Technische Universität Braunschweig

Almost Commuting Self-Adjoint Operators and Iterated Commutator Estimates

Jakob Geisler, June 19, 2025

Definition of the Problem

Problem 1: Let A, B be two hermitian (or normal, unitary, complex, resp.) $N \times N$ matrices with $\|A\|_{op}, \|B\|_{op} \leq 1$ and $\|[A, B]\|_{op} = \|AB - BA\|_{op} \leq \varepsilon \ll 1$. Are there two hermitian (or normal, unitary, complex, resp.) matrices A', B' with $[A', B'] = 0$ and

$$\|A - A'\|_{op} + \|B - B'\|_{op} \leq f(\varepsilon), \qquad (1)$$

where $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$?

Further Questions:

- How fast does f converge to 0 for $\varepsilon \to 0$?
- Does f depend on N?
- What about other norms? Hilbert-Schmidt/Frobenius, L^p-norms?

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Content

- **Almost Commuting Self-Adjoint Operators**
  - When do matrices almost commute?
  - Diagonalyzing Flows
- **Iterated Commutator Estimates**

When do hermitian Matrices almost commute?

- Let $P = P^* = P^2, Q = Q^* = Q^2 \in \mathbb{C}^{N \times N}$ be orthogonal projections.
- Lemma: If $\|[P, Q]\|_{\mathcal{L}^2} \leq \|[P, Q]\|_{op} \leq \varepsilon \ll 1$ then

(a) $\|PQ\|_{op} \geq 1 - 2\varepsilon$ (almost parallel vectors $x \in \mathrm{Ran}(P)$, $y \in \mathrm{Ran}(Q)$ exist) or

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- Construction of almost the same ONB of eigenvectors of P and Q:

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3. Observe $\|[P', Q']\|_{\mathcal{L}^2} \leq \|[P, Q]\|_{\mathcal{L}^2}$.
4. Repeat. $\Rightarrow$ Get two 'almost parallel ONB's' of eigenvectors of P and Q. $\Rightarrow$ There is an $R = R^* = R^2$ with $[R, Q] = 0$ and $\|R - P\|_{\mathcal{L}^2} \leq 2\varepsilon$.

- Now let $0 < \lambda, \mu \leq 1$, $A = \lambda \cdot P$, $B = \mu \cdot Q$ and $\|[A, B]\|_{\mathcal{L}^2} = \lambda\mu \|[P, Q]\|_{\mathcal{L}^2} \leq \varepsilon$.

$\Rightarrow$ **A and B almost commute if they have almost the same eigenvectors or if their eigenvalues are almost degenerate.**

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Theorem (V.B., J.B.B., '10: Brockett-Wegner-Flow)

Let $\mathcal{H}$ be a complex separable Hilbert space, $A = A^*, B = B^* \in \mathcal{L}^2(\mathcal{H})$ and $B > 0$, then

$$\dot{A}_t = [A_t, [A_t, B]], \quad A_{t=0} = A,$$

has a unique solution, $A_t \to A_\infty$ strongly, A_t is unitary equivalent to A for all $t \in [0, \infty]$ and $[A_\infty, B] = 0$.

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Theorem (1, JG 25'+)

Let $\mathcal{H}$ be a complex separable Hilbert space and $A, B \in \mathcal{L}^2(\mathcal{H})$ be two self-adjoint Hilbert-Schmidt operators. Then, the differential equation

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has a unique global solution $(A_t, B_t)_{t \geq 0}$ that satisfies

$$\lim_{t\to\infty} \sqrt{t} \cdot [A_t, B_t] = 0. \tag{3}$$

Moreover, if $\dim(\mathcal{H}) = N < \infty$, then the solution converges for $t \to \infty$ in Hilbert-Schmidt norm $(A_t, B_t) \to (A_\infty, B_\infty)$ and is close to its initial value, i.e.,

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- Local existence and uniqueness is clear.
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- Local existence and uniqueness is clear.
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$\Rightarrow$ $\dot{f}_t \leq -\frac{f_t^\gamma}{C}$ so f_t would be a subsolution to $\dot{g}_t = -\frac{g_t^\gamma}{C}$.

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$\Rightarrow$ Sufficient decay if $\gamma \in [1, 2)$.

