

THE PIERI RULE AT INFINITY

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ABSTRACT. We study the structure of tensor products of $\mathfrak{gl}(\infty) = \varinjlim \mathfrak{gl}(n)$ -modules $\mathbf{L}(\lambda) \otimes \mathbf{F}$ where $\mathbf{L}(\lambda)$ is a simple integrable highest weight module and \mathbf{F} is a simple integrable weight multiplicity-free module. Both $\mathbf{L}(\lambda)$ and \mathbf{F} are infinite dimensional, in particular \mathbf{F} can be a Fock module. Similar tensor products of $\mathfrak{gl}(n)$ -modules are semisimple and their simple constituents are described by the classical Pieri rule. We prove that a $\mathfrak{gl}(\infty)$ -module $\mathbf{M} := \mathbf{L}(\lambda) \otimes \mathbf{F}$ is semisimple only in relatively trivial cases, and is indecomposable otherwise. Our main results are a description of the simple constituents of \mathbf{M} , and the construction of a linkage filtration on \mathbf{M} that provides information on when two simple constituents of \mathbf{M} are linked. Using the linkage filtration, we compute the socle and radical filtrations of \mathbf{M} , and determine when \mathbf{M} is rigid.

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1. INTRODUCTION

The Pieri rule is a cornerstone of the representation theory of the Lie algebra $\mathfrak{gl}(n)$. It is fundamental for the study of the category of finite-dimensional $\mathfrak{gl}(n)$ -modules as a tensor category. In the simplest case of a tensor product $L(\lambda) \otimes V$, where $L(\lambda)$ is a simple $\mathfrak{gl}(n)$ -module with integral highest weight λ and V is the defining $\mathfrak{gl}(n)$ -module, the Pieri rule states that $L(\lambda) \otimes V$ decomposes as $\bigoplus_i L(\lambda + \varepsilon_i)$ where ε_j are the weights of V and the sum runs over all i such that $\lambda + \varepsilon_i$ is a dominant weight.

Our aim in this paper is to understand an analogue of Pieri's rule for the infinite-dimensional Lie algebra $\mathfrak{gl}(\infty) := \varinjlim \mathfrak{gl}(n)$. We consider a general tensor product $\mathbf{L}(\lambda) \otimes \mathbf{F}$, where $\mathbf{L}(\lambda)$ is any simple integrable highest weight $\mathfrak{gl}(\infty)$ -module with highest weight λ (with respect to some splitting Borel subalgebra of $\mathfrak{gl}(\infty)$) and \mathbf{F} is any simple weight multiplicity-free $\mathfrak{gl}(\infty)$ -module admitting a highest weight. Both \mathbf{F} and $\mathbf{L}(\lambda)$ are infinite dimensional.

Since the category of integrable finite-dimensional $\mathfrak{gl}(n)$ -modules is semisimple according to a celebrated theorem of H. Weyl from 1925, the classical Pieri rule is usually considered as a combinatorial rule in the language of Young diagrams. However, in the infinite-dimensional setting we consider, a priori there is no reason for the tensor products $\mathbf{L}(\lambda) \otimes \mathbf{F}$ to be semisimple, even when \mathbf{F} is the defining representation \mathbb{V} of $\mathfrak{gl}(\infty)$. Moreover, we show that in most interesting cases modules of the form $\mathbf{L}(\lambda) \otimes \mathbf{F}$, or even $\mathbf{L}(\lambda) \otimes \mathbb{V}$, are actually indecomposable. In the particular case when $\mathbf{L}(\lambda)$ is a tensor module (see [PS11]) and $\mathbf{F} = \mathbf{S}^d \mathbb{V}$, this has been proved earlier by A. Chirvasitu as part of an unpublished study undertaken jointly with the first author.

As already mentioned, in the current paper we take a much broader approach by letting $\mathbf{L}(\lambda)$ be any simple integrable highest weight $\mathfrak{gl}(\infty)$ -module and \mathbf{F} be any simple highest weight multiplicity-free $\mathfrak{gl}(\infty)$ -module. The classification of simple weight multiplicity-free $\mathfrak{sl}(\infty)$ -modules in [GP20] implies that, up to tensor multiplication with a one-dimensional module, a simple highest weight multiplicity-free $\mathfrak{gl}(\infty)$ -module is a symmetric or exterior power of one of the defining representations \mathbb{V} and \mathbb{V}_* , or is a Fock module. The latter are spaces of semi-infinite wedge vectors endowed with natural $\mathfrak{gl}(\infty)$ -actions.

To study the modules $\mathbf{L}(\lambda) \otimes \mathbf{F}$ we proceed as follows. Suppose we are given a sequence of finite-dimensional Lie subalgebras $\mathfrak{g}_n \subset \mathfrak{gl}(\infty)$ such that $\mathfrak{gl}(\infty) = \bigcup_{n \geq 1} \mathfrak{g}_n$. Then there exist simple finite-dimensional \mathfrak{g}_n -modules L_n and F_n , with \mathfrak{g}_n -linear maps $L_n \rightarrow L_{n+1}$ and $F_n \rightarrow F_{n+1}$ such that $\varinjlim L_n \cong \mathbf{L}(\lambda)$ and $\varinjlim F_n = \mathbf{F}$, so $\mathbf{L}(\lambda) \otimes \mathbf{F} = \varinjlim L_n \otimes F_n$. The maps $L_n \rightarrow L_{n+1}$ and $F_n \rightarrow F_{n+1}$ are easy to describe since the source is a simple \mathfrak{g}_n -module and the map is unique up to a constant. Understanding the tensor product of the maps is harder. Our strategy is to single out one particular exhaustion of $\mathbf{L}(\lambda) \otimes \mathbf{F}$ making a complete understanding of the tensor product maps possible.

Our main results are a characterization of the simple constituents of $\mathbf{L}(\lambda) \otimes \mathbf{F}$ and the construction of a natural filtration which we call the linkage filtration. This filtration shows when two simple constituents are linked and enables us to compute the socle and radical filtration of $\mathbf{L}(\lambda) \otimes \mathbf{F}$. In particular we give a necessary and sufficient condition for the socle and radical filtrations to be finite, as well for both filtrations to coincide, i.e., for the module to be rigid. We provide also several examples showing that the type and length of the linkage filtration can vary significantly.

The paper is structured as follows. In section 2 we recall some necessary general background on the representation theory of $\mathfrak{gl}(\infty)$. In section 3 we introduce some basic definitions, state our main results without proof, and give various examples of how they apply. Section 4 focuses on the transition maps $L_n \otimes F_n \rightarrow L_{n+1} \otimes F_{n+1}$ and shows that under certain hypotheses we can describe them completely. In section 5 we introduce exhaustions of $\mathfrak{gl}(\infty)$ which satisfy one of the hypotheses of the previous section at each step. Finally, the proofs of the main results are given in section 6.

As an outlook we would like to mention that the results of this paper should be instrumental in further studies of various tensor categories of integrable $\mathfrak{gl}(\infty)$ -modules. This applies in particular to the integrable part of the category $\mathcal{O}_{LA}^!$ introduced in [Zad22], in which certain indecomposable injective modules are of Pieri type.

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2. NOTATION AND BACKGROUND

2.1. Conventions on ordered sets. Let (P, \leq) be a partially ordered set (poset). Given $p, q \in P$ we denote by $\llbracket p, q \rrbracket$ the set $\{r \in P \mid p \leq r \leq q\}$. The set P is *locally finite* if for every $p, q \in P$ the interval $\llbracket p, q \rrbracket$ is a finite set. We say that P is *ranked* if there is a

function $f : P \rightarrow \mathbb{Z}$ such that $p < q$ implies $f(p) < f(q)$; P is *locally ranked* if every interval $\llbracket p, q \rrbracket$ is a ranked poset with the induced order. A *chain* in P is a subset which is totally ordered with the induced order, and the *rank* of P is the supremum of the cardinalities of all chains in P . The poset P is a *lattice* if any two elements have a least upper bound, called their *join*, and a maximal lower bound, called their *meet*. In the sequel we write $\llbracket n \rrbracket = \llbracket 1, n \rrbracket$ for any $n \in \mathbb{Z}_{>0}$.

The partial order \leq on P is *total* or *linear* if every pair of elements is comparable. Let P be a linearly ordered set. An *initial subset* is a subset $I \subset P$ such that $a \in I$ and $a' < a$ implies $a' \in I$. A *terminal subset* is a subset $F \subset P$ with the opposite property: if $a \in F$ and $a' \in P$ satisfies $a' > a$ then $a' \in F$. If \leq is a linear order on P we denote by P^{op} the set P with the opposite order. If P is a partially ordered set isomorphic to a subset of \mathbb{Z} , and $p \in P$, we write $p + 1$ and $p - 1$ for the immediate successor and the immediate predecessor of p , assuming these exist.

Given linearly ordered sets P and Q , we denote by $P \sqcup Q$ the set $P \times \{0\} \cup Q \times \{1\}$ with the total lexicographic order. More generally, given a linearly ordered set I and a family of linearly ordered sets $(P_i)_{i \in I}$, we denote by $\bigsqcup_{i \in I} P_i$ the set $\bigcup_{i \in I} P_i \times \{i\}$ with the total lexicographic order. We will sometimes identify the set P_i with $P_i \times \{i\}$ when the context makes this clear, in particular when P and Q are disjoint.

2.2. The Lie algebra $\mathfrak{gl}(\infty)$. The base field is the field of complex numbers \mathbb{C} . We fix a countable-dimensional space \mathbb{V} and a countable-dimensional subspace \mathbb{V}_* of $\mathbb{V}^* := \text{hom}_{\mathbb{C}}(\mathbb{V}, \mathbb{C})$ such that the pairing $\mathbb{V} \otimes \mathbb{V}_* \rightarrow \mathbb{C}$ (induced by the pairing $\mathbb{V} \otimes \mathbb{V}^* \rightarrow \mathbb{C}$) is non-degenerate. The space $\mathbb{V} \otimes \mathbb{V}_*$ is an associative \mathbb{C} -algebra: the multiplication is given by

$$\left(\sum_i v_i \otimes \varphi_i \right) \left(\sum_j v'_j \otimes \varphi'_j \right) = \sum_{i,j} \varphi_i(v'_j) v_i \otimes \varphi_j$$

for $v_i, v'_j \in \mathbb{V}$ and $\varphi_i, \varphi'_j \in \mathbb{V}_*$. By definition, $\mathfrak{gl}(\infty)$ is the Lie algebra determined by the associative algebra $\mathbb{V} \otimes \mathbb{V}_*$, and \mathbb{V} and \mathbb{V}_* are the *defining representations* of $\mathfrak{gl}(\infty)$. Denoting the pairing $\mathbb{V} \otimes \mathbb{V}_* \rightarrow \mathbb{C}$ by tr , we define the Lie algebra $\mathfrak{sl}(\infty)$ as $\ker \text{tr}$. In this way we obtain a simple Lie subalgebra of codimension 1 in $\mathfrak{gl}(\infty) = \mathbb{V} \otimes \mathbb{V}_*$. Note however that the exact sequence of $\mathfrak{sl}(\infty)$ -modules

$$0 \rightarrow \mathfrak{sl}(\infty) \rightarrow \mathbb{V} \otimes \mathbb{V}_* \xrightarrow{\text{tr}} \mathbb{C} \rightarrow 0$$

does not split.

According to a theorem of G. Mackey [Mac45], there exists a basis $\{e_k\}_{k \geq 1}$ of \mathbb{V} such that its dual system $\{e_k^*\}_{k \geq 1}$ is a basis of \mathbb{V}_* . Setting $V_n := \text{span}\{e_k \mid 1 \leq k \leq n\}$ and $V_n^* := \text{span}\{e_k^* \mid 1 \leq k \leq n\}$, we obtain $\mathfrak{gl}(\infty)$ as the inductive limit of the finite dimensional subalgebras $\mathfrak{gl}(V_n) \subset \mathfrak{gl}(\infty)$. A *splitting Borel subalgebra* \mathfrak{b} of $\mathfrak{gl}(\infty)$ is the inductive limit of a chain of Borel subalgebras $\mathfrak{b}_n \subset \mathfrak{gl}(V_n)$ for any choice of V_n as above. It is well known that any such \mathfrak{b} contains the limit $\mathfrak{h} = \varinjlim \mathfrak{h}_n$ of some Cartan subalgebras $\mathfrak{h}_n \subset \mathfrak{b}_n$. We fix the pair $\mathfrak{h} \subset \mathfrak{gl}(\infty)$, and will consider various splitting

Borel subalgebras \mathfrak{b} containing \mathfrak{h} . In particular we will assume that V_n are chosen so that $\mathfrak{h}_n = \mathfrak{h} \cap \mathfrak{gl}(V_n)$ is a Cartan subalgebra for all n .

A *weight $\mathfrak{gl}(\infty)$ -module* \mathbf{N} is an $\mathfrak{gl}(\infty)$ -module which is semisimple over \mathfrak{h} :

$$\mathbf{N} = \bigoplus_{\mu \in \mathfrak{h}^*} \mathbf{N}_\mu$$

where $\mathfrak{h}^* = \text{hom}(\mathfrak{h}, \mathbb{C})$ and

$$\mathbf{N}_\mu := \{v \in \mathbf{N} \mid hv = \mu(h)v \ \forall h \in \mathfrak{h}\}.$$

The elements of \mathfrak{h}^* are the *weights* of $\mathfrak{gl}(\infty)$, and the *support* $\text{supp } \mathbf{N}$ of the module \mathbf{N} is defined by setting $\text{supp } \mathbf{N} := \{\mu \in \mathfrak{h}^* \mid \mathbf{N}_\mu \neq 0\}$. The nonzero spaces \mathbf{N}_μ are the *weight spaces* of the module \mathbf{N} . The defining representation \mathbb{V} is a weight $\mathfrak{gl}(\infty)$ -module and its weights are $\{\varepsilon_i\}_{i \in \mathbb{I}}$, where \mathbb{I} is a countable set which will be fixed throughout this paper. As usual we denote a weight $\lambda \in \mathfrak{h}^*$ by $\sum_{i \in \mathbb{I}} \lambda_i \varepsilon_i$, where $\lambda_i = \lambda(E_{i,i})$; notice that λ is well defined even when the sum is infinite. We also fix the following notation: we let \mathfrak{h}° be the \mathbb{C} -span of the set $\{\varepsilon_i \mid i \in \mathbb{I}\}$, and given $A \subset \mathbb{I}$ put $\varepsilon_A := \sum_{i \in A} \varepsilon_i$.

Suitably chosen outer automorphisms of $\mathfrak{gl}(V_n)$ extend to an outer automorphism τ of $\mathfrak{gl}(\infty)$ acting by $-\text{Id}$ on \mathfrak{h} , and any $\mathfrak{gl}(\infty)$ -module \mathbf{N} can be twisted by this automorphism. We denote the result by \mathbf{N}_* . If \mathbf{N} is a weight module then $\text{supp } \mathbf{N}_* = -\text{supp } \mathbf{N}$. In particular, \mathbb{V}_* is nothing but \mathbb{V} twisted by the automorphism τ , and $\text{supp } \mathbb{V}_* = \{-\varepsilon_i\}_{i \in \mathbb{I}}$.

The adjoint representation of $\mathfrak{gl}(\infty)$ is a weight module, and its weights are $\{\varepsilon_i - \varepsilon_j\}_{i,j \in \mathbb{I}}$. The nonzero weights of the adjoint representation are the *roots* of $\mathfrak{gl}(\infty)$, and the zero weight space of $\mathfrak{gl}(\infty)$ is \mathfrak{h} . A splitting Borel subalgebra $\mathfrak{b} \supset \mathfrak{h}$ is determined by a linear order \leq on the set \mathbb{I} :

$$\mathfrak{b}_\leq := \mathfrak{h} \oplus \bigoplus_{i < j} \mathfrak{gl}(\infty)_{\varepsilon_i - \varepsilon_j}.$$

The *positive roots* of \mathfrak{b} are the roots $\varepsilon_i - \varepsilon_j$ with $i < j$, and the *negative roots* are $\varepsilon_i - \varepsilon_j$ with $i > j$; the *simple roots* of \mathfrak{b} are the positive roots which can not be written as a sum of two positive roots. In contrast with the finite-dimensional case, not every positive root is in the $\mathbb{Z}_{>0}$ -span of the simple roots of \mathfrak{b} . The linear order \leq which defines \mathfrak{b} determines a partial order on \mathfrak{h}^* , again denoted \leq , where $\lambda \leq \mu$ if and only if $\mu - \lambda$ is a sum of positive roots. Given a splitting Borel \mathfrak{b} , we denote by \mathfrak{b}^{op} the unique splitting Borel subalgebra whose positive roots are the negative roots of \mathfrak{b} .

2.3. Integrable highest weight $\mathfrak{gl}(\infty)$ -modules. A weight $\lambda \in \mathfrak{h}^*$ is *integral* if $\lambda_i \in \mathbb{Z}$ for all $i \in \mathbb{I}$. A $\mathfrak{gl}(\infty)$ -module \mathbf{N} is *integrable* if its restriction to $\mathfrak{gl}(V_n)$ is a direct sum of finite-dimensional $\mathfrak{gl}(V_n)$ -modules with integral weights. This means that \mathbf{N} integrates to a $\text{GL}(V_n)$ -module for any n .

A weight module \mathbf{N} is a *\mathfrak{b} -highest weight module* for a given Borel subalgebra \mathfrak{b} if \mathbf{N} is generated by a single weight vector v such that $[\mathfrak{b}, \mathfrak{b}]v = 0$. The weight of the vector v is the *highest weight* of \mathbf{N} . The usual construction shows that given a linear order \leq

on \mathbb{I} and a weight $\lambda \in \mathfrak{h}^*$ then there exists a \mathfrak{b}_{\leq} -highest weight module with highest weight λ , which we denote by $\mathbf{L}_{\mathfrak{b}_{\leq}}(\lambda)$.

We say that $\lambda \in \mathfrak{h}^*$ is \mathfrak{b}_{\leq} -dominant if $\lambda_i - \lambda_j \in \mathbb{Z}_{\geq 0}$ whenever $i < j$. The simple integrable modules which are \mathfrak{b}_{\leq} -highest weight modules are precisely the modules $\mathbf{L}_{\mathfrak{b}_{\leq}}(\lambda)$ such that λ is integral and \mathfrak{b}_{\leq} -dominant. In the rest of the paper we assume automatically that all weights are integral. We only consider integrable simple highest weight modules, and simplify the notation from $\mathbf{L}_{\mathfrak{b}_{\leq}}(\lambda)$ to $\mathbf{L}(\lambda)$ by assuming that \leq is any linear order on \mathbb{I} for which λ is \mathfrak{b}_{\leq} -dominant. In addition, it is easy to check that there is an isomorphism $\mathbf{L}(\lambda) \cong \mathbf{L}(\mu)$ if and only if there is a bijective map $\sigma : \mathbb{I} \rightarrow \mathbb{I}$ which is the identity outside of a finite set and such that $\lambda_i = \mu_{\sigma(i)}$.

2.4. Tensor modules. A $\mathfrak{gl}(\infty)$ -module is a *tensor module* if it is isomorphic to a submodule of a direct sum of finitely copies of the tensor algebra $T(\mathbb{V} \oplus \mathbb{V}_*)$. For instance, $\mathfrak{sl}(\infty)$ is a simple tensor module, being a submodule of $\mathbb{V} \otimes \mathbb{V}_*$. Moreover, all simple tensor modules are highest weight modules with respect to the Borel subalgebra corresponding to the *programmers order* on \mathbb{I} , defined as $\mathbb{Z}_{>0} \sqcup \mathbb{Z}_{<0}$ where the orders of $\mathbb{Z}_{>0}$ and $\mathbb{Z}_{<0}$ are the usual ones and $i < j$ whenever $i \in \mathbb{Z}_{>0}, j \in \mathbb{Z}_{<0}$. The highest weight of a simple tensor module is of the form $\lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n - \mu_k \varepsilon_{-k} - \cdots - \mu_1 \varepsilon_{-1}$ for some $n, k \in \mathbb{Z}_{\geq 0}$ and $\lambda_1 \geq \cdots \geq \lambda_n, \mu_1 \geq \cdots \geq \mu_k$. In particular the highest weight of $\mathfrak{sl}(\infty)$ is $\varepsilon_1 - \varepsilon_{-1}$ [PS11, PH22]. More generally, it follows from the results in the previous section that the tensor module $\mathbf{L}(\lambda)$ is a highest weight module for a Borel subalgebra \mathfrak{b}_{\leq} if and only if the order \leq has an initial subset isomorphic to $\llbracket n \rrbracket$ and a terminal subset isomorphic to $\llbracket k \rrbracket$.

2.5. Weight multiplicity-free $\mathfrak{gl}(\infty)$ -modules. A weight $\mathfrak{gl}(\infty)$ -module \mathbf{N} is *weight multiplicity-free* if $\dim \mathbf{N}_{\mu} = 1$ for all $\mu \in \text{supp } \mathbf{N}$. In [GP20] simple weight multiplicity free $\mathfrak{sl}(\infty)$ -modules with a highest weight are classified. Up to isomorphism, these are the symmetric and exterior powers $\mathbf{S}^d \mathbb{V}, \wedge^d \mathbb{V}, \mathbf{S}^d \mathbb{V}_*, \wedge^d \mathbb{V}_*$, and the Fock modules $\wedge^A \mathbb{V}$ which we now define. Any infinite set $A \subset \mathbb{I}$ with infinite complement $B = \mathbb{I} \setminus A$ determines a family of Borel subalgebras \mathfrak{b}_{\leq} for which A is an initial subset of (\mathbb{I}, \leq) . For all such \mathfrak{b}_{\leq} , the simple \mathfrak{b}_{\leq} -highest weight modules $\mathbf{L}_{\mathfrak{b}_{\leq}}(\varepsilon_A)$ are canonically isomorphic, and we define the module $\wedge^A \mathbb{V}$ as any of those modules. A weight in the support of $\wedge^A \mathbb{V}$ is of the form $\sum_{i \in B} \varepsilon_i$, where $B \subset \mathbb{I}$ and $\#A \setminus B = \#B \setminus A < \infty$. The module $\wedge^A \mathbb{V}$ is minuscule in the sense that every weight is a highest weight for a suitable \mathfrak{b}_{\leq} .

