On some properties of HKT manifolds

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HYPERCOMPLEX STRUCTURES

Three complex structures I, J and K on a smooth manifold form a hypercomplex structure if

$$I^2 = J^2 = K^2 = -Id, \text{ and } IJ = K = -JI,$$

where Id is the identity map. Then

$$I_{\vec{a}} = a_1 I + a_2 J + a_3 K$$

defines a complex structure for any point $\vec{a} = (a_1, a_2, a_3)$ in the unit 2-sphere S^2

The smooth manifold $Z = X \times S^2$ is called *twistor space* and is endowed with an integrable almost complex structure $\mathcal{I}$ defined by $\mathcal{I}_{(x,\vec{a})} = I_{\vec{a}} \oplus J_{S^2}$.

Suppose that M has a Riemannian metric g such that $g(IX, IY) = g(JX, JY) = g(KX, KY)$. Then

$$\omega_I(X, Y) = g(IX, Y),\ \omega_J(X, Y) = g(JX, Y),\ \omega_K(X, Y) = g(KX, Y)$$

define non-degenerate 2-forms. When $d\omega_I = d\omega_J = d\omega_K = 0$, (M, I, J, K, g) is called *hyperkähler* structure. The closed form $\Omega = \omega_J + \sqrt{-1}\omega_K$ defines a holomorphic symplectic structure on (M, I).

Definition (M, I, J, K, g) *is called HKT (hyperkähler with torsion) manifold, if*

$$Id\omega_I = Jd\omega_J = Kd\omega_K \tag{1}$$

or equivalently

$$\partial\Omega = 0 \tag{2}$$

Here $Id\omega_I(X, Y, Z) = -d\omega_I(IX, IY, IZ)$ and ∂ is the ∂-operator for I.

The 3-form $c(X, Y, Z) = -Id\omega_{=} - Jd\omega_J = Kd\omega_K$ defines a connection ∇ by

$$g(\nabla_X Y, Z) = g(\nabla_X^{LC} Y, Z) + \frac{1}{2}c(X, Y, Z)$$

where ∇^{LC} is the Levi-Civita connection of g. Then $g(T^{\nabla}(X, Y), Z) = c(X, Y, Z)$ and $\nabla I = \nabla J = \nabla K = \nabla g = 0$.

Remarks

1. HKT manifolds first appeared as target spaces of (4,0)-SUSY sigma models with Wess-Zumino term in string theory (Howe-Papadopoulos(1992)).

2. From (1) *follows the integrability of* I, J, K. *From* $Id\omega_I = Jd\omega_J$ *follows* (1) *for integrable structures.*

3. There is a unique torsion-free connection preserving given hypercomplex structure - the Obata connection.

Quaternionic plurisubharmonic functions

On (M, I, J, K) there are operators additional to the exterior derivatives for every $\vec{a} \in S^2$:

$$d^c_{\vec{a}}\alpha = (-1)^p I_{\vec{a}} d(I_{\vec{a}}\alpha)$$

for p-form α where $I_{\vec{a}}\alpha(X_1, ..., X_p) = (-1)^p \alpha(I_{\vec{a}}X_1, ..., I_{\vec{a}}X_p)$. Similarly $\partial_{\vec{a}} = \frac{1}{2}(d + \sqrt{-1}d^c_{\vec{a}})$ and $\overline{\partial_{\vec{a}}} = \frac{1}{2}(d - \sqrt{-1}d^c_{\vec{a}})$. For any oriented orthonormal triple $\vec{a}, \vec{b}, \vec{c} \in S^2$ we have:

$$dd^c_{\vec{a}} + d^c_{\vec{a}}d = d^c_{\vec{a}}d^c_{\vec{b}} + d^c_{\vec{b}}d^c_{\vec{a}} = 0, (d^c_{\vec{a}})^2 = 0$$

