Note first that $\mathfrak{k}$ acts semisimply on M as M is a quotient of the $\mathfrak{g}$-module induced by any irreducible $\mathfrak{k}$-submodule of M. Therefore M is of finite type over $\mathfrak{k}$ if and only if all $\mathfrak{k}$-isotypic components of M are finite-dimensional.

Consider now any $\mathfrak{k}$-isotypic component M_0 of M and represent M as a quotient of the induced module

$$M' := U(\mathfrak{g}) \otimes_{U_{\mathfrak{k}}} M_0,$$

where $U_{\mathfrak{k}}$ is the subalgebra of $U(\mathfrak{g})$ generated by $\mathfrak{k}$ and $U(\mathfrak{g})^{\mathfrak{k}} := \{y \in U(\mathfrak{g}) \colon \operatorname{ad}_{\mathfrak{k}}(y) = 0\}$. If M is not of infinite type over $\mathfrak{k}$, we can choose M_0 to be finite-dimensional. Therefore, to prove that M is of finite type over $\mathfrak{k}$, it suffices to show that M' is of finite type over $\mathfrak{k}$.

Let J be the left ideal in $U(\mathfrak{g})$ generated by all elements of $U_{\mathfrak{k}}$ with zero constant term. Then there is an isomorphism of $\mathfrak{k}$-modules

$$M' \simeq (U(\mathfrak{g})/J) \otimes_{\mathbb{C}} M_0,$$

where the $\mathfrak{k}$-module structure on $U(\mathfrak{g})$ comes from the adjoint action. As M_0 is finite-dimensional, it is enough to prove that $U(\mathfrak{g})/J$ is of finite type over $\mathfrak{k}$. The Poincaré–Birkhoff–Witt theorem gives an isomorphism of $\mathfrak{k}$-modules

$$U(\mathfrak{g})/J \simeq S^{\cdot}(\mathfrak{g}/\mathfrak{k})/I,$$

I being the ideal in $S^{\cdot}(\mathfrak{g}/\mathfrak{k})$ generated by the $\mathfrak{k}$ -invariant polynomials on $\mathfrak{g}^*/\mathfrak{k}^*$ of non-zero degree. Furthermore, $S^{\cdot}(\mathfrak{g}/\mathfrak{k})/I \simeq \operatorname{gr}\Gamma(\mathcal{O}_X)$, where X is a generic closed K-orbit in $\mathfrak{g}^*/\mathfrak{k}^*$, $\mathcal{O}_X$ is the sheaf of regular functions on X, and gr stands for graded ring. Since passing to the graded ring commutes with the $\mathfrak{k}$-action, all it remains to show is that $\Gamma(\mathcal{O}_X)$ is of finite type over $\mathfrak{k}$. But, for any $x \in X$, we have $X \simeq K/K^x$, where K^x is the stabilizer of x. Thus $\Gamma(\mathcal{O}_X)$ can be identified with the subspace of regular functions on K which are invariant with respect to right multiplication by elements from K^x. As K is a linear algebraic group, the $\mathfrak{k}$-module of all regular functions on K is of finite type over $\mathfrak{k}$. Therefore $\Gamma(\mathcal{O}_X)$ is also of finite type over $\mathfrak{k}$. □

In the rest of the paper we assume that $\mathfrak{k} \supset \mathfrak{h}$. In this case the reductive part $\mathfrak{k}_{\mathrm{red}}$ is canonically embedded into $\mathfrak{k}$ and acts semisimply on any irreducible $(\mathfrak{g}, \mathfrak{k})$-module M. Therefore M is of finite type over $\mathfrak{k}$ if and only if all $\mathfrak{k}_{\mathrm{red}}$-isotypic components of M are finite-dimensional. In particular, when $\mathfrak{k}$ is solvable, a $(\mathfrak{g}, \mathfrak{k})$-module is of finite type over $\mathfrak{k}$ if and only if it has finite weight multiplicities. Furthermore, there is a simple criterion for a solvable $\mathfrak{k}$ to be of finite type. Namely, $\mathfrak{k}$ is of finite type if and only if $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for some parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ with $\mathfrak{p} \cap \mathfrak{k} = \mathfrak{h}$ such that $\mathfrak{p}_{ss}$ is a direct sum of simple Lie algebras of types A and C. In one direction this is a direct corollary of Lemma 1, and in the other direction it follows from the fact that, given $\mathfrak{p}$, there always exists an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module M of finite type with $\mathfrak{p}_M = \mathfrak{p}$. M can be taken as the module induced from a suitable $\mathfrak{k} + \mathfrak{p}_{ss}$-module M_0 with trivial action of $[\mathfrak{k}, \mathfrak{k}]$. We leave it to the reader to check this.