Since we work with $\mathfrak{gl}(\infty)$ and not $\mathfrak{sl}(\infty)$ we need to adapt the classification from [GP20] to $\mathfrak{gl}(\infty)$. It is easy to check that a simple highest weight $\mathfrak{gl}(\infty)$ -module which is multiplicity-free is isomorphic to one of the modules $\mathbf{S}^d \mathbb{V}, \wedge^d \mathbb{V}, \mathbf{S}^d \mathbb{V}_*, \wedge^d \mathbb{V}_*$ or $\wedge^A \mathbb{V}$, possibly tensored by $\mathbf{D}^{\otimes n}$ for any $n \in \mathbb{Z}$. Here \mathbf{D} is the one-dimensional $\mathfrak{gl}(\infty)$ -module of weight $\sum_i \varepsilon_i$. We use the convention that $\mathbf{D}^{\otimes -n} = \mathbf{D}_*^{\otimes n}$. Note that \mathbf{D} is not a tensor module.

2.6. Compatible orders. The *support* $\text{supp } \lambda$ of a weight λ is the set of all $i \in \mathbb{I}$ such that $\lambda_i \neq 0$. We will often identify λ with the function $i \in \mathbb{I} \mapsto \lambda_i \in \mathbb{Z}$, and so use notation like $\lambda^{-1}(a) = \{i \in \mathbb{I} \mid \lambda_i = a\}$ for $a \in \mathbb{Z}$ and $\lambda(Z) = \{\lambda_i \mid i \in Z\}$ for $Z \subset \mathbb{I}$. Note that λ is \mathfrak{b}_{\leq} -dominant if and only if $\mathbb{I} = \bigsqcup_{a \in \mathbb{Z}^{\text{op}}} \lambda^{-1}(a)$, where the order on $\lambda^{-1}(a)$ is induced from \mathbb{I} .

Definition 2.1. Let $\lambda \in \mathfrak{h}^*$ and let \mathbf{F} be one of $\mathbf{S}^d \mathbb{V}$, $\bigwedge^d \mathbb{V}$ or $\bigwedge^A \mathbb{V}$. We say that a total order \leq on \mathbb{I} is (λ, \mathbf{F}) -compatible if it satisfies the following conditions:

- (i) λ is \mathfrak{b}_{\leq} -dominant.
- (ii) For every $a \in \mathbb{Z}$ the poset $\lambda^{-1}(a)$ is order-isomorphic to a subset of \mathbb{Z} .
- (iii) For each $\mu \in \text{supp } F$ the set $\lambda^{-1}(a) \cap \text{supp } \mu$ has a maximum and the set $\lambda^{-1}(a) \setminus \text{supp } \mu$ has a minimum.

Lemma 2.2. For $\lambda \in \mathfrak{h}^*$ and \mathbf{F} be as above, there exists a (λ, \mathbf{F}) -compatible total order \leq . Furthermore, if $\mathbf{L}(\lambda)$ and \mathbf{F} are both $\mathfrak{b}_{\leq'}$ -highest weight modules for some total order \leq' then \mathbf{F} is also a \mathfrak{b}_{\leq} -highest weight module.

Proof. We first show the existence of a (λ, \mathbf{F}) -compatible order \leq on \mathbb{I} , i.e., a linear order \leq such that $\mathbb{I} = \bigsqcup_{a \in \mathbb{Z}^{\text{op}}} \lambda^{-1}(a)$. For this it is enough to assign a total order on each $\lambda^{-1}(a)$ so that conditions (ii) and (iii) of Definition 2.1 are satisfied.

For $\mathbf{F} = \mathbf{S}^d \mathbb{V}$, $\bigwedge^d \mathbb{V}$ we impose on $\lambda^{-1}(a)$ an order making it isomorphic to a subset of $\mathbb{Z}_{>0}$; then $\lambda^{-1}(a) \cap \text{supp } \mu$ is a finite set and hence always has a maximum, while its complement has a minimum by the well-ordering principle. For $\mathbf{F} = \bigwedge^A \mathbb{V}$ we impose on $\lambda^{-1}(a) \cap A$ an order-isomorphic to that of a subset of $\mathbb{Z}_{<0}$, on $\lambda^{-1}(a) \setminus A$ an order-isomorphic to that of a subset of $\mathbb{Z}_{>0}$, and put $\lambda^{-1}(a) = (\lambda^{-1}(a) \cap A) \sqcup (\lambda^{-1}(a) \setminus A)$. If $\mu \in \text{supp } F$ then $\text{supp } \mu$ differs from A in finitely many elements, and since $\lambda^{-1}(a) \cap A$ has a minimum and $\lambda^{-1}(a) \setminus A$ has a maximum, the sets $\lambda^{-1}(a) \cap \text{supp } \mu$ and $\lambda^{-1}(a) \setminus \text{supp } \mu$ also have respectively a minimum and a maximum. This completes the proof of the existence of a (λ, \mathbf{F}) -compatible order.

Suppose now $\mathbf{L}(\lambda)$ and \mathbf{F} are both $\mathfrak{b}_{\leq'}$ -highest weight modules. If $\mathbf{F} = \mathbf{S}^d \mathbb{V}$ then (\mathbb{I}, \leq') must have a minimum, which implies that $\lambda(\mathbb{I}) \subset \mathbb{Z}$ must have a maximum $m \in \mathbb{Z}$. Then, if i is the minimal element of $\lambda^{-1}(m)$, any nonzero vector in $\mathbf{S}^d \mathbb{V}$ of weight $d\varepsilon_i$ is a \mathfrak{b}_{\leq} -highest weight vector. If $\mathbf{F} = \bigwedge^d \mathbb{V}$ then \leq' has an initial sequence of d elements $i'_1 <' i'_2 <' \dots <' i'_d$. Since λ is $\mathfrak{b}_{\leq'}$ -dominant we have $\lambda_{i'_1} \geq \lambda_{i'_2} \geq \dots \geq \lambda_{i'_d}$, and the order \leq constructed above also has an initial sequence of d elements $i_1 < i_2 < \dots < i_d$. Now it is clear that any nonzero vector of weight $\varepsilon_{i_1} + \varepsilon_{i_2} + \dots + \varepsilon_{i_d}$ is a \mathfrak{b}_{\leq} -highest weight vector of $\bigwedge^d \mathbb{V}$.

Finally, let $\mathbf{F} = \bigwedge^A \mathbb{V}$. This is a $\mathfrak{b}_{\leq'}$ -highest weight module if and only if A differs from an initial subset of (\mathbb{I}, \leq') by finitely many elements; without loss of generality we can and will assume that A is an initial subset of (\mathbb{I}, \leq') . It follows that $A = (A \cap \lambda^{-1}(r)) \cup \bigcup_{a < r} \lambda^{-1}(a)$ for some $r \in \mathbb{Z}$. By construction this is an initial subset in (\mathbb{I}, \leq) , so \mathbf{F} is a \mathfrak{b}_{\leq} -highest weight module. \square

2.7. Filtrations. We adopt the following terminology concerning filtrations. A *filtration* of a module \mathbf{N} is a chain of distinct submodules $\{F_\ell \mathbf{N}\}_{\ell \in \mathbb{L}}$ where \mathbb{L} is a linearly ordered set isomorphic to a subset of \mathbb{Z} . The quotients $\bar{F}_{\ell+1} \mathbf{N} := F_{\ell+1} \mathbf{N} / F_\ell \mathbf{N}$ are the *layers* of the filtration. A filtration is *exhaustive* if $\bigcup_{\ell \in \mathbb{L}} F_\ell \mathbf{N} = \mathbf{N}$ and *separated* if $\bigcap_{\ell \in \mathbb{L}} F_\ell \mathbf{N} = \{0\}$. If \mathbf{L} is a simple module its *multiplicity in \mathbf{N}* , denoted by $[\mathbf{N} : \mathbf{L}]$, is the supremum of the number of times \mathbf{L} appears as a layer in a finite filtration of \mathbf{N} . By definition $[\mathbf{N} : \mathbf{L}] \in \mathbb{Z}_{\geq 0} \cup \{\infty\}$.

The module \mathbf{N} is said to be *multiplicity free* if $[\mathbf{M} : \mathbf{L}] \leq 1$ for any simple module \mathbf{L} . A *composition series* is an exhaustive and discrete filtration in which every layer is simple, and such that for every simple subquotient \mathbf{L} of \mathbf{N} , there are $[\mathbf{N} : \mathbf{L}]$ layers isomorphic to \mathbf{L} . If \mathbf{N} is multiplicity free we say that a simple constituent \mathbf{L} is *linked* to a simple constituent \mathbf{L}' if for every submodule $\mathbf{S} \subset \mathbf{N}$ such that $[\mathbf{S} : \mathbf{L}] = 1$ then $[\mathbf{S} : \mathbf{L}'] = 1$.

Recall that the *socle* of \mathbf{N} , denoted $\text{soc } \mathbf{N}$, is the maximal semisimple submodule of \mathbf{N} . The *socle filtration* of \mathbf{N} is defined via induction by setting $\text{soc}^{(0)} \mathbf{N} := \{0\}$ and $\text{soc}^{(i+1)} \mathbf{N} := p^{-1}(\text{soc}(\mathbf{N} / \text{soc}^{(i)} \mathbf{N}))$, where $p : \mathbf{N} \rightarrow \mathbf{N} / \text{soc}^{(i)} \mathbf{N}$ is the canonical projection.

The *radical* of \mathbf{N} , denoted $\text{rad } \mathbf{N}$, is the intersection of the kernels of all maps $\pi : \mathbf{N} \rightarrow \mathbf{L}$ with \mathbf{L} a simple $\text{gl}(\infty)$ -module. The *radical filtration* of \mathbf{N} is similarly defined by induction, setting $\text{rad}^{(0)} \mathbf{N} = \mathbf{N}$ and $\text{rad}^{(i+1)} \mathbf{N} = \text{rad}(\text{rad}^{(i)} \mathbf{N})$. A module \mathbf{N} is *rigid* if its socle filtration is exhaustive, its radical filtration is separated, and they coincide.

We say that a module \mathbf{N} has a well-defined *Loewy length* if it has at least one exhaustive and discrete filtration whose layers are semisimple. The Loewy length of the module is then the minimum of the lengths of these filtrations, and is denoted by $\mathcal{L}\ell(\mathbf{N})$. By a classical result the socle filtration of the module \mathbf{N} is finite and exhaustive if and only if its radical filtration is finite and discrete, and in this case both filtrations have the same length, which is the Loewy length of the module. A proof of this fact for finite length modules can be found in [ASSo6, Chapter V, Proposition 1.3], and the proof adapts easily to our more general case.

3. MAIN STATEMENTS

Fix a weight $\lambda \in \mathfrak{h}^*$. From this point on, we assume that \mathbf{F} is either $\bigwedge^d \mathbb{V}, \mathbf{S}^d \mathbb{V}$ or $\bigwedge^A \mathbb{V}$, and that the fixed total order \leq on \mathbb{I} is (λ, \mathbf{F}) -compatible. We set $\mathfrak{b} = \mathfrak{b}_\leq$ and $\mathbf{M} := \mathbf{L}(\lambda) \otimes \mathbf{F}$.

3.1. The sets \mathbb{J} and \mathbb{J}_∞ . We introduce an equivalence relation \sim in \mathbb{I} : we write $i \sim i'$ if and only if $\lambda_i = \lambda_{i'}$ and set $\mathbb{J} := \mathbb{I} / \sim$. By definition every equivalence class $j \in \mathbb{J}$ is a set $\lambda^{-1}(a)$ for some $a \in \mathbb{Z}$, and thus an interval in the order \leq . This implies that \mathbb{J} inherits a total order from \mathbb{I} , and that the map $i \in \mathbb{I} \mapsto \lambda_i \in \mathbb{Z}^{\text{op}}$ induces an order-preserving injection of \mathbb{J} into \mathbb{Z}^{op} .

Denote by $\pi_{\mathbb{J}} : \mathbb{I} \longrightarrow \mathbb{J}$ the natural projection, and let $s : \mathbb{J} \longrightarrow \mathbb{I}$ be any section. We define a further equivalence relation \sim_{∞} in \mathbb{J} : if $j \leq j'$ then $j \sim_{\infty} j'$ if and only if the interval $[[s(j), s(j')]] \subset \mathbb{I}$ is finite. The definition of a (λ, \mathbf{F}) -compatible order guarantees that if $i < i'$ and $\pi_{\mathbb{J}}(i) = \pi_{\mathbb{J}}(i')$ then $[[i, i']]$ is finite, hence the equivalence relation \sim_{∞} is independent of the choice of s . Similarly to \mathbb{J} , the set $\mathbb{J}_{\infty} := \mathbb{J} / \sim_{\infty}$ inherits a total order and is order-isomorphic to a subset of \mathbb{Z} .

Denote by $\mathbb{C}^{(\mathbb{J})}$ the set of all finitely supported functions $\mathbb{J} \longrightarrow \mathbb{C}$. We define a linear map from \mathfrak{h}° to $\mathbb{C}^{(\mathbb{J})}$ by assigning to each $\gamma = \sum_i a_i \varepsilon_i$ the function δ_{γ} where $\delta_{\gamma}(j) = \sum_{i \in j} a_i$. Furthermore, we set $\mathbb{C}_0^{(\mathbb{J})} = \text{span} \{ \delta_j - \delta_{j'} \mid j, j' \in \mathbb{J} \}$, and introduce the linear map $h : \mathbb{C}_0^{(\mathbb{J})} \longrightarrow \mathbb{C}$, $h(\delta_j - \delta_{j'}) = \#[j, j'] - 1$. If $\gamma, \nu \in \mathfrak{h}^*$ and $\nu - \gamma$ is in the root lattice of $\mathfrak{gl}(\infty)$, we put $h(\nu, \gamma) = h(\delta_{\nu - \gamma})$.

In an analogous way we assign to every $\gamma \in \mathfrak{h}^{\circ}$ a map $\delta_{\gamma}^{\infty} : \mathbb{J}_{\infty} \longrightarrow \mathbb{C}$, define a linear map $h_{\infty} : \mathbb{C}_0^{(\mathbb{J}_{\infty})} \longrightarrow \mathbb{C}$, and a function $h_{\infty}(\nu, \gamma) = h_{\infty}(\delta_{\nu - \gamma}^{\infty})$ defined on pairs γ, ν such that $\nu - \gamma$ is in the root lattice. If $\nu > \gamma$ then $0 \leq h_{\infty}(\nu, \gamma) \leq h(\nu, \gamma)$.

Definition 3.1. Given $\nu, \gamma \in \mathfrak{h}^*$ we write $\nu \gg \gamma$ if $\nu = \gamma$ or $\nu - \gamma$ can be written as a sum of roots $\varepsilon_i - \varepsilon_j$ with $h_{\infty}(\varepsilon_i, \varepsilon_j) > 0$. Equivalently, $\nu \gg \gamma$ if and only if $\nu > \gamma$ and $h_{\infty}(\nu, \gamma) = h(\nu, \gamma)$.

3.2. Pieri sets and filtrations.

Definition 3.2. If $\mathbf{F} = \bigwedge^d \mathbb{V}, \bigwedge^A \mathbb{V}$ we define $\text{Pieri}(\lambda, \mathbf{F})$ to be the set of all $\gamma \in \text{supp } \mathbf{F}$ such that $\lambda + \gamma$ is \mathfrak{b}_{\leq} -dominant, i.e., such that $\lambda_i + \gamma_i \geq \lambda_{i'} + \gamma_{i'}$ whenever $i < i'$. If $\mathbf{F} = \mathbf{S}^d \mathbb{V}$ we define $\text{Pieri}(\lambda, \mathbf{F})$ to be the set of all $\gamma \in \text{supp } \mathbf{F}$ such that $\lambda_i \geq \lambda_{i'} + \gamma_{i'}$ whenever $i < i'$.

We point out that $\lambda + \gamma$ is \mathfrak{b}_{\leq} -dominant whenever $\gamma \in \text{Pieri}(\lambda, \mathbf{S}^d \mathbb{V})$. Since $\text{Pieri}(\lambda, \mathbf{F})$ is a set of weights, it is a poset with the order induced by the order \leq of \mathfrak{h}^* .

We now give a different characterization of the Pieri posets. Recall that the support of $\mathbf{S}^d \mathbb{V}$ consists of weights ε_I with $I = \{i_1 \leq i_2 \leq \dots \leq i_d\}$, while that of $\bigwedge^d \mathbb{V}$ is similar but with strict inequalities. If I is any such sequence and $X \subset \mathbb{I}$ then $I \cap X$ denotes the subsequence $\{i_{r_1} \leq \dots \leq i_{r_s} \mid i_{r_k} \in X\}$. Finally, the support of $\bigwedge^A \mathbb{V}$ consists of weights of the form $\varepsilon_A - \varepsilon_I + \varepsilon_{I'}$ with $I \subset A, I' \subset B$ and $\#I = \#I'$.

Lemma 3.3. *The poset $\text{Pieri}(\lambda, \mathbf{F})$ is locally finite, and for any $\mu \in \text{Pieri}(\lambda, \mathbf{F})$ the map $\gamma \in \text{Pieri}(\lambda, \mathbf{F}) \mapsto f_{\gamma - \mu} \in \mathbb{C}_0^{(\mathbb{J})}$ is injective.*

Proof. Suppose first $\mathbf{F} = \mathbf{S}^d \mathbb{V}, \bigwedge^d \mathbb{V}$. Consider the set \mathbb{J}^d with the product order. This is a locally finite poset since \mathbb{J} is isomorphic to a subset of \mathbb{Z} . Given $\varepsilon_I \in \text{Pieri}(\lambda, \mathbf{F})$ with $I = (i_1, \dots, i_d)$, we have $\pi_{\mathbb{J}}(I) = (\pi_{\mathbb{J}}(i_1), \dots, \pi_{\mathbb{J}}(i_d)) \in \mathbb{J}^d$. If $\mathbf{F} = \mathbf{S}^d \mathbb{V}$ we recover I from $\pi_{\mathbb{J}}(I)$ by taking i_k to be the first element in $\pi_{\mathbb{J}}(i_k)$. If $\mathbf{F} = \bigwedge^d \mathbb{V}$ we recover I from J as follows: let i_1 to be the first element of $\pi_{\mathbb{J}}(i_1)$, and once i_k has been determined let i_{k+1} to be the minimal element in $\pi_{\mathbb{J}}(i_{k+1})$ that is larger than i_k . Thus the map $I \mapsto \pi_{\mathbb{J}}(I)$ is injective in both cases. Notice also that $\varepsilon_I - \varepsilon_{I'} \leq 0$ if and only if I is

less than I' seen as elements of \mathbb{J}^d with the product order, which happens if and only if $\pi_{\mathbb{J}}(I) < \pi_{\mathbb{J}}(I')$. It follows that the map $I \mapsto \pi_{\mathbb{J}}(I)$ is monotone, so $\text{Pieri}(\lambda, \mathbf{F})$ is isomorphic to a subposet of \mathbb{J}^d and therefore locally finite.

Consider now the map that sends $J \in \mathbb{J}^d$ to $\delta_J = \sum_{k=1}^d \delta_{j_k} \in \mathbb{C}^{(\mathbb{J})}$. If we restrict the map to increasing sequences in \mathbb{J}^d then we get an injective map, and so $\varepsilon_I \mapsto f_\gamma = \delta_{\pi_{\mathbb{J}}(I)}$ is an injective map. It follows that, if we fix $\mu \in \text{Pieri}(\lambda, \mathbf{F})$, the map $\gamma \mapsto f_\gamma - f_\mu$ is also injective.

Now suppose $\mathbf{F} = \bigwedge^A \mathbb{V}$ with $A \subset \mathbb{I}$ and $B = \mathbb{I} \setminus A$ infinite sets. By definition $\varepsilon_A \in \text{Pieri}(\lambda, \mathbf{F})$. Also, $\gamma = \varepsilon_A - \varepsilon_I + \varepsilon_{I'} \in \text{supp } \mathbf{F}$ if and only if $I \cap j$ is a terminal subset of $A \cap j$ and $I' \cap j$ is an initial subset of $B \cap j$ for all $j \in \mathbb{J}$. The assignment $\gamma \in \text{Pieri}(\lambda, \mathbf{F}) \mapsto (\pi_{\mathbb{J}}(I), \pi_{\mathbb{J}}(I')) \in \mathbb{J}^{(\mathbb{Z}_{>0})} \times \mathbb{J}^{(\mathbb{Z}_{>0})}$ is once again injective and monotone, so $\text{Pieri}(\lambda, \mathbf{F})$ is locally finite. We can assume $\mu = \varepsilon_A$, and in this case $f_{\gamma - \mu} = \delta_{\pi_{\mathbb{J}}(I)} - \delta_{\pi_{\mathbb{J}}(I')}$, from which we can recover $(\pi_{\mathbb{J}}(I), \pi_{\mathbb{J}}(I'))$, and hence γ . This completes the proof. \square

Definition 3.4. Given $\gamma, \nu \in \text{Pieri}(\lambda, \mathbf{F})$ we write $\gamma \equiv_{\infty} \nu$ if and only if $h_{\infty}(\gamma, \nu) = 0$, and set $\mathbb{L}_{\infty} = \text{Pieri}(\lambda, \mathbf{F}) / \equiv_{\infty}$. We denote by \leq_{∞} the linear order on \mathbb{L}_{∞} induced by the order opposite of \leq .

