Ultraviolet Renormalization by Functional Integration

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The Spin-Boson Model

- Hilbert Space:

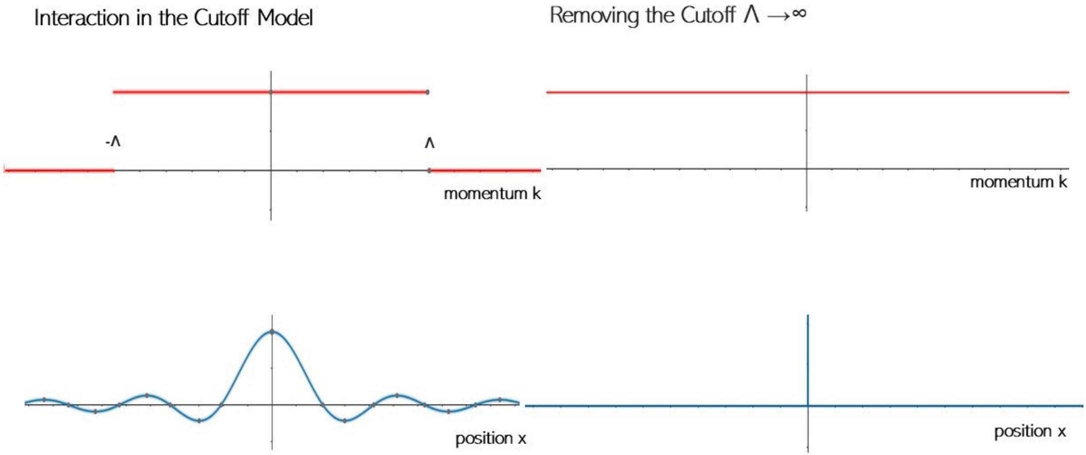
$$\underbrace{\mathbb{C}^2}_{\text{spin system}} \otimes \underbrace{\mathscr{F}_b}_{\text{bosonic radiation field}} \simeq \mathscr{L}^2(\{\pm 1\}, \mathscr{F}_b)$$

- Regularized Hamiltonian:

$$H_\Lambda = \underbrace{-\sigma_x \otimes \mathbb{1}}_{\text{free spin energy}} + \underbrace{\mathbb{1} \otimes d\Gamma(\omega)}_{\text{free radiation energy}} + \underbrace{\sigma_z \otimes \phi(\nu_\Lambda)}_{\text{interaction}}$$

H_Λ is well-defined self-adjoint on $D(d\Gamma(\omega))$.

Why do we need UV-Renormalization?



Self Energy Renormalization

Goal:

Find E_Λ such that $\lim_{\Lambda\to\infty} H_\Lambda - E_\Lambda$ exists as a self-adjoint operator.

Strategy

Develop a stochastic description of e^{-tH_Λ} and, extract E_Λ and show

$$\lim_{\Lambda\to\infty} e^{-t(H_\Lambda - E_\Lambda)}$$

exists as a self-adjoint semigroup.

Feynman-Kac Formula

Classical Feynman-Kac Formula

$\psi \in \mathscr{L}^2(\mathbb{R}^d, \mathbb{C})$ and $V \in \mathscr{L}^2(\mathbb{R}^d, \mathbb{R}) + \mathscr{L}^\infty(\mathbb{R}^d, \mathbb{R})$, then

$$e^{-t(-\Delta+V)}\psi(x) = \mathbb{E}^x[e^{-\int_0^t V(B_s)ds}\psi(B_t)].$$

Operator Theory ↔ Stochastic Processes

$$\begin{aligned} \Delta \ &\leftrightarrow \ B_t \ \text{Brownian Motion} \\ 1-\sigma_x \ &\leftrightarrow \ S_t \ \text{"Spin Process".} \end{aligned} \tag{1}$$