The above observation leads to the following necessary condition for a general $\mathfrak{k}$ (with $\mathfrak{h} \subset \mathfrak{k}$) to be of finite type. Recall that the reductive subalgebra $\mathfrak{c} \subset \mathfrak{g}$ is defined as the centralizer of $\mathfrak{k}_{ss}$ in $\mathfrak{g}$.

Proposition 4. *Let $\mathfrak{k}$ be of finite type. Then the solvable subalgebra $\mathfrak{k} \cap \mathfrak{c}$ is of finite type in $\mathfrak{c}$, i. e., $\mathfrak{c} = (\mathfrak{k} \cap \mathfrak{c}) + \mathfrak{p}$ for some parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{c}$ with $\mathfrak{p} \cap \mathfrak{k} = \mathfrak{h}$ and such that $\mathfrak{p}_{ss}$ is a direct sum of simple Lie algebras of types A and C. In addition, $[\mathfrak{k}, \mathfrak{p}_{ss}] \subset \mathfrak{k}$.*

Proof. Let M be an irreducible strict $(\mathfrak{g}, \mathfrak{k})$-module of finite type and M' be a $\mathfrak{k}_{ss}$-isotypic component of M. Fix a Borel subalgebra $\mathfrak{b}' \subset \mathfrak{k}_{red}$ with $\mathfrak{b}' \supset \mathfrak{h}$ and set $M'' := \{m' \in M' \colon [\mathfrak{b}', \mathfrak{b}'] \cdot m' = 0\}$. An immediate verification shows that M'' is an isotropic strict $(\mathfrak{c}, \mathfrak{k} \cap \mathfrak{c})$-module of finite type. Therefore $\mathfrak{c} = (\mathfrak{k} \cap \mathfrak{c}) + \mathfrak{p}$ for a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{c}$ as in Lemma 1.

It remains to show that $[\mathfrak{k}, \mathfrak{p}_{ss}] \subset \mathfrak{k}$. Assume the contrary, i. e., that there are $\gamma \in \Delta_{\mathfrak{k}} \backslash \Delta_{\mathfrak{k}_{red}}$ and $\beta \in \Delta_{\mathfrak{p}_{ss}}$ with $\alpha = \beta + \gamma \in \Delta \backslash \Delta_{\mathfrak{k}}$. Moreover, γ can be assumed to be a $\mathfrak{b}'$-maximal weight of the $\mathfrak{k}_{red}$-module $\mathfrak{k}$. Let $\alpha_1, \ldots, \alpha_s$ be the simple roots of $\mathfrak{b}'$. Then $-\gamma, \alpha_1, \ldots, \alpha_s$ is a system of simple roots of a reductive subalgebra $\mathfrak{s}$ containing $\mathfrak{k}_{red}$. The subalgebra $\mathfrak{q} := \mathfrak{k} \cap \mathfrak{s}$ is a parabolic subalgebra of $\mathfrak{s}$, and its reductive part equals $\mathfrak{k}_{red}$. As $\mathfrak{q} \subset \mathfrak{k}$, there is an irreducible finite-dimensional $\mathfrak{q}$-submodule $L_\lambda \subset M$ of $\mathfrak{b}'$-highest weight λ. Furthermore, since $\Delta_{\mathfrak{q}} \backslash \Delta_{\mathfrak{k}_{red}} \subset \Gamma_{\mathfrak{k}}$, M has an infinite family of finite-dimensional irreducible $\mathfrak{q}$-submodules L_{λ_k} with highest weights λ_k belonging to $\{\lambda + \langle \Delta_{\mathfrak{q}} \backslash \Delta_{\mathfrak{k}_{red}} \rangle_{\mathbb{Z}_+}\}$. Let $M_k := U(\mathfrak{s}) \cdot L_{\lambda_k}$. For almost all k, M_k contains a $\mathfrak{k}_{red}$-submodule isomorphic to L_λ. Thus the multiplicity of L_λ in M is infinite. Contradiction. □

Proposition 4 implies the existence of a large class of reductive subalgebras $\mathfrak{k}$ of infinite type.

Corollary 1. *Let $\mathfrak{g}$ be simple of types B_n for $n > 4$, D_n for $n > 4$, E_7, E_8, and let $\mathfrak{k}$ be reductive and such that $\mathfrak{c}_{ss}$ has a simple component not of type A or C. Then $\mathfrak{k}$ is of infinite type.*

Example 2. The Lie algebra $\mathfrak{g} = \mathfrak{sp}(6)$ is the simple Lie algebra of smallest dimension which admits a subalgebra $\mathfrak{k}$ of infinite type with nonzero nilpotent radical and nonzero semisimple part. Let $\mathfrak{k} = \mathfrak{h} \oplus \mathfrak{g}^{\alpha} \oplus \mathfrak{g}^{-\alpha} \oplus \mathfrak{g}^{\beta}$ where α and β are respectively a long and a short orthogonal roots. Then there is no $\mathfrak{p}$ as in Proposition 5, and hence $\mathfrak{k}$ is of infinite type.

