

Infinite Kostant cascades and centrally generated primitive ideals of $U(\mathfrak{n})$ in types A_∞, C_∞

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ABSTRACT. We study the center of $U(\mathfrak{n})$, where \mathfrak{n} is the locally nilpotent radical of a splitting Borel subalgebra of a simple complex Lie algebra $\mathfrak{g} = \mathfrak{sl}_\infty(\mathbb{C}), \mathfrak{so}_\infty(\mathbb{C}), \mathfrak{sp}_\infty(\mathbb{C})$. There are infinitely many isomorphism classes of Lie algebras \mathfrak{n} , and we provide explicit generators of the center of $U(\mathfrak{n})$ in all cases. We then fix \mathfrak{n} with “largest possible” center of $U(\mathfrak{n})$ and characterize the centrally generated primitive ideals of $U(\mathfrak{n})$ for $\mathfrak{g} = \mathfrak{sl}_\infty(\mathbb{C}), \mathfrak{sp}_\infty(\mathbb{C})$ in terms of the above generators. As a preliminary result, we provide a characterization of the centrally generated primitive ideals in the enveloping algebra of the nilradical of a Borel subalgebra of $\mathfrak{sl}_n(\mathbb{C}), \mathfrak{sp}_{2n}(\mathbb{C})$.

Keywords: locally nilpotent Lie algebra, Dixmier map, Kostant cascade, center of enveloping algebra, centrally generated primitive ideal.

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1. Introduction

The theory of primitive ideals in enveloping algebras of Lie algebras has its roots in the representation theory of Lie algebras. However, classifying irreducible representations of Lie algebras is not feasible except in few very special cases, while a classification of annihilators of irreducible representations, i.e., of primitive ideals, can be achieved in much greater generality. This idea goes back to J. Dixmier and his seminar, and for semisimple or solvable finite-dimensional Lie algebras there is an extensive theory of primitive ideals.

In the case when \mathfrak{n} is a finite-dimensional nilpotent Lie algebra, the primitive ideals in the universal enveloping algebra $U(\mathfrak{n})$ can be described in terms of the Dixmier map assigning to any linear form $f \in \mathfrak{n}^*$ a primitive ideal $J(f)$ of $U(\mathfrak{n})$. If \mathfrak{n} is abelian, $J(f)$ is simply the annihilator of f . For a general finite-dimensional nilpotent Lie algebra \mathfrak{n} , the theory of primitive ideals retains many properties from the abelian case: in particular, $J(f)$ is always a maximal ideal and every primitive ideal in $U(\mathfrak{n})$ is of the form $J(f)$ for some $f \in \mathfrak{n}^*$. Moreover, $J(f) = J(f')$ if and only if f and f' belong to the same coadjoint orbit in \mathfrak{n}^* .

The idea of classifying primitive ideals rather than irreducible representations makes even more sense for infinite-dimensional Lie algebras, and in this paper we make some first steps in this direction for a natural class of locally nilpotent infinite-dimensional Lie algebras. These are the locally nilpotent radicals \mathfrak{n} of splitting Borel subalgebras of the three simple finitary complex Lie algebras $\mathfrak{sl}_\infty(\mathbb{C}), \mathfrak{so}_\infty(\mathbb{C}), \mathfrak{sp}_\infty(\mathbb{C})$.

A comprehensive theory of primitive ideals in $U(\mathfrak{n})$ remains to be built. In this paper we concentrate on centrally generated primitive ideals in $U(\mathfrak{n})$. We first provide a description of the center of the enveloping algebra $U(\mathfrak{n})$ for any locally nilpotent radical \mathfrak{n} as above, and then use the result to describe the centrally generated primitive ideals in $U(\mathfrak{n})$ for some interesting choices of \mathfrak{n} .

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The splitting Borel subalgebras of $\mathfrak{g} = \mathfrak{sl}_\infty(\mathbb{C}), \mathfrak{so}_\infty(\mathbb{C}), \mathfrak{sp}_\infty(\mathbb{C})$ are not conjugate, and there are infinitely many isomorphism classes of locally nilpotent radicals \mathfrak{n} . In the finite-dimensional case Kostant cascades of orthogonal roots play an important role in describing the center of $U(\mathfrak{n})$. In the infinite-dimensional case the center of $U(\mathfrak{n})$ is described in terms of a possibly infinite Kostant cascade. A significant difference with the finite-dimensional case is that the Kostant cascade depends in an essential way on the isomorphism class of \mathfrak{n} , and that in most cases the cascade is finite rather than infinite.

In order to obtain an explicit form of the generators of the center of $U(\mathfrak{n})$, we first recall such an explicit form of the generators in the finite-dimensional case due to A. Joseph, F. Fauquant-Millet, R. Lipsman, J.A. Wolf, and A. Panov. This enables us to give an explicit description of the center of $U(\mathfrak{n})$ in all cases.

We then concentrate on the case when $\mathfrak{g} = \mathfrak{sl}_\infty(\mathbb{C}), \mathfrak{sp}_\infty(\mathbb{C})$ and $U(\mathfrak{n})$ has the ‘‘largest possible’’ center. This latter requirement singles out only one isomorphism class of subalgebras \mathfrak{n} for a fixed \mathfrak{g} . For such \mathfrak{n} we construct a Dixmier map defined for certain linear forms $f \in \mathfrak{n}^*$, closely related to the Kostant cascade of \mathfrak{n} . We refer to these forms as Kostant forms. Our main result implies then that the Dixmier map establishes a one-to-one correspondence between Kostant forms and centrally generated primitive ideals in $U(\mathfrak{n})$. This provides an explicit description of the centrally generated primitive ideals of $U(\mathfrak{n})$. As a corollary we obtain that centrally generated primitive ideals J of $U(\mathfrak{n})$ are maximal ideals, and that the quotient $U(\mathfrak{n})/J$ is a Weyl algebra with infinitely many generators.

To the best of our knowledge, the analogous description, via Kostant forms, of the centrally generated primitive ideals in $U(\mathfrak{n})$ for $\mathfrak{n} \subset \mathfrak{sl}_n(\mathbb{C}), \mathfrak{sp}_{2n}(\mathbb{C})$ is also new.

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2. The center of $U(\mathfrak{n})$

2.1. Finite-dimensional case. Let $n \in \mathbb{Z}_{>0}$. Throughout this subsection \mathfrak{g} denotes one of the Lie algebras $\mathfrak{sl}_n(\mathbb{C}), \mathfrak{so}_{2n}(\mathbb{C}), \mathfrak{so}_{2n+1}(\mathbb{C})$ or $\mathfrak{sp}_{2n}(\mathbb{C})$. The algebra $\mathfrak{so}_{2n}(\mathbb{C})$ (respectively, $\mathfrak{so}_{2n+1}(\mathbb{C})$ and $\mathfrak{sp}_{2n}(\mathbb{C})$) is realized as the subalgebra of $\mathfrak{sl}_{2n}(\mathbb{C})$ (respectively, $\mathfrak{sl}_{2n+1}(\mathbb{C})$ and $\mathfrak{sl}_{2n}(\mathbb{C})$) consisting of all x such that $\beta(u, xv) + \beta(xu, v) = 0$ for all u, v in \mathbb{C}^{2n} (respectively, in \mathbb{C}^{2n+1} and \mathbb{C}^{2n}), where

$$\beta(u, v) = \begin{cases} \sum_{i=1}^n (u_i v_{-i} + u_{-i} v_i) & \text{for } \mathfrak{so}_{2n}(\mathbb{C}), \\ u_0 v_0 + \sum_{i=1}^n (u_i v_{-i} + u_{-i} v_i) & \text{for } \mathfrak{so}_{2n+1}(\mathbb{C}), \\ \sum_{i=1}^n (u_i v_{-i} - u_{-i} v_i) & \text{for } \mathfrak{sp}_{2n}(\mathbb{C}). \end{cases}$$

Here for $\mathfrak{so}_{2n}(\mathbb{C})$ (respectively, for \mathfrak{so}_{2n+1} and $\mathfrak{sp}_{2n}(\mathbb{C})$) we denote by $e_1, \dots, e_n, e_{-n}, \dots, e_{-1}$ (respectively, by $e_1, \dots, e_n, e_0, e_{-n}, \dots, e_{-1}$ and $e_1, \dots, e_n, e_{-n}, \dots, e_{-1}$) the standard basis of \mathbb{C}^{2n} (respectively, of \mathbb{C}^{2n+1} and \mathbb{C}^{2n}), and by x_i the coordinate of a vector x corresponding to e_i .

The set of all diagonal matrices from \mathfrak{g} is a Cartan subalgebra of \mathfrak{g} ; we denote it by \mathfrak{h} . Let Φ be the root system of \mathfrak{g} with respect to \mathfrak{h} . Note that Φ is of type A_{n-1} (respectively, D_n, B_n and C_n) for $\mathfrak{sl}_n(\mathbb{C})$ (respectively, for $\mathfrak{so}_{2n}(\mathbb{C}), \mathfrak{so}_{2n+1}(\mathbb{C})$ and $\mathfrak{sp}_{2n}(\mathbb{C})$). The set of all upper-triangular matrices from \mathfrak{g} is a Borel subalgebra of \mathfrak{g} containing \mathfrak{h} ; we denote it by \mathfrak{b} . Let Φ^+ be the set of positive roots with respect to \mathfrak{b} . As usual, we identify Φ^+ with the following subset of \mathbb{R}^n :

$$\begin{aligned} A_{n-1}^+ &= \{\epsilon_i - \epsilon_j, 1 \leq i < j \leq n\}, \\ B_n^+ &= \{\epsilon_i - \epsilon_j, 1 \leq i < j \leq n\} \cup \{\epsilon_i + \epsilon_j, 1 \leq i < j \leq n\} \cup \{\epsilon_i, 1 \leq i \leq n\}, \\ C_n^+ &= \{\epsilon_i - \epsilon_j, 1 \leq i < j \leq n\} \cup \{\epsilon_i + \epsilon_j, 1 \leq i < j \leq n\} \cup \{2\epsilon_i, 1 \leq i \leq n\}, \\ D_n^+ &= \{\epsilon_i - \epsilon_j, 1 \leq i < j \leq n\} \cup \{\epsilon_i + \epsilon_j, 1 \leq i < j \leq n\}. \end{aligned}$$

Here $\{\epsilon_i\}_{i=1}^n$ is the standard basis of \mathbb{R}^n .

Denote by \mathfrak{n} the algebra of all strictly upper-triangular matrices from \mathfrak{g} . Then \mathfrak{n} has a basis consisting of root vectors e_α , $\alpha \in \Phi^+$, where

$$\begin{aligned} e_{\epsilon_i} &= \sqrt{2}(e_{0,i} - e_{-i,0}), \quad e_{2\epsilon_i} = e_{i,-i}, \\ e_{\epsilon_i - \epsilon_j} &= \begin{cases} e_{i,j} & \text{for } A_{n-1}, \\ e_{i,j} - e_{-j,-i} & \text{for } B_n, C_n \text{ and } D_n, \end{cases} \\ e_{\epsilon_i + \epsilon_j} &= \begin{cases} e_{i,-j} - e_{j,-i} & \text{for } B_n \text{ and } D_n, \\ e_{i,-j} + e_{j,-i} & \text{for } C_n, \end{cases} \end{aligned}$$

and $e_{i,j}$ are the usual elementary matrices. For $\mathfrak{so}_{2n}(\mathbb{C})$ (respectively, for $\mathfrak{so}_{2n+1}(\mathbb{C})$ and $\mathfrak{sp}_{2n}(\mathbb{C})$) we index the rows (from left to right) and the columns (from top to bottom) of matrices by the numbers $1, \dots, n, -n, \dots, -1$ (respectively, by the numbers $1, \dots, n, 0, -n, \dots, -1$ and $1, \dots, n, -n, \dots, -1$). Note that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n} \oplus \mathfrak{n}_-$, where $\mathfrak{n}_- = \langle e_{-\alpha}, \alpha \in \Phi^+ \rangle_{\mathbb{C}}$, and, by definition, $e_{-\alpha} = e_\alpha^T$. (The superscript T always stands for transposed.) The set $\{e_\alpha, \alpha \in \Phi\}$ can be extended to a unique Chevalley basis of \mathfrak{g} .

Let G be one of the following classical Lie groups: $\mathrm{SL}_n(\mathbb{C})$, $\mathrm{SO}_{2n}(\mathbb{C})$, $\mathrm{SO}_{2n+1}(\mathbb{C})$ or $\mathrm{Sp}_{2n}(\mathbb{C})$. The group $\mathrm{SO}_{2n}(\mathbb{C})$ (respectively, SO_{2n+1} and $\mathrm{Sp}_{2n}(\mathbb{C})$) is realized as the subgroup of $\mathrm{SL}_{2n}(\mathbb{C})$ (respectively, of $\mathrm{SL}_{2n+1}(\mathbb{C})$ and $\mathrm{SL}_{2n}(\mathbb{C})$) which preserves the form β . Let H (respectively, B and N) be the set of all diagonal (respectively, upper-triangular and upper-triangular with 1 on the diagonal) matrices from G . Then H is a maximal torus of G , B is a Borel subgroup of G containing H , N is the unipotent radical of B , and \mathfrak{g} (respectively, \mathfrak{h} , \mathfrak{b} and \mathfrak{n}) is the Lie algebra of G (respectively, of H , B and N).

Denote by $U(\mathfrak{n})$ the enveloping algebra of \mathfrak{n} , and by $S(\mathfrak{n})$ the symmetric algebra of \mathfrak{n} . Then \mathfrak{n} and $S(\mathfrak{n})$ are B -modules as B normalizes N . Denote by $Z(\mathfrak{n})$ the center of $U(\mathfrak{n})$. It is well-known that the restriction of the symmetrization map

$$\sigma: S(\mathfrak{n}) \rightarrow U(\mathfrak{n}), \quad x^k \mapsto x^k, \quad x \in \mathfrak{n}, \quad k \in \mathbb{Z}_{\geq 0},$$

to the algebra $S(\mathfrak{n})^N$ of N -invariants is an algebra isomorphism between $S(\mathfrak{n})^N$ and $Z(\mathfrak{n})$.

We next present a canonical set of generators of $Z(\mathfrak{n})$ (or, equivalently, of $S(\mathfrak{n})^N$), whose description goes back to J. Dixmier, A. Joseph and B. Kostant [Di3], [Jo], [Ko1], [Ko2]. Denote by \mathcal{B} the following subset of Φ^+ :

$$\mathcal{B} = \begin{cases} \bigcup_{1 \leq i \leq [n/2]} \{\epsilon_i - \epsilon_{n-i+1}\} & \text{for } A_{n-1}, \\ \bigcup_{1 \leq i \leq n/2} \{\epsilon_{2i-1} - \epsilon_{2i}, \epsilon_{2i-1} + \epsilon_{2i}\} & \text{for } B_n, n \text{ even}, \\ \bigcup_{1 \leq i \leq [n/2]} \{\epsilon_{2i-1} - \epsilon_{2i}, \epsilon_{2i-1} + \epsilon_{2i}\} \cup \{\epsilon_n\} & \text{for } B_n, n \text{ odd}, \\ \bigcup_{1 \leq i \leq n} \{2\epsilon_i\} & \text{for } C_n, \\ \bigcup_{1 \leq i \leq [n/2]} \{\epsilon_{2i-1} - \epsilon_{2i}, \epsilon_{2i-1} + \epsilon_{2i}\} & \text{for } D_n. \end{cases}$$

Note that \mathcal{B} is a maximal strongly orthogonal subset of Φ^+ , i.e., \mathcal{B} is maximal with the property that if $\alpha, \beta \in \mathcal{B}$ then neither $\alpha - \beta$ nor $\alpha + \beta$ belongs to Φ^+ . We call \mathcal{B} the *Kostant cascade* of orthogonal roots in Φ^+ .