If $\Omega_{\vec{c}} = \omega_{\vec{a}} + \sqrt{-1}\omega_{\vec{b}}$ then (1) and (2) are equivalent to

$$c = d^c_{\vec{a}}\omega_a, \qquad \partial_{\vec{c}}\Omega_{\vec{c}} = 0$$

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Theorem(G.-Poon (2000)) *Let $Z = M \times S^2$ be the twistor space of (M, I, J, K). A hyperhermitian metric on M is HKT if and only if the $(2,0)$-form on Z given by $G_{\vec{a}} = \pi^*(\omega_I)^{(2,0)}_{\vec{a}}$ satisfies $\partial G = 0$.*

Denote by d^c, ∂ and $\overline{\partial}$ the corresponding operators for $I_{(1,0,0)} = I$. Define

$$\partial^J \alpha = (-1)^p J \overline{\partial}(J\alpha)$$

for a p-form α. Since $J : \Lambda_I^{(p,q)} \to \Lambda_I^{(q,p)}$,

$$\partial, \partial^J : \Lambda_I^{(p,q)}(M) \to \Lambda_I^{(p+1,q)}(M)$$

and

$$\partial\partial^J + \partial^J\partial = (\partial^J)^2 = 0$$

So ∂, ∂^J play on $\Lambda_I^{(*,q)}(M)$ similar role as d, d^c on $\Lambda^*(M)$.

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Definition

1. The form $\alpha \in \Lambda_I^{(p,q)}(M)$ is called $real$ if $\alpha = J\overline{\alpha}$.

2. A real form $\alpha \in \Lambda_I^{(2,0)}(M)$ is called $positive$, if $\alpha(X, JX) > 0$ for any non-zero $(1,0)$-vector field $X \in T_I^{(1,0)}(M)$.

For example the form Ω is real and positive and $\partial\Omega = \partial^J\Omega = 0$.

Definition
A real function ϕ on M is called *quaternionic plurisubharmonic* *if $\partial\partial^J\phi$ is positive.*

Subharmonic $\subset$ Plurisubharmonic $\subset$ Q-Plurisubharmonic.

If (locally) $\Omega = \partial\partial^J\mu$, then μ is called (local) *HKT potential* of the structure (g, I, J, K)

Lemma
A function μ is HKT potential iff $\omega_{\vec{a}} = (dd^c_{\vec{a}} + d^c_{\vec{b}}d^c_{\vec{c}})\mu$ for some (and hence any) oriented orthonormal triple $(\vec{a}, \vec{b}, \vec{c}) \in S^2$.

Local $\partial\partial^J$-Lemma(Banos-Swann(2004))
If Ω is real $(2,0)$-form on hypercomplex manifold and $\partial\Omega = 0$, then locally $\Omega = \partial\partial^J f$ for a local real function f. In particular every HKT structure locally arises from a HKT potential.

EXAMPLES

Quaternionic Hopf surface:
Consider $\mathbb{Z}$ action on $\mathbb{C}^2 - \{0\}$ generated by $(z, w) \to (re^{i\theta}z, re^{-i\theta}w)$ for $r > 0$. Using the identification $\mathbb{H} \cong \mathbb{C}^2$ we see that the left multiplication by i, j, k induces a hypercomplex structure on the quotient $\mathbb{C}^2 - \{0\}/\mathbb{Z} \cong S^1 \times S^3$. The metric $\frac{|dz|^2+|dw|^2}{|z|^2+|w|^2}$ induces HKT metric on $S^1 \times S^3$. Any 4-dimensional hyperhermitian structure is HKT. Also $S^1 \times S^3 \cong S^1 \times Sp(1)$ is a Lie group and the HKT structure is left-invariant.

Compact Lie groups:(Joyce(1991))
Let G be a compact semi-simple Lie group and U a maximal torus. Let $\mathfrak{g}$ and $\mathfrak{u}$ be their algebras. Choose a system of ordered roots with respect to $\mathfrak{u}_{\mathbf{C}}$. Let α_1 be a maximal positive root, and $\mathfrak{h}_1$ the dual space of α_1. Let ∂_1 be the $\mathfrak{sp}(1)$-subalgebra of $\mathfrak{g}$ such that its complexification is isomorphic to $\mathfrak{h}_1 \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{-\alpha_1}$ where $\mathfrak{g}_{\alpha_1}$ and $\mathfrak{g}_{-\alpha_1}$ are the root spaces for α_1 and $-\alpha_1$ respectively. Let $\mathfrak{b}_1$ be the centralizer of ∂_1.