For a fixed $\mu \in \text{Pieri}(\lambda, \mathbf{F})$ the map $\gamma \in \text{Pieri}(\lambda, \mathbf{F}) \mapsto h_{\infty}(\mu, \gamma) \in \mathbb{Z}$ is injective and monotone, so \mathbb{L}_{∞} is isomorphic to a subset of \mathbb{Z} . If \mathbf{F} is a \mathfrak{b}_{\leq} -highest weight module then we can take μ to be its highest weight. Notice that in this case the class of μ in \mathbb{L}_{∞} is the minimum of \mathbb{L}_{∞} .

Definition 3.5. The *linkage filtration* of \mathbf{M} is $\mathcal{F} = \{F_{\ell} \mathbf{M}\}_{\ell \in \mathbb{L}_{\infty}}$ where $F_{\ell} \mathbf{M}$ is the $\mathfrak{gl}(\infty)$ -module generated by $\{v_{\lambda} \otimes e_{\gamma} \mid \gamma \in \ell' \leq_{\infty} \ell\}$. If \mathbb{L}_{∞} has a minimum ℓ we set $F_{\ell-1} \mathbf{M} = \{0\}$, and if \mathbb{L}_{∞} has a maximum ℓ' we set $F_{\ell'+1} \mathbf{M} = \mathbf{M}$.

3.3. The statements. We introduce a technical condition.

Definition 3.6. Let $\pi = \pi_{\infty} \circ \pi_{\mathbb{J}} : \mathbb{I} \rightarrow \mathbb{J}_{\infty}$. We say A is \mathbb{J}_{∞} -initial if for every $j \in \mathbb{J}_{\infty}$ the set $A \cap \pi^{-1}(j)$ is an initial subset of $\pi^{-1}(j)$.

In the following statements we assume that \mathbf{F} is either $\mathbf{S}^d \mathbb{V}, \bigwedge^d \mathbb{V}$ or a Fock module $\bigwedge^A \mathbb{V}$ where A is \mathbb{J}_{∞} -initial.

Theorem (A). *The representation \mathbf{M} is multiplicity free, has a composition series, and every simple constituent is of the form $\mathbf{L}(\lambda + \gamma)$ for some $\gamma \in \text{Pieri}(\lambda, \mathbf{F})$. Furthermore, a simple constituent $\mathbf{L}(\lambda + \nu)$ is linked to $\mathbf{L}(\lambda + \gamma)$ if and only if $\nu \gg \gamma$.*

Corollary (A). *The length of \mathbf{M} is equal to the cardinality of $\text{Pieri}(\lambda, \mathbf{F})$. Moreover, if $\mathbf{F} = \mathbf{S}^d \mathbb{V}, \bigwedge^d \mathbb{V}$ then the length of \mathbf{M} is finite if and only if \mathbb{J} is finite, while if $\mathbf{F} = \bigwedge^A \mathbb{V}$ then \mathbf{M} has finite length if and only if $\mathbf{L}(\lambda)$ is of the form $\mathbf{D}_n \otimes \mathbf{L}'$, with \mathbf{L}' a simple tensor module.*

Recall that a module \mathbf{N} is said to be *uniform* if any two nontrivial submodules have nontrivial intersection. In particular a uniform module is indecomposable.

Theorem (B). *The module \mathbf{M} is either indecomposable or semisimple. More precisely:*

- (i) *If \mathbf{F} is a \mathfrak{b} -highest weight module and \mathbb{L}_∞ is trivial, then \mathbf{M} is semisimple.*
- (ii) *If \mathbf{F} is a \mathfrak{b} -highest weight module and \mathbb{L}_∞ is not trivial, then \mathbf{M} is indecomposable.*
- (iii) *If \mathbf{F} is not a \mathfrak{b} -highest weight module, then \mathbf{M} is uniform and has no socle.*

Theorem (C). (i) *The module \mathbf{M} has an exhaustive socle filtration if and only if \mathbf{F} is a \mathfrak{b} -highest weight module. If this happens then the socle filtration and the linkage filtration coincide.*

- (ii) *The module \mathbf{M} has a separated radical filtration if and only if \mathbb{L}_∞ has a maximum. If this happens then the radical filtration and the linkage filtration coincide.*
- (iii) *The module \mathbf{M} has finite Loewy length if and only if \mathbb{L}_∞ is finite, and in this case $\mathcal{L}\ell(\mathbf{M}) = \#\mathbb{L}_\infty - 1$. Furthermore, \mathbf{M} is rigid.*

We say that a linear order \leq on \mathbb{I} is (λ, \mathbf{F}_*) -compatible if it is $(-\lambda, \mathbf{F})$ -compatible. We also put $\text{Pieri}(\lambda, \mathbf{F}_*) = -\text{Pieri}(-\lambda, \mathbf{F})$, and define \mathbb{L}_∞ as before.

Theorem (D). *Let $\mathbf{F} = \mathbf{S}^d \mathbb{V}$ or $\mathbf{F} = \bigwedge^d \mathbb{V}$, and let $\mathbf{M}' := \mathbf{L}_\mathfrak{b}(\lambda) \otimes \mathbf{F}_*$ for a Borel subalgebra arising from a (λ, \mathbf{F}_*) -compatible linear order \leq on \mathbb{I} . Then all above statements hold for \mathbf{M}' if one replaces in them \mathbf{F} with \mathbf{F}_* .*

Before we proceed to the proofs we illustrate the above theorems with examples.

3.4. Examples. In 3.4.1–3.4.4 the module \mathbf{F} equals $\mathbf{S}^d \mathbb{V}$ or $\bigwedge^d \mathbb{V}$ for a fixed $d \in \mathbb{Z}_{>0}$.

3.4.1. Tensor modules. Let $\mathbb{I} = \mathbb{Z}_{>0} \sqcup \mathbb{Z}_{<0}$, and $\lambda = \sum_{i \in \mathbb{I}} \lambda_i \varepsilon_i$ with $\lambda_i = 0$ for all $i \in \llbracket n+1, -m-1 \rrbracket$ with $n, m \in \mathbb{Z}_{\geq 0}$. Then $\lambda = (\lambda_1, \dots, \lambda_n)$ and $\mu = (-\lambda_{-1}, \dots, -\lambda_{-m})$ are nonempty partitions, and $\mathbf{L}(\lambda)$ is isomorphic to the tensor module $\mathbf{V}^{\lambda; \mu}$ from [PS11]. The set \mathbb{J} is a finite set of cardinality at most $n+1+m$, and \mathbb{J}_∞ has exactly two elements. The module \mathbf{M} is semisimple if and only if $\mu = \emptyset$ and simple if and only if $\mathbf{L}(\lambda) = \mathbb{C}$.

If $\mathbf{F} = \mathbf{S}^d \mathbb{V}$ then \mathbf{F} has highest weight $d\varepsilon_1$, and $\varepsilon_I \in \text{supp } \mathbf{F}$ belongs to $\text{Pieri}(\lambda, \mathbf{F})$ if and only if $I \cap j = \{i_1 \leq \dots \leq i_r\}$ with $i_r = \min j$ and $r \in \mathbb{Z}_{\geq 0}$. If $\mathbf{F} = \bigwedge^d \mathbb{V}$ its highest weight is $\varepsilon_1 + \dots + \varepsilon_d$, and $\varepsilon_I \in \text{supp } \mathbf{F}$ belongs to $\text{Pieri}(\lambda, \mathbf{F})$ if and only if for each $j \in \mathbb{J}$ the $I \cap j \subset j$ is an initial subset.

Set $q = \min\{\sum_{i=1}^m \lambda_{-i} - \lambda_{-i-1}, d\}$ when $\mathbf{F} = \mathbf{S}^d \mathbb{V}$ and $q = \min\{-\sum_{i=1}^m \lambda_{-i}, d\}$ when $\mathbf{F} = \bigwedge^d \mathbb{V}$. Then the set \mathbb{L}_∞ is isomorphic to $\llbracket 0, q \rrbracket$, and through this identification the class of $\varepsilon_I \in \text{Pieri}(\lambda, \mathbf{F})$ in \mathbb{L}_∞ is $\#(I \cap \mathbb{Z}_{<0})$. In particular the Loewy length of \mathbf{M} is q and \mathbf{M} is rigid. The r -th layer of the socle filtration is isomorphic to the direct sum of all $\mathbf{L}(\lambda + \varepsilon_I)$ with $\#I \cap \mathbb{Z}_{<0} = r$, and $\mathbf{L}(\lambda + \varepsilon_I)$ is linked to $\mathbf{L}(\lambda + \varepsilon_J)$ if and only if $I \setminus J \subset \mathbb{Z}_{>0}$ and $J \setminus I \subset \mathbb{Z}_{<0}$. In particular $\text{soc } \mathbf{M}$ is simple if and only if $\lambda = \emptyset$.

3.4.2. The case $\mathbb{I} \subset \mathbb{Z}$. Suppose $\mathbb{I} = \mathbb{Z}_{>0}$. Then $\lambda = \sum_{i=1}^\infty \lambda_i \varepsilon_i$ with $\lambda_i \geq \lambda_{i+1}$. Thus \mathbb{J} is order isomorphic to a subset of $\mathbb{Z}_{>0}$. Moreover \mathbb{J}_∞ has a single element, so \mathbf{M} is semisimple. The module \mathbf{M} has finite length if and only if \mathbb{J} is finite, i.e., if and only if $\lambda_i = a$ for some $a \in \mathbb{Z}$ and all $i \gg 0$. Suppose now that $\mathbb{I} = \mathbb{Z}_{<0}$ or $\mathbb{I} = \mathbb{Z}$. Then \mathbb{J}

cannot have a minimal element and the module \mathbf{M} is uniform, has infinite length and its socle is trivial. However, every nontrivial quotient of \mathbf{M} is semisimple. All such quotients have finite length if \mathbb{J} has a maximum, and have infinite length if \mathbb{J} does not have a maximum.

3.4.3. *Infinite staircase module.* In this example we take $\mathbb{I} = \bigsqcup_{n \in \mathbb{Z}_{>0}} \mathbb{Z}_{\geq 0}$ and $\lambda_{k,n} = -n$. Then $\mathbb{J} = \mathbb{J}_\infty \cong \mathbb{Z}_{>0}$. For $\mathbf{F} = \mathbf{S}^d \mathbb{V}$ the poset $\text{Pieri}(\lambda, \mathbf{F})$ is isomorphic to the poset of sequences $(i_1, i_{k+2}, \dots, i_d)$ such that $1 = i_1 = \dots = i_k < i_{k+1} < i_{k+2} < \dots < i_d$ for some $k \leq d$, with the lexicographic order. The correspondence assigns to each sequence the weight $k\varepsilon_{0,1} + \varepsilon_{0,i_{k+1}} + \dots + \varepsilon_{0,i_d}$. For $\mathbf{F} = \bigwedge^d \mathbb{V}$ the poset $\text{Pieri}(\lambda, \mathbf{F})$ is isomorphic to the poset of all non-decreasing sequences $(i_1, \dots, i_d) \in \mathbb{Z}_{\geq 0}^d$ with the lexicographic order, and the correspondence assigns to a sequence the weight $\varepsilon_{0,i_1} + \dots + \varepsilon_{0,i_d}$.

In both cases $\text{Pieri}(\lambda, \mathbf{F})$ has a minimum, namely $(1, 1, \dots, 1)$. In addition, $\text{Pieri}(\lambda, \mathbf{F})$ is a lattice and every maximal chain ending at a sequence has the same length, so the poset $\text{Pieri}(\lambda, \mathbf{F})$ is ranked; the rank function coincides with $\gamma \mapsto h_\infty(\mu, \gamma)$, where μ is the highest weight of \mathbf{F} . The linkage filtration on \mathbf{M} is thus infinite and coincides with the socle filtration. A simple $\mathbf{L}(\lambda + \gamma)$ appears in the r -th layer of the filtration if and only if r is the rank of the sequence corresponding to γ . In particular the socle of \mathbf{M} is simple and isomorphic to $\mathbf{L}(\lambda + d\varepsilon_{0,1})$ for $\mathbf{F} = \mathbf{S}^d \mathbb{V}$ and to $\mathbf{L}(\lambda + \varepsilon_{0,1} + \dots + \varepsilon_{0,d})$ for $\mathbf{F} = \bigwedge^d \mathbb{V}$. Since the poset $\text{Pieri}(\lambda, \mathbf{F})$ has finitely many elements of a fixed rank, each layer of the socle filtration has finitely many simple summands.

3.4.4. *Double-infinite staircase module.* Take $\mathbb{I} = \bigsqcup_{n \in \mathbb{Z}} \mathbb{Z}_{>0}$ and $\lambda_{k,n} = -n$. Then $\mathbb{J} = \mathbb{J}_\infty \cong \mathbb{Z}$. Here $\text{Pieri}(\lambda, \mathbf{F})$ can be identified with the poset of strictly increasing sequences of integers of length d when $\mathbf{F} = \mathbf{S}^d \mathbb{V}$, and with the poset of non-decreasing sequences of integers of length d when $\mathbf{F} = \bigwedge^d \mathbb{V}$. In both cases the poset is locally ranked so we can define a rank function on $\text{Pieri}(\lambda, \mathbf{F})$ taking values in \mathbb{Z} . The socle and radical filtration of \mathbf{M} are trivial. However, the linkage filtration has semisimple layers, and two simples appear in the same layer if and only if the rank of the corresponding sequences is equal. This implies in particular that every layer has infinitely many simple summands.

3.4.5. *Dual Fock modules.* Let $\mathbf{F} = \bigwedge^A \mathbb{V}$ and $\lambda = -\varepsilon_C$ for infinite sets $A, C \subset \mathbb{I}$, and put $B = \mathbb{I} \setminus A, D = \mathbb{I} \setminus C$. To describe the (λ, \mathbf{F}) -compatible order on \mathbb{I} we let $W = A \cap D, X = B \cap D, Y = A \cap C$ and $Z = B \cap C$. By the definition of a (λ, \mathbf{F}) -compatible order, X and Z are isomorphic to final subsets of $\mathbb{Z}_{<0}$, Y and W are isomorphic to initial subsets of $\mathbb{Z}_{\geq 0}$, and $\mathbb{I} = W \sqcup X \sqcup Y \sqcup Z$.

Identifying $X \sqcup Y$ and $Z \sqcup W$ with the corresponding subsets of \mathbb{Z} , we write the elements of \mathbb{I} as (r, i) with $r \in \mathbb{Z}$ and $i = 0, 1$. The simple constituents of $\mathbf{M} = \mathbf{L}(\lambda) \otimes \mathbf{F}$ have highest weights μ_r , where

$$\mu_r = \sum_{i \geq r} \varepsilon_{(-i,0)} - \varepsilon_{(i,1)}$$

for all r such that this sum makes sense. For example, if $r = 1$ we recover the simple constituent with highest weight $\varepsilon_A - \varepsilon_C$. The module \mathbf{M} always has infinite length, and is semisimple if and only if $\#Y \sqcup Z < \infty$. If $\#Y \sqcup Z = \infty$ then \mathbb{L}_∞ can be identified as an ordered set with $\{r \in \mathbb{Z} \mid \mu_r \in \text{Pieri}(\lambda, \mathbf{F})\}$, and the r -th layer of the linkage filtration is isomorphic to $\mathbf{L}(\mu_r)$. In the case where $\#A \triangle C < \infty$, the existence of the linkage filtration and the description of its layers are due to Serganova, see [Ser21, Theorem 1.1].

3.4.6. *Modules with finite \mathbb{J} .* Set again $\mathbf{F} = \bigwedge^A \mathbb{V}$ and let us consider the more general case where λ takes finitely many different values. This is equivalent to \mathbb{J} being finite and hence \mathbb{J}_∞ being also finite. Suppose $\mathbb{J}_\infty = \{j_1 < j_2 < \dots < j_r\}$, and let $A_k = A \cap j_k, B_k = B \cap j_k$. We can always assume that A_k, B_k is either infinite or empty. The definition of a (λ, \mathbf{F}) -compatible order shows that $j_k = A_k \sqcup B_k$. The set $\text{Pieri}(\lambda, \mathbf{F})$ consists of those weights $\varepsilon_A + \varepsilon_I - \varepsilon_J$ such that $J \cap A_k$ is a terminal subset of $A_k, I \cap B_k$ is an initial subset of B_k , and for each k at least one of $J \cap A_k$ and $I \cap B_k$ is empty.

The module \mathbf{M} has a socle if and only if \mathbf{F} is a \mathfrak{b} -highest weight module, i.e., if and only if A is an initial subset of \mathbb{I} . In this case the linkage filtration coincides with the socle filtration of \mathbf{M} .

Suppose at least two elements of \mathbb{J} are infinite classes. Then there exist two distinct infinite classes $j, j' \in \mathbb{J}$ such that $A \cap j$ and $B \cap j'$ are also infinite; for simplicity we assume that $j < j'$. If I is a terminal subset of $A \cap j$ and J is an initial subset of $B \cap j'$ then $\varepsilon_{A \setminus I \cup J} \in \text{Pieri}(\lambda, \mathbf{F})$ and

$$h_\infty(\varepsilon_A, \varepsilon_A - \varepsilon_I + \varepsilon_J) = h_\infty(\varepsilon_I, \varepsilon_J) = (\#I)(\#[j, j'])$$

Thus if \mathbb{J} has two infinite classes the length of \mathbf{M} and its Loewy length are both infinite.

On the other hand if \mathbb{J} has exactly one infinite class then $\mathbf{L}(\lambda)$ is of the form $\mathbf{D}_n \otimes \mathbf{V}^{\lambda, \mu}$ with λ, μ finite partitions. In this case we can assume $\mathbb{I} = \llbracket r \rrbracket \sqcup \mathbb{Z} \sqcup \llbracket s \rrbracket$, where r is the number of distinct entries of λ and s the number of distinct entries in μ . Up to finite exchanges, the only infinite subsets of \mathbb{I} with infinite complement are either initial or terminal. Since the order on \mathbb{I} is (λ, \mathbf{F}) -compatible it follows that A is an initial subset and \mathbf{F} is a highest weight module. Without loss of generality we can assume $A = \llbracket r \rrbracket \cup \mathbb{Z}_{\leq 0}$. The set \mathbb{L}_∞ is isomorphic to $\llbracket 0, r+s \rrbracket$, with the class of a weight ε_I equal to $2\#(I \cap \llbracket s \rrbracket) + \max I \cap \mathbb{Z}$. The socle of \mathbf{M} is simple and isomorphic to $\mathbf{L}(\lambda + \varepsilon_A)$, and the radical of \mathbf{M} is also simple and isomorphic to $\mathbf{L}(\lambda + \varepsilon_{A'})$ with $A' = \mathbb{Z}_{\leq r-s} \cup \llbracket s \rrbracket$.

3.4.7. *A non-example.* We finish this section with an example in which our results do not apply. Let $\mathbb{I} = \mathbb{Z}_{>0}, \lambda = \sum_{i=1}^{\infty} (-i)\varepsilon_i$ and $\mathbf{F} = \bigwedge^A \mathbb{V}$ with $A = 2\mathbb{Z}_{>0}$. Then \mathbf{F} is not a \mathfrak{b} -highest weight module since \mathbb{I} has no infinite initial subsets other than itself. Also $\mathbb{J} \cong \mathbb{Z}_{>0}$ and $\#\mathbb{J}_\infty = 1$ so A is not \mathbb{J}_∞ -initial. As we show in 5.3.1 and 6.1.1 below, the module \mathbf{M} has a simple subquotient isomorphic to $\mathbf{L}(\lambda + \varepsilon_B)$, and since ε_B is not in the support of \mathbf{F} Theorem A does not hold.

The rest of the paper is devoted to proving of Theorems A–D.

4. TRANSITION MAPS

We start the build-up for the proofs by addressing some key issues in the finite-dimensional theory. Given a finite set $Y \subset \mathbb{I}$ we set $V_Y := \text{span}\{e_i \mid i \in Y\}$, $V_Y^* := \text{span}\{e_i^* \mid i \in Y\}$. Then $\mathfrak{gl}(Y) := V_Y^* \otimes V_Y \subset \mathfrak{gl}(\infty)$ is isomorphic to $\mathfrak{gl}(\#Y)$. For a subspace $W \subset \mathfrak{gl}(\infty)$ we write $W_Y = W \cap \mathfrak{gl}(Y)$. In particular, \mathfrak{h}_Y is a Cartan subalgebra of $\mathfrak{gl}(Y)$ and all weights $\lambda \in \mathfrak{h}_Y^*$ are assumed to be integral. We will assume that \mathbb{I} has a total order \leq so $\mathfrak{b}_Y = \mathfrak{b}_{\leq} \cap \mathfrak{gl}(Y)$ is a Borel subalgebra of $\mathfrak{gl}(Y)$, and we will refer to \mathfrak{b}_Y -dominant weights simply as dominant weights.

4.1. Weight multiplicity-free representations and Pieri rules. Let $Y \subset \mathbb{I}$ be a finite subset. It is well known that the modules $\mathbf{S}^d V_Y$ with $d \geq 0$ and $\bigwedge^d V_Y$ with $0 \leq d \leq \#Y$ are weight multiplicity-free, and every other weight multiplicity free $\mathfrak{gl}(Y)$ -module can be obtained by dualizing and tensoring these modules with one-dimensional modules. In what follows, a skew Young diagram is called a vertical strip if it contains at most one box in each row, and a horizontal strip if it contains at most one box in each column.