Remark: Sadly, $\gamma = 2$ is again the critical case. Here, it is easy to prove

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Section 2

Iterated Commutator Estimates

General Observations

Consider inequalities of the form

$$\|[A,B]\|_{\mathcal{L}^2}^{2\gamma} \leq C(N)\Big(\|[A,[A,B]]\|_{\mathcal{L}^2}^2 + \|[B,[B,A]]\|_{\mathcal{L}^2}^2\Big), \tag{6}$$

for a fixed constant $C(N)$ and all $A, B \in \mathbb{C}^{N\times N}$.

- In general, $\gamma = \frac{3}{2}$ is necessary, even for A, B self-adjoint.
- For all $N \geq 2$ and $C(N) < \infty$ there are diagonalizable $A, B \in \mathbb{C}^{N\times N}$ that violate (6).
- There are self-adjoint $A, B \in \mathbb{C}^{N\times N}$ saturating (6) with $C(N) = (N/4)^{3/2}$. These matrices also saturate (6) if $\|\cdot\|_{\mathcal{L}^2}$ is changed to $\|\cdot\|_{op}$ and $C(N) \sim \sqrt{N}$.

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Existence Results

Theorem (2, JG 25'+)

Let $N \in \mathbb{N}$, then there is a constant $C(N) < \frac{N^3}{2}$ such that for all self-adjoint Hilbert-Schmidt operators $A, B \in \mathcal{L}^2(\mathbb{C}^N)$ one has

$$\|[A,B]\|_{\mathcal{L}^2}^3 \leq C(N) \cdot (\|[A,[A,B]]\|_{\mathcal{L}^2}^2 + \|[B,[A,B]]\|_{\mathcal{L}^2}^2). \quad (7)$$

- If $A = A^*$ or $B = B^*$ has just two eigenvalues, i.e., $\sigma(A) = \{a_1, a_2\}$ then $C(N) \leq \sqrt{N}/4$.
- With spectral gap $\Delta := \min\{|a - a'| : a \neq a' \in \sigma(A)\}$, one has $\|[A,B]\|_{\mathcal{L}^2}^2 \leq \frac{\|[A,[A,B]]\|_{\mathcal{L}^2}^2}{\Delta^2}$. $\Rightarrow$ In this case $\dot{B}_t = [A,[B_t,A]]$ has a unique solution, $B_t \to B_\infty$ exponentially and $\|B_\infty - B\|_{\mathcal{L}^2} \leq \frac{\|[A,B]\|_{\mathcal{L}^2}}{\Delta}$.

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Definition of the Problem

Problem 1: Let A, B be two hermitian (or normal, unitary, complex, resp.) $N \times N$ matrices with $\|A\|_{op}, \|B\|_{op} \leq 1$ and $\|[A, B]\|_{op} = \|AB - BA\|_{op} \leq \varepsilon$. Are there two hermitian (or normal, unitary, complex, resp.) matrices A', B' with $[A', B'] = 0$ and

$$\|A - A'\|_{op} + \|B - B'\|_{op} \leq f(\varepsilon), \tag{1}$$

where $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$?

Further Questions:

- How fast does f converge to 0 for $\varepsilon \to 0$?
- Does f depend on N?
- What about other norms? Hilbert-Schmidt/Frobenius, L^p-norms?

History of the Problem I

- First formulated by John v. Neumann, 1929.
- Communicated as 'Open Problem' by P. Rosenthal, '69, and P.R. Halmos '70.
- Affirmative Answers if f is **allowed to depend on N**:
  - W. Luxemburg and R. Taylor, '70 (complex matrices): For every $N \in \mathbb{N}$, there is a function $f_N(\varepsilon)$.
  - J. Bastian and K. Harrison, '74 (normal operators).
  - C. Pearcy and A. Shields, '79 (A self-adjoint, B complex): $f_N(\varepsilon) = \sqrt{\frac{(N-1)\varepsilon}{2}}$
- Negative Answers for f **independent of N**:
  - D. Voiculescu, 81' (triplets of hermitian matrices) and 83' (unitary operators).
  - K. R. Davidson, 85': A hermitian, B normal, A' hermitian, B' arbitrary.