Our final step is to establish a sufficient condition for a reductive $\mathfrak{k}$ to be of finite type.

Proposition 5. *Let* $\mathfrak{k}$ *be reductive. Assume that* $\langle\Delta_{\mathfrak{c}}\rangle_{\mathbb{C}} \cap \Delta = \Delta_{\mathfrak{c}}$ (*or equivalently that* $\mathfrak{h} + \mathfrak{c}_{\mathrm{ss}}$ *is the reductive part of a parabolic subalgebra of* $\mathfrak{g}$) *and that the simple components of* $\mathfrak{c}_{\mathrm{ss}}$ *are of types* A *and* C *only. Then* $\mathfrak{k}$ *is of finite type.*

Proof. Let $\mathfrak{h}'$ be the centralizer of $\mathfrak{c}_{\mathrm{ss}}$ in $\mathfrak{h}$, and $\mathfrak{k}' := \mathfrak{k}_{\mathrm{ss}} + \mathfrak{h}'$. The condition $\langle\Delta_{\mathfrak{c}}\rangle_{\mathbb{C}} \cap \Delta = \Delta_{\mathfrak{c}}$ enables us to choose a $\mathfrak{k}$-preferable Borel subalgebra $\mathfrak{b}$ for which the projection of $\Delta^- \backslash \Delta_{\mathfrak{c}}$ onto $(\mathfrak{h}')^*$ is contained in some open half-space of some real subspace of $(\mathfrak{h}')^*$. Put $Q_1 := K'/(K' \cap B)$, $Q_2 := C_{\mathrm{ss}}/(C_{\mathrm{ss}} \cap B)$ and $Q := (K' \times C_{\mathrm{ss}}) \cdot B$. Then $Q = Q_1 \times Q_2$ is a closed subvariety of G/B, and Lemma 5 implies that $\mathrm{Stab}_{\mathfrak{g}}\, Q = \mathfrak{k}' \oplus \mathfrak{c}_{\mathrm{ss}}$. Let F_2 be an irreducible strict $(\mathfrak{c}_{\mathrm{ss}}, \mathfrak{h}'')$-module with finite weight multiplicities, where $\mathfrak{h}'' := \mathfrak{c}_{\mathrm{ss}} \cap \mathfrak{h}$. The existence of F_2 is well-known, see for instance [M]. Denote by $\mathcal{F}_2$ the localization of F_2 on Q_2. Set finally $\mathcal{F} := \mathcal{O}_{Q_1} \boxtimes \mathcal{F}_2$, $\mathcal{M} := i_*\mathcal{F}$ and $M := \Gamma(\mathcal{M})$, i being the closed embedding of Q into G/B.

Arguments similar to ones in the proof of Proposition 3 imply that M is a strict irreducible $(\mathfrak{g}, \mathfrak{k})$-module. We will show that M is of finite type. Consider the filtration of $\mathcal{M}$ with successive quotients

$$\Lambda^{\max}(\mathcal{N}_Q) \otimes_{\mathcal{O}_Q} S^i(\mathcal{N}_Q) \otimes_{\mathcal{O}_Q} \mathcal{F},$$