Let $\lambda = \sum_{i \in Y} \lambda_i \varepsilon_i \in \mathfrak{h}_Y^*$ be a dominant weight, and let $F = \mathbf{S}^d V_Y$ or $F = \bigwedge^d V_Y$. The tensor product $L_Y(\lambda) \otimes F$ is semisimple and multiplicity free, i.e., every simple submodule appears exactly once in its direct sum decomposition. We denote by $\text{Pieri}(\lambda, F)$ the set of weights $\gamma \in \text{supp } F$ such that $\lambda + \gamma$ is a highest weight in $L_Y(\lambda) \otimes F$. In this language, the classical Pieri rules [FH91, formulas (6.8) and (6.9)] state that $\gamma \in \text{Pieri}(\lambda, \bigwedge^d V_Y)$ if and only if $\lambda + \gamma$ is dominant, while $\gamma \in \text{Pieri}(\lambda, \mathbf{S}^d V_Y)$ if and only if $\gamma_{i+1} \leq \lambda_i - \lambda_{i+1}$. If $\lambda_i \geq 0$ for all $i \in Y$ then λ can be read as a partition with $\#Y$ parts, which we identify it with its Young diagram. Moreover $\gamma \in \text{supp } F$ belongs to $\text{Pieri}(\lambda, F)$ if and only if the skew diagram $(\lambda + \gamma)/\lambda$ is a horizontal strip for $F = \mathbf{S}^d V_Y$, or a vertical strip for $F = \bigwedge^d V_Y$ (see for example [Ful97, section 2.2 (4) and (5)]). In both cases, given $\nu, \gamma \in \text{Pieri}(\lambda, F)$ we have $\nu > \gamma$ if and only if the k -th box of the strip $(\lambda + \nu)/\lambda$ is located above and to the right of the k -th box of the strip $(\lambda + \gamma)/\lambda$ for all $k \in \llbracket d \rrbracket$.

Lemma 4.1. *Let F be a simple weight multiplicity-free $\mathfrak{gl}(Y)$ -module and λ be a dominant weight.*

- (i) *The poset $\text{Pieri}(\lambda, F)$ is a lattice.*
- (ii) *For every $\gamma \in \text{supp } F$ the set $\{\mu \in \text{Pieri}(\lambda, F) \mid \mu \geq \gamma\}$ has a least element, which we denote γ' .*
- (iii) *Assume $\gamma \in \text{Pieri}(\lambda, F)$ and $\nu \in \text{supp } F$ are such that $\nu = \gamma + \varepsilon_I - \varepsilon_J$ and $\max I < \min J$. Then $\nu' = \gamma + \varepsilon_{I'} - \varepsilon_{J'}$ and $\max I' \leq \max I < \min J \leq \min J'$.*

Proof. It is enough to prove the results when $\lambda_i \geq 0$ and F equals $\mathbf{S}^d V_Y$ or $\bigwedge^d V_Y$.

Let $\nu, \gamma \in \text{Pieri}(\lambda, F)$. The skew diagram obtained by superimposing the strips $(\lambda + \nu)/\lambda$ and $(\lambda + \gamma)/\lambda$ is again a horizontal or vertical strip, with at most $2d$ boxes. We let $\tilde{\mu}$ to be the strip obtained by taking the first d boxes in this superposition, counting from bottom to top and from right to left. Then $\lambda \cup \tilde{\mu} = \lambda + \mu$ for some

$\mu \in \text{Pieri}(\lambda, F_Y)$, and by construction any vertical or horizontal strip $(\lambda + \sigma)/\sigma$ will have its k -th box below the k -th box of $\tilde{\mu}$ if and only if it is below the k -th box of both $(\lambda + \nu)/\lambda$ and $(\lambda + \gamma)/\lambda$. Thus every pair of elements in $\text{Pieri}(\lambda, F)$ has a maximal lower bound. Since the highest weight of F is the maximum of $\text{Pieri}(\lambda, F)$, [Sta12, 3.3.1 Proposition] implies $\text{Pieri}(\lambda, F)$ is a lattice and item (i) is proved.

Every subset of a finite lattice has a unique maximal lower bound. Denoting γ' the maximal lower bound of $\{\mu \in \text{Pieri}(\lambda, F) \mid \mu \geq \gamma\}$, we see that $\gamma' \geq \gamma$ since γ is tautologically a lower bound for this set.

We will prove item (iii) by induction on $r = \#I = \#J$. First assume $\nu = \gamma + \varepsilon_i - \varepsilon_j$ with $\lambda_i > \lambda_j$. By the previous item there is a maximal $j' \leq j$ such that $(\lambda + \gamma - \varepsilon_{j'})/\lambda$ is a horizontal or vertical strip and $\gamma - \varepsilon_{j'} \geq \gamma - \varepsilon_j$. There is also a minimal $i' \geq i$ such that $(\gamma - \varepsilon_{j'} + \varepsilon_{i'})/\lambda$ is a horizontal or vertical strip and $\gamma - \varepsilon_{j'} + \varepsilon_{i'} \geq \gamma - \varepsilon_{j'} + \varepsilon_i \geq \gamma - \varepsilon_j + \varepsilon_i$. Since $\nu' = \gamma + \varepsilon_{i'} - \varepsilon_{j'}$ the base case is complete. Now suppose $I = (i_1, \dots, i_r)$ and $J = (j_1, \dots, j_r)$. Set $\tilde{I} = (i_1, \dots, i_{r-1})$, $\tilde{J} = (j_1, \dots, j_{r-1})$ and $\mu = \gamma + \varepsilon_{\tilde{I}} - \varepsilon_{\tilde{J}}$. Then using the induction hypothesis and the base case we have

$$\nu' = (\mu' + \varepsilon_{i_r} - \varepsilon_{j_r})' = \mu' + \varepsilon_{i_r} - \varepsilon_{j_r} = \gamma + \varepsilon_{\tilde{I}} - \varepsilon_{\tilde{J}} + \varepsilon_{i_r} - \varepsilon_{j_r}.$$

Set $I' = \tilde{I}' \cup \{i_r'\}$ and $J' = \tilde{J}' \cup \{j_r'\}$. By the induction hypothesis $\max \tilde{I}' \leq i_{r-1} \leq i_r$ and $\min J' = j_1 \leq \min \tilde{J}'$, and by the base step $i_r' \leq i_r$ and $j_1 \leq j_r \leq j_r'$. Combining both observations we see that $\max I' \leq i_r = \max I$ and $j_1 = \min J \leq \min J'$. \square

4.2. Transition maps. Let $X \subset Y \subset \mathbb{I}$ be fixed finite sets, and let $\lambda \in \mathfrak{h}_Y^*$ be a dominant weight. The simple module $L_Y(\lambda)$ is a $\mathfrak{gl}(X)$ -module by restriction, and the $\mathfrak{gl}(X)$ -module generated by its highest weight vector is isomorphic to $L_X(\lambda_X)$. We thus get an injective $\mathfrak{gl}(X)$ -homomorphism $\iota : L_X(\lambda_X) \rightarrow L_Y(\lambda)$. Let F_Y be a weight multiplicity-free $\mathfrak{gl}(Y)$ -module, and let F_X be a simple $\mathfrak{gl}(X)$ -submodule of F_Y (not necessarily generated by the highest weight vector of F_Y). Then F_X is also weight multiplicity free, and thus we get a $\mathfrak{gl}(X)$ -homomorphism $\tau : F_X \rightarrow F_Y$.

Set $\varphi(X, Y) := \iota \otimes \tau : L_X(\lambda_X) \otimes F_X \rightarrow L_Y(\lambda) \otimes F_Y$. By the Pieri rule this homomorphism decomposes as

$$\begin{array}{ccc} L_X(\lambda_X) \otimes F_X & \xrightarrow{\cong} & \bigoplus_{\gamma \in \text{Pieri}(\lambda_X, F_X)} L_X(\lambda_X + \gamma) \\ \varphi(X, Y) \downarrow & & \downarrow \varphi(X, Y, \gamma, \nu) \\ L_Y(\lambda) \otimes F_Y & \xrightarrow{\cong} & \bigoplus_{\nu \in \text{Pieri}(\lambda, F_Y)} L_Y(\lambda + \nu). \end{array}$$

Since the domain of each homomorphism $\varphi(X, Y, \gamma, \nu)$ is a simple $\mathfrak{gl}(X)$ -module, $\varphi(X, Y, \gamma, \nu)$ is either zero or injective. Notice that $\varphi(X, Y, \gamma, \nu) \neq 0$ if and only if the $\mathfrak{gl}(Y)$ -submodule generated by $\varphi(X, Y)(L_X(\lambda + \gamma))$ contains $L_Y(\lambda + \nu)$. We will devote the rest of this section to give necessary and sufficient conditions for $\varphi(X, Y, \gamma, \nu)$ to be nonzero.

4.3. Necessary conditions. Recall we have fixed $X \subset Y \subset \mathbb{I}$. In addition we suppose that there is a decomposition $Y = X^- \sqcup Z \sqcup X^+$ where $Z = Y \setminus X$. We also assume $(F_X, F_Y) = (\mathbf{S}^d V_X, \mathbf{S}^d V_Y)$ or $(F_X, F_Y) = (\bigwedge^d V_X, \bigwedge^d V_Y)$, with the map $\tau : F_X \rightarrow F_Y$ induced by the natural inclusion $V_X \rightarrow V_Y$, and fix a dominant $\mathfrak{gl}(Y)$ -weight λ . For any $X \subset Y$ we denote by λ_X its restriction to \mathfrak{h}_X . We have an injection $\mathfrak{h}_X^* \rightarrow \mathfrak{h}_Y^*$ mapping $\varepsilon_i \in \mathfrak{h}_X^*$ to $\varepsilon_i \in \mathfrak{h}_Y^*$, which allows us to consider a weight $\gamma \in \mathfrak{h}_X^*$ as a weight in \mathfrak{h}_Y^* .

Lemma 4.2. *Let $\gamma \in \text{Pieri}(\lambda_X, F_X)$ and $\nu \in \text{Pieri}(\lambda, F_Y)$. If $\varphi(X, Y, \gamma, \nu) \neq 0$ then there exist sequences $I \subset X^- \sqcup Z$ and $J \subset X^+$ such that $\nu - \gamma = \varepsilon_I - \varepsilon_J$.*

Proof. We proceed by induction on $\#Z$. Suppose first that $Z = \{z\}$, and that $\varphi(X, Y, \gamma, \nu) \neq 0$. Then $L_Y(\lambda + \nu)$ has a $\mathfrak{gl}(X)$ -submodule isomorphic to $L_X(\lambda_X + \gamma)$. The classical branching rule [Zhe73, Theorem 2, section 66] (see also [FH91, Exercise 6.12]) tells us that $\lambda_i + \nu_i \geq \lambda_i + \gamma_i \geq \lambda_{i+1} + \nu_{i+1}$ for all $i \in X^-$ and $\lambda_{i-1} + \nu_{i-1} \geq \lambda_i + \gamma_i \geq \lambda_i + \nu_i$ for all $i \in X^+$ (here the successor and predecessor are taken in Y). In particular $\nu_i \geq \gamma_i$ if $i \in X^-$ and $\gamma_i \geq \nu_i$ if $i \in X^+$, so

$$\nu - \gamma = \sum_{i \in X^-} (\nu_i - \gamma_i) \varepsilon_i + \nu_z \varepsilon_z - \sum_{i \in X^+} (\gamma_i - \nu_i) \varepsilon_i.$$

This proves the base case.

For the induction step, assume $Z = Z' \sqcup \{z\}$ with Z' nonempty. Set $Y' = X^- \sqcup Z' \sqcup X^+$ and $F_{Y'} = \mathbf{S}^d V_{Y'}$ or $F_{Y'} = \bigwedge^d V_{Y'}$. Since $\varphi(X, Y, \gamma, \nu) \neq 0$ there exists $\mu \in \text{Pieri}(\lambda_{Y'}, F_{Y'})$ such that $\varphi(X, Y', \gamma, \mu) \neq 0$ and $\varphi(Y', Y, \mu, \nu) \neq 0$. Thus by the induction hypothesis

$$\nu - \gamma = (\nu - \mu) + (\mu - \gamma) = (\varepsilon_K - \varepsilon_L) + (\varepsilon_{I'} - \varepsilon_{J'})$$

with $K \subset X^- \sqcup Z, I' \subset X^- \sqcup Z'$ and $L, J' \subset X^+$, and the proof is complete. \square

4.4. Nonzero transition maps. The following is a partial converse of Lemma 4.2.

Proposition 4.3. *Let $\gamma \in \text{Pieri}(\lambda_X, F_X)$ and $\nu \in \text{Pieri}(\lambda, F_Y)$ and let I, J be sequences such that $\nu - \gamma = \varepsilon_I - \varepsilon_J$.*

- (i) *Suppose $X^+ = \emptyset$. Then $\varphi(X, Y, \gamma, \nu) \neq 0$ if and only if $I = J = \emptyset$.*
- (ii) *Suppose $X^- = \emptyset$. Then $\varphi(X, Y, \gamma, \nu) \neq 0$ if and only if $I \subset Z$ and $J \subset X^+$.*
- (iii) *Suppose $X^+ \neq \emptyset \neq X^-$, $\lambda(Z) = \{a\}$ with $a > \lambda_{\min X^+}$, and $\#Z \geq d$. Then $\varphi(X, Y, \gamma, \nu) \neq 0$ if and only if $I \subset X^- \sqcup Z$ and $J \subset X^+$.*

We will prove each item separately. The first one is easy.

Proof of item (i) of Proposition 4.3. The condition is necessary by Lemma 4.2, and as a consequence $\varphi(X, Y, \gamma, \nu) = 0$ whenever $\nu \neq \gamma$. On the other hand, since $Y = X \sqcup Z$ the definition of $\text{Pieri}(\lambda, F_Y)$ implies that $\gamma \in \text{Pieri}(\lambda, F_Y)$. Finally, since at least one of the transition maps $\varphi(X, Y, \gamma, \nu)$ must be nonzero, it follows that $\varphi(X, Y, \gamma, \gamma) \neq 0$. \square

4.5. **Highest weight vectors and the case $Y = Z \sqcup X$.** Our strategy to prove items (ii) and (iii) of Proposition 4.3 is to take the highest weight vector $v_{\lambda+\gamma}^X \in L_X(\lambda_X + \gamma) \subset L_X(\lambda_X) \otimes F_X$ and show that the $\mathfrak{gl}(Y)$ -module generated by $\varphi(X, Y)(v_{\lambda+\gamma}^X)$ has a $\mathfrak{gl}(Y)$ -submodule isomorphic to $L_Y(\lambda + \nu)$. For this we will need some technical results on highest weight vectors.

Let $\mu \in \text{supp } F_X$. We denote by $M_X(> \mu)$ the sum of all submodules $L_X(\lambda_X + \gamma)$ with $\gamma \in \text{Pieri}(\lambda, F_X)$ and $\gamma > \mu$. The submodules $M_X(\geq \mu)$ and $M_X(\not\geq \mu)$ are defined analogously.

Lemma 4.4. *Let $\gamma \in \text{Pieri}(\lambda, F_X)$ and let $v_{\lambda+\gamma}^X$ be the highest weight vector of $L_X(\lambda_X + \gamma) \subset L_X(\lambda_X) \otimes F_X$. Then*

$$v_{\lambda+\gamma}^X \equiv \sum_{\mu \geq \eta \geq \gamma} w_{\lambda+\gamma-\eta} \otimes e_\eta \pmod{M_X(\not\geq \mu)}.$$

Furthermore, $w_{\lambda+\gamma-\eta} \neq 0$ for all η , and their linear span is an \mathfrak{n}_X -submodule of $L_X(\lambda_X)$ isomorphic to $\bigoplus_{\mu \geq \eta \geq \gamma} (F_X^*)_{-\eta}$.

Proof. We start with the case where μ is the highest weight vector of F_X , and so $M_X(\not\geq \mu) = 0$. Then $v_{\lambda+\gamma}^X = \sum_{\eta \in \text{supp } F_X} w_{\lambda+\gamma-\eta} \otimes e_\eta$, with $w_{\lambda+\gamma-\eta} \in L_X(\lambda_X)$ of weight $\lambda_X + \gamma - \eta$. Since λ_X is the highest weight of this module we see that $w_{\lambda+\gamma-\eta} = 0$ unless $\gamma \leq \eta$, so $v_{\lambda+\gamma}^X = \sum_{\mu \geq \eta \geq \gamma} w_{\lambda+\gamma-\eta} \otimes e_\eta$.

By standard Lie theory we have isomorphisms

$$(L_X(\lambda) \otimes F_X)^{\mathfrak{n}_X} \cong \text{hom}_{\mathfrak{n}_X}(F_X^*, L_X(\lambda)) \quad (L_X(\lambda) \otimes F_X)^{\mathfrak{n}_X} \cong \text{hom}_{\mathfrak{n}_X}(L_X(\lambda)^*, F_X).$$

The first one sends $v_{\lambda+\gamma}^X$ to the \mathfrak{n}_X -homomorphism ψ given by $\psi(e_\eta^*) = w_{\lambda+\gamma-\eta}$, and so the span of the vectors $w_{\lambda+\gamma-\eta}$ is the image of the \mathfrak{n}_X -submodule $\bigoplus_{\mu \geq \eta \geq \gamma} (F_X^*)_{-\eta}$ by ψ . The statement will be proved once we show that the restriction of ψ to $\bigoplus_{\mu \geq \eta \geq \gamma} (F_X^*)_{-\eta}$ is injective, which is equivalent to the fact that $\psi(e_\gamma^*) = w_\lambda$ is nonzero. Now, the second isomorphism sends $v_{\lambda+\gamma}^X$ to the unique \mathfrak{n}_X -homomorphism defined by $v_\lambda^* \mapsto v_\lambda^*(w_\lambda)e_\gamma$. Since $v_{\lambda+\gamma}^X \neq 0$, it must be the case that $v_\lambda^*(w_\lambda) \neq 0$ and hence $w_\lambda \neq 0$. This completes the proof when μ is the highest weight of F_X .

We now proceed with the general case. A descending induction argument, again starting with the highest weight μ of F_X , shows that $M_X(\geq \mu)$ is generated by $v_\lambda^X \otimes e_\mu$ as a $\mathfrak{gl}(X)$ -module. It follows that if $\eta \not\geq \mu$ then $v_\lambda^X \otimes e_\eta \in M_X(\not\geq \mu)$. Now, since $L_X(\lambda) \otimes F_X$ is generated as an \mathfrak{n}_X^- -module by the set $\{v_\lambda^X \otimes e_\eta \mid \eta \in \text{supp } F_X\}$, the quotient $M/M_X(\not\geq \mu)$ is generated as an \mathfrak{n}_X^- -module by $\{v_\lambda^X \otimes e_\eta \mid \eta \leq \mu\}$. If we act by any element of \mathfrak{n}_X^- on any such generator, we obtain a linear combination of vectors $\sum_{\mu \geq \eta \geq \gamma} w_{\lambda+\gamma-\eta} \otimes e_\eta$ for some $w_{\lambda+\gamma-\eta}$. In particular $v_{\lambda+\gamma}^X \equiv \sum_{\mu \geq \eta \geq \gamma} w_{\lambda+\gamma-\eta} \otimes e_\eta \pmod{M_X(\not\geq \mu)}$, and the second part of the statement follows from the previous case. \square

Proof of item (ii) of Proposition 4.3. That the condition is necessary follows from Lemma 4.2. To see that it is sufficient we proceed by induction on $\#Z$. Suppose that $Z = \{z\}$

and that $\nu - \gamma = \sum_{i \in X} a_i(\varepsilon_z - \varepsilon_i)$ for some $a_i \in \mathbb{Z}_{\geq 0}$. Let $E = \prod_{j \in X} E_{z,i}^{a_i} \in U(\mathfrak{gl}(Y))_{\nu-\gamma}$; this product is well defined since all the factors commute. By Lemma 4.4

$$\varphi(X, Y)(v_{\lambda+\gamma}^X) \equiv \sum_{\nu \geq \eta \geq \gamma} w_{\lambda+\gamma-\eta} \otimes e_\eta \pmod{M_Y(\not\leq \nu)}.$$

Applying E to the sum on the right hand side of the congruence we obtain a sum of terms of the form $E_\sigma w_{\lambda+\gamma-\eta} \otimes E_\tau e_\eta$, where $\sigma + \tau = \nu - \gamma$. Notice that $\sigma \neq 0$ implies $\sigma_z \neq 0$.

By weight considerations, $E_\sigma w_{\lambda+\gamma-\eta} = 0$ unless $\sigma + \lambda + \gamma - \eta \leq \lambda$, or equivalently $\sigma + \gamma \leq \eta$. On the other hand, $E_\sigma w_{\lambda+\gamma-\eta} \equiv 0$ modulo $M(\not\leq \nu)$ unless $\tau + \eta \leq \nu$, which implies $\eta \leq \nu - \tau = \gamma + \sigma$. Thus $\sigma = \eta - \lambda_X$. Since $\text{supp } \eta - \lambda_X \subset X$, it must be the case that $\sigma = 0$. In conclusion

$$E\varphi(X, Y)(v_{\lambda+\gamma}^X) \equiv w_\lambda \otimes Ee_\gamma \equiv cw_\lambda \otimes e_\nu \pmod{M_Y(\not\leq \nu)}.$$

for some nonzero constant c . This vector is a nonzero multiple of $v_\lambda \otimes e_\nu$ which generates $L_Y(\lambda + \nu)$ modulo $M_Y(\not\leq \nu)$, and so the direct summand $L_Y(\lambda + \nu)$ is contained in the $\mathfrak{gl}(Y)$ -module generated by $\varphi(X, Y)(L_X(\lambda + \gamma))$. This shows that $\varphi(X, Y, \gamma, \nu) \neq 0$ and completes the proof of the base case.