We can consider $\mathbb{Z}\Phi$, the \mathbb{Z} -linear span of Φ , as a subgroup of the group \mathfrak{X} of rational multiplicative characters of H by putting $\pm\epsilon_i(h) = h_{i,i}^{\pm 1}$, where $h_{i,j}$ is the i -th diagonal element of a matrix $h \in H$. Recall that a vector $\lambda \in \mathbb{R}^n$ is called a *weight* of H if $c(\alpha, \lambda) = 2(\alpha, \lambda)/(\alpha, \alpha)$ is an integer for any $\alpha \in \Phi^+$, where (\cdot, \cdot) is the standard inner product on \mathbb{R}^n . A weight λ is called *dominant* if $c(\alpha, \lambda) \geq 0$ for all $\alpha \in \Phi^+$. An element a of an H -module is called an *H -weight vector*, if there exists $\nu \in \mathfrak{X}$ such that $h \cdot a = \nu(h)a$ for all $h \in H$. By [Ko2, Theorems 6, 7], there exist unique (up to scalars) prime polynomials $\xi_1, \dots, \xi_m \in S(\mathfrak{n})^N$, $m = |\mathcal{B}|$, such that each ξ_i is an H -weight polynomial of a dominant weight μ_i belonging to the \mathbb{Z} -linear span $\mathbb{Z}\mathcal{B}$ of \mathcal{B} . A remarkable fact is that

$$\xi_1, \dots, \xi_m \text{ are algebraically independent generators of } S(\mathfrak{n})^N, \quad (1)$$

so $S(\mathfrak{n})^N$ and $Z(\mathfrak{n})$ are polynomial rings. We call ξ_i the *i -th canonical generator* of $S(\mathfrak{n})^N$.

Let \mathfrak{n}^* be the dual space of \mathfrak{n} , and let $\{e_\alpha^*, \alpha \in \Phi^+\}$ be the basis of \mathfrak{n}^* dual to the basis $\{e_\alpha, \alpha \in \Phi^+\}$ of \mathfrak{n} . Put $R = \left\{t = \sum_{\beta \in \mathcal{B}} t_\beta e_\beta^*, t_\beta \in \mathbb{C}^\times\right\}$ and denote by X the union of all N -orbits in \mathfrak{n}^* of elements of R . In fact, X is a single B -orbit in \mathfrak{n}^* , and the N -orbits of two distinct point of R are disjoint. Kostant [Ko3, Theorems 1.1, 1.3] proves that X is a Zariski dense subset of \mathfrak{n}^* , and for $t \in R$, up to scalar multiplication,

$$\xi_i(t) = \prod_{\beta \in \mathcal{B}} t_\beta^{r_{\mu_i}(\beta)}, \quad 1 \leq i \leq m, \quad \text{where } r_{\mu_i}(\beta) = \frac{(\mu_i, \beta)}{(\beta, \beta)}. \quad (2)$$

The following representation-theoretic description of ξ_i is given in [Pa2], however, it can be found in slightly different terms also in [LW] and [FMJ]. Let $\Pi = \{\alpha_1, \dots, \alpha_n\}$ be the set of simple roots [Bo, Table I-IV], and let $\varpi_i, 1 \leq i \leq n$, be the i -th fundamental dominant weight of Φ . We define positive integers k_i for $1 \leq i \leq m = |\mathcal{B}|$ by the following rule:

$\Phi = A_{n-1}$	$m = [n/2]$	$k_i = 1$ for $1 \leq i \leq m$
$\Phi = B_n$ or D_n	$m = n$ for B_n $m = n$ for D_n when n even $m = n - 1$ for D_n when n odd	$k_i = 1$ for odd i , $k_i = 2$ for even $i < m$, $k_m = 1$ for even m
$\Phi = C_n$	$m = n$	$k_i = 1$ for $1 \leq i \leq m$

Let W be the Weyl group of Φ . Denote by w_0 the unique element of W such that $w_0(\alpha) \in -\Phi^+$ for all $\alpha \in \Phi^+$. Furthermore, set $\varpi'_i = \frac{(1 - w_0)\varpi_i}{k_i}$. Then the weights ϖ'_i 's have the following form:

$\Phi = A_{n-1}$	$\varpi'_i = 2\epsilon_1 + \dots + 2\epsilon_{i-1} + \epsilon_i$ for $1 \leq i \leq m$
$\Phi = B_n$	$\varpi'_i = \begin{cases} 2\epsilon_1 + \dots + 2\epsilon_i & \text{for odd } i < m, \\ \epsilon_1 + \dots + \epsilon_i & \text{otherwise} \end{cases}$
$\Phi = C_n$	$\varpi'_i = 2\epsilon_1 + \dots + 2\epsilon_i$ for $1 \leq i \leq m$
$\Phi = D_n$	$\varpi'_i = \begin{cases} 2\epsilon_1 + \dots + 2\epsilon_i & \text{for odd } i < m - 1, \\ 2\epsilon_1 + \dots + 2\epsilon_{n-1} & \text{for } i = n - 2 = m - 1 \text{ when } n \text{ is odd,} \\ \epsilon_1 + \dots + \epsilon_{n-1} - \epsilon_n & \text{for } i = n - 1 = m - 1 \text{ when } n \text{ is even,} \\ \epsilon_1 + \dots + \epsilon_i & \text{otherwise} \end{cases}$

Now, let V_i be the i -th fundamental representation of the group G for $1 \leq i \leq m$, and let V_i^* be its dual representation. Fix highest-weight vectors v_i and l_i respectively of V_i and V_i^* . Let S_i be the regular function on G defined by $S_i(g) = l_i(g \cdot v_i)$. By $\exp: \mathfrak{g} \rightarrow G$ we denote the usual exponential map. Using the \mathfrak{g} -invariant form $\mathfrak{g} \ni x, y \mapsto \text{tr } xy$ we identify the space \mathfrak{n}_- of all lower-triangular matrices from \mathfrak{g} with \mathfrak{n}^* . Then

$$e_\alpha^* = \begin{cases} e_\alpha^T & \text{if } \Phi = A_{n-1}, \text{ or } \Phi = C_n \text{ and } \alpha = 2\epsilon_i, \\ e_\alpha^T/4 & \text{if } \Phi = B_n \text{ and } \alpha = \epsilon_i, \\ e_\alpha^T/2 & \text{otherwise.} \end{cases} \quad (3)$$

To each regular function S on G we assign the sequence of regular functions S^j on \mathfrak{n}_- (or, equivalently, of elements of $S(\mathfrak{n})$) defined as the coefficients in the expansion

$$S(\exp tx) = t^k(S^0(x) + tS^1(x) + t^2S^2(x) + \dots), \quad k \geq 0, \quad x \in \mathfrak{n}_-.$$

In particular, regular functions S_i^j on G are now defined.

Theorem 2.1. [Pa2, Theorem 2.12]

- i) If $k_i = 2$ for $1 \leq i \leq m$, then S_i^0 is a square in $S(\mathfrak{n})$.
- ii) Set $K_i = S_i^0$ if $k_i = 1$, and $(K_i)^2 = S_i^0$ if $k_i = 2$. Then K_i is a prime H -weight polynomial of weight ϖ'_i contained in $S(\mathfrak{n})^N$ for $1 \leq i \leq m$.

Corollary 2.2. *After suitable reordering of indices, we have (up to scalar multiplication) $\mu_i = \varpi'_i$ and $\xi_i = K_i$ for $1 \leq i \leq m$.*

PROOF. It is easy to check that ϖ'_i is a dominant weight in $\mathbb{Z}\mathcal{B}$. Then our claim follows from Kostant's characterization of ξ_1, \dots, ξ_m as the unique (up to scalars) prime polynomials in $S(\mathfrak{n})^N$ which are H -weight polynomials of dominant weights belonging to $\mathbb{Z}\mathcal{B}$. \square

We now fix explicit expressions for ξ_i in all classical root systems.

i) $\Phi = A_{n-1}$. Let $V = \mathbb{C}^n$ be the natural representation of $G = \mathrm{SL}_n(\mathbb{C})$. Then $V_i = \bigwedge^i V$ for all i from 1 to $n-1$. If e_1, \dots, e_n is the standard basis of V , and e_1^*, \dots, e_n^* is the dual basis of V^* , then $v_i = e_1 \wedge \dots \wedge e_i$ is a highest-weight vector of V_i , and $l_i = e_{n-i+1}^* \wedge \dots \wedge e_n^*$ is a highest-weight vector of V_i^* . This implies that $S_i^0(x)$ is the lower left $(i \times i)$ -minor of a matrix $x \in \mathfrak{n}_-$. Using (3) we can set

$$\xi_i = \begin{vmatrix} e_{1,n-i+1} & \cdots & e_{1,n-1} & e_{1,n} \\ e_{2,n-i+1} & \cdots & e_{2,n-1} & e_{2,n} \\ \vdots & \ddots & \vdots & \vdots \\ e_{i,n-i+1} & \cdots & e_{i,n-1} & e_{i,n} \end{vmatrix} = \begin{vmatrix} e_{\epsilon_1 - \epsilon_{n-i+1}} & \cdots & e_{\epsilon_1 - \epsilon_{n-1}} & e_{\epsilon_1 - \epsilon_n} \\ e_{\epsilon_2 - \epsilon_{n-i+1}} & \cdots & e_{\epsilon_2 - \epsilon_{n-1}} & e_{\epsilon_2 - \epsilon_n} \\ \vdots & \ddots & \vdots & \vdots \\ e_{\epsilon_i - \epsilon_{n-i+1}} & \cdots & e_{\epsilon_i - \epsilon_{n-1}} & e_{\epsilon_i - \epsilon_n} \end{vmatrix}. \quad (4)$$

ii) $\Phi = C_n$. Let $V = \mathbb{C}^{2n}$ be the natural representation of $G = \mathrm{Sp}_{2n}(\mathbb{C})$. Then $V_i = \bigwedge^i V$ for all i from 1 to $n-1 = m-1$. If $e_1, \dots, e_n, e_{-n}, \dots, e_{-1}$ is the standard basis of V , and $e_1^*, \dots, e_n^*, e_{-n}^*, \dots, e_{-1}^*$ is the dual basis of V^* , then $v_i = e_1 \wedge \dots \wedge e_i$ is a highest-weight vector of V_i , and $l_i = e_{-i}^* \wedge \dots \wedge e_{-1}^*$ is a highest-weight vector of V_i^* . Consequently, $S_i^0(x)$ is again the lower left $(i \times i)$ -minor of a matrix $x \in \mathfrak{n}_-$, and for $1 \leq i \leq m-1$ we can set

$$\xi_i = \begin{vmatrix} e_{\epsilon_1 + \epsilon_i} & \cdots & e_{\epsilon_1 + \epsilon_3} & e_{\epsilon_1 + \epsilon_2} & 2e_{2\epsilon_1} \\ e_{\epsilon_2 + \epsilon_i} & \cdots & e_{\epsilon_2 + \epsilon_3} & 2e_{2\epsilon_2} & e_{\epsilon_1 + \epsilon_2} \\ e_{\epsilon_3 + \epsilon_i} & \cdots & 2e_{2\epsilon_3} & e_{\epsilon_2 + \epsilon_3} & e_{\epsilon_1 + \epsilon_3} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 2e_{2\epsilon_i} & \cdots & e_{\epsilon_3 + \epsilon_i} & e_{\epsilon_2 + \epsilon_i} & e_{\epsilon_1 + \epsilon_i} \end{vmatrix}. \quad (5)$$

We claim that ξ_m can also be defined via formula (5) for $i = m$. To verify this it suffices to check that ξ_m is proportional to the m -th canonical generator for $\mathfrak{sp}_{2m+2}(\mathbb{C})$. But the latter is obvious because this generator is a prime N -invariant H -weight polynomial of weight $\varpi'_m \in \mathbb{Z}\mathcal{B}$.

iii) $\Phi = D_n$. Let $V = \mathbb{C}^{2n}$ be the natural representation of $G = \mathrm{SO}_{2n}(\mathbb{C})$. Then $V_i = \bigwedge^i V$ for all i from 1 to $n-2$. If $e_1, \dots, e_n, e_{-n}, \dots, e_{-1}$ is the standard basis of V , and $e_1^*, \dots, e_n^*, e_{-n}^*, \dots, e_{-1}^*$ is the dual basis of V^* , then $v_i = e_1 \wedge \dots \wedge e_i$ is a highest-weight vector of V_i , and $l_i = e_{-i}^* \wedge \dots \wedge e_{-1}^*$ is a highest-weight vector of V_i^* . First, if $i \leq n-2$ is even, then $k_i = 2$, so $S_i^0 = (K_i)^2$. It follows that $K_i(x)$, for $x \in \mathfrak{n}_-$, is the Pfaffian of the skew-symmetric matrix obtained from the lower left $i \times i$ submatrix of x by reordering the columns. Therefore, for even $i \leq n-2$ we set

$$\xi_i^2 = \pm \begin{vmatrix} e_{\epsilon_1 + \epsilon_i} & \cdots & e_{\epsilon_1 + \epsilon_3} & e_{\epsilon_1 + \epsilon_2} & 0 \\ e_{\epsilon_2 + \epsilon_i} & \cdots & e_{\epsilon_2 + \epsilon_3} & 0 & -e_{\epsilon_1 + \epsilon_2} \\ e_{\epsilon_3 + \epsilon_i} & \cdots & 0 & -e_{\epsilon_2 + \epsilon_3} & -e_{\epsilon_1 + \epsilon_3} \\ \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & -e_{\epsilon_3 + \epsilon_i} & -e_{\epsilon_2 + \epsilon_i} & -e_{\epsilon_1 + \epsilon_i} \end{vmatrix}. \quad (6)$$

Our normalization is such that the term $e_{\epsilon_1 + \epsilon_2} e_{\epsilon_3 + \epsilon_4} \cdots e_{\epsilon_{i-1} + \epsilon_i}$ enters ξ_i with coefficient 1.

If $i \leq n-2$ is odd and $x = \sum_{\alpha \in \Phi^+} x_\alpha e_\alpha^T \in \mathfrak{n}_-$, then

$$\begin{aligned} S_i(\exp tx) &= \begin{vmatrix} tx_{\epsilon_1 + \epsilon_i} + O(t^2) & \cdots & tx_{\epsilon_1 + \epsilon_2} + O(t^2) & 2t^2 a_1(x) + O(t^3) \\ tx_{\epsilon_2 + \epsilon_i} + O(t^2) & \cdots & 2t^2 a_2(x) + O(t^3) & -tx_{\epsilon_1 + \epsilon_2} + O(t^2) \\ \vdots & \ddots & \vdots & \vdots \\ 2t^2 a_i(x) + O(t^3) & \cdots & -tx_{\epsilon_2 + \epsilon_i} + O(t^2) & -tx_{\epsilon_1 + \epsilon_i} + O(t^2) \end{vmatrix} \\ &= t^k (S_i^0(x) + tS_i^1(x) + t^2 S_i^2(x) + \dots) \end{aligned} \quad (7)$$

for some $k \geq 0$, where $a_s(x) = \sum_{j=s+1}^n x_{\epsilon_s - \epsilon_j} x_{\epsilon_s + \epsilon_j}$. Let \mathcal{U}^i be the $(i \times i)$ -matrix defined by $(\mathcal{U}^i)_{a,b} = -(\mathcal{U}^i)_{a,b} = e_{\epsilon_a + \epsilon_{i-b+1}}$ for $a < i - b + 1$, $(\mathcal{U}^i)_{a,i-a+1} = 0$, and \mathcal{U}_s^i be the matrix obtained from \mathcal{U}^i by deleting the $(i - s + 1)$ -th row and column. Then $k = i + 1$ and we can set

$$\xi_i = \sum_{s=1}^i a_s \det \mathcal{U}_s^i, \quad (8)$$

where $a_s = \sum_{j=s+1}^n e_{\epsilon_s - \epsilon_j} e_{\epsilon_s + \epsilon_j}$.