Spin Process

- Probability space:

$$(\Omega, F, P)$$

- jump intervals:

$$(\tau_i)_{i\in\mathbb{N}} \text{ independent, exponentially distributed}$$

- absolute jumps times :

$$T_i := \sum_{j=1}^{i} \tau_J$$

- Spin process

$$S_t = (-1)^k \text{ if } t \in [T_k, T_{k+1}), k \in \mathbb{N}$$

- $\implies (S_t)_{t\geq 0}$ is a Markov process w.r.t. natural Filtration

Feynman-Kac Formula for the Cut-off Model

Feynman-Kac Formula

Encode interaction

- $U^{+}_{t,\Lambda}(S_\bullet) := \int_0^t e^{-(t-s)\omega} S_s \nu_\Lambda ds$
- $U^{-}_{t,\Lambda}(S_\bullet) := \int_0^t e^{-s\omega} S_s \nu_\Lambda ds$
- $u_{t,\Lambda}(S_\bullet) = \int_0^t \langle U^{+}_{t,\Lambda}(S_\bullet), S_s \nu_\Lambda \rangle ds$

then for all $f \in \mathscr{L}^2(\{\pm 1\}, \mathscr{F}_b)$,

$$e^{-tH_\Lambda} f(y) = e^t \mathbb{E}^y [\underbrace{e^{u_{t,\Lambda}(S_\bullet)} e^{a^\dagger(U^{-}_{t,\Lambda}(S_\bullet))} e^{-\frac{t}{2} d\Gamma(\omega)} e^{a(U^{+}_{t,\Lambda}(S_\bullet))} e^{-\frac{t}{2} d\Gamma(\omega)}}_{=: V_t(S_\bullet)} f(S_t)].$$

Self-Energy and Renormalization

Stochastic Identity

- $U^{\pm}_{t,\text{ren}}(S_\bullet) := \lim_{\Lambda\to\infty} U^{\pm}_{t,\Lambda}(S_\bullet)$ is well-defined since $||U_{t,\Lambda}(S_\bullet)^{\pm}|| \leq ||\nu_\Lambda \omega^{-1}||$.
- For each spin path in Ω we find

$$\underbrace{\int_0^t \langle U^{+}_{s,\Lambda}(S_\bullet), S_s \nu_\Lambda\rangle dt}_{=u_{t,\Lambda}(S_\bullet)} - \underbrace{||\nu_\Lambda \omega^{-1/2}||^2}_{=:E_\Lambda} t = \int_0^t \langle U^{+}_{s,\Lambda}(S_\bullet), \nu_\Lambda \omega^{-1}\rangle dS_s$$

$$- \langle U^{+}_{t,\Lambda}(S_\bullet), S_t \nu_\Lambda \omega^{-1}\rangle$$

$$u_{t,\text{ren}}(S_\bullet) := \int_0^t \langle U^{+}_{s}(S_\bullet), \nu\omega^{-1}\rangle dS_s - \langle U^{+}_{t}(S_\bullet), S_t \nu \omega^{-1}\rangle = \lim_{\Lambda\to\infty} (u_{t,\Lambda}(S_\bullet) - E_\Lambda)$$

Renormalization

Define the renormalized semigroup

$$T(t)_{\mathrm{ren}} f(y) := e^{t}\mathbb{E}^{y}[e^{u_{t,\mathrm{ren}}(S_\bullet)} e^{a^\dagger(U^-_{t,\mathrm{ren}}(S_\bullet))} e^{-\frac{t}{2}d\Gamma(\omega)} e^{a(U^+_{t,\mathrm{ren}}(S_\bullet))} e^{-\frac{t}{2}d\Gamma(\omega)} f(S_t)]$$

Renormalization

$$\lim_{\Lambda\to\infty} e^{-t(H_\Lambda - E_\Lambda)} = T_{ren}(t)$$

in operator norm and T_{ren} has a self-adjoint Generator H_{ren}.

Conclusion

Outlook

- Model atoms with multiple energy levels
- Feynman path integral for the unitary time evolution

References

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