We now proceed with the inductive step. Suppose $Z = \{z\} \sqcup Z'$ and $\nu - \gamma = \sum_{i \in Z, j \in X} a_{i,j}(\varepsilon_i - \varepsilon_j)$. Set $Y' = Z' \sqcup X$ and $\mu = \gamma + \sum_{i \in Z', j \in X} a_{i,j}(\varepsilon_i - \varepsilon_j)$. By the induction hypothesis $\varphi(X, Y', \gamma, \mu)$ and $\varphi(Y', Y, \mu, \nu) \neq 0$, which implies $\varphi(X, Y, \gamma, \nu) \neq 0$. \square

4.6. The case $X^+ \neq \emptyset \neq X^-$. For proving item (iii) of Proposition 4.3 we need some preparation. Suppose $A \subset X$ and let $C = X \setminus A$. Then $\mathfrak{gl}(A) \oplus \mathfrak{gl}(C)$ is a Lie subalgebra of $\mathfrak{gl}(X)$ and any $\mathfrak{gl}(X)$ -module is a $\mathfrak{gl}(A) \oplus \mathfrak{gl}(C)$ -module by restriction. In particular there are decompositions of $\mathfrak{gl}(A) \oplus \mathfrak{gl}(C)$ -modules

$$\mathbf{S}^d V_X = \bigoplus_{k+\ell=d} \mathbf{S}^k V_A \boxtimes \mathbf{S}^\ell V_C, \quad \bigwedge^d V_X = \bigoplus_{k+\ell=d} \bigwedge^k V_A \boxtimes \bigwedge^\ell V_C.$$

Thus, if $\gamma \in \text{supp } F_X$ there exist a unique weight multiplicity-free $\mathfrak{gl}(A)$ -module F_A and a unique weight multiplicity-free $\mathfrak{gl}(C)$ -module F_C such that $\gamma \in \text{supp } F_A \boxtimes F_C$. Since F_C is a trivial $\mathfrak{gl}(A)$ -module we have a $\mathfrak{gl}(A)$ -homomorphism $p_\gamma : F_A \otimes F_C \rightarrow F_A$ given by

$$p_\gamma(e_\eta) = \begin{cases} e_{\eta_A} & \text{if } \eta_C = \gamma_C \\ 0 & \text{otherwise.} \end{cases}$$

Now let $\mu \in \mathfrak{h}_X$ be a dominant weight. The highest weight vector $v_\mu^X \in L_X(\mu)$ is a $\mathfrak{gl}(A) \oplus \mathfrak{gl}(C)$ -highest weight vector and generates the unique $\mathfrak{gl}(A) \oplus \mathfrak{gl}(C)$ -submodule of $L_X(\mu)$ isomorphic to $L_A(\mu_A) \boxtimes L_C(\mu_C)$. Since $L_C(\mu_C)$ has a trivial $\mathfrak{gl}(A)$ -action there exists a unique $\mathfrak{gl}(A)$ -homomorphism $L_C(\mu_C) \rightarrow \mathbb{C}$ sending the highest weight vector $v_{\mu_C}^C$ to 1 and annihilating any weight space of weight lower than μ_C . We denote by p_A the $\mathfrak{gl}(A)$ -homomorphism given by the composition $L_X(\mu) \rightarrow L_A(\mu_A) \otimes L_C(\mu_C) \rightarrow L_A(\mu_A)$. With these preliminaries in place we can prove the following lemma.

Lemma 4.5. *Let $\gamma \in \text{Pieri}(\lambda_X, F_X)$ and $\nu \in \text{Pieri}(\lambda, F_Y)$ with $\nu - \gamma = \varepsilon_I - \varepsilon_J$. Suppose $B \subset Y$ is such that $I, J \subset B$, and let $A = B \cap X$. If $\varphi(A, B, \gamma_A, \nu_B) \neq 0$ then $\varphi(X, Y, \gamma, \nu) \neq 0$.*

Proof. We now prove that the diagram

$$\begin{array}{ccccccc} L_X(\lambda_X + \gamma) & \longrightarrow & L_X(\lambda_X) \otimes F_X & \longrightarrow & L_Y(\lambda) \otimes F_Y & \longrightarrow & L_Y(\lambda + \nu) \\ \iota_A \uparrow & & \downarrow p_A \otimes p_\gamma & & \downarrow p_B \otimes p_\nu & & \downarrow p_B \\ L_A(\lambda_A + \gamma_A) & \longrightarrow & L_A(\lambda_A) \otimes F_A & \longrightarrow & L_B(\lambda_B) \otimes F_B & \longrightarrow & L_B(\lambda_B + \nu_B) \end{array}$$

commutes. As a consequence we will have $\varphi(A, B, \gamma_A, \nu_B) = p_B \circ \varphi(X, Y, \gamma, \nu) \circ \iota_A$, and hence the statement.

To show that the left square commutes we need to prove that the highest weight vector $v_{\lambda_A + \gamma_A}^A \in L_A(\lambda_A + \gamma_A) \subset L_A(\lambda_A) \otimes F_A$ is mapped to itself by the up-right-down path in the diagram. By definition $v_{\lambda_A + \gamma_A}^A$ is mapped to $v_{\lambda_A + \gamma}^X \in L_X(\lambda_X + \gamma)$, which is then mapped to itself in $L_X(\lambda_X) \otimes F_X$. Since p_A is a $\mathfrak{gl}(A)$ -homomorphism, the vector $p_A(v_{\lambda_A + \gamma}^X)$ is a $\mathfrak{gl}(A)$ -highest weight vector of weight $\lambda_A + \gamma_A$ and hence equal to $v_{\lambda_A + \gamma_A}^A$.

To see that the middle square commutes it is enough to check the commutativity of the squares

$$(1) \quad \begin{array}{ccc} L_X(\lambda_X) & \longrightarrow & L_Y(\lambda) \\ \downarrow & & \downarrow \\ L_A(\lambda_A) & \longrightarrow & L_B(\lambda_B) \end{array} \quad (2) \quad \begin{array}{ccc} F_X & \longrightarrow & F_Y \\ \downarrow & & \downarrow \\ F_A & \longrightarrow & F_B \end{array}$$

The proof that square (2) commutes is just diagram chasing, so we focus on square (1). We denote by $U_{\bar{X}}$ the enveloping algebra of $\mathfrak{n}_{\bar{X}}$, and use similar notation for the sets Y, A, B . The PBW-theorem implies that $U_{\bar{X}}$ is free as a right $U_{\bar{A}}$ -modules, that $U_{\bar{Y}}$ is free as a right $U_{\bar{B}}$ -module, and that we can choose a $U_{\bar{B}}$ -basis \mathcal{B} of $U_{\bar{Y}}$ such that $\mathcal{B} \cap U_{\bar{X}}$ is a $U_{\bar{A}}$ -basis of $U_{\bar{X}}$. Without loss of generality we can assume $1 \in \mathcal{B}$. If $f = f_1 f_2 \in U_{\bar{X}}$ with $f_1 \in \mathcal{B}$ and f_2 then $\iota_\lambda(f_1 f_2 v_\lambda^X) = f_1 f_2 v_\lambda^X$ by definition, while

$$p_A(f_1 f_2 v_\lambda^X) = \begin{cases} f_2 v_\lambda^A & \text{if } f_1 = 1; \\ 0 & \text{otherwise.} \end{cases}$$

A similar formula holds for p_B . Now the proof that the first square commutes also reduces to diagram chasing.

To prove that the right square commutes, we observe that the map $L_Y(\lambda) \otimes F_Y \longrightarrow L_B(\lambda) \otimes F_B$ maps $v_\lambda^Y \otimes e_\eta$ to $v_\lambda^B \otimes e_{\eta_B}$ if $\eta_{Y \setminus B} = \nu_{Y \setminus B}$, and to zero otherwise. It follows

that $M_Y(\not\subseteq \nu)$ is mapped to $M_B(\not\subseteq \nu_B)$, so we can expand the square as follows

$$\begin{array}{ccccc} L_Y(\lambda) \otimes F_Y & \longrightarrow & \frac{L_Y(\lambda) \otimes F_Y}{M_Y(\not\subseteq \nu)} & \longrightarrow & L_Y(\lambda + \nu) \\ \downarrow & & \downarrow & & \downarrow \\ L_B(\lambda_B) \otimes F_B & \longrightarrow & \frac{L_B(\lambda_B) \otimes F_B}{M_B(\not\subseteq \nu_B)} & \longrightarrow & L_B(\lambda_B + \nu_B). \end{array}$$

The left square of this expanded diagram commutes by definition. On the other hand, the weight space $\left(\frac{L_Y(\lambda) \otimes F_Y}{M_Y(\not\subseteq \nu)}\right)_{\lambda + \nu}$ has dimension 1, which implies that $\frac{L_Y(\lambda) \otimes F_Y}{M_Y(\not\subseteq \nu)}$ contains exactly one copy of the $\mathfrak{gl}(B)$ -module $L_B(\lambda_B + \nu_B)$, generated by $v_X^\lambda \otimes e_\nu$. The image of this vector through either path in the right-hand square is equal to the projection of $v_B^\lambda \otimes e_{\nu_B}$ to $L_B(\lambda_B + \nu_B)$, which is equal to the highest weight vector $v_{\lambda_B + \nu_B}^B$. The result is proved. \square

We now establish a stronger result than the one stated in item (iii) Proposition 4.3.

Proposition 4.6. *Suppose that $X^+ \neq \emptyset \neq X^-$, and that $\lambda(Z) = \{a\}$ with $a > \lambda_{\min X^+}$. Let $\gamma \in \text{Pieri}(\lambda_X, F_X)$ and $\nu \in \text{Pieri}(\lambda, F_Y)$, and let $I \subset X^- \sqcup Z, J \subset X^+$ be sequences such that $\nu - \gamma = \varepsilon_I - \varepsilon_J$ and $\#I = \#J \leq \#Z$. Then $\varphi(X, Y, \gamma, \nu) \neq 0$.*

Proof. The proof goes by induction on $\#Z$, and we begin with the base case $Z = \{z\}$. Suppose first $\nu = \gamma + \varepsilon_z - \varepsilon_j$ for some $j \in X^-$, and put $B = \{z, j\}, A = B \cap X = \{j\}$. By Lemma 4.5 $\varphi(X, Y, \gamma, \nu) \neq 0$ whenever $\varphi(A, B, \gamma_A, \nu_B) \neq 0$. Since $B = \{z\} \sqcup A$, the fact that $\varphi(A, B, \gamma_A, \nu_B) \neq 0$ follows from item (i) of Proposition 4.3.

Suppose now that $\nu = \gamma + \varepsilon_i - \varepsilon_j$ for $i \in X^-$ and $j \in X^+$. Taking $B = \{i, z, j\}$ and $A = B \cap X = \{i, j\}$, a new application of Lemma 4.5 shows that it is enough to prove that $\varphi(A, B, \gamma_A, \nu_B) \neq 0$. Thus we may assume that $X = \{i, j\}$ and $Y = \{i, z, j\}$. Since $\mathfrak{gl}(X) \cong \mathfrak{gl}(2)$, Lemma 4.4 reduces to the statement

$$v_{\lambda + \gamma} = v_\lambda \otimes e_\gamma + \sum_{r=1}^n a_r E_{j,i}^r v_\lambda \otimes e_{\gamma + r(\varepsilon_i - \varepsilon_j)}.$$

Applying $E_{i,j}$ to this highest weight vector we obtain

$$\begin{aligned} 0 &= E_{i,j} v_{\lambda + \gamma} = (v_\lambda + a_1 E_{i,j} E_{j,i} v_\lambda) \otimes e_{\gamma + \varepsilon_i - \varepsilon_j} + \cdots \\ &= (1 + a_1(\lambda_i - \lambda_j)) v_\lambda \otimes e_{\gamma + \varepsilon_i - \varepsilon_j} + \cdots, \end{aligned}$$

so $a_1 = -\frac{1}{\lambda_i - \lambda_j}$. Notice that $\lambda_i - \lambda_j \neq 0$ since $\lambda_i \geq \lambda_z > \lambda_j$. On the other hand, Lemma 4.4 implies

$$\varphi(X, Y)(v_{\lambda + \gamma}) \equiv v_\lambda \otimes e_\gamma \pmod{M_Y(\not\subseteq \nu)}.$$

Applying $E_{z,j} E_{i,z}$ to $\varphi(X, Y)(v_{\lambda + \gamma})$, and using the fact that $E_{i,z} e_\gamma = 0$ since $\gamma_z = 0$, we have

$$E_{z,j} E_{i,z} \varphi(X, Y)(v_{\lambda + \gamma}) \equiv -\frac{1}{\lambda_i - \lambda_j} E_{z,j} E_{i,z} E_{j,i} v_\lambda \otimes e_\nu \pmod{M_Y(\not\subseteq \nu)}.$$

Now

$$E_{z,j}E_{i,z}E_{j,i}v_\lambda = E_{z,j}(E_{j,i}E_{i,z} - E_{j,z})v_\lambda = -E_{z,j}E_{j,z}v_\lambda = -(\lambda_j - \lambda_z)v_\lambda$$

so

$$E_{z,j}E_{i,z}\varphi(X, Y)(v_{\lambda+\gamma}) \equiv \frac{\lambda_j - \lambda_z}{\lambda_i - \lambda_j}v_\lambda \otimes e_\nu \pmod{M_Y(\not\leq \nu)}.$$

Since $v_\lambda \otimes e_\nu$ generates $L_Y(\lambda + \nu)$ modulo $M_Y(\not\leq \nu)$, it follows that $\varphi(X, Y, \gamma, \nu) \neq 0$. This completes the proof of the base case.

Let us now proceed with the inductive step. Let $Z = Z' \sqcup \{z\}$, $I = \{i_1 \leq \dots \leq i_r\}$, $J = \{j_1 \leq \dots \leq j_r\}$, and assume that the statement holds for Z' . Set $Y' = X^- \sqcup Z' \sqcup X^+$, $I' = I \setminus \{i_r\}$, $J' = J \setminus \{j_r\}$ and $\mu = \gamma + \varepsilon_{I'} - \varepsilon_{J'}$. Then $\mu \in \text{Pieri}(\lambda_{Y'}, F_{X \cup Z'})$ since $\lambda_i = a$ for all $i \in Z$. We can now apply the induction hypothesis to conclude that $\varphi(X, Y', \gamma, \mu) \neq 0$ and $\varphi(Y', Y, \mu, \nu) \neq 0$, which in turn implies $\varphi(X, Y, \gamma, \nu) \neq 0$. \square

Proof of item (iii) of Proposition 4.2. That the result is necessary follows once more from Lemma 4.2. Now by definition $\#I = \#J = d$, so if $\#Z \geq d$ we can invoke Proposition 4.6 to deduce the result. \square

4.7. Further nonzero transition maps. In this subsection we assume $d \leq \#X$ and set $r = \#Z$, $(F_X, F_Y) = \left(\bigwedge^d V_X, \bigwedge^{d+r} V_Y \right)$. We have a nonzero $\mathfrak{gl}(X)$ -homomorphism $\tau : F_X \rightarrow F_Y$ given by $e_\mu \mapsto e_\mu \wedge e_{\varepsilon_Z}$, and so we are in the context of subsection 4.2. We now establish a result analogous to Proposition 4.3 for the transition maps associated to the map $\varphi(X, Y) : L_X(\lambda) \otimes F_X \rightarrow L_Y(\lambda) \otimes F_Y$.

Proposition 4.7. *Let $\gamma \in \text{Pieri}(\lambda_X, F_X)$ and $\nu \in \text{Pieri}(\lambda, F_Y)$ and let I, J be sequences such that $\nu - \gamma = \varepsilon_I - \varepsilon_J$.*

- (i) *Suppose $X^- = \emptyset$. Then $\varphi(X, Y, \gamma, \nu) \neq 0$ if and only if $I = J = \emptyset$.*
- (ii) *Suppose $X^+ = \emptyset$. Then $\varphi(X, Y, \gamma, \nu) \neq 0$ if and only if $I \subset X^-$ and $J \subset Z$.*
- (iii) *Suppose $X^+ \neq \emptyset \neq X^-$, $\lambda(Z) = \{a\}$ with $a < \lambda_{\max X^-}$, and $\#Z \geq d' = d - \#X$. Then $\varphi(X, Y, \gamma, \nu) \neq 0$ if and only if $I \subset X^-$ and $J \subset Z \sqcup X^+$.*

Proof. It is possible to prove the statement by proving a series of intermediate results just as we did for Proposition 4.3. Instead, we will employ an involution χ of $\mathfrak{gl}(Y)$ such that twisting $\varphi(X, Y)$ by χ produces a map $\tilde{\varphi}(X', Y)$ to which Proposition 4.3 can be applied.

We denote by ω_0 the longest element of the Weyl group S_Y of $\mathfrak{gl}(Y)$. Let χ be the automorphism of $\mathfrak{gl}(Y)$ given by $\chi(E_{i,j}) = -E_{\omega_0(j), \omega_0(i)}$. Notice that $\chi(\mathfrak{b}_Y) = \mathfrak{b}_Y$. Then ${}^X L_Y(\lambda) \cong L_Y(-\omega_0(\lambda))$, and the highest weight vector $v_\lambda \in L_Y(\lambda)$ equals the highest weight vector of ${}^X L_Y(\lambda)$. Recall $F_Y = L_Y(\varepsilon_I)$ where I is the sequence formed by the first $d + r$ elements of Y . By definition $\omega_0(I)$ is the sequence formed by the last $d + r$ elements of Y , and

$${}^X F_Y \cong L_Y(-\varepsilon_{\omega_0(I)}) \cong L_Y(-\varepsilon_Y) \otimes L_Y(\varepsilon_{Y \setminus \omega_0(I)}) \cong L_Y(-\varepsilon_Y) \otimes \bigwedge^{d'} V_Y$$

where $d' = \#Y - d - r = \#X - d$. Set $\tilde{F}_Y = \bigwedge^{d'} V_Y$ and $\tilde{\lambda} = -\omega_0(\lambda) - \varepsilon_Y$. Since twisting by χ commutes with tensor products, there is an isomorphism

$${}^{\chi}(L_Y(\lambda) \otimes F_Y) \cong L_Y(-\omega_0(\lambda)) \otimes L_Y(-\varepsilon_Y) \otimes \bigwedge^{d'} V_Y \cong L_Y(\tilde{\lambda}) \otimes \tilde{F}_Y.$$

Furthermore, comparing the decompositions afforded by the Pieri rule, a summand $L_Y(\lambda + \nu)$ with $\nu \in \text{Pieri}(\lambda, F_Y)$ is twisted by χ into a summand $L_Y(-\omega_0(\lambda + \nu))$. Writing $-\omega_0(\lambda + \nu) = \tilde{\lambda} + \tilde{\nu}$ we have $\tilde{\nu} = \varepsilon_Y - \omega_0(\nu) \in \text{Pieri}(\tilde{\lambda}, \tilde{F}_Y)$.

Now set $X' = \omega_0(X) \subset Y$. Then $\chi(\mathfrak{gl}(X')) = \mathfrak{gl}(X)$ and every $\mathfrak{gl}(X)$ -module becomes a $\mathfrak{gl}(X')$ -module through χ . Similar arguments show that ${}^{\chi}L_X(\lambda_X) \cong L_{X'}(-\omega_0(\lambda)_{X'})$, with the highest weight vector of $L_X(\lambda_X)$ corresponding to the highest weight vector of $L_{X'}(-\omega_0(\lambda)_{X'})$. Taking $J \subset X$ to be the sequence formed by the first d elements

$${}^{\chi}F_X \cong L_{X'}(-\varepsilon_{\omega_0(J)}) \cong L_Y(-\varepsilon_{X'}) \otimes L_Y(\varepsilon_{X' \setminus \omega_0(J)}) \cong L_Y(-\varepsilon_{X'}) \otimes \bigwedge^{d'} V_X.$$

Thus setting $\tilde{F}_X = \bigwedge^{d'} V_X$

$${}^{\chi}(L_X(\lambda_X) \otimes F_X) \cong L_Y(-\omega_0(\lambda)_{X'}) \otimes L_{X'}(-\varepsilon_{X'}) \otimes L_{X'}(\varepsilon_J) \cong L_X(\tilde{\lambda}_{X'}) \otimes \tilde{F}_X.$$

Again comparing decompositions we see that $\gamma \in \text{Pieri}(\lambda_X, F_X)$ if and only if $\tilde{\gamma} = \varepsilon_{X'} - \omega_0(\gamma) \in \text{Pieri}(\tilde{\lambda}_{X'}, \tilde{F}_{X'})$.

Since χ preserves \mathfrak{b}_Y and sends \mathfrak{b}_X to $\mathfrak{b}_{X'}$, the highest weight vectors of $L_X(\lambda)$ and $L_Y(\lambda)$ remain highest weight vectors of the respective twisted modules ${}^{\chi}L_X(\lambda_X)$ and ${}^{\chi}L_Y(\lambda)$. Thus the χ -twist of the map ι_{λ} is the map $\iota_{-\omega_0(\lambda)} : L_{X'}(-\omega_0(\lambda)_{X'}) \rightarrow L_Y(-\omega_0(\lambda))$. Direct inspection shows that the χ -twist of $\tau : F_X \rightarrow F_Y$ is the map $\iota_{\varepsilon_Y} \otimes \tilde{\tau} : \mathbb{C}_{\varepsilon_X} \otimes \tilde{F}_X \rightarrow \mathbb{C}_{\varepsilon_X} \otimes \tilde{F}_Y$, where $\tilde{\tau} : \tilde{F}_X \rightarrow \tilde{F}_Y$ is the map induced by $V_{X'} \rightarrow V_Y$. It follows that twisting the map $\varphi(X, Y) : L_X(\lambda_X) \otimes F_X \rightarrow L_Y(\lambda) \otimes F_Y$ by χ we get the map $\varphi(X', Y) : L_X(\tilde{\lambda}_X) \otimes \tilde{F}_X \rightarrow L_Y(\tilde{\lambda}) \otimes \tilde{F}_Y$, and the transition map $\varphi(X, Y, \gamma, \nu)$ is twisted into the transition map $\varphi(X', Y, \tilde{\gamma}, \tilde{\nu})$. Thus $\varphi(X, Y, \gamma, \nu) \neq 0$ if and only if $\varphi(X', Y, \tilde{\gamma}, \tilde{\nu}) \neq 0$. Proposition 4.3 now implies the statement. \square

5. EXHAUSTIONS OF TENSOR PRODUCTS OF $\mathfrak{gl}(\infty)$ -MODULES

We now start our study of tensor products of $\mathfrak{gl}(\infty)$ -modules. An *exhaustion of \mathbb{I}* is a chain of finite subsets $\{X_n\}_{n \geq 1}$ such that $\mathbb{I} = \bigcup_{n \geq 1} X_n$. In this section we prove the existence of exhaustions such that every pair $X_n \subset X_{n+1}$ satisfies the hypotheses of Propositions 4.3 and 4.7, which allows us to determine exactly which transition maps $\varphi(X_n, X_{n+1}, \gamma, \nu)$ are nonzero. We fix a weight $\lambda \in \mathfrak{h}^*$, and for any finite subset X of \mathbb{I} we denote by λ_X the restriction of λ to \mathfrak{h}_X .