Then there is a vector subspace $\mathfrak{f}_1$ composed of root spaces such that $\mathfrak{g} = \mathfrak{b}_1 \oplus \mathfrak{d}_1 \oplus \mathfrak{f}_1$. If $\mathfrak{b}_1$ is not Abelian, apply the same decomposition to it. By inductively searching for $\mathfrak{sp}(1)$ subalgebras one obtains:

Lemma(Joyce(1991))

The Lie algebra $\mathfrak{g}$ of a compact Lie group G decomposes as

$$\mathfrak{g} = \mathfrak{b} \oplus_{j=1}^{n} \mathfrak{d}_j \oplus_{j=1}^{n} \mathfrak{f}_j, \tag{3}$$

with the following properties. (1) *$\mathfrak{b}$ is Abelian and $\mathfrak{d}_j$ is isomorphic to $\mathfrak{sp}(1)$.* (2) *$\mathfrak{b} \oplus_{j=1}^{n} \mathfrak{d}_j$ contains $\mathfrak{u}$.* (3) *Set $\mathfrak{b}_0 = \mathfrak{g}$, $\mathfrak{b}_n = \mathfrak{b}$ and $\mathfrak{b}_k = \mathfrak{b} \oplus_{j=k+1}^{n} \mathfrak{d}_j \oplus_{j=k+1}^{n} \mathfrak{f}_j$. Then $[\mathfrak{b}_k, \mathfrak{d}_j] = 0$ for $k \geq j$.* (4) *$[\mathfrak{d}_l, \mathfrak{f}_l] \subset \mathfrak{f}_l$.* (5) *The adjoint representation of $\mathfrak{d}_l$ on $\mathfrak{f}_l$ is reducible to a direct sum of the irreducible 2-dimensional representations of $\mathfrak{sp}(1)$.* (6)*The decomposition is an orthogonal decomposition with respect to the Killing-Cartan form.*

Let G be a compact semi-simple Lie group with rank r. Then

$$(2n - r)\mathfrak{u}(1) \oplus \mathfrak{g} \cong \mathbf{R}^n \oplus_{j=1}^{n} \mathfrak{d}_j \oplus_{j=1}^{n} \mathfrak{f}_j. \tag{4}$$

At the tangent space of the identity element of $T^{2n-r} \times G$, i.e. the Lie algebra $(2n-r)\mathfrak{u}(1) \oplus \mathfrak{g}$, a hypercomplex structure $\{I_1, I_2, I_3\}$ is defined as follows:
First $\mathbf{R}^n \oplus_{j=1}^n \partial_j \cong \oplus_{j=1}^n (u(1) + sp(1))$ so it carries hypercomplex structure. Then each $\mathfrak{f}_j$ is a representation of ∂_j and has an induced hypercomplex structure from the action of $i, j, k \in sp(1)$. The induced structures are integrable. Basic example of such group is $SU(3)$.

Higher dimensional Hopf manifolds
For any real number r, with $0 < r < 1$, and $\theta_1, \ldots, \theta_n$ modulo 2π, we consider the integer group $\langle r \rangle$ generated by the following action on$(\mathbf{C}^n \oplus \mathbf{C}^n) \backslash \{0\}$.

$$(z_\alpha, w_\alpha) \mapsto (re^{i\theta_\alpha} z_\alpha, re^{-i\theta_\alpha} w_\alpha). \tag{5}$$

One can check that the group $\langle r \rangle$ is a group of hypercomplex transformations. The quotient space of $(\mathbf{C}^n \oplus \mathbf{C}^n)\backslash\{0\}$ with respect to $\langle r \rangle$ is the manifold $S^1 \times S^{4n-1} = S^1 \times Sp(n)/Sp(n-1)$. Since the group $\langle r \rangle$ is also a group of isometries with respect to an HKT-metric $\hat{g}$ determined by a potential $\mu = \ln(|z|^2 + |w|^2)$, the HKT-structure descends from $(\mathbf{C}^n \oplus \mathbf{C}^n)\backslash\{0\}$ to a HKT-structure on $S^1 \times S^{4n-1}$. The deformation space of hypercomplex structures on $S^1 \times S^{4n-1}$ is parametrized by $(r, \theta_1, \ldots, \theta_n)$ and a generic hypercomplex structure in this family is inhomogeneous.