Next, assume $i = m$ (in other words, $i = n - 1$ and n is odd, or $i = n$ and n is even). As the m -th canonical generator for $\mathfrak{so}_{2n+2}(\mathbb{C})$ is a prime N -invariant H -weight polynomial of weight $\varpi'_m \in \mathbb{Z}\mathcal{B}$, it follows that ξ_m can be also defined via formula (6) for $i = m$.

Finally, assume n is even and $i = m - 1 = n - 1$. In this case ξ_{m-1} can be defined by

$$\xi_{m-1}^2 = \pm \begin{vmatrix} e_{\epsilon_1 - \epsilon_n} & e_{\epsilon_1 + \epsilon_{n-1}} & \cdots & e_{\epsilon_1 + \epsilon_2} & 0 \\ e_{\epsilon_2 - \epsilon_n} & e_{\epsilon_2 + \epsilon_{n-1}} & \cdots & 0 & -e_{\epsilon_1 + \epsilon_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e_{\epsilon_{n-1} - \epsilon_n} & 0 & \cdots & -e_{\epsilon_2 + \epsilon_{n-1}} & -e_{\epsilon_1 + \epsilon_{n-1}} \\ 0 & -e_{\epsilon_{n-1} - \epsilon_n} & \cdots & -e_{\epsilon_2 - \epsilon_n} & -e_{\epsilon_1 - \epsilon_n} \end{vmatrix}$$

(our normalization is such that the term $e_{\epsilon_{n-1} - \epsilon_n} \xi_{m-3}$ enters ξ_{m-1} with coefficient 1) as it is easy to check that this polynomial is a prime N -invariant H -weight polynomial of weight ϖ'_{m-1} .

iv) Case $\Phi = B_n$. Let $V = \mathbb{C}^{2n+1}$ be the natural representation of $G = \mathrm{SO}_{2n+1}(\mathbb{C})$. Then $V_i = \bigwedge^i V$ for all i from 1 to $n - 1$. If $e_1, \dots, e_n, e_0, e_{-n}, \dots, e_{-1}$ is the standard basis of V , and $e_1^*, \dots, e_n^*, e_0^*, e_{-n}^*, \dots, e_{-1}^*$ is the dual basis of V^* , then $v_i = e_1 \wedge \dots \wedge e_i$ is a highest-weight vector of V_i , and $l_i = e_{-i}^* \wedge \dots \wedge e_{-1}^*$ is a highest-weight vector of V_i^* .

First, if $i \leq n - 1$ is even, then $k_i = 2$, so $S_i^0 = (K_i)^2$. It follows that ξ_i can be defined via formula (6). Second, if $i = n = m$ is even, then, arguing as above, we can define ξ_m again by formula (6) for $i = n$. Next, if $i \leq n - 1$ is odd and $x = \sum_{\alpha \in \Phi^+} x_\alpha e_\alpha^T \in \mathfrak{n}_-$, then

$$\begin{aligned} S_i(\exp tx) &= \begin{vmatrix} tx_{\epsilon_1 + \epsilon_i} + O(t^2) & \cdots & tx_{\epsilon_1 + \epsilon_2} + O(t^2) & 2t^2 b_1(x) + O(t^3) \\ tx_{\epsilon_2 + \epsilon_i} + O(t^2) & \cdots & 2t^2 b_2(x) + O(t^3) & -tx_{\epsilon_1 + \epsilon_2} + O(t^2) \\ \vdots & \ddots & \vdots & \vdots \\ 2t^2 b_i(x) + O(t^3) & \cdots & -tx_{\epsilon_2 + \epsilon_i} + O(t^2) & -tx_{\epsilon_1 + \epsilon_i} + O(t^2) \end{vmatrix} \\ &= t^k (S_i^0(x) + tS_i^1(x) + t^2 S_i^2(x) + \dots) \end{aligned} \quad (9)$$

for some $k \geq 0$, where $b_s(x) = \sum_{j=s+1}^n x_{\epsilon_s - \epsilon_j} x_{\epsilon_s + \epsilon_j} + x_{\epsilon_s}^2$. It turns out that $k = i + 1$ and we can define ξ_i via formula (8) for $b_s = \sum_{j=s+1}^n e_{\epsilon_s - \epsilon_j} e_{\epsilon_s + \epsilon_j} + e_{\epsilon_s}^2/4$. Finally, assume $i = n = m$ is odd. Arguing as above, we see that ξ_m can be defined by

$$\xi_m^2 = \pm \begin{vmatrix} e_{\epsilon_1} & e_{\epsilon_1 + \epsilon_n} & \cdots & e_{\epsilon_1 + \epsilon_2} & 0 \\ e_{\epsilon_2} & e_{\epsilon_2 + \epsilon_n} & \cdots & 0 & -e_{\epsilon_1 + \epsilon_2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ e_{\epsilon_n} & 0 & \cdots & -e_{\epsilon_2 + \epsilon_n} & -e_{\epsilon_1 + \epsilon_n} \\ 0 & -e_{\epsilon_n} & \cdots & -e_{\epsilon_2} & -e_{\epsilon_1} \end{vmatrix}$$

(our normalization is such that the term $e_{\epsilon_n} \xi_{m-2}$ enters ξ_m with coefficient 1).

For A_{n-1} and C_n , denote $\Delta_i = \sigma(\xi_i)$ for $1 \leq i \leq m$. We call Δ_i the i -th canonical generator of $Z(\mathfrak{n})$. Since all e_α involved in ξ_i (i.e., e_α which appear in a term of ξ_i) commute, we conclude that Δ_i is defined as an element of $U(\mathfrak{n})$ again by formulas (4), (5) for A_{n-1} , C_n respectively. For B_n and D_n , then denote $P_{i/2} = \sigma(\xi_i)$ when i is even, and $\mathcal{D}_{(i+1)/2} = \sigma(\xi_i)$ when i is odd. If i is even, all e_α involved in ξ_i commute, so we can conclude that P_i is defined as an element of $U(\mathfrak{n})$ again by formula (6).

2.2. Infinite-dimensional case. Let $\mathfrak{sl}_\infty(\mathbb{C})$, $\mathfrak{so}_\infty(\mathbb{C})$, $\mathfrak{sp}_\infty(\mathbb{C})$ be the three simple complex finitary countable dimensional Lie algebras as classified by A. Baranov [Ba]. Each of them can be described as follows (see for example [DP2]). Consider an infinite chain of inclusions

$$\mathfrak{g}_1 \subset \mathfrak{g}_2 \subset \dots \subset \mathfrak{g}_n \subset \dots$$

of simple Lie algebras, where $\text{rk } \mathfrak{g}_n = n$ and all \mathfrak{g}_n are of the same type A , B , C or D . Then the union $\mathfrak{g} = \bigcup \mathfrak{g}_n$ is isomorphic to $\mathfrak{sl}_\infty(\mathbb{C})$, $\mathfrak{so}_\infty(\mathbb{C})$ or $\mathfrak{sp}_\infty(\mathbb{C})$. It is always possible to choose nested Cartan subalgebras $\mathfrak{h}_n \subset \mathfrak{g}_n$, $\mathfrak{h}_n \subset \mathfrak{h}_{n+1}$, so that each root space of \mathfrak{g}_n is mapped to exactly one root space of \mathfrak{g}_{n+1} . The union $\mathfrak{h} = \bigcup \mathfrak{h}_n$ acts semisimply on \mathfrak{g} , and is by definition a *splitting Cartan subalgebra* of \mathfrak{g} . We have a root decomposition $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}^\alpha$ where Φ is the *root system* of \mathfrak{g} with respect to \mathfrak{h} and \mathfrak{g}^α are the *root spaces*. The root system Φ is simply the union of the root systems of \mathfrak{g}_n and equals one of the following infinite root systems:

$$\begin{aligned} A_\infty &= \pm\{\epsilon_i - \epsilon_j, i, j \in \mathbb{Z}_{>0}, i < j\}, \\ B_\infty &= \pm\{\epsilon_i - \epsilon_j, i, j \in \mathbb{Z}_{>0}, i < j\} \cup \pm\{\epsilon_i + \epsilon_j, i, j \in \mathbb{Z}_{>0}, i < j\} \cup \pm\{\epsilon_i, i \in \mathbb{Z}_{>0}\}, \\ C_\infty &= \pm\{\epsilon_i - \epsilon_j, i, j \in \mathbb{Z}_{>0}, i < j\} \cup \pm\{\epsilon_i + \epsilon_j, i, j \in \mathbb{Z}_{>0}, i < j\} \cup \pm\{2\epsilon_i, i \in \mathbb{Z}_{>0}\}, \\ D_\infty &= \pm\{\epsilon_i - \epsilon_j, i, j \in \mathbb{Z}_{>0}, i < j\} \cup \pm\{\epsilon_i + \epsilon_j, i, j \in \mathbb{Z}_{>0}, i < j\}. \end{aligned}$$

A *splitting Borel subalgebra* of \mathfrak{g} is a subalgebra \mathfrak{b} such that for every n , $\mathfrak{b}_n = \mathfrak{b} \cap \mathfrak{g}_n$ is a Borel subalgebra of \mathfrak{g}_n . It is well-known that any splitting Borel subalgebra is conjugate via $\text{Aut } \mathfrak{g}$ to a splitting Borel subalgebra containing \mathfrak{h} . Therefore, in what follows we restrict ourselves to considering only such Borel subalgebras \mathfrak{b} .

Recall [DP1] that a linear order on $\{0\} \cup \{\pm\epsilon_i\}$ is \mathbb{Z}_2 -linear if multiplication by -1 reverses the order. By [DP1, Proposition 3], there exists a bijection between splitting Borel subalgebras of \mathfrak{g} containing \mathfrak{h} and certain linearly ordered sets as follows.

- For A_∞ : linear orders on $\{\epsilon_i\}$;
- for B_∞ and C_∞ : \mathbb{Z}_2 -linear orders on $\{0\} \cup \{\pm\epsilon_i\}$;
- for D_∞ : \mathbb{Z}_2 -linear orders on $\{0\} \cup \{\pm\epsilon_i\}$ with the property that a minimal positive element (if it exists) belongs to $\mathbb{Z}_{>0}$.

In the sequel we denote these linear orders by \prec . To write down the above bijection, denote $\vartheta_i = \epsilon_i$, if $i > 0$, and $\vartheta_i = -\epsilon_i$, if $\epsilon_i \prec 0$ (for A_∞ , $\vartheta_i = \epsilon_i$ for all i). Then put $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, where $\mathfrak{n} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}^\alpha$ and

$$\begin{aligned} A_\infty^+ &= \{\vartheta_i - \vartheta_j, i, j \in \mathbb{Z}_{>0}, \vartheta_i \succ \vartheta_j\}, \\ B_\infty^+ &= \{\vartheta_i - \vartheta_j, i, j \in \mathbb{Z}_{>0}, \vartheta_i \succ \vartheta_j\} \cup \{\vartheta_i + \vartheta_j, i, j \in \mathbb{Z}_{>0}, \vartheta_i \succ \vartheta_j\} \cup \{\vartheta_i, i \in \mathbb{Z}_{>0}\}, \\ C_\infty^+ &= \{\vartheta_i - \vartheta_j, i, j \in \mathbb{Z}_{>0}, \vartheta_i \succ \vartheta_j\} \cup \{\vartheta_i + \vartheta_j, i, j \in \mathbb{Z}_{>0}, \vartheta_i \succ \vartheta_j\} \cup \{2\vartheta_i, i \in \mathbb{Z}_{>0}\}, \\ D_\infty^+ &= \{\vartheta_i - \vartheta_j, i, j \in \mathbb{Z}_{>0}, \vartheta_i \succ \vartheta_j\} \cup \{\vartheta_i + \vartheta_j, i, j \in \mathbb{Z}_{>0}, \vartheta_i \succ \vartheta_j\}. \end{aligned}$$

Our goal in this subsection is to describe the center $Z(\mathfrak{n})$ of the enveloping algebra $U(\mathfrak{n})$. Fix \mathfrak{n} , i.e., fix an order \prec as above. Define the subset $\mathcal{N} \subseteq \mathbb{Z}_{>0}$ by setting $\mathcal{N} = \bigcup_{k \geq 0} \mathcal{N}_k$, where $\mathcal{N}_0 = \emptyset$ and \mathcal{N}_k for $k \geq 1$ is defined inductively in the following table.

Φ	\mathcal{N}_k
A_∞	$\mathcal{N}_{k-1} \cup \{i, j\}$ if there exists a maximal element ϑ_i and a minimal element ϑ_j of $\{\vartheta_s, s \in \mathbb{Z}_{>0} \setminus \mathcal{N}_{k-1}\}$, \mathcal{N}_{k-1} otherwise
C_∞	$\mathcal{N}_{k-1} \cup \{i\}$ if there exists a maximal element ϑ_i of $\{\vartheta_s, s \in \mathbb{Z}_{>0} \setminus \mathcal{N}_{k-1}\}$, \mathcal{N}_{k-1} otherwise
B_∞ , D_∞	$\mathcal{N}_{k-1} \cup \{i, j\}$ if there exists a maximal element ϑ_i of $\{\vartheta_s, s \in \mathbb{Z}_{>0} \setminus \mathcal{N}_{k-1}\}$ and a maximal element ϑ_j of $\{\vartheta_s, s \in \mathbb{Z}_{>0} \setminus (\mathcal{N}_{k-1} \cup \{i\})\}$, \mathcal{N}_{k-1} otherwise

Example 2.3. i) Let $\Phi = A_\infty$. If $\epsilon_1 \succ \epsilon_3 \succ \dots \succ \epsilon_4 \succ \epsilon_2$, then $\mathcal{N} = \mathbb{Z}_{>0}$. If $\epsilon_1 \succ \epsilon_2 \succ \epsilon_3 \succ \dots$, then $\mathcal{N} = \emptyset$. ii) Let $\Phi \neq A_\infty$ and $\epsilon_1 \succ \epsilon_2 \succ \dots \succ 0 \succ \dots \succ -\epsilon_2 \succ -\epsilon_1$. Then $\mathcal{N} = \mathbb{Z}_{>0}$.

Now we can define the (possibly infinite) Kostant cascade corresponding to \mathfrak{n} . Namely, to each \mathcal{N}_k such that $\mathcal{N}_{k-1} \subsetneq \mathcal{N}_k$, we assign the root

$$\beta_k = \begin{cases} \vartheta_i - \vartheta_j, & \text{if } \Phi = A_\infty \text{ and } \mathcal{N}_k \setminus \mathcal{N}_{k-1} = \{i, j\}, i \succ j, \\ \vartheta_i + \vartheta_j, & \text{if } \Phi = B_\infty \text{ or } D_\infty \text{ and } \mathcal{N}_k \setminus \mathcal{N}_{k-1} = \{i, j\}, i \succ j, \\ 2\vartheta_i, & \text{if } \Phi = C_\infty \text{ and } \mathcal{N}_k \setminus \mathcal{N}_{k-1} = \{i\}, \end{cases}$$

and put $\mathcal{B} = \{\beta_k, k \geq 1, \mathcal{N}_{k-1} \subsetneq \mathcal{N}_k\}$. Note that \mathcal{B} is a strongly orthogonal subset of Φ^+ ; however it is not necessarily maximal with this property.