5.1. Exhaustions of \mathbb{I} for $\mathbf{F} = \mathbf{S}^d \mathbb{V}, \bigwedge^d \mathbb{V}$. For this subsection we assume that \mathbf{F} is either $\mathbf{S}^d \mathbb{V}$ or $\bigwedge^d \mathbb{V}$ for some $d \in \mathbb{Z}_{>0}$, and for every finite set $X \subset \mathbb{I}$ we take $F_X = \mathbf{S}^d V_X$ or $F_X = \bigwedge^d V_X$ accordingly. Whenever $X \subset Y$ we have maps $F_X \rightarrow F_Y$ induced by the natural maps $V_X \rightarrow V_Y$, and $\mathbf{F} = \varinjlim F_X$.

We fix a (λ, \mathbf{F}) -compatible order \leq on \mathbb{I} , and set $\mathfrak{b} = \mathfrak{b}_{\leq}$. Given $\gamma \in \text{supp } F_X$ we denote by γ^Y the image of γ through the embedding $\mathfrak{h}_X^* \rightarrow \mathfrak{h}_Y^*$, and by $\bar{\gamma}$ the image of γ through the embedding $\mathfrak{h}_X^* \rightarrow \mathfrak{h}^*$.

Definition 5.1. We say a finite set $X \subset \mathbb{I}$ is a d -block if the following conditions hold:

- (i) $\pi_{\mathbb{J}}(X)$ is an interval in \mathbb{J} .
- (ii) If $j \in \pi_{\mathbb{J}}(X)$ and j is finite, then $j \subset X$.
- (iii) If $j \in \pi_{\mathbb{J}}(X)$ and j is infinite, then $j \cap X$ is an initial subset of j and contains at least d elements.

Recall from Lemma 3.3 that for every $\gamma \in \mathfrak{h}_X^*$ there is a least element $\gamma' \in \text{Pieri}(\lambda_X, F_X)$ such that $\gamma' \geq \gamma$. The following is immediate from the definitions and Lemma 3.3.

Lemma 5.2. Let $X \subset \mathbb{I}$ be a d -block, and suppose $\gamma \in \text{Pieri}(\lambda_X, F_X)$.

- (i) If $\mathbf{F} = \bigwedge^d \mathbb{V}$ then $\bar{\gamma} \in \text{Pieri}(\lambda, \mathbf{F})$.
- (ii) If $\mathbf{F} = \mathbf{S}^d \mathbb{V}$ then there exists $Y \supset X$ such that $(\overline{\gamma^Y})' \in \text{Pieri}(\lambda, \mathbf{F})$. If $\min \text{supp } \gamma < \min X$ or $\min X = \min \mathbb{I}$ then we can take $Y = X$.

Definition 5.3. Suppose $X \subset Y \subset \mathbb{I}$ are finite d -blocks. We will say that (X, Y) is a d -step if the set $Z = Y \setminus X$ satisfies the following conditions: (i) $\pi_{\mathbb{J}}(Z) = \{j\}$, and (ii) if j is infinite then $\#Z \geq d$. We call an exhaustion $\{X_n\}_{n \geq 1}$ of \mathbb{I} a d -exhaustion if each pair (X_n, X_{n+1}) is a d -step.

Suppose (X, Y) is a d -step with $Z = Y \setminus X$ and $\{j\} = \pi_{\mathbb{J}}(Z)$. Setting

$$X^- = \{x \in X \mid \pi_{\mathbb{J}}(x) \leq j\} \quad \text{and} \quad X^+ = \{x \in X \mid \pi_{\mathbb{J}}(x) > j\},$$

we have $Y = X^- \sqcup Z \sqcup X^+$. It follows from the definitions that if $X^- \neq \emptyset \neq X^+$ then j is an infinite class and every element of Z has at least d predecessors in j . The definition of a d -step is made so that we can apply Proposition 4.3, and thus determine exactly which transition maps are nonzero in a d -step (X, Y) .

Proposition 5.4. Let (X, Y) be a d -step with $Y = X^- \sqcup Z \sqcup X^+$ and $\pi_{\mathbb{J}}(Z) = \{j\}$. Let $\gamma \in \text{Pieri}(\lambda_X, F_X)$ and $\nu \in \text{Pieri}(\lambda_Y, F_Y)$, with $\bar{\nu} - \bar{\gamma} = \varepsilon_I - \varepsilon_J$ and I, J of minimal length.

- (i) Suppose $X^+ = \emptyset$. Then $\varphi(X, Y, \gamma, \nu) \neq 0$ if and only if $I = J = \emptyset$.
- (ii) Suppose $X^- = \emptyset$. Then $\varphi(X, Y, \gamma, \nu) \neq 0$ if and only if $I \subset Z$ and $J \subset X^+$.
Furthermore

$$h_{\infty}(\bar{\nu}, \bar{\gamma}) = \begin{cases} h(\bar{\nu}, \bar{\gamma}) & \text{if } j \text{ is infinite;} \\ 0 & \text{if } j \text{ is finite.} \end{cases}$$

- (iii) Suppose $X^- \neq \emptyset \neq X^+$. Then $\varphi(X, Y, \gamma, \nu) \neq 0$ if and only if $I \subset X^-$ and $J \subset X^+$.
Furthermore $h_{\infty}(\bar{\nu}, \bar{\gamma}) = h(\bar{\nu}, \bar{\gamma})$.

Proof. Items (i) and (ii) follow from the corresponding items of Proposition 4.3. For the third item, the proposition only guarantees that $I \subset X^- \sqcup Z$. Notice however that $I \subset \text{supp } \nu$, and the Pieri rule guarantees that $I \cap j$ must be an initial subset of j

with at most d elements. Since the first d elements of j belong to X^- , it follows that $I \subset X^-$. \square

We now show that d -exhaustions exist.

Proposition 5.5. *There exists a d -exhaustion $\{X_n\}_{n \geq 1}$. If \mathbb{J} has a minimum then the exhaustion can be chosen so that $\min X_n = \min \mathbb{J}$ for all n .*

Proof. We first fix some notation. Whenever $j \in \mathbb{J}$ is an infinite class $j = \{i_1 < i_2 < \dots\}$ we denote by $j^{(k)}$ the subset $\{i_{(k-1)d+1}, \dots, i_{kd}\}$, so $j = j^{(1)} \sqcup j^{(2)} \sqcup \dots$.

Since \mathbb{J} is order-isomorphic to a subset of \mathbb{Z} , there exists a function $\theta : \mathbb{Z}_{>0} \rightarrow \mathbb{J}$ such that $\theta(\llbracket n \rrbracket)$ is an interval in \mathbb{J} for all $n \in \mathbb{Z}_{>0}$ and $\theta^{-1}(j)$ is infinite for all j . Set $Y_0 = \emptyset$ and for each $m \in \mathbb{Z}_{>0}$ define Y_m as follows: if $j = \theta(m)$ is finite then $Y_m = Y_{m-1} \cup j$, and if j is infinite then $Y_m = Y_{m-1} \cup j^{(k+1)}$, where $k = \max\{r \mid j^{(r)} \subset Y_{m-1}\}$. By construction each Y_m is a d -block and either $Y_m = Y_{m+1}$ or (Y_m, Y_{m+1}) is a d -step. The exhaustion is obtained by setting X_n to be the n -th different set in the sequence of the Y_m . If \mathbb{J} has a minimum then we can take $\theta(1) = \min \mathbb{J}$, which guarantees that $\min \mathbb{J} \in X_1 \subset X_n$. \square

Corollary 5.6. *Let $\{X_n\}_{n \geq 1}$ be a d -exhaustion. Suppose $\gamma \in \text{Pieri}(\lambda_{X_n}, X_n)$, $\nu \in \text{Pieri}(\lambda_{X_m}, X_m)$, and $\varphi(X_n, X_m, \gamma, \nu) \neq 0$. Then there exist $\mu \in \mathfrak{h}^\circ$ with $h_\infty(\bar{\nu}, \bar{\gamma}) = h_\infty(\mu, 0) = h(\mu, 0)$, and sequences $J \subset \text{supp } \gamma$ and $I \subset X_m \setminus X_n$ such that $\bar{\nu} - \bar{\gamma} = \mu + \varepsilon_I - \varepsilon_J$.*

Proof. The proof is an easy induction using Proposition 5.4. \square

5.2. Exhaustions of \mathbb{I} for $\mathbf{F} = \bigwedge^A \mathbb{V}$. Throughout this subsection $A \subset \mathbb{I}$ denotes an infinite set with infinite complement $B = \mathbb{I} \setminus A$, and $\mathbf{F} = \bigwedge^A \mathbb{V}$. For any finite set $X \subset \mathbb{I}$ we set $d(X) = \#A \cap X$ and $F_X = \bigwedge^{d(X)} V_X$. If $X \subset Y$ then there is a map $F_X \rightarrow F_Y$ given by $e_\eta \mapsto e_\eta \wedge e_{\varepsilon_K}$ where $K = (Y \setminus X) \cap A$, and $\mathbf{F} = \varinjlim_X F_X$. We have maps $\mathfrak{h}_X^* \rightarrow \mathfrak{h}_Y^*$ given by $\gamma = \sum_{i \in X} \gamma_i \varepsilon_i \mapsto \gamma^Y = \sum_{i \in X} \gamma_i \varepsilon_i + \varepsilon_K$, and $\mathfrak{h}_X^* \rightarrow \mathfrak{h}^*$ given by $\gamma \mapsto \bar{\gamma} = \sum_{i \in X} \gamma_i \varepsilon_i + \varepsilon_{A \setminus X}$. Notice that $\bar{\gamma}_Y = \gamma^Y$.

We fix on \mathbb{I} a (λ, \mathbf{F}) -compatible linear order \leq and set $\mathfrak{b} = \mathfrak{b}_\leq$. In particular, for every $j \in \mathbb{J}$ the set $j \cap A$ is order-isomorphic to a subset of $\mathbb{Z}_{<0}$ and $j \cap B$ is order-isomorphic to a subset of $\mathbb{Z}_{>0}$.

Definition 5.7. Let $A \subset \mathbb{I}$ be an infinite set with infinite complement $B = \mathbb{I} \setminus A$. Let X be a finite subset of \mathbb{I} . We say X is (A, B) -block if the following conditions hold:

- (i) $\pi_{\mathbb{J}}(X)$ is an interval in \mathbb{J} .
- (ii) If $j \in \pi_{\mathbb{J}}(X)$ is finite then $j \subset X$.
- (iii) If $j \in \pi_{\mathbb{J}}(X)$ is infinite then $j \cap A \cap X$ is a terminal subset of $j \cap A$, and $j \cap B \cap X$ is an initial subset of $j \cap B$.

The following Lemma is immediate from Lemma 3.3.

Lemma 5.8. *Let $X \subset \mathbb{I}$ be an (A, B) -block, and let $\gamma \in \text{Pieri}(\lambda_X, F_X)$. Then $\bar{\gamma} \in \text{Pieri}(\lambda, \mathbf{F})$.*

Definition 5.9. Suppose $X \subset Y \subset \mathbb{I}$ are (A, B) -blocks. We say that (X, Y) is an A -step if the set $Z = Y \setminus X$ satisfies the following conditions:

- (i) $Z \subset j \cap A$ for some $j \in \mathbb{J}$;
- (ii) if j is infinite then $\#Z \geq \#(X \cap B)$.

We say (X, Y) is a B -step if the same conditions hold interchanging A and B . An exhaustion $\{X_n\}_{n \geq 1}$ of \mathbb{I} by finite sets is an (A, B) -exhaustion if every pair (X_n, X_{n+1}) is either an A -step or a B -step.

Suppose (X, Y) is a B -step. Then $d(X) = d(Y)$, and (X, Y) is also a $d(X)$ -step. In particular $Y = X^+ \sqcup Z \sqcup X^-$ just as in the previous subsection, and $F_X = \bigwedge^{d(X)} V_X, F_Y = \bigwedge^{d(X)} V_Y$. Proposition 5.4 gives necessary and sufficient conditions for a transition map to be nonzero.

Now suppose (X, Y) is an A -step with $Z = Y \setminus X$ and $\pi_{\mathbb{J}}(Z) = \{j\}$. Then $d(Y) = d(X) + d(Z)$, so $F_X = \bigwedge^{d(X)} V_X, F_Y = \bigwedge^{d(X)+d(Z)} V_Y$, and the map $F_X \rightarrow F_Y$ is given by $-\wedge e_{\varepsilon_Z}$. Setting

$$X^- = \{x \in X \mid \pi_{\mathbb{J}}(x) < j\} \quad \text{and} \quad X^+ = \{x \in X \mid \pi_{\mathbb{J}}(x) \geq j\}$$

we have $Y = X^- \sqcup Z \sqcup X^+$, and $X^- \neq \emptyset \neq X^+$ implies that j is an infinite class. The next proposition shows that we know exactly which transition maps in an A -step are nonzero.

Proposition 5.10. *Let (X, Y) be an A -step with $Y = X^- \sqcup Z \sqcup X^+$ and $\pi_{\mathbb{J}}(Z) = \{j\}$. Let $\gamma \in \text{Pieri}(\lambda_X, F_X)$ and $\nu \in \text{Pieri}(\lambda, F_Y)$, with $\bar{\nu} - \bar{\gamma} = \varepsilon_I - \varepsilon_J$ and I, J of minimal length.*

- (i) *Suppose $X^- = \emptyset$. Then $\varphi(X, Y, \gamma, \nu) \neq 0$ if and only if $I = J = \emptyset$.*
- (ii) *Suppose $X^+ = \emptyset$. Then $\varphi(X, Y, \gamma, \nu) \neq 0$ if and only if $I \subset X^-$ and $J \subset Z$.*

Furthermore

$$h_{\infty}(\bar{\nu}, \bar{\gamma}) = \begin{cases} h(\bar{\nu}, \bar{\gamma}) & \text{if } j \text{ is infinite;} \\ 0 & \text{if } j \text{ is finite.} \end{cases}$$

- (iii) *Suppose $X^+ \neq \emptyset \neq X^-$. Then $\varphi(X, Y, \gamma, \nu) \neq 0$ if and only if $I \subset X^-$ and $J \subset Z \sqcup X^+$. Furthermore $h_{\infty}(\bar{\nu}, \bar{\gamma}) = h(\bar{\nu}, \bar{\gamma})$.*

Proof. These are all special cases of Proposition 4.7. □

Proposition 5.11. *There exists an (A, B) -exhaustion. If A is an initial subset of \mathbb{I} then X_n can be chosen so that $\max A, \min B \in X_n$ for all n .*

Proof. We consider the set $\tilde{\mathbb{J}} = \{j \cap A, j \cap B \mid j \in \mathbb{J}\}$ with the unique total order \leq such that $j \cap A < j \cap B$ for all $j \in \mathbb{J}$, and such that $j \cap B < j' \cap A$ for all $j, j' \in \mathbb{J}$ with $j < j'$. Since \mathbb{J} is isomorphic to a subset of \mathbb{Z} , so is $\tilde{\mathbb{J}}$. We now proceed as in the proof of Proposition 5.5, using a map $\theta : \mathbb{Z}_{>0} \rightarrow \tilde{\mathbb{J}}$. □

Corollary 5.12. *Let $\{X_n\}_{n \geq 1}$ be an (A, B) -exhaustion. Suppose $\gamma \in \text{Pieri}(\lambda_{X_n}, X_n), \nu \in \text{Pieri}(\lambda_{X_m}, X_m)$ and $\varphi(X_n, X_m, \gamma, \nu) \neq 0$. Then there exist $\mu \in \mathfrak{h}^{\circ}$ with $h_{\infty}(\mu, 0) = h_{\infty}(\bar{\nu}, \bar{\gamma}) = h(\mu, 0)$ and sequences $I \subset X_m \setminus (A \cup X_n), J \subset X_n \cap \text{supp } \gamma, K \subset X_n \setminus \text{supp } \gamma$ and $L \subset X_m \cap A$ such that $\bar{\nu} - \bar{\gamma} = \mu + \varepsilon_I - \varepsilon_J + \varepsilon_K - \varepsilon_L$.*

Proof. The proof is by induction using Proposition 5.10. □

5.3. **On sequences of weights.** We finish this section with a result that will be fundamental in the sequel.

Lemma 5.13. *Suppose (X, Y) is a d -step, an A -step or a B -step. Suppose we have $\gamma, \sigma \in \text{Pieri}(\lambda_X, F_X)$ and $\nu \in \text{Pieri}(\lambda_Y, F_Y)$ with $\gamma - \sigma = \varepsilon_I - \varepsilon_J \geq 0$ and $\varphi(X, Y, \gamma, \nu) \neq 0$. Then there exists $\tau \in \text{Pieri}(\lambda_Y, F_Y)$ such that $\nu - \tau = \varepsilon_{I'} - \varepsilon_{J'} \leq \varepsilon_I - \varepsilon_J$, $I' \subset X^-, J' \subset Z \sqcup X^+$ and $\varphi(X, Y, \sigma, \tau) \neq 0$.*

Proof. Suppose $I = \{i_1 \leq \dots \leq i_s\}, J = \{j_1 \leq \dots \leq j_s\}$ and let K, L be sequences of minimal length such that $\bar{\nu} - \bar{\gamma} = \varepsilon_K - \varepsilon_L$.

Assume first that (X, Y) is a d -step or a B -step. We have $K \subset X^-$ and $L \subset X^+$ by Proposition 5.4, and we set $R = L \cap I = \{i_{u_1} \leq \dots \leq i_{u_r}\} \subset X^+$ and $S = \{j_{u_1} \leq \dots \leq j_{u_r}\}$; since $i_k < j_k$ it follows that $S \subset X^+$. Now take a, b such that

$$J \cap X^+ = \{j_a \leq \dots \leq j_s\}, \quad I \cap X^- = \{i_1 \leq \dots \leq i_b\},$$

and set $P = \{i_a \leq \dots \leq i_b\} \subset X^-$ and $Q = \{j_a \leq \dots \leq j_b\} \subset X^+$. It follows that $b < u_1$, so P and R are disjoint subsequences of I , and Q and S are disjoint subsequences of J . We now set

$$\tau_0 = \sigma^Y + \varepsilon_K - \varepsilon_{(L \setminus R) \cup S} + \varepsilon_P - \varepsilon_Q$$

and $\tau := \tau'_0$. Since $K, P \subset X^-$ and $L, S, Q \subset X^+$, item (iii) of Lemma 4.1 implies $\tau - \sigma^Y = \varepsilon_U - \varepsilon_V$ for sequences $U \subset X^-$ and $V \subset X^+$. Now item (iii) of Proposition 4.3 implies $\varphi(X, Y, \sigma, \tau) \neq 0$. We also have

$$\bar{\nu} - \bar{\tau} \leq \bar{\nu} - \bar{\tau}_0 = \bar{\gamma} + \varepsilon_K - \varepsilon_L - \bar{\nu} - \varepsilon_K + \varepsilon_{(L \setminus R) \cup S} - \varepsilon_P + \varepsilon_Q = \varepsilon_{I \setminus (P \cup R)} - \varepsilon_{J \setminus (Q \cup S)}$$

This concludes the proof of the first statement when (X, Y) is a d -step or a B -step. If in particular $I \subset X^-$ and $J \subset X^+$ then $I = P, J = Q$ and $R, S = \emptyset$, and so $\nu = \tau$.

Now suppose (X, Y) is an A -step, so $K \subset X^-$ and $L \subset Z \sqcup X^+$ by Proposition 5.10. We take $a, b \in \llbracket s \rrbracket$ such that

$$J \cap (X^+ \sqcup Z) = \{j_a \leq \dots \leq j_s\} \quad I \cap X^- = \{i_1 \leq \dots \leq i_b\}$$

and define P, Q as above. We also set $S = J \cap X^- = \{j_{u_1} \leq \dots \leq j_{u_r}\}$ and $R = \{i_{u_1} \leq \dots \leq i_{u_r}\}$. Set

$$\tau_0 = \sigma^Y + \varepsilon_{(K \setminus S) \cup R} - \varepsilon_L + \varepsilon_P - \varepsilon_Q$$

and $\tau = \tau'_0$. The rest of the proof proceeds as in the case where (X, Y) is a d -step or a B -step. \square

We now prove two results on sequences of weights $\{\mu_n\}_{n \geq r}$ with $\mu_n \in \text{Pieri}(\lambda_n, F_n)$.

Lemma 5.14. *Let $\mathbf{F} = \bigwedge^A \mathbb{V}$ with A a \mathbb{J}_∞ -initial set, and let $(\mu_n)_{n \geq r}$ be a sequence with $\mu_n \in \text{Pieri}(\lambda_n, F_n)$ and $\overline{\mu_{n+1}} \geq \overline{\mu_n}$. If the sequence $\overline{\mu_n}$ does not stabilize then there exists n such that $\overline{\mu_{n+1}} \gg \overline{\mu_n}$.*

Proof. Recall that $\pi : \mathbb{I} \longrightarrow \mathbb{J}_\infty$ denotes the natural projection. Let $\mu \in \text{Pieri}(\lambda, \mathbf{F})$. For every $j \in \mathbb{J}_\infty$ we have a decomposition $(\text{supp } \mu) \cap \pi^{-1}(j) = A_j \sqcup I_j$, where A_j is an initial subset of $\pi^{-1}(j)$, I_j is finite, and the cardinality of I_j is minimal among all such decompositions. Furthermore, since $\text{supp } \mu$ and A differ in finitely many elements, I_j is empty for all but finitely many $j \in \mathbb{J}_\infty$ and the set $\bigcup_{j \in \mathbb{J}_\infty} I_j$ is finite. For each $i \in I_j$ we set $t(i) = \#\llbracket \max A_j, i \rrbracket - 1$, and take $t(\mu)$ to be the sum of all $t(i)$ with $i \in \bigcup_{j \in \mathbb{J}_\infty} I_j$. We will prove the statement by induction on $t = t(\overline{\mu}_r)$.