Theorem(G.-Poon(2000))

Every hypercomplex deformation of the homogeneous hypercomplex structure on $S^1 \times S^{4n-1}$ admits a HKT-metric.

REDUCTION

The reduction for HKT structures is modeled on the symplectic reduction of Marsden-Weinstein and extends the hypercomplex reduction of Joyce. Let G be a compact group of hypercomplex automorphisms of M. Denote the algebra of hyper-holomorphic vector fields by $\mathfrak{g}$. Suppose that $\nu = (\nu_1, \nu_2, \nu_3) : M \longrightarrow \mathbf{R}^3 \otimes \mathfrak{g}$ is a G-equivariant map satisfying the following the Cauchy-Riemann condition, $I_1 d\nu_1 = I_2 d\nu_2 = I_3 d\nu_3$, and the transversality condition, $I_a d\nu_a(X) \neq 0$ for all $X \in \mathfrak{g}$. Any map satisfying these conditions is called a G-moment map. Given a point $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ in $\mathbf{R}^3 \otimes \mathfrak{g}$, denote the level set $\nu^{-1}(\zeta)$ by P. Assuming that the level set P is invariant, and the action of G on P is free, then the quotient space $N = P/G$ is a smooth manifold.

The quotient space $N = P/G$ inherits a natural hypercomplex structure as follows. For each point m in the space P, its tangent space is

$$T_m P = \{t \in T_m M : d\nu_1(t) = d\nu_2(t) = d\nu_3(t) = 0\}.$$

Consider the vector subspace

$$\mathcal{U}_m = \{t \in T_mP : I_1 d\nu_1(t) = I_2 d\nu_2(t) = I_3 d\nu_3(t) = 0\}. \qquad (6)$$

Due to the transversality condition, this space is transversal to the vectors generated by elements in $\mathfrak{g}$. Due to the Cauchy-Riemann condition, this space is a vector subspace of T_mP with co-dimension $\dim \mathfrak{g}$, hence it is a vector subspace of T_mM with co-dimension $4 \dim \mathfrak{g}$.

The same condition implies that, as a subbundle of $TM_{|P}$, $\mathcal{U}$ is closed under I_a. Moreover there is a G-invariant splitting

$$TP = \mathcal{U} \oplus \mathcal{V} \qquad (7)$$

where $\mathcal{V}$ is the tangent space to the orbits of G and coincides with the bundle of kernels of $d\pi$. Again, we use the terms "horizontal" and "vertical" for $\mathcal{U}$ and $\mathcal{V}$, although the two spaces are not necessarily orthogonal. Then a hypercomplex structure on N is defined by (Joyce(1992)):

$$I_a\hat{A} = d\pi(I_aA^u), \quad \text{i.e.} \quad (I_aA)^u = I_aA^u. \qquad (8)$$

Theorem(G.-Papadopoulos-Poon(2001))
Let $(M, \mathcal{I}, g)$ *be a HKT-manifold. Suppose that G is a compact group of hypercomplex isometries admitting a G-moment map* ν*. Then hypercomplex reduced space* $N = M//G$ *inherits an HKT structure.*

One consequence is that the instanton moduli space over the quaternionic Hopf surace has HKT structure. Another example is construction of HKT structures on all small deformations of the left-invariant hypercomplex structure on $SU(3)$, similar to the higher dimensional Hopf manifolds above.

SL(n,H) MANIFOLDS

The Obata connection has holonomy in $GL(n,\mathbb{H})$. An important subgroup inside $GL(n,\mathbb{H})$ is its commutator $SL(n,\mathbb{H})$. When the holonomy of the Obata connection is in $SL(n,\mathbb{H})$, the manifold is called an $SL(n,\mathbb{H})$-manifold. To characterize such space we recall a notion form Hermitian geometry.