Definition 2.4. The subset \mathcal{B} is called the *Kostant cascade* corresponding to \mathfrak{n} .

Example 2.5. i) If $\Phi = A_\infty$ and $\epsilon_1 \succ \epsilon_3 \succ \dots \succ \epsilon_4 \succ \epsilon_2$, then $\mathcal{B} = \{\epsilon_1 - \epsilon_2, \epsilon_3 - \epsilon_4, \epsilon_5 - \epsilon_6, \dots\}$. ii) If $\Phi \neq A_\infty$ and $\epsilon_1 \succ \epsilon_2 \succ \dots \succ 0 \succ \dots \succ -\epsilon_2 \succ -\epsilon_1$, then

$$\mathcal{B} = \begin{cases} \{\epsilon_1 + \epsilon_2, \epsilon_3 + \epsilon_4, \epsilon_5 + \epsilon_6, \dots\} & \text{for } B_\infty \text{ and } D_\infty, \\ \{2\epsilon_1, 2\epsilon_2, 2\epsilon_3, \dots\} & \text{for } C_\infty. \end{cases}$$

To each finite non-empty subset $M \subset \mathbb{Z}_{>0}$, one can assign a root subsystem Φ_M of Φ and a subalgebra \mathfrak{n}_M of \mathfrak{n} by putting $\Phi_M = \Phi \cap \langle \epsilon_i, i \in M \rangle_{\mathbb{R}}$, $\mathfrak{n}_M = \bigoplus_{\alpha \in \Phi_M^+} \mathfrak{g}^\alpha$, $\Phi_M^+ = \Phi_M \cap \Phi^+$. Then the subsystem Φ_M is isomorphic to the root system Φ_n of \mathfrak{g}_n for $n = |M|$; we denote this isomorphism by $j_M: \Phi_n \rightarrow \Phi_M$, $\epsilon_i \mapsto \vartheta_{a_i}$, where $M = \{a_1, \dots, a_n\}$, $\vartheta_{a_1} \succ \dots \succ \vartheta_{a_n}$. Besides, \mathfrak{n}_M is isomorphic as a Lie algebra to the maximal nilpotent subalgebra \mathfrak{n}_n of \mathfrak{g}_n considered in the previous subsection. Note also that $\mathfrak{n} = \varinjlim \mathfrak{n}_M$. Here, for $M \subseteq M'$, the monomorphism $i_{M, M'}: \mathfrak{n}_M \hookrightarrow \mathfrak{n}_{M'}$ is just the inclusion. Further, it is easy to see that there exist isomorphisms $\phi_M: \mathfrak{n}_n \rightarrow \mathfrak{n}_M$, $M \subset \mathbb{Z}_{>0}$, $n = |M|$, such that, for $M \subseteq M'$, $i_{M, M'} \circ \phi_M$ is just the restriction of $\phi_{M'}$ to $\mathfrak{n}_n \subset \mathfrak{n}_{n+1}$, and, for $\alpha \in \Phi_n^+$, $\phi_M(e_\alpha)$ is a root vector corresponding to the root $j_M(\alpha)$; we denote it by $f_{j_M(\alpha)}$.

We are now ready to write down a set of generators of $Z(\mathfrak{n})$. Namely, suppose that $q \geq 1$ and $|\mathcal{B}| \geq q$. Let M be a finite subset of $\mathbb{Z}_{>0}$ such that $\mathcal{N}_q \subseteq M$. The isomorphism ϕ_M gives rise to an isomorphism $U(\mathfrak{n}_n) \rightarrow U(\mathfrak{n}_M)$, $n = |M|$. Slightly abusing notation, we denote the images of Δ_q and P_q (as elements of $U(\mathfrak{n}_n)$ whenever defined) in $U(\mathfrak{n}_M)$ by the same letters. Then for A_∞ (respectively, for C_∞), $\Delta_q \in U(\mathfrak{n}_M)$ is given by formula (4) (respectively, (5)) for $i = q$ with $f_{j_M(\alpha)}$ instead of e_α . Similarly, for B_∞ and D_∞ , $P_q \in U(\mathfrak{n}_M)$ is given by formula (6) for $i = 2q$ with $f_{j_M(\alpha)}$ instead of e_α . It is important that $\Delta_q, P_q \in U(\mathfrak{n}_M)$ depend only on q but not on M . Moreover, it is clear from the finite-dimensional theory that Δ_q (respectively, P_q) belong to the center of $U(\mathfrak{n})$ for A_∞ and C_∞ (respectively, for B_∞ and D_∞).

Our first main result is as follows.

Theorem 2.6. *If $\Phi = A_\infty, C_\infty$ (respectively, $\Phi = B_\infty, D_\infty$), then Δ_q (respectively, P_q), $q \leq |\mathcal{B}|$, generate $Z(\mathfrak{n})$ as an algebra. In particular, $Z(\mathfrak{n})$ is a polynomial ring in $|\mathcal{B}|$ variables.*

PROOF. Let $\Phi = A_\infty, C_\infty$, and $t \in Z(\mathfrak{n})$. We need to prove that t is a polynomial in Δ_q , $q \leq |\mathcal{B}|$. Let M be the minimal subset of $\mathbb{Z}_{>0}$ for which $t \in U(\mathfrak{n}_M)$ (in particular, if $\Phi = A_\infty$, then $|M|$ is even), and k be the maximal number such that $\mathcal{N}_k \subseteq M$ and $\mathcal{N}_{k-1} \subsetneq \mathcal{N}_k$. If $M = \mathcal{N}_k$, there is nothing to prove as the center of $U(\mathfrak{n}_M)$ is generated by $\Delta_1, \dots, \Delta_k$. Therefore, assume that $\mathcal{N}_k \subsetneq M$. More precisely, let

$$\mathcal{N}_k = \begin{cases} \{i_1, \dots, i_k, j_1, \dots, j_k\}, & \text{if } \Phi = A_\infty, \\ \{i_1, \dots, i_k\}, & \text{if } \Phi = C_\infty, \end{cases}$$

$M \setminus \mathcal{N}_k = \{s_1, \dots, s_r\}$, $n = |M|$, where $\vartheta_{i_1} \succ \dots \succ \vartheta_{i_k} \succ \vartheta_{s_1} \succ \dots \succ \vartheta_{s_r} \succ \vartheta_{j_k} \succ \dots \succ \vartheta_{j_1}$. In the rest of the proof we show that the assumption $\mathcal{N}_k \subsetneq M$ is contradictory.

Put $l = n/2 = k + r/2$ (respectively, $l = n = k + r$) for A_∞ (respectively, for C_∞) and extend the set $\{\Delta_1, \dots, \Delta_k\}$ to a set $\{\Delta_1, \dots, \Delta_k, \Delta'_{k+1}, \Delta'_{k+2}, \dots, \Delta'_l\}$ of generators of $Z(\mathfrak{n}_M)$ by letting Δ'_i be the image of the i -th canonical generator of $Z(\mathfrak{n}_n)$ under the isomorphism $\phi_M: U(\mathfrak{n}_n) \rightarrow U(\mathfrak{n}_M)$. For instance, let $k = 2, r = 4$ for A_∞ , or $k = 2, r = 2$ for C_∞ . Then Δ'_4 has the following form.

$\Phi = A_\infty, k = 2, r = 4$	$\Phi = C_\infty, k = 2, r = 2$
$\begin{vmatrix} f_{\vartheta_{i_1} - \vartheta_{s_3}} & f_{\vartheta_{i_1} - \vartheta_{s_4}} & f_{\vartheta_{i_1} - \vartheta_{j_2}} & f_{\vartheta_{i_1} - \vartheta_{j_1}} \\ f_{\vartheta_{i_2} - \vartheta_{s_3}} & f_{\vartheta_{i_2} - \vartheta_{s_4}} & f_{\vartheta_{i_2} - \vartheta_{j_2}} & f_{\vartheta_{i_2} - \vartheta_{j_1}} \\ f_{\vartheta_{s_1} - \vartheta_{s_3}} & f_{\vartheta_{s_1} - \vartheta_{s_4}} & f_{\vartheta_{s_1} - \vartheta_{j_2}} & f_{\vartheta_{s_1} - \vartheta_{j_1}} \\ f_{\vartheta_{s_2} - \vartheta_{s_3}} & f_{\vartheta_{s_2} - \vartheta_{s_4}} & f_{\vartheta_{s_2} - \vartheta_{j_2}} & f_{\vartheta_{s_2} - \vartheta_{j_1}} \end{vmatrix}$	$\begin{vmatrix} f_{\vartheta_{i_1} + \vartheta_{s_2}} & f_{\vartheta_{i_1} + \vartheta_{s_1}} & f_{\vartheta_{i_1} + \vartheta_{i_2}} & 2f_{2\vartheta_{i_1}} \\ f_{\vartheta_{i_2} + \vartheta_{s_2}} & f_{\vartheta_{i_2} + \vartheta_{s_1}} & 2f_{2\vartheta_{i_2}} & f_{\vartheta_{i_1} + \vartheta_{i_2}} \\ f_{\vartheta_{s_1} + \vartheta_{s_2}} & 2f_{2\vartheta_{s_1}} & f_{\vartheta_{i_2} + \vartheta_{s_1}} & f_{\vartheta_{i_1} + \vartheta_{s_1}} \\ 2f_{2\vartheta_{s_2}} & f_{\vartheta_{s_1} + \vartheta_{s_2}} & f_{\vartheta_{i_2} + \vartheta_{s_2}} & f_{\vartheta_{i_1} + \vartheta_{s_2}} \end{vmatrix}$

Now we write $t = t' + t''$, where t' belongs to the ideal of $Z(\mathfrak{n}_M)$ generated by $\Delta'_{k+1}, \dots, \Delta'_l$, and t'' belongs to the subalgebra of $Z(\mathfrak{n}_M)$ generated by $\Delta_1, \dots, \Delta_k$. Note that $t'' \in Z(\mathfrak{n})$, hence $t' \in Z(\mathfrak{n})$. Moreover, $t' \neq 0$ by the assumption that $\mathcal{N}_k \subsetneq M$.

The definition of k implies that if $\Phi = C_\infty$ then ϑ_{s_1} is not a \prec -maximal element of the set $\{\vartheta_s, s \in \mathbb{Z}_{>0} \setminus \mathcal{N}_k\}$. Similarly, if $\Phi = A_\infty$ then at least one of the following holds: ϑ_{s_1} is not a \prec -maximal element of $\{\vartheta_s, s \in \mathbb{Z}_{>0} \setminus \mathcal{N}_k\}$, or ϑ_{s_r} is not a \prec -minimal element of $\{\vartheta_s, s \in \mathbb{Z}_{>0} \setminus \mathcal{N}_k\}$. In the sequel we assume that the former condition is satisfied (the case of the latter condition is similar). Then, in both cases $\Phi = A_\infty$ and $\Phi = C_\infty$, there exists $s_0 \in \mathbb{Z}_{>0} \setminus M$ such that $\vartheta_{i_k} \succ \vartheta_{s_0} \succ \vartheta_{s_1}$. Denote $M_0 = M \cup \{s_0\}$, $\mathfrak{n}_0 = \mathfrak{n}_{M_0}$ and $N_0 = \exp \mathfrak{n}_0$. We have $t' \in Z(\mathfrak{n}_0)$. Put $\xi'_i = \sigma_0^{-1}(\Delta'_i)$, $i = k+1, \dots, l$, where $\sigma_0: S(\mathfrak{n}_0) \rightarrow U(\mathfrak{n}_0)$ is the symmetrization map. Note that $d' = \sigma_0^{-1}(t') \neq 0$ as $t' \neq 0$, and that d' belongs to the ideal of $S(\mathfrak{n}_0)^{N_0}$ generated by $\xi'_{k+1}, \dots, \xi'_l$.

Given $\alpha \in \Phi^+$, denote by f_α^* the linear form on \mathfrak{n} (or on \mathfrak{n}_M , if $\alpha \in \Phi_M$) such that $f_\alpha^*(f_\beta) = \delta_{\alpha, \beta}$ (the Kronecker delta). Furthermore, let R_0 be the subset of \mathfrak{n}_0^* consisting of all elements of the form $\lambda = \sum_{\beta \in \mathcal{B}_0} \lambda_\beta f_\beta^*$, $\lambda_\beta \in \mathbb{C}^\times$, where

$$\mathcal{B}_0 = \begin{cases} \{\vartheta_{i_1} - \vartheta_{j_1}, \dots, \vartheta_{i_k} - \vartheta_{j_k}, \vartheta_{s_0} - \vartheta_{s_r}, \vartheta_{s_1} - \vartheta_{s_{r-1}}, \dots, \vartheta_{s_{r/2-1}} - \vartheta_{s_{r/2+1}}\} & \text{for } A_\infty, \\ \{2\vartheta_{i_1}, \dots, 2\vartheta_{i_k}, 2\vartheta_{s_0}, 2\vartheta_{s_1}, \dots, 2\vartheta_{s_r}\} & \text{for } C_\infty. \end{cases}$$

Denote by X_0 the union of the coadjoint N_0 -orbits of all linear forms from R_0 . As pointed out in Subsection 2.1, X_0 is a Zariski dense subset of \mathfrak{n}_0^* by a result of Kostant.

If $\Phi = A_\infty$ and $\lambda \in R_0$, then $\lambda(f_\alpha) = 0$ for each f_α from the last row of ξ'_i for $i = k+1, \dots, l$. Therefore, $\xi'_i(\lambda) = 0$ for all $\lambda \in R_0$. Thus, $d'(\lambda) = 0$ for all $\lambda \in R_0$. However, since d' is N_0 -invariant, we obtain that the restriction of d' to X_0 equals zero. As X_0 is Zariski dense in \mathfrak{n}_0^* , we conclude that $d' = 0$, and consequently that $t' = 0$. This contradiction completes the proof for $\Phi = A_\infty$.

Here is an illustration of the vanishing of ξ'_i on R_0 . Let $k = 2, r = 4$. On the picture below the boxes from \mathcal{B}_0 are marked by \otimes , and the boxes corresponding to the variables involved in ξ'_4 are grey:

	i_1	i_2	s_0	s_1	s_2	s_3	s_4	j_2	j_1
i_1									
i_2									
s_0									
s_1									
s_2									
s_3									
s_4									
j_2									
j_1									

$$\xi'_4 = \begin{vmatrix} f_{\vartheta_{i_1} - \vartheta_{s_3}} & f_{\vartheta_{i_1} - \vartheta_{s_4}} & f_{\vartheta_{i_1} - \vartheta_{j_2}} & f_{\vartheta_{i_1} - \vartheta_{j_1}} \\ f_{\vartheta_{i_2} - \vartheta_{s_3}} & f_{\vartheta_{i_2} - \vartheta_{s_4}} & f_{\vartheta_{i_2} - \vartheta_{j_2}} & f_{\vartheta_{i_2} - \vartheta_{j_1}} \\ f_{\vartheta_{s_1} - \vartheta_{s_3}} & f_{\vartheta_{s_1} - \vartheta_{s_4}} & f_{\vartheta_{s_1} - \vartheta_{j_2}} & f_{\vartheta_{s_1} - \vartheta_{j_1}} \\ f_{\vartheta_{s_2} - \vartheta_{s_3}} & f_{\vartheta_{s_2} - \vartheta_{s_4}} & f_{\vartheta_{s_2} - \vartheta_{j_2}} & f_{\vartheta_{s_2} - \vartheta_{j_1}} \end{vmatrix}.$$

If $\Phi = C_\infty$, we write $t' = t_1 + t_2$ where t_2 belongs to the subalgebra of $Z(\mathfrak{n}_0)$ generated by $\Delta_1, \dots, \Delta_k$, and $t_1 \neq 0$ belongs to the ideal of $Z(\mathfrak{n}_0)$ generated by all other generators of $Z(\mathfrak{n}_0)$. Then $t_2 \in Z(\mathfrak{n})$, whence $t_1 \in Z(\mathfrak{n})$. Denote $d_1 = \sigma_0^{-1}(t_1) \neq 0$, $d_2 = \sigma_0^{-1}(t_2)$. By definition, d_2 belongs to the ideal of the algebra $S(\mathfrak{n}_0)^{N_0}$ generated by the first k canonical generators of $S(\mathfrak{n}_0)^{N_0}$. At the same time, d' depends only on the f_α 's from \mathfrak{n}_M . Since $d_1 \neq 0$, there exists $\lambda \in R_0$ such that $d_1(\lambda) \neq 0$. On the other hand, set $\lambda_0 = \lambda - \lambda_{2\vartheta_{s_0}} f_{2\vartheta_{s_0}}^*$. Then $d'(\lambda) = d'(\lambda_0)$ and $d_2(\lambda) = d_2(\lambda_0)$ by the definitions of d' and d_2 . Therefore $d_1(\lambda) = d_1(\lambda_0)$. However, $d_1(\lambda_0) = 0$, because if $i > k$, then the value of the i -th canonical generator of $S(\mathfrak{n}_0)^{N_0}$ on λ_0 equals zero. This contradiction completes the proof for $\Phi = C_\infty$.