History of the Problem II

- H. Lin, ’95: first N-independent result for hermitian matrices.
- P. Friis and M. Rørdam, ’96: shorter proof and generalization to certain classes of C^*-Algebras.
- M. B. Hastings, ’09: $f(\varepsilon) \leq \varepsilon^{1/5} \cdot E(1/\varepsilon)$.
- N. Filonov and I. Kachkovskiy, ’10, and L. Glebsky, ’10: Analogue for normalized Hilbert-Schmidt-norm $\|A\|^2 = \frac{1}{N}\mathrm{Tr}(A^*A)$.

Open Questions: Results for regular Hilbert-Schmidt-norm/ other norms, (efficient) construction of A' and B'.

Applications

- G. A. Elliott, G. Gong, H. Lin, and C. Pasnicu. Abelian C^*-subalgebras of C^*-algebras of real rank zero and inductive limit C^*-algebras. Duke Math. J., 85:511–554, 1996.
- H. Lin. Classification of simple C^*-algebras and higher dimensional noncommutative tori. Ann. of Math., 157:521–544, 2003.
- M. B. Hastings. Topology and phases in fermionic systems. J. Stat. Mech., 2008:L01001, 2008.
- M. B. Hastings and T. A. Loring. Almost commuting matrices, localized Wannier functions, and the quantum Hall effect. J. Math. Phys., 51:015214, 2010.
- M. B. Hastings and T. A. Loring. Topological insulators and C^*-algebras: Theory and numerical practice. Annals of Physics, 326:1699–1759, 2011.
- Y. Ogata. Approximating macroscopic observables in quantum spin systems with commuting matrices. Journal of Functional Analysis, 264:2005–2033, 2013.

Content

When do hermitian Matrices almost commute?

- Let $P = P^* = P^2, Q = Q^* = Q^2 \in \mathbb{C}^{N\times N}$ be orthogonal projections.
- Lemma: If $\|[P,Q]\|_{\mathcal{L}^2} \leq \|[P,Q]\|_{op} \leq \varepsilon \ll 1$ then

(a) $\|PQ\|_{op} \geq 1 - 2\varepsilon$ (almost parallel vectors $x \in \mathrm{Ran}(P)$, $y \in \mathrm{Ran}(Q)$ exist) or
(b) $\|PQ\|_{op} \leq 2\varepsilon$ (almost orthogonal subspaces).

- Construction of almost the same ONB of eigenvectors of P and Q:
  1. If $\|PQ\|_{op} \geq 1 - 2\varepsilon$ then choose normalized $x \in \mathrm{Ran}(P)$, $y \in \mathrm{Ran}(Q)$ such that $\|PQ\|_{op} = \langle x|y\rangle$.
  2. Set $P' = P - |x\rangle\langle x|$, $Q' = Q - |y\rangle\langle y|$.
  3. Observe $\|[P',Q']\|_{\mathcal{L}^2} \leq \|[P,Q]\|_{\mathcal{L}^2}$.
  4. Repeat. $\Rightarrow$ Get two 'almost parallel ONB's' of eigenvectors of P and Q. $\Rightarrow$ There is an $R = R^* = R^2$ with $[R,Q] = 0$ and $\|R - P\|_{\mathcal{L}^2} \leq 2\varepsilon$.
- Now let $0 < \lambda, \mu \leq 1$, $A = \lambda \cdot P$, $B = \mu \cdot Q$ and $\|[A,B]\|_{\mathcal{L}^2} = \lambda\mu\|[P,Q]\|_{\mathcal{L}^2} \leq \varepsilon$.

$\Rightarrow$ **A and B almost commute if they have almost the same eigenvectors or if their eigenvalues are almost degenerate.**

Unitary Diagonalyzing Flows

- Proposed by R. W. Brockett, 91' and F. Wegner, '91.
- Further developed by V. Bach, J.B. Bru, '10.

Theorem (V.B., J.B.B., '10: Brockett-Wegner-Flow)

Let $\mathcal{H}$ be a complex separable Hilbert space, $A = A^*, B = B^* \in \mathcal{L}^2(\mathcal{H})$ and $B > 0$, then

$$\dot{A}_t = [A_t, [A_t, B]], \quad A_{t=0} = A,$$

has a unique solution, $A_t \to A_\infty$ strongly, A_t is unitary equivalent to A for all $t \in [0, \infty]$ and $[A_\infty, B] = 0$.

- Problem: In general, A_∞ is not close to A because of two reasons:
  1. Only A is changing, not B.
  2. $\|A_t - B\|$ is monotonically decreasing.