Since the sequence $\overline{\mu}_n$ does not stabilize we can assume that $\overline{\mu}_{r+1} > \overline{\mu}_r$, and so $\overline{\mu}_{r+1} - \overline{\mu}_r = \varepsilon_I - \varepsilon_{I'}$. Suppose first that $t = 0$, and let $i = \min I, i' = \min I'$ and $j = \pi_{\mathbb{J}}(i), j' = \pi_{\mathbb{J}}(i')$. Since $i' \in \text{supp } \overline{\mu}_r$ it follows that $i' \in A_{j'}$ and, since $A_{j'}$ is an initial subset of $\pi^{-1}(j')$, that $j < j'$. Thus $h_\infty(\overline{\mu}_{r+1}, \overline{\mu}_r) > 0$ and by Propositions 5.4 and 5.10 this can only happen if $\overline{\mu}_{r+1} \gg \overline{\mu}_r$.

Suppose now that the statement holds for all sequences starting with a weight μ such that $t(\overline{\mu}) < t$. As we have just seen, $\overline{\mu}_{r+1} \gg \overline{\mu}_r$ unless $h_\infty(\overline{\mu}_{r+1}, \overline{\mu}_r) = 0$. If $\overline{\mu}_{r+1} \gg \overline{\mu}_r$ holds we are done. If $h_\infty(\overline{\mu}_{r+1}, \overline{\mu}_r) = 0$, taking $I = \{i_1 < \dots < i_s\}$ and $I' = \{i'_1 < \dots < i'_s\}$ it must be the case that $i'_k < i_k$ and $\pi(i'_k) = \pi(i_k)$. Consequently, $t(i'_k) < t(i_k)$ and hence $t(\overline{\mu}_{r+1}) < t(\overline{\mu}_r)$. The result now follows by the induction hypothesis. \square

We now fix a d -exhaustion or (A, B) -exhaustion $\{X_n\}_{n \geq 1}$. In order to lighten notation we write F_n for F_{X_n} .

Proposition 5.15. *Let $r \geq 0$. Let $(\mu_n)_{n \geq r}$ be a sequence with $\mu_n \in \text{Pieri}(\lambda_n, F_n)$ and $\varphi(X_n, X_{n+1}, \mu_n, \mu_{n+1}) \neq 0$.*

- (i) *If \mathbf{F} is a \mathfrak{b} -highest weight module then the sequence $(\overline{\mu}_n)_{n \geq r}$ stabilizes.*
- (ii) *If the sequence $\overline{\mu}_n$ does not stabilize then there exist $m > n$ such that $(\mu_n^{X_{n+1}})' < \mu_{n+1}$ and $\varphi(X_{n+1}, X_m, (\mu_n^{X_{n+1}})', \mu_m) \neq 0$.*

Proof. Suppose \mathbf{F} is a \mathfrak{b} -highest weight module with highest weight μ . By Propositions 5.5 and 5.11 we can assume that $X_n^- = \emptyset$ whenever (X_n, X_{n+1}) is a d -step or a B -step, and that $X_n^+ = \emptyset$ whenever (X_n, X_{n+1}) is an A -step. Thus by Propositions 5.4 and 5.10, $\varphi(X_n, X_{n+1}, \mu_n, \mu_{n+1}) \neq 0$ implies

$$h_\infty(\mu, \overline{\mu}_{n+1}) = h_\infty(\mu, \overline{\mu}_n) - h_\infty(\overline{\mu}_{n+1}, \overline{\mu}_n) \leq h_\infty(\mu, \overline{\mu}_n).$$

Since $h_\infty(\mu, \overline{\mu}_n) \in \mathbb{Z}_{>0}$ it must be the case that $h_\infty(\overline{\mu}_{n+1}, \overline{\mu}_n) = 0$ for $n \gg 0$. Thus the sequence $\overline{\mu}_n$ must eventually stabilize at γ . This takes care of item (i).

Suppose now that the sequence $\overline{\mu}_n$ does not stabilize, so there are infinitely many $n \in \mathbb{Z}_{>0}$ such that $\overline{\mu}_{n+1} > \overline{\mu}_n$. Let v_n be the sequence defined by $v_r = \mu_r$ and $v_{n+1} = (v_n^{X_{n+1}})'$. By Lemmas 5.2 and 5.8 the sequence $(\overline{v}_n)_{n \geq r}$ stabilizes. Since $\overline{\mu}_n$ does not stabilize, there must be infinitely n for which $\mu_{n+1} > (\mu_n^{X_{n+1}})'$.

We consider now the subcase where $\mu_{n+1} > v_{n+1} = (\mu_n^{X_{n+1}})'$ and $h_\infty(\overline{\mu}_{n+1}, \overline{v}_{n+1}) > 0$ for some n . Using Lemma 5.13 we define recursively $v_{n+k} \in \text{Pieri}(\lambda_{n+k}, F_{n+k})$ such that $\varphi(X_{n+1}, X_{n+k}, v_{n+1}, v_{n+k}) \neq 0$ and $\overline{\mu}_{n+k} - \overline{v}_{n+k} = \varepsilon_{I^{n+k}} - \varepsilon_{J^{n+k}}$ with $I^{n+k} \subset X^-, J^{n+k} \subset$

$X^+ \sqcup Z$, so $\mu_{n+k} \gg v_{n_k}$. By construction $h_\infty(\overline{v_{m+1}}, \overline{\mu_{m+1}}) = 0$, and so $v_{m+1} = \mu_{m+1}$ and the result follows.

We are left with the case where the sequence $\overline{\mu}_n$ does not stabilize and $h_\infty(\overline{\mu_{n+1}}, \overline{\mu}_n) = 0$ for all $n \geq r$. By the previous lemma we only have to consider the cases $\mathbf{F} = \mathbf{S}^d \mathbb{V}$ or $\mathbf{F} = \bigwedge^d \mathbb{V}$, and so $\#\text{supp } \mu_n = d$. Furthermore, whenever $\overline{\mu_{n+1}} > \overline{\mu}_n$ we must have $X_{n+1} = Z_n \sqcup X_n$ and $\overline{\mu_{n+1}} - \overline{\mu}_n = \varepsilon_I - \varepsilon_J$ with $I \subset Z_n$ and $J \subset X_n$. In particular, $\text{supp } \mu_{n+1} \cap Z_n \neq \emptyset$ and $\text{supp } \mu_{n+1} \cap X_n \subsetneq \text{supp } \mu_n$.

Consider a sequence n_0, n_1, \dots, n_d such that $\mu_{n_s} > (\mu_{n_{s-1}}^{X_{n_s}})'$ and set $m = n_d$ and $\nu = \mu_{n_d}$. By the pigeonhole principle either $\text{supp } \nu \cap X_{n_0} = \emptyset$, in which case we take $s = 0$, or there exists an $s \in \llbracket d \rrbracket$ such that $\text{supp } \nu \cap Z_s = \emptyset$. Setting $n = n_s$ and $\gamma = (\mu_{n_s}^{X_{n_s+1}})'$ we have $\nu - \gamma = \varepsilon_P - \varepsilon_Q$ for sequences Q and P such that $Q \subset X_{n_s}$ and $\min P > \max Z_s = \max X_{n_s+1}$. Thus $\varphi(X_n, X_m, \gamma, \nu) \neq 0$ by item (ii) of Proposition 4.3. \square

5.3.1. The non-example. We remark that Proposition 5.15 does not apply when $\mathbf{F} = \bigwedge^A \mathbb{V}$ but A is not \mathbb{J}_∞ -initial. To see this, let $\mathbb{I} = \mathbb{Z}_{>0}$, $\lambda = \sum_{i=1}^\infty -i\varepsilon_i$ and $A = 2\mathbb{Z}_{>0}$. We get an (A, B) -exhaustion by taking $X_n = \llbracket n \rrbracket$ with (X_n, X_{n+1}) a B -step whenever n is even, and an A -step whenever n is odd.

Set $\mu_n = \varepsilon_1 + \varepsilon_3 + \dots + \varepsilon_{2k-1}$ with $k = \#(A \cap \llbracket n \rrbracket)$. Then $\mu_n \in \text{Pieri}(\lambda, F_n)$ and furthermore

$$\overline{\mu}_{n+1} - \overline{\mu}_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \varepsilon_n - \varepsilon_{n+1} & \text{if } n \text{ is odd.} \end{cases}$$

In both cases $\varphi(X_n, X_{n+1}, \mu_n, \mu_{n+1}) \neq 0$ and $(\mu_n^{X_{n+1}})' = \mu_n^{X_n}$, while $\mu_{n+1} > \mu_n^{X_{n+1}}$ if and only n is odd. On the other hand, using Propositions 5.4 and 5.10 it is easy to prove that $\varphi(\mu_n^{X_{n+1}}, \nu, X_{n+1}, X_m) \neq 0$ implies $k+1 \in \text{supp } \nu$, while $k+1 \notin \text{supp } \mu_m$ for any $m > n+1$. The consequences of this failure will be seen in 6.1.1.

6. PROOFS OF THE MAIN THEOREMS

We are now ready to establish the necessary results on the structure of the modules $\mathbf{L}(\lambda) \otimes \mathbf{F}$ in order to prove theorems A through D.

Fix a weight $\lambda \in \mathfrak{h}^*$, and denote by \mathbf{F} one of $\mathbf{S}^d \mathbb{V}, \bigwedge^d \mathbb{V}$ for $d \in \mathbb{Z}_{>0}$, or $\bigwedge^A \mathbb{V}$ for some infinite set A with $B = \mathbb{I} \setminus A$ infinite. When \mathbf{F} is a Fock module, we assume that A is \mathbb{J}_∞ -initial. We fix a (λ, \mathbf{F}) -compatible total order \leq on \mathbb{I} , and set $\mathfrak{b} = \mathfrak{b}_\leq$ and $\mathbf{M} = \mathbf{L}(\lambda) \otimes \mathbf{F}$.

We fix a d -exhaustion or (A, B) -exhaustion $\{X_n\}_{n \geq 1}$ of \mathbb{I} . To lighten up notation we write γ_n for γ_{X_n} and $L_n(\lambda_n)$ for $L_{X_n}(\lambda_{X_n})$. We fix inductive systems of $\mathfrak{gl}(X_n)$ -modules F_n such that $\varinjlim F_n = \mathbf{F}$ as in subsections 5.1 and 5.2. With this setup $\mathbf{M} = \varinjlim L_n(\lambda_n) \otimes F_n$, and we identify $M_n := L_n(\lambda_n) \otimes F_n$ with its image in \mathbf{M} .

Definition 6.1. Let $\gamma \in \text{Pieri}(\lambda, F_n)$. We denote by $\mathbf{M}(n, \gamma)$ the submodule of \mathbf{M} generated by $L_n(\lambda_n + \gamma) \subset M_n$. Then we observe that $\mathbf{M}(n, \gamma)$ equals the sum of all

$L_m(\lambda_m + \nu)$ such that $\varphi(X_n, X_m, \gamma, \nu) \neq 0$. We denote by $\mathbf{K}(n, \gamma) \subset \mathbf{M}(n, \gamma)$ the sum of all $L_m(\lambda_m + \nu)$ with $\varphi(X_n, X_m, \gamma, \nu) \neq 0$ and $\nu \neq \gamma^{X_m}$.

It is immediate from the observation above that $\mathbf{M}(m, \nu) \subset \mathbf{M}(n, \gamma)$ if and only if $\varphi(X_n, X_m, \gamma, \nu) \neq 0$. If $\mathbf{S} \subset \mathbf{M}$ is a submodule then $\mathbf{S} = \sum \mathbf{M}(n, \gamma)$, where the sum is over all pairs (n, γ) such that $L_n(\lambda_n + \gamma) \subset \mathbf{S}$. If $\gamma \in \text{Pieri}(\lambda, \mathbf{F})$ then $\gamma_{X_n} \in \text{Pieri}(\lambda_n, F_n)$ for $n \gg 0$, and we set $\mathbf{M}(n, \gamma) := \mathbf{M}(n, \gamma_{X_n})$ and $\mathbf{K}(n, \gamma) := \mathbf{K}(n, \gamma_{X_n})$.

Lemma 6.2. *Let $\gamma \in \text{Pieri}(\lambda_n, F_n)$. Then $\mathbf{K}(n, \gamma)$ is a proper submodule of $\mathbf{M}(n, \gamma)$ if and only if $\bar{\gamma} \in \text{Pieri}(\lambda, \mathbf{F})$, and in that case $\mathbf{M}(n, \gamma)/\mathbf{K}(n, \gamma) \cong \mathbf{L}(\lambda + \bar{\gamma})$.*

Proof. For every $m > n$ we have

$$\frac{\mathbf{M}(n, \gamma) \cap M_m}{\mathbf{K}(n, \gamma) \cap M_m} = \begin{cases} L_m(\lambda + \gamma^{X_m}) & \text{if } \gamma^{X_m} \in \text{Pieri}(\lambda_m, F_m) \\ \{0\} & \text{if } \gamma^{X_m} \notin \text{Pieri}(\lambda_m, F_m). \end{cases}$$

By the definition of d -exhaustions and (A, B) -exhaustions, $\bar{\gamma} \in \text{Pieri}(\lambda, \mathbf{F})$ if and only if $\gamma^{X_m} \in \text{Pieri}(\lambda, F_m)$ for all $m \geq n$. On the other hand, since direct limits of vector spaces preserve exactness there is an isomorphism

$$\frac{\mathbf{M}(n, \gamma)}{\mathbf{K}(n, \gamma)} \cong \varinjlim_m \frac{\mathbf{M}(n, \gamma) \cap M_m}{\mathbf{K}(n, \gamma) \cap M_m}$$

and the result follows. \square

Lemma 6.3. *Suppose $\gamma, \nu \in \text{Pieri}(\lambda, \mathbf{F})$ and $\nu \gg \gamma$. Then for every n there exists $m > n$ such that $\mathbf{M}(m, \nu) \subset \mathbf{M}(n, \gamma)$.*

Proof. We proceed by induction on $h_\infty(\nu, \gamma)$. If $h_\infty(\nu, \gamma) = 0$ then $\nu = \gamma$ and we can take $m = n + 1$. Now suppose $\nu - \gamma = \varepsilon_I - \varepsilon_J$ with $I \neq \emptyset \neq J$, and let i be the last element of I and i' be the last element of J . It follows that $h_\infty(\varepsilon_i, \varepsilon_{i'}) > 0$, hence there exists some infinite class $j \in \mathbb{J}$ such that $\pi_{\mathbb{J}}(i) < j \leq \pi_{\mathbb{J}}(i')$.

Choose $k > n$ such that $\pi_{\mathbb{J}}(Z_k) = \{j\}$, and set $\gamma_1 = (\gamma_{X_{k+1}} + \varepsilon_i - \varepsilon_{i'})'$. By Propositions 5.4 and 5.10 we have $\varphi(X_k, X_{k+1}, \gamma_{X_k}, \gamma_1) \neq 0$ and so

$$\mathbf{M}(k+1, \gamma_1) \subset \mathbf{M}(k, \gamma) \subset \mathbf{M}(n, \gamma).$$

By construction $h_\infty(\nu, \bar{\gamma}_1) < h_\infty(\nu, \gamma)$, and the induction hypothesis implies there exists $m > k$ for which $\mathbf{M}(m, \nu) \subset \mathbf{M}(k, \gamma_1) \subset \mathbf{M}(n, \gamma)$. \square

6.1. Semisimple submodules and subquotients.

Lemma 6.4 (Separation lemma). *Let $\mathbf{T} \subsetneq \mathbf{S} \subset \mathbf{M}$ be submodules. There exists $\gamma \in \text{Pieri}(\lambda, \mathbf{F})$ such that $\mathbf{M}(n, \gamma) \subset \mathbf{S}$ and $\mathbf{T} \cap \mathbf{M}(n, \gamma) \subset \mathbf{K}(n, \gamma)$.*

Proof. Let $T_n := \mathbf{T} \cap M_n$ and $S_n := \mathbf{S} \cap M_n$ for all $n \geq 0$. Then there exists $r \in \mathbb{Z}_{>0}$ such that $T_n \subsetneq S_n$ for all $n \geq r$, and in particular there exists $\mu_r \in \text{Pieri}(\lambda_r, F_r)$ such that $L_r(\lambda_r + \mu_r) \subset S_r \setminus T_r$. Thus any $v \in L_r(\lambda_r + \mu_r)$ is in \mathbf{S} but not in \mathbf{T} , and hence $\varphi(X_r, X_n)(v) \in S_n \setminus T_n$. In particular there is at least one $\mu_n \in \text{Pieri}(\lambda_n, X_n)$ with $L_n(\lambda_n + \mu_n) \subset S_n \setminus T_n$ and $\varphi(X_r, X_n, \mu_r, \mu_n) \neq 0$.

We now define recursively a sequence $(\mu_n)_{n \geq r}$ with $\mu_n \in \text{Pieri}(\lambda_n, F_n)$. The first term of the sequence is μ_r as above, and assuming μ_n is defined we let μ_{n+1} be a minimal element in the set of all weights in $\text{Pieri}(\lambda_{n+1}, F_n)$ such that $\varphi(X_n, X_{n+1}, \mu_n, \mu_{n+1}) \neq 0$ and $L_{n+1}(\lambda_{n+1} + \mu_{n+1}) \subset S_{n+1} \setminus T_{n+1}$. We claim that there exists $\gamma \in \text{Pieri}(\lambda, F)$ with $\mu_m = \gamma_{X_m}$ for $m \gg 0$. Indeed, Proposition 5.15 states that if there is no such γ then there exist n, m such that $\varphi(X_{n+1}, X_m, (\mu_n^{X_{n+1}})', \mu_m) \neq 0$ and $(\mu_n^{X_{n+1}})' < \mu_n$. But $(\mu_n^{X_{n+1}})' < \mu_n$ implies that $L_{n+1}(\lambda_{n+1} + (\mu_n^{X_{n+1}})') \subset T_{n+1}$ and hence $L_m(\lambda_m + \mu_m) \subset T_m$, which contradicts the choice of μ_m . \square

As an immediate consequence we see that every simple subquotient of \mathbf{M} is of the form $\mathbf{L}(\lambda + \gamma)$ with $\gamma \in \text{Pieri}(\lambda, F)$.

Proposition 6.5. *Let $\mathbf{T} \subset \mathbf{S} \subset \mathbf{M}$. The quotient \mathbf{S}/\mathbf{T} has a submodule isomorphic to $\mathbf{L}(\lambda + \gamma)$ for some $\gamma \in \text{Pieri}(\lambda, F)$ if and only if $\mathbf{M}(n, \gamma) \subset \mathbf{S}$ and $\mathbf{M}(n, \gamma) \cap \mathbf{T} = \mathbf{K}(n, \gamma)$ for $n \gg 0$. Moreover, if \mathbf{S}/\mathbf{T} is simple then γ is unique and $\mathbf{S}/\mathbf{T} \cong \mathbf{L}(\lambda + \gamma)$.*

Proof. For the “if part”, the hypothesis implies that there is a nonzero map $\mathbf{L}(\lambda + \gamma) \cong \mathbf{M}(n, \gamma)/\mathbf{K}(n, \gamma) \rightarrow \mathbf{S}/\mathbf{T}$. For the “only if part”, suppose $s \in \mathbf{S}$ is a vector of weight $\lambda + \gamma$ which is a highest weight vector modulo \mathbf{T} . Then $s \in S_n = \mathbf{S} \cap M_n$ for all $n \gg 0$, and it must be a highest weight vector modulo $T_n = \mathbf{T} \cap M_n$. Thus the module generated by s is equal to $L_n(\lambda_n + \gamma_n) \oplus T'_n$, with $T'_n \subset T_n$ and $L_n(\lambda_n + \gamma_n) \subset S_n/T'_n$ for all $n \gg 0$. Consequently $\mathbf{M}(n, \lambda) \subset \mathbf{S}$, while $\mathbf{M}(n, \lambda) \cap \mathbf{T} \subset \mathbf{K}(n, \gamma)$.

The image of $L_n(\lambda_n + \gamma_n)$ under the map $\varphi(X_n, X_{n+1})$ is contained in $L_{n+1}(\lambda_{n+1} + \gamma_{n+1}) \oplus T'_{n+1} \subset M_{n+1}$. It follows that if $\varphi(X_n, X_{n+1}, \gamma, \nu) \neq 0$ and $\nu \neq \gamma$ then the image of $\varphi(X_n, X_{n+1}, \gamma, \nu)$ lies in $T'_{n+1} \subset \mathbf{T}$. Therefore, by definition $\mathbf{K}(n, \gamma) \subset \mathbf{T}$. It follows now from Lemma 6.4 that every simple subquotient of \mathbf{M} is isomorphic to $\mathbf{L}(\lambda + \gamma)$ for some $\gamma \in \text{Pieri}(\lambda, F)$. \square

We now show that a subquotient of \mathbf{M} is characterized by its simple constituents.