Assume now that $\Phi = D_n$ and t is a central element of $U(\mathfrak{n})$. We claim that t belongs to the subalgebra of $U(\mathfrak{n})$ generated by P_q for $q \leq |\mathcal{B}|$. Let $M \subset \mathbb{Z}_{>0}$ be a minimal finite set for which $t \in U(\mathfrak{n}_M)$ and $|M| = 2n$ is even, and let k be maximal such that $\mathcal{N}_k \subseteq M$ and $\mathcal{N}_{k-1} \not\subseteq \mathcal{N}_k$. Denote $M \setminus \mathcal{N}_k = \{i_{k+1}, j_{k+1}, \dots, i_n, j_n\}$, where $\vartheta_{i_{k+1}} \succ \vartheta_{j_{k+1}} \succ \dots \succ \vartheta_{i_n} \succ \vartheta_{j_n}$ (if $n = k$, then $M = \mathcal{N}_k$). Set $N_M = \exp \mathfrak{n}_M$,

$$\begin{aligned} \mathcal{B}_M &= \{\vartheta_{i_1} - \vartheta_{j_1}, \vartheta_{i_1} + \vartheta_{j_1}, \dots, \vartheta_{i_n} - \vartheta_{j_n}, \vartheta_{i_n} + \vartheta_{j_n}\}, \\ R_M &= \{t = \sum_{\beta \in \mathcal{B}} t_\beta f_\beta^* \mid t_\beta \neq 0 \text{ for all } \beta\}. \end{aligned}$$

Let X_M be the union of all N_M -orbits of elements from R_M . Then X_M is Zariski dense in \mathfrak{n}_M^* .

Denote the canonical generators of $S(\mathfrak{n}_M)^{N_M}$ by ξ_1, \dots, ξ_{2n} . Then $Z(\mathfrak{n}_M)$ is generated as an algebra by $P_i = \sigma_M(\xi_{2i})$, $\mathcal{D}_i = \sigma_M(\xi_{2i-1})$ for $1 \leq i \leq n$, where $\sigma: S(\mathfrak{n}_M) \rightarrow U(\mathfrak{n}_M)$ is the symmetrization map. Further, set for simplicity $d_i = \xi_{2i-1}$, $1 \leq i \leq n$. Using (8) one checks that for $t \in R_M$

$$\begin{aligned} d_s(t) &= t_{\vartheta_{i_1} + \vartheta_{j_1}}^2 \dots t_{\vartheta_{i_{s-1}} + \vartheta_{j_{s-1}}}^2 t_{\vartheta_{i_s} - \vartheta_{j_s}} t_{\vartheta_{i_s} + \vartheta_{j_s}}, \quad 1 \leq s \leq n-1, \\ d_n(t) &= t_{\vartheta_{i_1} + \vartheta_{j_1}} \dots t_{\vartheta_{i_{n-1}} + \vartheta_{j_{n-1}}} t_{\vartheta_{i_n} - \vartheta_{j_n}}. \end{aligned} \tag{10}$$

Assume b is a positive integer such that there exists $a \in \mathbb{Z}_{>0} \setminus M$ satisfying $\vartheta_b \succ \vartheta_a$. We can express t as $t = t' + t''$, where t' belongs to the ideal of $Z(\mathfrak{n}_M)$ generated by $\mathcal{D}_1, \dots, \mathcal{D}_b$, and t'' belongs to the subalgebra of $Z(\mathfrak{n}_M)$ generated by all remaining generators of $Z(\mathfrak{n}_M)$. Let $d = \sigma_M^{-1}(t)$, $d' = \sigma_M^{-1}(t')$, $d'' = \sigma_M^{-1}(t'')$, so $d = d' + d''$. If $t' \neq 0$, then $d' \neq 0$. Let c be the minimal among all numbers from 1 to b such that the variable d_c appears in d' . Since $d' \neq 0$, there exists $\lambda = \sum_{\beta \in \mathcal{B}_M} \lambda_\beta f_\beta^* \in R_M$ for which $d'(\lambda) \neq 0$. Consider the set $Y = \lambda + \mathbb{C} f_{\vartheta_{i_c} - \vartheta_{j_c}}^*$. Obviously, Y is a one-dimensional affine subspace of \mathfrak{n}_M^* , and the restriction of d' to Y is a nonzero polynomial in one variable. Clearly, we can choose λ so that this polynomial is of positive degree.

Now, put $M_a = M \cup \{a\}$, $N_a = \exp \mathfrak{n}_{M_a}$, $\mu = \lambda + f_{\vartheta_{i_c} + \vartheta_a}^* \in \mathfrak{n}_{M_a}^*$. Pick $s \in \mathbb{C}$ and put also $g = \exp(s f_{\vartheta_{j_c} + \vartheta_a}) \in N_a$, $\mu' = g \cdot \mu$. Set $\Phi_a = \Phi_{M_a}$. One can easily check that $\mu'(f_\alpha) = \mu(f_\alpha)$ for all $\alpha \in \Phi_a^+ \setminus \{\vartheta_{i_c} - \vartheta_{j_c}\}$, and $\mu'(f_{\vartheta_{i_c} - \vartheta_{j_c}}) = \mu(f_{\vartheta_{i_c} - \vartheta_{j_c}}) + s$. Since $d \in S(\mathfrak{n}_{M_a})^{N_a}$, we obtain $d(\mu') = d(\mu)$ for all $s \in \mathbb{C}$. On the other hand, from the definition of d'' we see that $d''(\mu') = d''(\mu)$. Therefore, $d'(\mu') = d'(\mu)$. Define μ'' as the restriction of μ' to \mathfrak{n}_M . Then μ'' belongs to Y , and $d'(\mu'') = d'(\mu') = d'(\mu) = d'(\lambda)$ as $d' \in S(\mathfrak{n}_M)$. Thus, the restriction of d' to Y is constant, a contradiction. We conclude that $d' = 0$, and consequently that $t' = 0$.

The above implies that it is sufficient to show that $M = \mathcal{N}_k$. Indeed, if $M = \mathcal{N}_k$, then b can be chosen as k , and by the above t is a polynomial in P_i , $1 \leq i \leq k$. Assume, to the contrary, that $M \neq \mathcal{N}_k$, so $n > k$. Then there exists $s_0 \in \mathbb{Z}_{>0} \setminus M$ such that $\vartheta_{j_k} \succ \vartheta_{s_0} \succ \vartheta_{j_{k+1}}$. As we already know, t belongs to the subalgebra of $Z(\mathfrak{n}_M)$ generated by P_1, \dots, P_k , and by P_i, \mathcal{D}_i for $k+1 \leq i \leq n$. We can express t as $t = \tilde{t} + t_0$, where $\tilde{t} \neq 0$ lies in the ideal of $Z(\mathfrak{n}_M)$ generated by P_i, \mathcal{D}_i for $k+1 \leq i \leq n$, and t_0 lies in the subalgebra of $Z(\mathfrak{n}_M)$ generated by P_1, \dots, P_k . If $M_0 = M \cup \{s_0\}$ and $\mathfrak{n}_0 = \mathfrak{n}_{M_0}$, then $\tilde{t} \in Z(\mathfrak{n}_0)$.

Next, we write \tilde{t} as $\tilde{t} = \tilde{t}_1 + \tilde{t}_2$, where \tilde{t}_2 lies in the subalgebra of $Z(\mathfrak{n}_0)$ generated by its first $2k$ canonical generators, and \tilde{t}_1 lies in the ideal of $Z(\mathfrak{n}_0)$ generated by the remaining generators of $Z(\mathfrak{n}_0)$. If $\sigma_0: S(\mathfrak{n}_0) \rightarrow U(\mathfrak{n}_0)$ is the symmetrization map, we put $\tilde{d} = \sigma_0^{-1}(\tilde{t})$, $\tilde{d}_1 = \sigma_0^{-1}(\tilde{t}_1)$, $\tilde{d}_2 = \sigma_0^{-1}(\tilde{t}_2)$. Let

R_0 be the subset of \mathfrak{n}_0^* consisting of all elements of the form $\lambda = \sum_{\beta \in \mathcal{B}_0} \lambda_\beta f_\beta^*$, $\lambda_\beta \in \mathbb{C}^\times$, where \mathcal{B}_0 is the Kostant cascade for \mathfrak{n}_0 . Note that the $(2k+1)$ -th root in \mathcal{B}_0 equals $\vartheta_{s_0} + \vartheta_{i_{k+1}}$.

Denote by X_0 the union of the coadjoint N_0 -orbits of all linear forms from R_0 . Then X_0 is a Zariski dense subset of \mathfrak{n}_0^* . Assume $\tilde{d}_1 \neq 0$. It is easy to check that there exists $\lambda \in R_0$ such that $\tilde{d}_1(\lambda) \neq 0$. Set $\lambda_0 = \lambda - \lambda_{\vartheta_{s_0} + \vartheta_{i_{k+1}}} f_{\vartheta_{s_0} + \vartheta_{i_{k+1}}}^*$. Then $\tilde{d}(\lambda) = \tilde{d}(\lambda_0)$ and $\tilde{d}_2(\lambda) = \tilde{d}_2(\lambda_0)$ by definition of \tilde{d} and \tilde{d}_2 . Hence $\tilde{d}_1(\lambda) = \tilde{d}_1(\lambda_0) = 0$, a contradiction. Thus, $\tilde{d}_1 = 0$, so $\tilde{d} = \tilde{d}_2$ belongs to the subalgebra of $Z(\mathfrak{n}_0)$ generated by its first $2k$ canonical generators. Finally, consider the affine space $Z = \sum_{r=1}^k (\mathbb{C}e_{\vartheta_{i_r} - \vartheta_{j_r}} + \mathbb{C}e_{\vartheta_{i_r} + \vartheta_{j_r}}) = \langle e_\beta, \beta \in \mathcal{B}_M \cap \mathcal{B}_0 \rangle_{\mathbb{C}}$. Let \tilde{d}_Z be the restriction of \tilde{d} to Z . Since $\tilde{d} = \tilde{d}_2$, if $\tilde{d} \neq 0$ then \tilde{d}_Z is a nonzero polynomial of positive degree. But it follows from the definition of \tilde{d} that \tilde{d}_Z is zero (see (10)). This shows that $M = \mathcal{N}_k$, and the proof for $\Phi = D_\infty$ is complete.

The proof for $\Phi = B_\infty$ is similar and we skip it. \square

3. Centrally generated ideals of $U(\mathfrak{n})$

3.1. Finite-dimensional case. Throughout this subsection \mathfrak{g} and \mathfrak{n} are as in Subsection 2.1. By definition, an ideal $J \subseteq U(\mathfrak{n})$ is *primitive* if J is the annihilator of a simple \mathfrak{n} -module. Here we describe all primitive centrally generated ideals of $U(\mathfrak{n})$, i.e., all primitive ideals J generated (as ideals) by its intersection $J \cap Z(\mathfrak{n})$ with the center $Z(\mathfrak{n})$ of $U(\mathfrak{n})$.

In the 1960s A. Kirillov, B. Kostant and J.-M. Souriau discovered that the orbits of the coadjoint action play a crucial role in the representation theory of B and N (see, e.g., [Kil], [Ki2]). The orbit method has a number of applications in the theory of integrable systems, symplectic geometry, etc. Work of J. Dixmier, M. Duflo, M. Vergne, O. Mathieu, N. Conze and R. Rentschler led to the result that the orbit method provides a nice description of primitive ideals of the universal enveloping algebra of a nilpotent Lie algebra (in particular, of \mathfrak{n}). Let us describe this in detail.

To any linear form $\lambda \in \mathfrak{n}^*$ one can assign a bilinear form β_λ on \mathfrak{n} by putting $\beta_\lambda(x, y) = \lambda([x, y])$. A subalgebra $\mathfrak{p} \subseteq \mathfrak{n}$ is a *polarization of \mathfrak{n} at λ* if it is a maximal β_λ -isotropic subspace. By [Ve], such a subalgebra always exists. Let \mathfrak{p} be a polarization of \mathfrak{n} at λ , and W be the one-dimensional representation of \mathfrak{p} defined by $x \mapsto \lambda(x)$. Then the induced representation $V = U(\mathfrak{n}) \otimes_{U(\mathfrak{p})} W$ of \mathfrak{n} is irreducible. Hence, the annihilator $J(\lambda) = \text{Ann}_{U(\mathfrak{n})} V$ is a primitive two-sided ideal of $U(\mathfrak{n})$. It turns out that $J(\lambda)$ depends only on λ and not on the choice of polarization. Further, $J(\lambda) = J(\mu)$ if and only if the coadjoint N -orbits of λ and μ coincide. Finally, the *Dixmier map*

$$\mathcal{D}: \mathfrak{n}^* \rightarrow \text{Prim } U(\mathfrak{n}), \quad \lambda \mapsto J(\lambda),$$

induces a homeomorphism between \mathfrak{n}^*/N and $\text{Prim } U(\mathfrak{n})$, where the latter set is endowed with the Jacobson topology. (See [Di2], [Di4], [BGR] for the details.)

In addition, it is well known that the following conditions on an ideal $J \subset U(\mathfrak{n})$ are equivalent [Di4, Proposition 4.7.4, Theorem 4.7.9]:

- i) J is primitive;
 - ii) J is maximal;
 - iii) the center of $U(\mathfrak{n})/J$ is trivial;
 - iv) $U(\mathfrak{n})/J$ is isomorphic to a Weyl algebra of finitely many variables.
- (11)

Recall that the Weyl algebra \mathcal{A}_r of r variables is the unital associative algebra with generators p_i, q_i for $1 \leq i \leq r$, and relations $[p_i, q_i] = 1$, $[p_i, q_j] = 0$ for $i \neq j$, $[p_i, p_j] = [q_i, q_j] = 0$ for all i, j . Furthermore, in condition (11) we have $U(\mathfrak{n})/J \cong \mathcal{A}_r$ where r equals one half of the dimension of the coadjoint N -orbit of λ , given that $J = J(\lambda)$.