Let (M, I, g) be a complex Hermitian manifold, $\dim|_{\mathbb{C}} M = n$, and $\omega \in \Lambda^{1,1}(M)$ its Hermitian form. One says that M is **balanced** if $d(\omega^{n-1}) = 0$.

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Theorem(Barberis-Dotti-Verbitsky(2009))

Let (M, I, J, K, Ω) be an HKT-manifold, $\dim_{\mathbb{H}} M = n$. Then the following conditions are equivalent.

(i) $\overline{\partial}(\Omega^n) = 0$

(ii) $\nabla(\Omega^n) = 0$, *where ∇ is the Obata connection*

(iii) *The manifold (M, I) with the induced quaternionic Hermitian metric is balanced as a Hermitian manifold:*

$$d(\omega_I^{2n-1}) = 0.$$

Let Φ_I be a nowhere degenerate holomorphic section of $\Lambda_I^{2n,0}(M)$. Assume that Φ_I is real, that is, $J(\Phi_I) = \bar{\Phi}_I$. Existence of such a form on compact M is equivalent to $Hol(M) \subset SL(n, \mathbb{H})$. The above result shows that balanced HKT manifold is $SL(n, \mathbb{H})$.
On balanced HKT manifold with a fixed form $\Phi_I = \Omega^n$ one can define operators $L_\Omega(\alpha) = \Omega \wedge \alpha$ and its Hermitian adjoint $\Lambda_\Omega = *L_\Omega*$ acting on $\Lambda_I^{(*,0)}$. The following identities hold

$$[L_\Omega, \partial^*] = -\partial_J, [\Lambda_\Omega, \partial] = \partial_J^*, [L_\Omega, \partial] = 0$$

$$[L_\Omega, \partial_J^*] = -\partial, [\Lambda_\Omega, \partial_J] = \partial^*, [L_\Omega, \Lambda_\Omega] = (n-p)Id$$

where $n = dim_{\mathbb{H}} M$. In combination with $\partial_J \partial^* + \partial^* \partial_J = \partial_J \partial + \partial \partial_J = 0$ follows that

$$\Delta = \partial^* \partial + \partial^* \partial = \partial_J^* \partial_J + \partial_J^* \partial_J = \Delta_J$$

As a consequence one obtains Hodge-type decomposition as well as:

Global $\partial\partial^J$-Lemma(G.-Lejmi-Verbitsky)
If Ω is ∂-exact $(2,0)$-form on balanced HKT manifold and $\partial^J\Omega = 0$, then $\Omega = \partial\partial^J f$ for a real function f

For a compact balanced HKT-manifold, we denote by $\mathfrak{h}$ the (finite dimensional) Lie algebra of hyper-holomorphic vector fields and

$$\mathfrak{h}_0 = \{X \in \mathfrak{h} \mid \Omega(X, \cdot) = \partial f + \partial_J h \text{ for some functions } f \text{ and } h\}.$$

Corollary(G.-Lejmi-Verbitsky)
If X is a hyper-holomorphic vector field with non-empty zero set then $X \in \mathfrak{h}_0$. Moreover $\mathfrak{h}_0$ is an ideal of $\mathfrak{h}$ such that

$$[\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}_0.$$

For $SL(n\mathbb{H})$ manifold with any HKT metric $\Phi_I = e^f(\Omega)^n$ for some f and there is a *quaternionic Monge-Ampere equation* (Alesker-Verbitsky(2010))

$$(\Omega + \partial\partial^J\phi)^n = e^f(\Omega)^n$$

Example of hypercomplex manifold without HKT metric (Fino-G.(2004))

Consider the nilpotent Lie algebra $\mathbb{R} \times \mathfrak{h}_7$ where $\mathfrak{h}_7$ is the algebra of the quaternionic Heisenberg group H_7. It is defined by the following relation on a basis of left-invariant 1-forms:

$$\begin{array}{l} de^i = 0, i = 1, \dots, 5 \\ de^6 = e^1 \wedge e^2 + e^3 \wedge e^4, \\ de^7 = e^1 \wedge e^3 - e^2 \wedge e^4, \\ de^8 = e^1 \wedge e^4 + e^2 \wedge e^3 \end{array}$$