and choose a finer filtration of $\mathcal{M}$ such that its successive quotients are all sheaves $\mathcal{O}(\lambda) \otimes_{\mathcal{O}_Q} \mathcal{F}$, where λ runs over the multiset D of sums of roots from $\Delta^- \backslash \Delta_{\mathfrak{k}+\mathfrak{c}}$ and $\mathcal{O}(\lambda)$ stands for the invertible $K' \times C_{\mathrm{ss}}$-sheaf on Q induced by λ. Then $\mathcal{O}(\lambda) \otimes_{\mathcal{O}_Q} \mathcal{F} \simeq \mathcal{O}(\lambda') \boxtimes \mathcal{F}(\lambda'')$, where λ' (respectively, λ'') is the projection of λ on $(\mathfrak{h}')^*$ (resp., $(\mathfrak{h}'')^*$) and $\mathcal{F}(\lambda'') := \mathcal{F}_2 \otimes_{\mathcal{O}_{Q_2}} \mathcal{O}(\lambda'')$. Thus M is a $\mathfrak{k} + \mathfrak{c}$-submodule of $\bigoplus_{\lambda \in D}(\Gamma(\mathcal{O}(\lambda')) \boxtimes \Gamma(\mathcal{F}(\lambda'')))$. The Borel–Weil–Bott theorem implies that $\Gamma(\mathcal{O}(\lambda')) \neq 0$ if and only if λ' is $\mathfrak{b} \cap \mathfrak{k}$-antidominant. In the latter case $\Gamma(\mathcal{O}(\lambda')) = V(-\lambda')^*$, where $V(-\lambda')$ is the irreducible finite-dimensional $\mathfrak{k}'$-module of $\mathfrak{b} \cap \mathfrak{k}'$-highest weight $-\lambda'$. Furthermore $\Gamma(\mathcal{F}(\lambda''))$ is a $(\mathfrak{c}_{\mathrm{ss}}, \mathfrak{h}'')$-module of finite type over $\mathfrak{h}''$. Therefore to show that M has finite type it suffices to check that for each $\omega \in (\mathfrak{h}')^*$ the set $D_\omega := \{\lambda \in D \colon \lambda' = \omega\}$ is finite. The latter is the direct corollary of the fact that the projection of D onto $(\mathfrak{h}')^*$ is contained in an open half-space of a real subspace of $(\mathfrak{h}')^*$. □

Corollary 2. *Let* $\mathfrak{k}$ *be reductive.*

(a) *If* $\mathfrak{c}$ *is abelian, then* $\mathfrak{k}$ *is of finite type.*

(b) *Assume that* $\mathfrak{g}$ *has no simple components of type* B_n *for* $n > 2$ *or* F_4*. Then* $\mathfrak{k}$ *is of finite type if and only if all simple components of* $\mathfrak{c}_{\mathrm{ss}}$ *are of types* A *and* C.

Proof. (a) is obvious. In proving (b) one can assume that $\mathfrak{g}$ is simple. If $\mathfrak{g}$ is not of type C_n, B_n for $n > 2$ or F_4, for any root subsystem $\Xi \subset \Delta$ we have $\langle\Xi\rangle_{\mathbb{C}} \cap \Delta = \Xi$, and therefore Propositions 4 and 5 imply the statement. Furthermore, for $\mathfrak{g}$ of type C_n it is also always true that $\langle\Delta_{\mathfrak{c}}\rangle_{\mathbb{C}} \cap \Delta = \Delta_{\mathfrak{c}}$, but it is essential that $\mathfrak{c}$ is the centralizer of $\mathfrak{k}_{\mathrm{ss}}$. Here the equality $\langle\Delta_{\mathfrak{c}}\rangle_{\mathbb{C}} \cap \Delta = \Delta_{\mathfrak{c}}$ follows from the observation that, for any two orthogonal long roots α, $\beta \in \Delta_{\mathfrak{c}}$, we have also $(\alpha + \beta)/2 \in \Delta_{\mathfrak{c}}$. This is easily verified explicitly. □

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Erratum to the paper On generalized Harish-Chandra modules

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Part of the argument in the proof of Lemma 6 is erroneous. Starting with the sentence "The Poincare-Birkhoff-Witt..." and until the end of the proof, the text should be replaced by the following.

Let R be a $U(\mathfrak{k})$-module complement to J in $U(\mathfrak{g})$. We will show that R has finite type over $\mathfrak{k}$. Indeed, for any simple finite-dimensional $\mathfrak{k}$-module W,

$$\mathrm{Hom}_{\mathfrak{k}}(W, U(\mathfrak{g})) = \mathrm{Hom}_{\mathfrak{k}}(W, R) \oplus \mathrm{Hom}_{\mathfrak{k}}(W, J).$$

Furthermore, it is a fact of classical invariant theory that $\mathrm{Hom}_{\mathfrak{k}}(W, S(\mathfrak{g}))$ is a finitely generated $S(\mathfrak{g})^{\mathfrak{k}}$-module, see for instance [1]. Therefore $\mathrm{Hom}_{\mathfrak{k}}(W, U(\mathfrak{g}))$ is a finitely generated $U(\mathfrak{g})^{\mathfrak{k}}$-module. Finally, if $m_1, \ldots, m_n$ are generators of $\mathrm{Hom}_{\mathfrak{k}}(W, U(\mathfrak{g}))$ over $U(\mathfrak{g})^{\mathfrak{k}}$, then clearly their projections in $\mathrm{Hom}_{\mathfrak{k}}(W, R)$ span $\mathrm{Hom}_{\mathfrak{k}}(W, R)$, hence $\dim \mathrm{Hom}_{\mathfrak{k}}(W, R) < \infty$.

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