Recall the definition of the Kostant cascade \mathcal{B} (Subsection 2.1) and set

$$\mathcal{B}' = \begin{cases} \mathcal{B} & \text{for } \Phi = A_{n-1}, n \text{ odd,} \\ \mathcal{B} \setminus \{\epsilon_m - \epsilon_{n-m+1}\} & \text{for } \Phi = A_{n-1}, n \text{ even, } m = n/2, \\ \bigcup_{1 \leq i < n/2} \{\epsilon_{2i-1} + \epsilon_{2i+1}\} & \text{for } \Phi = B_n \text{ or } \Phi = D_n, \\ \mathcal{B} \setminus \{2\epsilon_n\} & \text{for } \Phi = C_n. \end{cases}$$

To a map $\xi: \mathcal{B} \rightarrow \mathbb{C}$ we assign the linear form $f_\xi = \sum_{\beta \in \mathcal{B}} \xi(\beta) e_\beta^* \in \mathfrak{n}^*$. We call a form f_ξ a *Kostant form* if $\xi(\beta) \neq 0$ for any $\beta \in \mathcal{B}'$.

Let V be a simple \mathfrak{n} -module and $J = \text{Ann}_{U(\mathfrak{n})} V$ be the corresponding primitive ideal of $U(\mathfrak{n})$. By a version of Schur's Lemma [Di1], each central element of $U(\mathfrak{n})$ acts on V as a scalar operator. For A_{n-1} and C_n , let c_k be the scalar corresponding to Δ_k . For B_n and D_n , let c_k (respectively, \tilde{c}_k) be the scalar corresponding to P_k (respectively, to \mathcal{D}_k). Note that these scalars do not depend on V and are determined by J . Denote by J_c the ideal of $U(\mathfrak{n})$ generated by all $\Delta_k - c_k$ (respectively, by all $P_k - c_k$ and $\mathcal{D}_k - \tilde{c}_k$) for A_{n-1} and C_n (respectively, for B_n and D_n). Clearly, $J_c \subseteq J$. Further, since $Z(\mathfrak{n})$ is a polynomial ring and the center of $U(\mathfrak{n})/J$ is trivial, J is centrally generated if and only if $J = J_c$. Put $m' = |\mathcal{B}'|$, $m = |\mathcal{B}|$.

Our second main result is as follows.

Theorem 3.1. *Suppose Φ is of type A_{n-1} or C_n . The following conditions on a primitive ideal $J \subset U(\mathfrak{n})$ are equivalent:*

- i) J is centrally generated (or, equivalently, $J = J_c$);
- ii) the scalars $c_1, \dots, c_{m'}$ are nonzero;
- iii) $J = J(f_\xi)$ for a Kostant form $f_\xi \in \mathfrak{n}^*$.

If these conditions are satisfied, then the map ξ is reconstructed by J :

$$\xi(\beta) = (-1)^{k+1} c_k / c_{k-1}, \quad (12)$$

where $c_0 = 1$, and $\beta = \epsilon_k - \epsilon_{n-k+1}$ for $\Phi = A_{n-1}$, $\beta = 2\epsilon_k$ for $\Phi = C_n$.

We expect this theorem to be true also for B_n and D_n .

Before we prove Theorem 3.1 we prove few lemmas. We define the maps $\text{row}: \Phi^+ \rightarrow \mathbb{Z}$ and $\text{col}: \Phi^+ \rightarrow \mathbb{Z}$ by putting $\text{row}(\epsilon_i - \epsilon_j) = \text{row}(\epsilon_i + \epsilon_j) = \text{row}(2\epsilon_i) = i$, $\text{col}(\epsilon_i + \epsilon_j) = \text{col}(2\epsilon_j) = -j$, $\text{col}(\epsilon_i - \epsilon_j) = j$. Let $\mathcal{R}_i = \{\alpha \in \Phi^+ \mid \text{row}(\alpha) = i\}$. For $\alpha \in \Phi^+$, set

$$A(\alpha) = \begin{cases} \bigcup_{j+1 \leq k \leq n-i+1} \{\epsilon_j - \epsilon_k\}, & \text{if } \Phi = A_{n-1}, \alpha = \epsilon_i - \epsilon_j, j < n - i + 1, \\ \bigcup_{n-j+1 \leq k \leq i-1} \{\epsilon_k - \epsilon_i\}, & \text{if } \Phi = A_{n-1}, \alpha = \epsilon_i - \epsilon_j, j > n - i + 1, \\ \bigcup_{i \leq k \leq j-1} \{\epsilon_k + \epsilon_j\} \cup \mathcal{R}_j, & \text{if } \Phi = C_n, \alpha = \epsilon_i - \epsilon_j, \\ \bigcup_{i \leq k \leq j-1} \{\epsilon_k - \epsilon_j\}, & \text{if } \Phi = C_n, \alpha = \epsilon_i + \epsilon_j, \end{cases}$$

$$\mathcal{B}(\alpha) = \{\alpha\} \cup \{\beta \in \mathcal{B} \mid \text{row}(\beta) < \text{row}(\alpha)\},$$

$$R(\alpha) = \{\text{row}(\gamma), \gamma \in \mathcal{B}(\alpha)\}, \quad C(\alpha) = \{\text{col}(\gamma), \gamma \in \mathcal{B}(\alpha)\}.$$

Define a matrix \mathcal{U} with entries from $U(\mathfrak{n})$ by the following rule.

Φ	Size of \mathcal{U}	\mathcal{U}
A_{n-1}	$n \times n$	$\mathcal{U}_{i,j} = e_{\epsilon_i - \epsilon_j}$ for $1 \leq i < j \leq n$, $\mathcal{U}_{i,j} = 0$ otherwise
C_n	$2n \times 2n$	$\mathcal{U}_{i,j} = -\mathcal{U}_{-j,-i} = e_{\epsilon_i - \epsilon_j}$, $\mathcal{U}_{i,-j} = \mathcal{U}_{j,-i} = e_{\epsilon_i + \epsilon_j}$ for $1 \leq i < j \leq n$, $\mathcal{U}_{i,-i} = 2e_{\epsilon_{2i}}$, $1 \leq i \leq n$, $\mathcal{U}_{i,j} = 0$ otherwise

Denote by Δ_α the element of $U(\mathfrak{n})$, which equals the minor of \mathcal{U} with rows $R(\alpha)$ and columns $C(\alpha)$. Note that the variables involved in each Δ_α commute. For example, let $\Phi = A_{n-1}$, $n = 8$, $\alpha = \epsilon_3 - \epsilon_4$. On the picture below α is marked by \bullet , the roots from \mathcal{B} are marked by \otimes 's, and the roots γ such that e_γ is involved in Δ_α are grey:

	1	2	3	4	5	6	7	8
1								\otimes
2							\otimes	
3				\bullet		\otimes		
4					\otimes			
5								
6								
7								
8								

$$\Delta_{\epsilon_3 - \epsilon_4} = \begin{vmatrix} e_{\epsilon_1 - \epsilon_4} & e_{\epsilon_1 - \epsilon_7} & e_{\epsilon_1 - \epsilon_8} \\ e_{\epsilon_2 - \epsilon_4} & e_{\epsilon_2 - \epsilon_7} & e_{\epsilon_2 - \epsilon_8} \\ e_{\epsilon_3 - \epsilon_4} & e_{\epsilon_3 - \epsilon_7} & e_{\epsilon_3 - \epsilon_8} \end{vmatrix}.$$

Lemma 3.2. *Let $\alpha \in \Phi^+ \setminus \mathcal{B}$. If $\gamma \notin A(\alpha)$ then $[\Delta_\alpha, e_\gamma] = 0$. If $\gamma \in A(\alpha)$ then $[\Delta_\alpha, e_\gamma] = \pm \Delta_{\alpha+\gamma}$. More precisely, for A_{n-1} , if $j < n - i + 1$ then $[\Delta_{\epsilon_i - \epsilon_j}, e_{\epsilon_j - \epsilon_s}] = \Delta_{\epsilon_i - \epsilon_s}$ for all $\epsilon_j - \epsilon_j \in A(\epsilon_i - \epsilon_j)$, and if $j > n - i + 1$ then $[e_{\epsilon_r - \epsilon_i}, \Delta_{\epsilon_i - \epsilon_j}] = \Delta_{\epsilon_r - \epsilon_j}$ for all $\epsilon_r - \epsilon_i \in A(\epsilon_i - \epsilon_j)$. For C_n , if $\alpha = \epsilon_i \pm \epsilon_j$ then $[\Delta_\alpha, e_\gamma] = \mp \Delta_{\alpha+\gamma}$ for all $\gamma \in A(\alpha)$.*

PROOF. Let $\Phi = A_{n-1}$ (the proof for C_n is similar). Suppose $\alpha = \epsilon_i - \epsilon_j$. Consider the case $j < n - i + 1$ (the case $j > n - i + 1$ can be considered similarly). If $e_{\epsilon_p - \epsilon_q}$ is involved in Δ_α then $p \in \{1, 2, \dots, i\} = R(\alpha)$ and $q \in \{j, n - i + 2, n - i + 3, \dots, n\} = C(\alpha)$. Assume that $e_{\epsilon_r - \epsilon_s}$ and $e_{\epsilon_p - \epsilon_q}$ do not commute. Then either $s \in R(\alpha)$ or $r \in C(\alpha)$. Consider these two cases separately.

First, if $s \in R(\alpha)$, i.e., $1 \leq s \leq i$, then also $r \in R(\alpha)$ as $r < s$. Denote by $A_{p,q}$ the algebraic complement in Δ_α to an element $e_{\epsilon_p - \epsilon_q}$. Then

$$\Delta_\alpha = e_{\epsilon_s - \epsilon_j} A_{s,j} + e_{\epsilon_s - \epsilon_{n-i+2}} A_{s,n-i+2} + e_{\epsilon_s - \epsilon_{n-i+3}} A_{s,n-i+3} \dots + e_{\epsilon_s - \epsilon_n} A_{s,n}.$$

Since $e_{\epsilon_r - \epsilon_s}$ commutes with each e_γ involved in each algebraic complement, we have

$$\begin{aligned} [e_{\epsilon_r - \epsilon_s}, \Delta_\alpha] &= [e_{\epsilon_r - \epsilon_s}, e_{\epsilon_s - \epsilon_j}] A_{s,j} + [e_{\epsilon_r - \epsilon_s}, e_{\epsilon_s - \epsilon_{n-i+2}}] A_{s,n-i+2} + \dots + [e_{\epsilon_r - \epsilon_s}, e_{\epsilon_s - \epsilon_n}] A_{s,n} \\ &= e_{\epsilon_r - \epsilon_j} A_{s,j} + e_{\epsilon_r - \epsilon_{n-i+2}} A_{\epsilon_s - \epsilon_{n-i+2}} + \dots + e_{\epsilon_r - \epsilon_n} A_{s,n}. \end{aligned}$$

In other words, $[e_{\epsilon_r - \epsilon_s}, \Delta_\alpha]$ equals the minor obtained from Δ_α by replacing the s -th row by the r -th row. Thus $[e_{\epsilon_r - \epsilon_s}, \Delta_\alpha] = 0$.

Second, assume $r \in C(\alpha) = \{j, n - i + 2, n - i + 3, \dots, n\}$. Clearly,

$$\Delta_\alpha = e_{\epsilon_1 - \epsilon_r} A_{1,r} + e_{\epsilon_2 - \epsilon_r} A_{2,r} + \dots + e_{\epsilon_i - \epsilon_r} A_{i,r},$$

so

$$\begin{aligned} [\Delta_\alpha, e_{\epsilon_r - \epsilon_s}] &= [e_{\epsilon_1 - \epsilon_r}, e_{\epsilon_r - \epsilon_s}] A_{1,r} + [e_{\epsilon_2 - \epsilon_r}, e_{\epsilon_r - \epsilon_s}] A_{2,r} + \dots + [e_{\epsilon_i - \epsilon_r}, e_{\epsilon_r - \epsilon_s}] A_{i,r} \\ &= e_{\epsilon_1 - \epsilon_s} A_{1,r} + e_{\epsilon_2 - \epsilon_s} A_{2,r} + \dots + e_{\epsilon_i - \epsilon_s} A_{i,r}. \end{aligned}$$

Hence $[\Delta_\alpha, e_{\epsilon_r - \epsilon_s}]$ equals the minor obtained from Δ_α by replacing its r -th column by the column $e_{\epsilon_1 - \epsilon_s}, e_{\epsilon_2 - \epsilon_s}, \dots, e_{\epsilon_i - \epsilon_s}$. If $r \neq j$, then the latter column is a column of Δ_α , so the commutator $[\Delta_\alpha, e_{\epsilon_r - \epsilon_s}]$ is zero. If $r = j$ (and so $\epsilon_r - \epsilon_s = \epsilon_j - \epsilon_s \in A(\alpha)$), then the commutator equals $\Delta_{\epsilon_i - \epsilon_s}$ as required. \square

Denote $\Delta_0 = 1$.

Lemma 3.3. *Let $\alpha, \beta \in \Phi^+$. Then $[\Delta_\alpha, \Delta_\beta] = 0$, except the following cases:*

$$\begin{aligned} [\Delta_{\epsilon_i - \epsilon_j}, \Delta_{\epsilon_j - \epsilon_{n-i+1}}] &= (-1)^{i+1} \Delta_i \Delta_{i-1} \text{ if } \Phi = A_{n-1}, j < n - i + 1, \\ [\Delta_{\epsilon_i - \epsilon_j}, \Delta_{\epsilon_i + \epsilon_j}] &= (-1)^{i+1} \Delta_i \Delta_{i-1} \text{ if } \Phi = C_n. \end{aligned}$$

PROOF. Suppose $\Phi = A_{n-1}$ (the proof for C_n is similar). Consider the case $j < n - i + 1$ (the case $j > n - i + 1$ can be considered similarly). If $\beta = \epsilon_r - \epsilon_s \in \mathcal{B}$, then $\Delta_\beta = \Delta_r$ belongs to $Z(\mathfrak{n})$, hence Δ_α and Δ_β commute. So we may assume that $\beta \notin \mathcal{B}$, i.e., $s \neq n - r + 1$. According to Lemma 3.2, if $[\Delta_\alpha, e_\gamma] \neq 0$ then $\gamma \in A(\alpha)$. This implies that if $[\Delta_\alpha, \Delta_\beta] \neq 0$ then $\beta \in A(\alpha)$, because if $\beta \notin A(\alpha)$ then no e_γ involved in Δ_β are contained in $A(\alpha)$. Hence $\beta = \epsilon_j - \epsilon_s$ for some s such that $j+1 \leq s \leq n-i+1$.

Suppose Δ_α and Δ_β do not commute. Then, arguing as above, we see that $\alpha = \epsilon_i - \epsilon_j \in A(\beta)$. If $s < n - j + 1$ then $B(\beta)$ consists of certain roots of the form $\epsilon_s - \epsilon_t$ for $s < t$, but $i < j < s$ so $\alpha \notin A(\beta)$. If $s = n - j + 1$, then $\Delta_\beta = \Delta_j$ is a central element of $U(\mathfrak{n})$, so it commutes with Δ_α . Finally, if $s > n - j + 1$, then $A(\beta)$ consists of certain roots of the form $\epsilon_k - \epsilon_j$, $n - s + 1 \leq k \leq j - 1$. Hence $n - s + 1 \leq i$, but $s \leq n - i + 1$, so $s = n - i + 1$. Thus, if $\beta \neq \epsilon_j - \epsilon_{n-i+1}$ then $[\Delta_\alpha, \Delta_\beta] = 0$.