On a compact quotient $M = \mathbb{R} \times H_7/\Gamma$ consider the family of complex structures defined via:

$$\begin{array}{l} I_t(e^1) = \frac{t-1}{t} e^2, I_t(e^3) = e^4, J_t(e^5) = \frac{1}{t} e^6, J_t(e^7) = e^8, \\ J_t(e^1) = \frac{t-1}{t} e^3, J_t(e^2) = -e^4, J_t(e^5) = \frac{1}{t} e^7, J_t(e^6) = -e^8. \end{array}$$

for $t \in (0, 1)$. Then for each t, $I_t J_t = -J_t I_t = K_t$ defines a hypercomplex structure on M. Using averaging argument it was shown that for $t = \frac{1}{2}$ the structure is HKT and for $t \neq \frac{1}{2}$ there is no HKT metric.

BOTT-CHERN TYPE COHOMOLOGY AND CURRENTS

Define $H^{p,0}_{\partial,\partial^J}(M)$ to be the group $(p > 1)$

$$H^{p,0}_{\partial,\partial_J}(M) = \frac{\{\phi \in \Lambda^{p,0}(M,I) | \partial\phi = \partial_J\phi = 0\}}{\partial\partial_J\Lambda^{p-2,0}(M,I)}.$$

and $H^{p,0}_{\partial\partial_J}(M)$ to be the group

$$H^{p,0}_{\partial\partial_J} = \frac{\{\phi \in \Lambda^{p,0}(M,I) | \partial\partial_J\phi = 0\}}{(\partial\Lambda^{p-1,0}(M,I) + \partial_J\Lambda^{p-1,0}(M,I))}.$$

Theorem(G.-Lejmi-Verbitsky)
The groups $H^{p,0}_{\partial,\partial_J}(M)$ and $H^{p,0}_{\partial\partial_J}(M)$ are finite dimensional for any hypercomplex manifold M. If M is also $SL(n,\mathbb{H})$ manifold with nondegenerate form Φ_I, then the two groups are dual to each other via the pairing

$$([\alpha],[\beta]) \mapsto \int_M \alpha \wedge \beta \wedge \bar{\Phi}_I.$$

Global $\partial\partial^J$ Lemma in dimension 8(G.-Lejmi-Verbitsky)
Every $SL(n,\mathbb{H})$ compact manifold M admits a hyperhermitian metric with $\partial\partial^J(\Omega^{n-1})=0$. If $n=2$ and $dim H^1(\mathcal{O}_{(\mathcal{M},\mathcal{I})})$ is even, then for each ∂^J-exact real $(2,0)$-form η, $\eta=\partial\partial^J\phi$.

Currents on $SL(n,\mathbb{H})$ manifolds

Currents of type (p,q) on complex manifold are defined as the continuous functionals on the space of $(n-p,n-q)$ smooth forms with compact support endowed with the Frechet topology. They can be considered as (p,q) forms with distribution coefficients in local patch. The operators ∂, J and ∂^J are extended naturally to act by duality on currents. Similarly a $(2n-2,0)$-current T is called *real* if $T=J\overline{T}$ and *positive*, if it is real and $T(\alpha\wedge\overline{\Phi_I})\geq 0$ for any real $(2,0)$ form $\alpha>0$. It is called a real component of a boundary, if $T=\partial\alpha+\partial^J(J\alpha)$ for some $(2n-1,0)$-current α.

Theorem(G.-Lejmi-Verbitsky)
Let (M, I, J, K) be a compact $SL(n, \mathbb{H})$-manifold. Then M admits no HKT metrics if and only if M admits a positive $(2n-2, 0)$-current which is real component of a boundary.

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Now we provide an example of simply connected $SL(n, \mathbb{H})$ manifold without HKT structure(Swann(2010)).