Definition 6.6. Let \mathbf{Q} be a subquotient of \mathbf{M} . We set

$$\mathcal{L}(\mathbf{Q}) := \{\gamma \in \text{Pieri}(\lambda, F) \mid [\mathbf{Q} : \mathbf{L}(\lambda + \gamma)] \neq 0\}.$$

Lemma 6.7. *Let $\mathbf{S}, \mathbf{T} \subset \mathbf{M}$ be submodules.*

- (i) *If $\gamma \in \mathcal{L}(\mathbf{S})$ and $\nu \gg \gamma$ then $\nu \in \mathcal{L}(\mathbf{S})$.*
- (ii) *$\mathcal{L}(\mathbf{T}) \subset \mathcal{L}(\mathbf{S})$ if and only if $\mathbf{T} \subset \mathbf{S}$.*
- (iii) *If $\mathbf{T} \subset \mathbf{S}$ then $\mathcal{L}(\mathbf{S}/\mathbf{T}) = \mathcal{L}(\mathbf{S}) \setminus \mathcal{L}(\mathbf{T})$.*

Proof. Suppose $\gamma \in \mathcal{L}(\mathbf{S})$. By Proposition 6.5 we have $\mathbf{M}(n, \gamma) \subset \mathbf{S}$ for some $n \gg 0$ and by Lemma 6.3 there exists $m > n$ such that $\mathbf{M}(m, \nu) \subset \mathbf{M}(n, \gamma)$. Hence $\nu \in \mathcal{L}(\mathbf{S})$ by Proposition 6.5.

The “if” part of the second item is clear. For the “only if” part, suppose $\mathcal{L}(\mathbf{T}) \subset \mathcal{L}(\mathbf{S})$ but $\mathbf{S} \cap \mathbf{T} \subsetneq \mathbf{T}$. By the separation lemma 6.4 there exists $\gamma \in \mathcal{L}(\mathbf{T})$ such that $\mathbf{M}(n, \gamma) \subset \mathbf{T}$ and $\mathbf{M}(n, \gamma) \cap \mathbf{T} \cap \mathbf{S} = \mathbf{M}(n, \gamma) \cap \mathbf{S} = \mathbf{K}(n, \gamma)$ for all $n \gg 0$. This

implies $\mathbf{M}(m, \gamma) \cap \mathbf{S} = \mathbf{K}(m, \gamma)$ for all $m \geq n$, and so $\gamma \notin \mathcal{L}(\mathbf{S})$ by Proposition 6.5, a contradiction.

We now prove the third item. By the separation lemma $\gamma \in \mathcal{L}(\mathbf{S}/\mathbf{T})$ if and only if $\mathbf{M}(n, \gamma) \subset \mathbf{S}$ and $\mathbf{M}(n, \gamma) \cap \mathbf{T} \subset \mathbf{K}(n, \gamma)$ for $n \gg 0$, and by Proposition 6.5 this is equivalent to $\gamma \in \mathcal{L}(\mathbf{S}) \setminus \mathcal{L}(\mathbf{T})$. \square

The following result gives necessary and sufficient conditions for a quotient \mathbf{S}/\mathbf{T} to be semisimple.

Proposition 6.8. *Let $\mathbf{T} \subset \mathbf{S} \subset \mathbf{M}$ be a chain of submodules.*

- (i) *If \mathbf{S}/\mathbf{T} is semisimple and $\nu, \gamma \in \mathcal{L}(\mathbf{S})$ with $\nu \gg \gamma$ and $\nu \neq \gamma$, then $\nu \in \mathcal{L}(\mathbf{T})$.*
- (ii) *Suppose \mathbf{F} is a \mathfrak{b} -highest weight module, and that $\nu \in \mathcal{L}(\mathbf{T})$ for all $\nu, \gamma \in \mathcal{L}(\mathbf{S})$ with $\nu \gg \gamma$ and $\nu \neq \gamma$. Then \mathbf{S}/\mathbf{T} is semisimple.*

Proof. Suppose \mathbf{S}/\mathbf{T} is semisimple. Then by Proposition 6.5 for every $\gamma \in \mathcal{L}(\mathbf{S}/\mathbf{T})$ there exists $n \gg 0$ such that $\mathbf{M}(n, \gamma) \subset \mathbf{S}$ and $\mathbf{K}(n, \gamma) \subset \mathbf{T}$. Lemma 6.3 implies $\mathbf{M}(m, \nu) \subset \mathbf{K}(n, \gamma)$ for some $m > n$, and so $\nu \in \mathcal{L}(\mathbf{T})$.

Suppose now \mathbf{F} is a \mathfrak{b} -highest weight module. If \mathbf{F} is $\mathbf{S}^d \mathbb{V}$ or $\bigwedge^d \mathbb{V}$ then by Proposition 5.5 every d -step (X_n, X_{n+1}) satisfies $X_n^- \neq \emptyset$. It follows from Proposition 4.3 that $\varphi(X_n, X_{n+1}, \gamma, \nu) \neq 0$ implies $\bar{\nu} \gg \bar{\gamma}$. Then $\mathbf{K}(n, \gamma) \subset \mathbf{T}$ whenever $\gamma \in \mathcal{L}(\mathbf{S}/\mathbf{T})$, and \mathbf{S}/\mathbf{T} has a submodule isomorphic to $\mathbf{M}(n, \gamma)/\mathbf{K}(n, \gamma) \cong \mathbf{L}(\lambda + \gamma)$. Therefore, $\bigoplus_{\gamma \in \mathcal{L}(\mathbf{S}/\mathbf{T})} \mathbf{L}(\lambda + \gamma) \subset \mathbf{S}/\mathbf{T}$, and by item (iii) or Lemma 6.7 this inclusion is an equality.

If $\mathbf{F} = \bigwedge^A \mathbb{V}$ then by Proposition 5.11 we can assume that each B -step (X_n, X_{n+1}) satisfies $X_n^- \neq \emptyset$, and every A -step satisfies $X_n^+ \neq \emptyset$. The reasoning is then completely analogous to the previous case, using Propositions 4.3 and 4.7. \square

Lemma 6.9. *For every $\ell \in \mathbb{L}_\infty$ we have $F_\ell \mathbf{M} = \text{rad } F_{\ell+1} \mathbf{M}$. Furthermore, if $F_\ell \mathbf{M}/F_{\ell-1} \mathbf{M}$ is semisimple then it is equal to the socle of $\mathbf{M}/F_{\ell-1} \mathbf{M}$.*

Proof. Let \mathbf{L} be a simple $\mathfrak{gl}(\infty)$ -module and let $\pi : F_{\ell+1} \mathbf{M} \rightarrow \mathbf{L}$ be a nonzero morphism. Then $\mathbf{L} \cong \mathbf{L}(\lambda + \gamma)$ for some $\gamma \in \text{Pieri}(\lambda, \mathbf{F})$ by Proposition 6.5. Furthermore $\{\gamma\} = \mathcal{L}(F_{\ell+1} \mathbf{M}) \setminus \mathcal{L}(\ker \pi)$, and by item (i) of Proposition 6.8 it follows that the \mathbb{L}_∞ class of γ is $d + 1$. Consequently $F_\ell \mathbf{M} \subset \ker \pi$ and, since π was arbitrary, that $F_\ell \mathbf{M} \subset \text{rad } F_{\ell+1} \mathbf{M}$. On the other hand for every $\gamma \in d + 1$ we have a map $F_{\ell+1} \mathbf{M} \rightarrow \mathbf{L}(\lambda + \gamma)$, which implies that $\mathcal{L}(\text{rad } F_{\ell+1} \mathbf{M}) \subset \mathcal{L}(F_\ell \mathbf{M})$. It follows from item (ii) of Lemma 6.7 that $\text{rad } F_{\ell+1} \mathbf{M} \subset F_\ell \mathbf{M}$, and hence these submodules are equal.

If $F_\ell \mathbf{M}/F_{\ell-1} \mathbf{M}$ is semisimple then it is a submodule of $\text{soc } \mathbf{M}/F_{\ell-1} \mathbf{M}$. On the other hand, if $\mathbf{L}(\lambda + \gamma)$ is a submodule of $\mathbf{M}/F_{\ell-1} \mathbf{M}$ then every $\nu \gg \gamma$ must be of \mathbb{L}_∞ -class strictly less than d thanks to item (i) of Proposition 6.8. Thus the \mathbb{L}_∞ class of γ is d and $\mathbf{L}(\lambda + \gamma) \subset F_\ell \mathbf{M}/F_{\ell-1} \mathbf{M}$. \square

6.1.1. The non-example revisited. We go back to the non-example 5.3.1. Taking $\mathbf{K} = \sum_{k=1}^{\infty} \mathbf{K}(2k + 1, \mu_{2k+1})$ and $\mathbf{N} = \sum_{k=1}^{\infty} \mathbf{M}(2k + 1, \mu_{2k+1})$, the quotient \mathbf{N}/\mathbf{K} is isomorphic to $\mathbf{L}(\lambda + \varepsilon_B)$. Since $\varepsilon_B \notin \text{supp } \mathbf{F}$, both the separation Lemma 6.4 and Proposition 6.5 fail.

6.2. Proofs of theorems A through D.

Proof of Theorem A. Proposition 6.5 implies that every simple constituent of \mathbf{M} is of the form $\mathbf{L}(\lambda + \gamma)$ with $\gamma \in \text{Pieri}(\lambda, \mathbf{F})$, and that \mathbf{M} is multiplicity free follows from item (iii) of Lemma 6.7.

By Lemma 3.3 the poset $\text{Pieri}(\lambda, \mathbf{F})$ is locally finite, and by [FM12, Theorem 1.3] its order can be extended to a total order \leq such that $(\text{Pieri}(\lambda, \mathbf{F}), \leq)$ is isomorphic to a subset of \mathbb{Z} . Let $G_\gamma \mathbf{M}$ be the submodule of \mathbf{M} generated by $\{v_\lambda \otimes e_\nu \mid \nu \geq \gamma\}$. Since $v_{\lambda_n} \otimes e_{\gamma_{X_n}} \in G_\gamma \mathbf{M}$ for all $n \gg 0$ it follows that $\mathbf{M}(n, \gamma) \subset G_\gamma \mathbf{M}$, and since $v_{\lambda_n} \otimes e_{\gamma_{X_n}} \notin G_{\gamma-1} \mathbf{M}$ it follows that $\mathcal{L}(G_\gamma \mathbf{M}/G_{\gamma-1} \mathbf{M}) = \{\gamma\}$, so $\{G_\gamma \mathbf{M}\}_{\gamma \in \text{Pieri}(\lambda, \mathbf{F})}$ is a composition series of \mathbf{M} .

If $\nu \gg \gamma$ then $\mathbf{L}(\lambda + \nu)$ is linked to $\mathbf{L}(\lambda + \gamma)$ by Lemma 6.3. Suppose now that $\mathbf{L}(\lambda + \nu)$ is linked to $\mathbf{L}(\lambda + \gamma)$, and choose $n \gg 0$ such that $\gamma_i = \nu_i$ for all $i \in \mathbb{I} \setminus X_n$. Then $[\mathbf{M}(n, \lambda) : \mathbf{L}(\lambda + \nu)] \neq 0$, so by Proposition 6.5 there exists $m \geq n$ such that $\mathbf{M}(m, \nu) \subset \mathbf{M}(n, \gamma)$, and hence $\varphi(n, m, \gamma_{X_n}, \nu_{X_m}) \neq 0$. The choice of n plus Corollaries 5.6 and 5.12 imply that $\nu - \gamma = \mu$ with $\mu \gg 0$, so $\nu \gg \gamma$. \square

Proof of Corollary A. It is an immediate corollary of Theorem A that the length of \mathbf{M} is equal to the cardinality of $\text{Pieri}(\lambda, \mathbf{F})$. As we have seen in Lemma 3.3, if $\mathbf{F} = \mathbf{S}^{d \vee}$ or $\mathbf{F} = \bigwedge^d \mathbb{V}$, we have an embedding $\text{Pieri}(\lambda, \mathbf{F}) \rightarrow \mathbb{J}^d$, and so $\text{Pieri}(\lambda, \mathbf{F})$ is finite whenever \mathbb{J} is finite. On the other hand, if \mathbb{J} is infinite then there are infinitely many sequences (j_1, \dots, j_d) with $j_k \in \mathbb{J}$ and $j_k < j_{k+1}$ for all $k \in \llbracket d-1 \rrbracket$. Taking i_k to be the first element of the class j_k and $I = (i_1, \dots, i_d)$ we have $\varepsilon_I \in \text{Pieri}(\lambda, \mathbf{F})$, and hence $\text{Pieri}(\lambda, \mathbf{F})$ is infinite.

Suppose now that $\mathbf{F} = \bigwedge^A \mathbb{V}$ and \mathbf{M} has finite length. We claim that \mathbb{J} must be a finite set. Let $I \subset A$ be any subset such that $I \cap A \cap j$ is a terminal subset of $A \cap j$ for every $j \in \mathbb{J}$. If \mathbb{J} is infinite then either $B \cap j$ is infinite for some infinite class $j \in \mathbb{J}$, in which case we set J to be an initial subset of $B \cap j$, or there is an infinite family of classes $\{j_k\}_{k \geq 1} \subset \mathbb{J}$ such that $B \cap j_k \neq \emptyset$, and we let J be the sequence formed by the first elements of $B \cap j_1, B \cap j_2, \dots, B \cap j_{\#I}$. In either case $\varepsilon_A - \varepsilon_I + \varepsilon_J \in \text{Pieri}(\lambda, \mathbf{F})$, and since there are infinitely many choices for I the set $\text{Pieri}(\lambda, \mathbf{F})$ is infinite, a contradiction. Thus if \mathbf{M} has finite length then \mathbb{J} must be finite. The case where \mathbb{J} is finite is analyzed in Example 3.4.6, where we showed that \mathbf{M} has finite length if and only if $\mathbf{L}(\lambda) \cong \mathbf{D}_n \otimes \mathbf{V}^{\lambda, \mu}$. \square

Proof of Theorem B. Suppose \mathbf{F} is a \mathfrak{b} -highest weight module. Then $\mu \in \text{Pieri}(\lambda, \mathbf{F})$ and its \mathbb{L}_∞ -class d is the minimal element of \mathbb{L}_∞ . It follows that the linkage filtration has a minimal layer, and by item (ii) of Proposition 6.8 its layers are semisimple. A simple induction using Lemma 6.9 now shows that the linkage filtration and the socle filtration coincide. In particular, \mathbf{M} is semisimple if and only if the linkage filtration has length one, i.e., if and only if \mathbb{L}_∞ is trivial. This proves item (i).

We now prove item (ii). Suppose $\mathbf{M} = \mathbf{S} + \mathbf{T}$, with \mathbf{S} and \mathbf{T} nontrivial submodules. Again the hypothesis that \mathbf{F} has a highest weight implies that \mathbf{M} has a nontrivial socle, and hence so do \mathbf{S} and \mathbf{T} . Let $\gamma \in \mathcal{L}(\text{soc } \mathbf{S})$ and $\nu \in \mathcal{L}(\text{soc } \mathbf{T})$, and let I, I' be sequences

of minimal length such that $\nu - \gamma = \varepsilon_I - \varepsilon_{I'}$. We will prove that $\mathbf{S} \cap \mathbf{T} \neq \{0\}$ by induction on the length r of I and I' .

If $r = 0$ then by definition $\mathbf{S} \cap \mathbf{T}$ contains the unique submodule of $\text{soc } \mathbf{M}$ isomorphic to $\mathbf{L}(\lambda + \gamma)$, and in particular $\mathbf{S} \cap \mathbf{T}$ is nonzero. If $r = 1$ then $\nu - \gamma = \varepsilon_i - \varepsilon_{i'}$ and without loss of generality we can assume $i < i'$. Since \mathbb{L}_∞ has at least two elements the same holds for \mathbb{J}_∞ , and hence there exists an infinite class $j \in \mathbb{J}$ such that its minimal element $s(j)$ is strictly larger than any element of $\text{supp } \gamma \cup \text{supp } \nu$. By construction $\sigma = \gamma - \varepsilon_{i'} + \varepsilon_{s(j)} = \nu - \varepsilon_i + \varepsilon_{s(j)}$ belongs to Pieri (λ, \mathbf{F}) , and $\gamma, \nu \gg \sigma$. As $\text{Pieri}(\lambda, \mathbf{F}) = \mathcal{L}(\mathbf{S}) \cup \mathcal{L}(\mathbf{T})$, we can assume without loss of generality $\sigma \in \mathcal{L}(\mathbf{S})$. Item (i) of Proposition 6.7 then implies that $\gamma, \nu \in \mathcal{L}(\mathbf{S})$ and so $\mathbf{L}(\lambda + \nu) \subset \mathbf{S} \cap \mathbf{T}$.

To proceed with the induction, we assume $r > 1$ and let $i = \max I, i' = \max I'$. Set $\sigma = \nu - \varepsilon_{i'} + \varepsilon_i$. Again $\sigma \in \mathcal{L}(\mathbf{S}) \cup \mathcal{L}(\mathbf{T})$. If $\sigma \in \mathcal{L}(\mathbf{S})$ we replace γ by σ and conclude from the case $r = 1$ that $\mathbf{S} \cap \mathbf{T} \neq \{0\}$. If $\sigma \in \mathcal{L}(\mathbf{T})$ we replace ν by σ and the result now follows from the induction hypothesis. This completes the proof of item (ii).

To prove item (iii), let $\mathbf{S}, \mathbf{T} \subset \mathbf{M}$ be arbitrary nontrivial submodules and let $\gamma \in \mathcal{L}(\mathbf{S}), \nu \in \mathcal{L}(\mathbf{T})$. Again take sequences I, I' of minimal length r such that $\nu - \gamma = \varepsilon_I - \varepsilon_{I'}$. For $n \gg 0$ we have $I, I' \subset X_n$, and

$$L_n(\lambda_{X_n} + \gamma_{X_n}) \subset \mathbf{S}, \quad L_n(\lambda_{X_n} + \nu_{X_n}) \subset \mathbf{T}.$$

The hypothesis that \mathbf{F} is not a \mathfrak{b} -highest weight module implies that there are infinitely many n such that $X_{n+1} = Z_n \sqcup X_n$ with $Z_n \cap \text{supp } \gamma = Z_n \cap \text{supp } \nu = \emptyset$. We fix such an n and let $i = \max I, i' = \max I'$ and $z = \min Z_n$. Item (ii) of Proposition 5.4 implies

$$L_{n+1}(\lambda_{X_{n+1}} + \gamma_{X_{n+1}} - \varepsilon_i + \varepsilon_z) \subset \mathbf{S}, \quad L_{n+1}(\lambda_{X_{n+1}} + \nu_{X_{n+1}} - \varepsilon_{i'} + \varepsilon_z) \subset \mathbf{T},$$

so $\gamma_1 = \gamma - \varepsilon_i + \varepsilon_z \in \mathcal{L}(\mathbf{S})$ and $\nu_1 = \nu - \varepsilon_{i'} + \varepsilon_z \in \mathcal{L}(\mathbf{T})$. By construction $\nu_1 - \gamma_1 = \varepsilon_{I_1} - \varepsilon_{I'_1}$ with I_1, I'_1 of length $r - 1$. Proceeding recursively we can find $\gamma_k \in \mathcal{L}(\mathbf{S})$ and $\nu_k \in \mathcal{L}(\mathbf{T})$ such that $\nu_k - \gamma_k = \varepsilon_{I_k} - \varepsilon_{I'_k}$ with the length of I_k, I'_k equal to $r - k$. After r steps we reach $\gamma_r = \nu_r \in \mathcal{L}(\mathbf{S}) \cap \mathcal{L}(\mathbf{T}) = \mathcal{L}(\mathbf{S} \cap \mathbf{T})$. Thus $\mathbf{S} \cap \mathbf{T}$ is nonzero.

In the course of the proof we showed that for every submodule $\mathbf{S} \subset \mathbf{M}$ with $L_n(\lambda_{X_n} + \gamma_{X_n}) \subset \mathbf{S}$ we also have $L_{n+1}(\lambda_{X_{n+1}} + \gamma_{X_{n+1}} - \varepsilon_i + \varepsilon_z) \subset \mathbf{S}$. This implies that $\mathbf{K}(n + 1, \gamma_{X_{n+1}} - \varepsilon_i + \varepsilon_z)$ is a strict submodule of \mathbf{S} . Thus every submodule of \mathbf{M} has a nontrivial submodule, and hence \mathbf{M} has no socle. \square

Proof of Theorem C. If \mathbf{F} is a \mathfrak{b} -highest weight module then the socle filtration and the linkage filtration of \mathbf{M} coincide by Theorem B, and since the linkage filtration is exhaustive so is the socle filtration. On the other hand, if \mathbf{F} is not a \mathfrak{b} -highest weight module then $\text{soc } \mathbf{M} = \{0\}$ by item (iii) of Theorem B. This proves item (i). Notice in particular that \mathbb{L}_∞ has a minimum if and only if \mathbf{F} is a \mathfrak{b} -highest weight module.

If \mathbb{L}_∞ has a maximum ℓ , Lemma 6.9 implies $F_{\ell-1} \mathbf{M} = \text{rad } \mathbf{M}$. On the other hand, if $\text{rad } \mathbf{M} \subset \mathbf{M}$ then the \mathbb{L}_∞ class of any $\gamma \in \mathcal{L}(\mathbf{M}/\text{rad } \mathbf{M})$ is the maximum of \mathbb{L}_∞ . Thus $\text{rad } \mathbf{M}$ is a strict submodule of \mathbf{M} if and only if \mathbb{L}_∞ has a maximum. Lemma 6.9 plus descending induction show that the linkage filtration and the radical filtration

coincide, and since the linkage filtration is separated, so is the radical filtration. This proves item (ii). Item (iii) follows from (i) and (ii). \square

Proof of Theorem D. The proof follows immediately from the fact that $-_*$ is an autoequivalence of the tensor category of integrable $\mathfrak{gl}(\infty)$ -modules. In particular note that $\mathbf{L}(\lambda) \otimes \mathbf{F}_* \cong (\mathbf{L}(-\lambda) \otimes \mathbf{F})_*$, and that \mathbf{F}_* is a highest weight module with respect to a Borel subalgebra arising from a (λ, \mathbf{F}_*) -compatible order if and only if \mathbf{F} is a highest weight module with respect to a Borel subalgebra arising from a $(-\lambda, \mathbf{F})$ -compatible order. \square

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