It remains to compute $[\Delta_\alpha, \Delta_{\epsilon_j - \epsilon_{n-i+1}}]$. One has

$$\Delta_\alpha = e_{\epsilon_1 - \epsilon_j} A_{1,j} + e_{\epsilon_2 - \epsilon_j} A_{2,j} + \dots + e_{\epsilon_i - \epsilon_j} A_{i,j}.$$

The minor $\Delta_{\epsilon_j - \epsilon_{n-i+1}}$ commutes with all variables involved in this expression except for $e_{\epsilon_i - \epsilon_j}$. Since $A_{i,j} = (-1)^{i+1} \Delta_{i-1}$, we obtain

$$[\Delta_\alpha, \Delta_{\epsilon_j - \epsilon_{n-i+1}}] = [(-1)^{i+1} e_{\epsilon_i - \epsilon_j} \Delta_{i-1}, \Delta_{\epsilon_j - \epsilon_{n-i+1}}] = (-1)^{i+1} [e_{\epsilon_i - \epsilon_j}, \Delta_{\epsilon_j - \epsilon_{n-i+1}}] \Delta_{i-1}.$$

By Lemma 3.2,

$$[e_{\epsilon_i - \epsilon_j}, \Delta_{\epsilon_j - \epsilon_{n-i+1}}] = \Delta_{\epsilon_i - \epsilon_{n-i+1}} = \Delta_i.$$

This concludes the proof. \square

Lemma 3.4. *Suppose that $c_i \neq 0$ for $1 \leq i \leq m' = |\mathcal{B}'|$. Then the ideal J_c is primitive.*

PROOF. Consider the case $\Phi = A_{n-1}$ (the proof for C_n is similar). Put $\mathcal{A} = U(\mathfrak{n})/J_c$. Given $x \in U(\mathfrak{n})$, denote by \tilde{x} its image in \mathcal{A} under the canonical projection. There is a natural partial order on Φ^+ : $\alpha > \beta$ if $\alpha - \beta$ is a sum of positive roots. Note that for $k \geq 2$ we have

$$\Delta_k = \pm e_{\epsilon_k - \epsilon_{n-k+1}} \Delta_{k-1} + \text{terms containing only } e_\alpha \text{ for } \alpha > \epsilon_k - \epsilon_{n-k+1}.$$

It follows that in \mathcal{A} we can write any \tilde{e}_β for $\beta \in \mathcal{B}$ as a polynomial in \tilde{e}_α for $\alpha \in \Phi^+ \setminus \mathcal{B}$. In other words, \mathcal{A} is generated as an algebra by \tilde{e}_α for $\alpha \in \Phi^+ \setminus \mathcal{B}$.

Similarly, given $\alpha = \epsilon_i - \epsilon_j \in \Phi^+ \setminus \mathcal{B}$, we have

$$\Delta_\alpha = \pm e_\alpha \Delta_{k-1} + \text{terms containing only } e_\gamma \text{ for } \gamma > \alpha, \text{ where}$$

$$k = \begin{cases} i, & \text{if } j < n - i + 1, \\ n - j + 1, & \text{if } j > n - i + 1. \end{cases}$$

This implies that in \mathcal{A} one can write \tilde{e}_α as a polynomial in $\tilde{\Delta}_\gamma$ for $\gamma \in \Phi^+ \setminus \mathcal{B}$. Thus, $\tilde{\Delta}_\gamma$ for $\gamma \in \Phi^+ \setminus \mathcal{B}$ generate \mathcal{A} as an algebra.

Now, given $\alpha = \epsilon_i - \epsilon_j \in \Phi^+ \setminus \mathcal{B}$, $j < n - i + 1$, let

$$p_\alpha = \tilde{\Delta}_\alpha, \quad q_\alpha = (-1)^{i+1} c_i^{-1} c_{i-1}^{-1} \tilde{\Delta}_{\epsilon_j - \epsilon_{n-i+1}}.$$

Lemma 3.3 shows that $[p_\alpha, q_\gamma] = 0$ for $\alpha \neq \gamma$, $[p_\alpha, p_\gamma] = [q_\alpha, q_\gamma] = 0$ for all α, γ , and $[p_\alpha, q_\alpha] = 1$. Hence \mathcal{A} is a quotient algebra of the Weyl algebra \mathcal{A}_N for

$$N = (n-2) + (n-4) + \dots = \#\{\epsilon_i - \epsilon_j \in \Phi^+ \setminus \mathcal{B} \mid j < n - i + 1\}.$$

But the Weyl algebra \mathcal{A}_N is simple, and $\mathcal{A} \neq 0$, so $\mathcal{A} \cong \mathcal{A}_N$. Thus J_c is primitive (see (11)). \square

PROOF OF THEOREM 3.1. (ii) \implies (iii). Put $c_0 = 1$ and define $\xi(\beta)$, $\beta \in \mathcal{B}$, by formula (12). Denote $\mathfrak{p} = \langle e_{\epsilon_i - \epsilon_j}, j \leq n - i + 1 \rangle_{\mathbb{C}}$ for A_{n-1} , and $\mathfrak{p} = \langle e_{\alpha}, \text{col}(\alpha) < 0 \rangle_{\mathbb{C}}$ for C_n . By [Pa1, Theorem 1.1] and [Ig1, Theorem 1.1] the space \mathfrak{p} is a polarization of \mathfrak{n} at f_{ξ} . Let V_{ξ} be the simple \mathfrak{n} -module corresponding to the linear form f_{ξ} and the polarization \mathfrak{p} , i.e., $V_{\xi} = U(\mathfrak{n}) \otimes_{U(\mathfrak{p})} W_{\xi}$, where W_{ξ} is a one-dimensional representation of \mathfrak{p} defined by $x \mapsto f_{\xi}(x)$. Then Δ_k acts on V_{ξ} via the scalar c_k for $1 \leq k \leq m$. Consequently $J = J_c \subseteq J(f_{\xi})$, and so $J = J(f_{\xi})$.

(iii) \implies (i). Let \mathfrak{p} , V_{ξ} be as in the previous paragraph. Let β_k be the k -th root from \mathcal{B} , i.e., $\beta_k = \epsilon_k - \epsilon_{n-k+1}$ for A_{n-1} , and $\beta_k = 2\epsilon_k$ for C_n . Then $c_k \neq 0$ for $1 \leq k \leq m'$. Moreover, $\xi(\beta) = (-1)^{k+1} c_k / c_{k-1}$ for $1 \leq k \leq m$, where $c_0 = 1$, because Δ_k acts on V_{ξ} via the scalar $(-1)^{k+1} \xi(\beta_1) \dots \xi(\beta_k)$. By Lemma 3.4, the ideal J_c is primitive. Since $J_c \subseteq J = J(f_{\xi})$, we have $J = J_c = J(f_{\xi})$, so J is centrally generated.

(i) \implies (ii). Assume, to the contrary, that some scalars c_k equal zero. Suppose that $i_1 \leq m'$ is the minimal number such that $c_{i_1} = 0$. Now, define inductively two (finite) sequences $\{i_j\}$ and $\{k_j\}$ of positive integers by the following rule. If i_j is already defined and there exists k such that $i_j < k \leq m$ and $c_k \neq 0$, then set k_j to be the minimal among all such k . Similarly, if k_j is already defined and there exists i such that $k_j < i \leq m'$ and $c_i = 0$, then set i_{j+1} to be the minimal among all such i .

To each j for which i_j exists we assign the root

$$\gamma_j = \begin{cases} \epsilon_{i_j} - \epsilon_{n-i_j}, & \text{if } \Phi = A_{n-1}, \\ \epsilon_{i_j} + \epsilon_{i_j+2}, & \text{if } \Phi = C_n \text{ and } i_j < m' = m - 1 = n - 1, \\ \epsilon_{n-1} - \epsilon_n, & \text{if } \Phi = C_n \text{ and } i_j = m'. \end{cases}$$

To each j such that both i_j and k_j exist we assign the set of roots

$$\Gamma_j = \begin{cases} \{\epsilon_{i_j} - \epsilon_{n-k_j+1}, \epsilon_{k_j} - \epsilon_{n-i_j+1}\}, & \text{if } \Phi = A_{n-1}, \\ \{\epsilon_{i_j} + \epsilon_{k_j}\}, & \text{if } \Phi = C_n. \end{cases}$$

Denote the lengths of the sequences $\{i_j\}$, $\{k_j\}$ by l_I , l_K respectively, and put

$$X = \begin{cases} \left(\mathcal{B} \setminus \left(\bigcup_{j=1}^r \{\beta_{i_j}, \beta_{k_j}\} \cup \bigcup_{j=i_{r+1}}^m \beta_j \right) \right) \cup \bigcup_{j=1}^r \Gamma_j \cup \{\gamma_{r+1}\}, & \text{if } l_I = r + 1, l_K = r, \\ \left(\mathcal{B} \setminus \left(\bigcup_{j=1}^r \{\beta_{i_j}, \beta_{k_j}\} \right) \right) \cup \bigcup_{j=1}^r \Gamma_j, & \text{if } l_I = l_K = r. \end{cases}$$

Let $\varphi: X \rightarrow \mathbb{C}$ be a map. Put $\mu_{\varphi} = \sum_{\alpha \in X} \varphi(\alpha) e_{\alpha}^*$. By [Di4, 6.6.9 (c)], $\Delta_k - c'_k \in J(\mu_{\varphi})$, $1 \leq k \leq m$, where $c'_k = \xi_k(f) = \sigma^{-1}(\Delta_k)(f)$. By the definition of Δ_k there exist at least two distinct maps φ_1 , φ_2 such that $c'_k = c_k$ for $1 \leq k \leq m$. It follows from [Pa1, Theorem 1.4] and from the proof of [Ig2, Theorem 3.1] that the orbits of μ_{φ_1} and μ_{φ_2} are disjoint, so $J(\mu_{\varphi_1}) \neq J(\mu_{\varphi_2})$. On the other hand, both $J(\mu_{\varphi_1})$ and $J(\mu_{\varphi_2})$ contain $J = J_c$, and this contradicts the maximality of J . The equivalence of (i), (ii), (iii) is now proved. The fact that the map ξ is reconstructed by J via formula (12) follows from the proof of the implication (iii) \implies (i). \square

Recall that $\lambda \in \mathfrak{n}^*$ is *regular* if the N -orbit $\Omega_{\lambda} \subset \mathfrak{n}^*$ of λ has maximal possible dimension. It follows from [Ko1, Theorem 2.3] that all Kostant forms are regular. Moreover, for $\Phi = A_{n-1}$, a form $\lambda \in \mathfrak{n}^*$ is a Kostant form if and only if it is regular. Since it is known that an orbit Ω_{λ} contains at most one Kostant form, Theorem 3.1 that the Dixmier map establishes a bijection between Kostant forms and centrally generated ideals of $U(\mathfrak{n})$ for $\Phi = A_{n-1}, C_n$.

For $\Phi = A_{n-1}$, Theorem 3.1 implies that a primitive ideal $J(\lambda)$ of $U(\mathfrak{n})$ is centrally generated if and only if it is ‘‘minimal’’ in the sense that the orbit Ω_{λ} has maximal dimension. This reminds us of Duflo’s famous theorem that if \mathfrak{a} is a semi-simple Lie algebra, then any minimal primitive ideal of $U(\mathfrak{a})$ is centrally generated [Du]. However, for $\Phi = C_n$, this analogy no longer holds as there exist regular forms λ such that $J(\lambda)$ is not centrally generated (due to the fact that not every regular form λ is a Kostant form).

3.2. Infinite-dimensional case. Throughout this subsection we use the notation from Subsection 2.2. We now restrict ourselves to the case $\mathcal{N} = \mathbb{Z}_{>0}$. This means that, up to isomorphism, \mathfrak{n} can be chosen to correspond to the linear order $\epsilon_1 \succ \epsilon_3 \succ \epsilon_5 \succ \dots \succ \epsilon_6 \succ \epsilon_4 \succ \epsilon_2$ for A_∞ (respectively, to the linear order $\epsilon_1 \succ \epsilon_2 \succ \epsilon_3 \succ \dots \succ 0 \succ \dots \succ -\epsilon_2 \succ -\epsilon_1$ for all other root systems). In particular, $\vartheta_i = e_i$ for all $i \in \mathbb{Z}_{>0}$, and $f_\alpha = e_\alpha$ for all $M \subset \mathbb{Z}_{>0}$, $\alpha \in \Phi_M^+$. For $\alpha \in \Phi^+$, denote by e_α^* the linear form on \mathfrak{n} such that $e_\alpha^*(e_\beta) = \delta_{\alpha,\beta}$ (the Kronecker delta) for all $\beta \in \Phi^+$. In this subsection we describe all centrally generated ideals of $U(\mathfrak{n})$ for A_∞ and C_∞ .

For our choice of \mathfrak{n} , the Kostant cascade has the following form:

$$\mathcal{B} = \begin{cases} \{\epsilon_1 - \epsilon_2, \epsilon_3 - \epsilon_4, \dots\} & \text{for } A_\infty, \\ \{\epsilon_1 + \epsilon_2, \epsilon_3 + \epsilon_4, \dots\} & \text{for } B_\infty \text{ and } D_\infty, \\ \{2\epsilon_1, 2\epsilon_2, \dots\} & \text{for } C_\infty. \end{cases}$$

The forms $f_\xi = \sum_{\beta \in \mathcal{B}} \xi(\beta) e_\beta^* \in \mathfrak{n}^*$ for all maps $\xi: \mathcal{B} \rightarrow \mathbb{C}^\times$ are by definition the *Kostant forms* on \mathfrak{n} .

Our goal is to construct a partial Dixmier map, which attaches to each Kostant form a primitive ideal of $U(\mathfrak{n})$. As in Subsection 3.1, define the maps $\text{row}: \Phi \rightarrow \mathbb{Z}$ and $\text{col}: \Phi \rightarrow \mathbb{Z}$ by putting $\text{row}(\epsilon_i - \epsilon_j) = \text{row}(\epsilon_i + \epsilon_j) = \text{row}(2\epsilon_i) = i$, $\text{col}(\epsilon_i + \epsilon_j) = \text{col}(2\epsilon_j) = -j$, $\text{col}(\epsilon_i - \epsilon_j) = j$, and set $\mathcal{R}_k = \{\alpha \in \Phi^+ \mid \text{row}(\alpha) = k\}$. Put $\mathfrak{p} = \langle e_\alpha, \alpha \in \Phi^+ \setminus \mathcal{M} \rangle_{\mathbb{C}}$, where

$$\mathcal{M} = \begin{cases} \{\epsilon_i - \epsilon_j, i \text{ odd}, j \text{ even}, j < i\} & \text{for } A_\infty, \\ \mathcal{R}_i, i \text{ even} & \text{for } B_\infty \text{ and } D_\infty, \\ \{\epsilon_i - \epsilon_j, 1 \leq i < j \leq n\} & \text{for } C_\infty. \end{cases}$$

Put also $\mathfrak{p}_n = \mathfrak{p} \cap \mathfrak{n}_n$, where $\mathfrak{n}_n = \mathfrak{n}_M$ for $M = \{1, \dots, n\}$. Fix a Kostant form $f = f_\xi$. By [Ig1, Theorem 1.1], \mathfrak{p}_n is a polarization of \mathfrak{n}_n at the linear form $f_n = f|_{\mathfrak{n}_n}$. Thus, $\mathfrak{p} = \varinjlim \mathfrak{p}_n$ is a polarization of \mathfrak{n} at f . Moreover, denote

$$V_\xi = U(\mathfrak{n}) \otimes_{U(\mathfrak{p})} W, \quad V_\xi^n = U(\mathfrak{n}_n) \otimes_{U(\mathfrak{p}_n)} W^n, \quad (13)$$

where W (respectively, W^n) is the one-dimensional representation of \mathfrak{p} (respectively, of \mathfrak{p}_n) given by $x \mapsto f_\xi(x)$. The \mathfrak{n}_n -modules V_n^ξ are simple and form a natural chain whose union is V_ξ . Hence, V_ξ is a simple \mathfrak{n} -module. We denote its annihilator in $U(\mathfrak{n})$ by $J(f_\xi)$.