Let (X, I, J, K, g) be a $K3$ surface equipped with a hyperkähler structure and large enough Picard group such that there are 4 independent integral classes defining a principal T^4-bundle M over $X = K3$ which is simply-connected up to a finite covering The bundle M admits a connection A given by 1-forms θ_i s.t. $d\theta_i = \pi^*(\alpha_i)$, where $\alpha_1, ..., \alpha_4$ are the characteristic classes of M. Assume $\alpha_i^2 \neq 0$ and define structures $\mathcal{I}, \mathcal{J}, \mathcal{K}$ on M by their action on T^*M given as:

$$\mathcal{I}(\theta_1) = \theta_2, \mathcal{I}(\theta_3) = \theta_4, \mathcal{J}(\theta_1) = \theta_3, \mathcal{J}(\theta_2) = -\theta_4,$$
$$\mathcal{I}(\pi^*\alpha) = \pi^*(I\alpha), \mathcal{J}(\pi^*\alpha) = \pi^*(J\alpha)$$

for any 1-form α on X.

The structure $\mathcal{I}$ is integrable if $\alpha_1 + \sqrt{-1}\alpha_2, \alpha_3 + \sqrt{-1}\alpha_4$ are of type $(2,0) + (1,1)$ with respect to I on X. Similarly $\mathcal{J}$ is integrable if $\alpha_1 + \sqrt{-1}\alpha_3, \alpha_2 - \sqrt{-1}\alpha_4$ are of type $(2,0) + (1,1)$ with respect to J.
Similarly one can define a hyperhermitian metric on M from g and a fixed hyper-Kähler metric on T^4 using the splitting of TM in horizontal and vertical subspaces. As A. Swann has shown the structure has a holonomy in $SL(n, \mathbb{H})$ and is HKT when the forms above have no $(2,0)$-components respectively. If one chooses $\alpha_1 + \sqrt{-1}\alpha_2$ to be $(2,0)$-form for I, but $(1,1)$ for J and K and the almost hypercomplex structure $\mathcal{I}, \mathcal{J}, \mathcal{K}$ is still integrable. Then M does not admit HKT structure. By a spectral sequence argument, one can see that all second cohomology classes of M are pull-backs from classes on the base $K3$-surface. Then $\pi^*(\alpha_1 + \sqrt{-1}\alpha_2)$ will define positive $(2,0)$-current which obstructs the existence of HKT structure.

CALIBRATIONS AND $SL(n, \mathbb{H})$ MANIFOLDS

Let $W \subset V$ be a p-dimensional subspace in a Euclidean space, and $Vol(W)$ denote the Riemannian volume form of $W \subset V$, defined up to a sign. For any p-form $\eta \in \Lambda^p V$, let $\textsc{comass}(\eta)$ be the maximum of $\frac{\eta(v_1, v_2, \ldots, v_p)}{|v_1||v_2|\ldots|v_p|}$, for all p-tuples $(v_1, \ldots, v_p)$ of vectors in V and *face* be the set of planes $W \subset V$ where $\frac{\eta}{Vol(W)} = \textsc{comass}(\eta)$. A *calibration* on a Riemannian manifold is a closed differential form η with comass ≤ 1 everywhere.Let $X \subset M$ be a k-dimensional subvariety. We say that X is *calibrated by* η if at any smooth point $x \in X$, the space $T_x X$ is a face of the calibration η.

Theorem(G.-Verbitsky(preprint(2010), to appear))
Let (M, I, J, K, Φ_I) be an $SL(n, \mathbb{H})$-manifold, and $(\Phi_I)^{n,n}_J$ the (n, n)-part of Φ_I taken with respect to J. Pick a hyperhermitian metric on M such that $|\Phi_I|_g = 2^n$ (such metric always exists). Then $Re((\Phi_I)^{n,n}_J)$ is a calibration, and it calibrates complex subvarieties of (M, J) which are Lagrangian with respect to the $(2, 0)$-form $\omega_K + \sqrt{-1}\omega_I$.

Theorem (A.Soldatenkov, M.Verbitky (preprint(2013))
Let M be a compact $SL(n, \mathbb{H})$-manifold, and $\phi : M \to X$ a smooth holomorphic Lagrangian fibration. If M admits HKT-structure, then X is Kähler.