Remark 3.5. Let $\mathcal{P}(f_\xi)$ be the set of all polarizations \mathfrak{a} of \mathfrak{n} at f_ξ such that $\mathfrak{a}_n = \mathfrak{a} \cap \mathfrak{n}_n$ is a polarization of \mathfrak{n}_n at f_n for large enough n . Define $V_{\xi,\mathfrak{a}}$ and $V_{\xi,\mathfrak{a}}^n$ by formula (13) in which \mathfrak{p} and \mathfrak{p}_n are replaced by $\mathfrak{a} \in \mathcal{P}(f_\xi)$ and \mathfrak{a}_n respectively. Then $V_{\xi,\mathfrak{a}} = \varinjlim V_{\xi,\mathfrak{a}}^n$ is a simple \mathfrak{n} -module. As the annihilator of $V_{\xi,\mathfrak{a}}^n$ in $U(\mathfrak{n}_n)$ does not depend on \mathfrak{a}_n , we conclude that the annihilator of $V_{\xi,\mathfrak{a}}$ does not depend on \mathfrak{a} . This shows that $J(f_\xi)$ can be defined via any polarization $\mathfrak{a} \in \mathcal{P}(f_\xi)$.

Lemma 3.6. *For A_∞ and C_∞ , the primitive ideal $J(f_\xi)$ is generated by $\Delta_k - c_k$ for $k \geq 1$.*

PROOF. It follows from the definition of \mathfrak{p} that Δ_k acts on V_ξ and on each V_ξ^n for $n \geq k$ via the scalar c_k . Let J_n be the annihilator of V_ξ^n in $U(\mathfrak{n}_n)$, $n \geq 1$. Theorem 3.1 implies that J_n is generated by $\Delta_k - c_k$ for $1 \leq k \leq m$, where $m = \lfloor n/2 \rfloor$ for A_∞ and $m = n$ for C_∞ . Hence $J_n = J(f_\xi) \cap U(\mathfrak{n}_n)$. The result follows. \square

Now let V be a simple \mathfrak{n} -module and J be its annihilator. By [Di1], for A_∞ and C_∞ (respectively, for B_∞ and D_∞), each Δ_k (respectively, P_k) acts on V via some scalar c_k for $k \geq 1$.

Our third main result is as follows.

Theorem 3.7. *Let $\Phi = A_\infty, C_\infty$, and \mathfrak{n} be as above. The following conditions on a primitive ideal $J \subset U(\mathfrak{n})$ are equivalent:*

- i) J is centrally generated;
- ii) all scalars c_k are nonzero;
- iii) $J = J(f_\xi)$ for a Kostant form f_ξ .

If these conditions are satisfied, then the scalars c_k reconstruct $\xi(\beta)$ exactly as in Theorem 3.1.

PROOF. (ii) \implies (iii). Define ξ by formula (12). Then $J(f_\xi) \subseteq J$ by Lemma 3.6. On the other hand, consider $J_n = J(f_\xi) \cap U(\mathfrak{n}_n)$ for $n \geq 1$. The ideal J_n of $U(\mathfrak{n}_n)$ contains $\Delta_k - c_k$ for $1 \leq k \leq m$, where $m = \lfloor n/2 \rfloor$ for A_∞ and $m = n$ for C_∞ . Hence, according to Theorem 3.1, J_n is a maximal ideal of $U(\mathfrak{n}_n)$ contained in $J \cap U(\mathfrak{n}_n)$. Thus, $J_n = J \cap U(\mathfrak{n}_n)$ for $n \geq 1$, i.e., $J = J(f_\xi)$.

(iii) \implies (i). Follows from Lemma 3.6.

(i) \implies (ii). Assume, to the contrary, that some scalars c_k are zero. Let i_1 be the minimal number for which $c_{i_1} = 0$. Define two (possibly, infinite) sequences $\{i_j\}$ and $\{k_j\}$ of positive integers inductively by the following rule. If i_j is already defined and there exists $k > i_j$ such that $c_k \neq 0$, then set k_j to be the minimal number for which $c_{k_j} \neq 0$. Similarly, if k_j is already defined and there exists $i > k_j$ such that $c_i = 0$, then set i_{j+1} to be the minimal number such that $c_{i_{j+1}} = 0$.

To each j for which i_j exists we assign the root

$$\gamma_j = \begin{cases} \epsilon_{2i_j-1} - \epsilon_{2i_j+2}, & \text{if } \Phi = A_\infty, \\ \epsilon_{i_j} + \epsilon_{i_j+1}, & \text{if } \Phi = C_\infty. \end{cases}$$

To each j such that both i_j and k_j exist we assign the set of roots

$$\Gamma_j = \begin{cases} \{\epsilon_{2i_j-1} - \epsilon_{2k_j}, \epsilon_{2k_j-1} - \epsilon_{2i_j}\}, & \text{if } \Phi = A_\infty, \\ \{\epsilon_{i_j} + \epsilon_{k_j}\}, & \text{if } \Phi = C_\infty. \end{cases}$$

Next, we define the subset $X \subset \Phi^+$ as in the proof of Theorem 3.1. Namely, let β_k be the k -th root from \mathcal{B} (i.e., $\beta_k = \epsilon_{2k-1} - \epsilon_{2k}$ for A_∞ , and $\beta_k = 2\epsilon_k$ for C_∞). Denote the lengths of the sequences $\{i_j\}$, $\{k_j\}$ by l_I, l_K respectively, and put

$$X = \begin{cases} \left(\mathcal{B} \setminus \left(\bigcup_{j \leq r} \{\beta_{i_j}, \beta_{k_j}\} \cup \bigcup_{j \geq i_{r+1}} \beta_j \right) \right) \cup \bigcup_{j \leq r} \Gamma_j \cup \{\gamma_{r+1}\}, & \text{if } l_I = r+1, l_K = r, \\ \left(\mathcal{B} \setminus \left(\bigcup_{j \leq r} \{\beta_{i_j}, \beta_{k_j}\} \right) \right) \cup \bigcup_{j \leq r} \Gamma_j, & \text{if } l_I = l_K = r, \\ \left(\mathcal{B} \setminus \left(\bigcup_{j \geq 1} \{\beta_{i_j}, \beta_{k_j}\} \right) \right) \cup \bigcup_{j \geq 1} \Gamma_j, & \text{if } l_I = l_K = \infty. \end{cases}$$

Let $\mu_\varphi = \sum_{\alpha \in X} \varphi(\alpha) e_\alpha^*$, where $\varphi: X \rightarrow \mathbb{C}^\times$ is a map. To each $\alpha \in \Phi^+$ we assign the subset $S(\alpha) \subset \Phi^+$ as follows:

$$S(\alpha) = \begin{cases} \bigcup_{\epsilon_l > \epsilon_i} \{\epsilon_l - \epsilon_j\}, & \text{if } \alpha = \epsilon_i - \epsilon_j, \\ \bigcup_{l=i+1}^{j-1} \{\epsilon_l + \epsilon_j\} \cup \mathcal{R}_j, & \text{if } \Phi = C_\infty, \alpha = 2\epsilon_i. \end{cases}$$

We then set $\mathcal{M} = \bigcup_{\beta \in X} \mathcal{M}_\beta$ where $\mathcal{M}_\beta = \{\gamma \in S(\beta) \mid \gamma, \beta - \gamma \notin \bigcup \mathcal{M}_\alpha\}$, the latter union being taken over all $\alpha \in X$ such that $\text{row}(\alpha) < \text{row}(\beta)$. Note that if $\beta \in X$, $\alpha, \gamma \in \Phi^+$, $\alpha \notin \mathcal{M}$ and $\alpha + \gamma = \beta$, then $\gamma \in \mathcal{M}$. This implies that $\mu_\varphi([x, y]) = 0$ for all $x, y \in \mathfrak{a} = \langle e_\alpha, \alpha \in \Phi^+ \setminus \mathcal{M} \rangle_{\mathbb{C}}$. Moreover, it is easy to see that \mathfrak{a} is a subalgebra of \mathfrak{n} , hence we can consider the \mathfrak{n} -module $V_\varphi = U(\mathfrak{n}) \otimes_{U(\mathfrak{a})} W_\varphi$, where W_φ is the one-dimensional representation of \mathfrak{a} given by $x \mapsto \mu_\varphi(x)$. Let J_φ be the annihilator of V_φ in $U(\mathfrak{n})$ (we do not assert that J_φ is a primitive ideal as we do not discuss the irreducibility of V_φ). One can check that the map φ can be chosen so that $J \subseteq J_\varphi$, so we assume in the rest of the proof that this condition is satisfied.

Let α be the unique root from $\mathcal{R}_{i_1} \cap X$. Explicitly,

$$\alpha = \begin{cases} \epsilon_{2i_1-1} - \epsilon_{2i_1+2}, & \text{if } \Phi = A_\infty \text{ and } c_k = 0 \text{ for all } k \geq i_1, \\ \epsilon_{2i_1-1} - \epsilon_{2k_1}, & \text{if } \Phi = A_\infty \text{ and } k_1 > i_1 \text{ is the minimal such that } c_{k_1} \neq 0, \\ \epsilon_{i_1} + \epsilon_{i_1+2}, & \text{if } \Phi = C_\infty \text{ and } c_k = 0 \text{ for all } k \geq i_1, \\ \epsilon_{i_1} + \epsilon_{k_1}, & \text{if } \Phi = C_\infty \text{ and } k_1 > i_1 \text{ is the minimal such that } c_{k_1} \neq 0. \end{cases}$$

Given $\gamma \in \Phi^+$, let M be a finite subset of $\mathbb{Z}_{>0}$ such that $\gamma \in \Phi_M^+$, $n = |M|$. Recall the definition of j_M and ϕ_M from Subsection 2.2. Let Δ_γ be the image in $U(\mathfrak{n}_M)$ of $\Delta_{j_M^{-1}(\gamma)} \in U(\mathfrak{n}_n)$ under the isomorphism ϕ_M . Note that Δ_γ depends only on γ and not on M . We now show that $\Delta_\alpha - c \in J$ for some scalar c . We prove this for $\Phi = A_\infty$ (the proof for $\Phi = C_\infty$ is similar). First, suppose $\text{col}(\alpha) = 2i_1 + 2$. By Lemma 3.2, Δ_α commutes with all e_γ , except $\gamma = \beta_{i_1} - \alpha$, and in the latter case $[\Delta_\alpha, e_\gamma] = \pm\Delta_{\alpha+\gamma} = \pm\Delta_{\beta_{i_1}} = \pm\Delta_{i_1}$. Hence the image of Δ_α belongs to the center of the image of $U(\mathfrak{n})$ in the algebra $\text{End}_{\mathbb{C}}V$. By [Di1], there exist $c \in \mathbb{C}$ such that $\Delta_\alpha - c \in J$.

Now we apply induction on $\text{col}(\alpha)$ (the base is $\text{col}(\alpha) = 2i_1 + 2$). By Lemma 3.2, for any α , Δ_α commutes with all e_γ for $\gamma \notin A(\alpha)$. On the other hand, if $\gamma \in A(\alpha)$ then $[\Delta_\alpha, e_\gamma] = \pm\Delta_{\alpha+\gamma}$. But $\gamma + \alpha \in \mathcal{R}_{i_1}$ and $2i_1 \leq \text{col}(\gamma + \alpha) < \text{col}(\alpha)$. If $\text{col}(\gamma + \alpha) = 2i_1$, then $\Delta_{\gamma+\alpha} = \Delta_{i_1} \in J$. By the inductive assumption, if $\text{col}(\gamma + \alpha) \geq 2i_1 + 2$ then there exists $c' \in \mathbb{C}$ such that $\Delta_{\alpha+\gamma} - c' \in J \subseteq J_\varphi$. It follows from the definition of V_φ that for $\delta \in \mathcal{R}_{i_1}$, where $\text{col}(\delta) \geq 2i_1$, Δ_δ acts on V_φ via the scalar $c_\delta = \pm\mu_\varphi(e_\delta) \prod_{k < i_1} \varphi(\beta_k)$. In other words, $\Delta_\delta - c_\delta \in J_\varphi$. In particular, $\Delta_{\alpha+\gamma} - c_{\alpha+\gamma} \in J_\varphi$. Since c_δ is uniquely determined by J_φ , we conclude that $c' = c_{\alpha+\gamma} = 0$ because $\mu_\varphi(e_{\gamma+\delta}) = 0$. Thus the image of Δ_α belongs to the center of the image of $U(\mathfrak{n})$ in the algebra $\text{End}_{\mathbb{C}}V$. By [Di1] there exists $c \in \mathbb{C}$ such that $\Delta_\alpha - c \in J$.

Further, we see that $c = \pm\varphi(\alpha) \prod_{k < i_1} \varphi(\beta_k)$ because $J \subseteq J_\varphi$ and $\mu_\varphi(e_\alpha) = \varphi(\alpha)$. Note also that Δ_k acts on V_φ via the scalar $c'_k = (-1)^{k+1} \prod_{i \leq k} \varphi(\beta_i)$. Thus there exist at least two maps φ_1, φ_2 from X to \mathbb{C}^\times such that $\varphi_1(\alpha) \neq \varphi_2(\alpha)$ and $c'_k = c_k$ for all $k \in \mathbb{Z}_{>0}$. This implies that both J_{φ_1} and J_{φ_2} contain J , which contradicts the uniqueness of c . The proof is complete. \square

Denote by \mathcal{A}_∞ the Weyl algebra with countably many generators p_i, q_i for $i \in \mathbb{Z}_{>0}$, and relations

$$[p_i, q_i] = 1, [p_i, q_j] = 0 \text{ for } i \neq j, [p_i, p_j] = [q_i, q_j] = 0 \text{ for all } i, j. \quad (14)$$

Note that the center of \mathcal{A}_∞ is trivial because the center of \mathcal{A}_r is trivial for any $r \geq 1$. Similarly, \mathcal{A}_∞ is a simple algebra. We have the following corollary (cf. (11)).

Corollary 3.8. *Let $\Phi = A_\infty, C_\infty, \mathfrak{n}$ be as above, and J be a primitive centrally generated ideal of $U(\mathfrak{n})$. Then*

- i) J is maximal;
- ii) the center of $U(\mathfrak{n})/J$ is trivial;
- iii) $U(\mathfrak{n})/J$ is isomorphic to the Weyl algebra \mathcal{A}_∞ .

PROOF. (i) By Theorem 3.7, $J = J_{f_\xi}$ for some Kostant form f_ξ . It follows from the proof of Lemma 3.6 that $J \cap U(\mathfrak{n}_n)$ is maximal for all $n \geq 1$. Hence J is maximal.

(ii) This follows immediately from (iii).

(iii) One can construct a set of generators of $U(\mathfrak{n})/J$ satisfying (14) as in the proof of Lemma 3.4. Since \mathcal{A}_∞ is simple, we have $U(\mathfrak{n})/J \cong \mathcal{A}_\infty$. \square

We expect Theorem 3.7 and Corollary 3.8 to hold also for B_∞ and D_∞ . Finally, we note that Theorem 3.7 establishes a one-to-one correspondence between centrally generated primitive ideals of $U(\mathfrak{n})$ for $\Phi = A_\infty, C_\infty$ and Kostant forms.

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