Non-Unitary Flows

Theorem (1, JG 25'+)

Let $\mathcal{H}$ be a complex separable Hilbert space and $A, B \in \mathcal{L}^2(\mathcal{H})$ be two self-adjoint Hilbert-Schmidt operators. Then, the differential equation

$$\begin{aligned} \dot{A}_t &= [B_t, [A_t, B_t]], \quad A_0 = A, \\ \dot{B}_t &= [A_t, [B_t, A_t]], \quad B_0 = B, \end{aligned} \tag{2}$$

has a unique global solution $(A_t, B_t)_{t \geq 0}$ that satisfies

$$\lim_{t \to \infty} \sqrt{t} \cdot [A_t, B_t] = 0. \tag{3}$$

Moreover, if $\dim(\mathcal{H}) = N < \infty$, then the solution converges for $t \to \infty$ in Hilbert-Schmidt norm $(A_t, B_t) \to (A_\infty, B_\infty)$ and is close to its initial value, i.e.,

$$\|A - A_\infty\|_{\mathcal{L}^2}^2 + \|B - B_\infty\|_{\mathcal{L}^2}^2 \leq 4C(N)\|[A, B]\|_{\mathcal{L}^2} < 2N^3\|[A, B]\|_{\mathcal{L}^2}, \tag{4}$$

where $C(N) < N^3/2$ is the optimal constant of

$$\|[D, E]\|_{\mathcal{L}^2}^3 \leq C(N)(\|[D, [D, E]]\|_{\mathcal{L}^2}^2 + \|[E, [E, D]]\|_{\mathcal{L}^2}^2), \ \forall D = D^*, E = E^* \in \mathcal{L}^2(\mathcal{H}).$$

Strategy of Proof

- Local existence and uniqueness is clear.
- Norm of A_t and B_t is monotonically decreasing because of

$$\frac{\mathrm{d}}{\mathrm{d}t}\|A_t\|_{\mathcal{L}^2}^2 = 2\mathrm{Tr}(A_t\dot{A}_t) = 2\mathrm{Tr}(A_t[B_t,[A_t,B_t]])$$
$$= 2\mathrm{Tr}([A_t,B_t][A_t,B_t]) = -2\|[A_t,B_t]\|_{\mathcal{L}^2}^2 \leq 0.$$

- With $f_t := \|[A_t,B_t]\|_{\mathcal{L}^2}^2/2$, we have $\dot{f}_t = -\|\dot{A}_t\|_{\mathcal{L}^2}^2 - \|\dot{B}_t\|_{\mathcal{L}^2}^2 \leq 0$ and $\frac{\mathrm{d}}{\mathrm{d}t}\|A_t\|_{\mathcal{L}^2}^2 = -4f_t$ and hence $\|A\|^2 - \|A_T\|^2 = 4\int_0^T f_t\,\mathrm{d}t \leq \|A\|^2$ for all $T > 0$. $\Rightarrow \lim_{t\to\infty} t\cdot f_t = 0 \Rightarrow \dot{f}_t \leq C/t^2$.
- For convergence of A_t, B_t for $t \to \infty$ we need integrability of

$$\|A_t - A_s\| + \|B_t - B_s\| \leq \int_s^t (\|\dot{A}_\tau\| + \|\dot{B}_\tau\|)\,\mathrm{d}\tau \leq \int_s^t \sqrt{-2\dot{f}_\tau}\,\mathrm{d}\tau.$$

- Get improved decay estimates for f_t by iterated commutator estimates.

Improved Decay Estimates

Recall that for $f_t = \frac{\|[A_t, B_t]\|_{\mathcal{L}^2}^2}{2}$ we have
$\dot{f}_t = -\|[A_t, [A_t, B_t]]\|_{\mathcal{L}^2}^2 - \|[B_t, [B_t, A_t]]\|_{\mathcal{L}^2}$.

- Suppose there is $\gamma \geq 1$ and a constant $C < \infty$ such that for all self-adjoint A, B we have

$$\|[A, B]\|_{\mathcal{L}^2}^{2\gamma} \leq C\Big(\|[A, [A, B]]\|_{\mathcal{L}^2}^2 + \|[B, [B, A]]\|_{\mathcal{L}^2}^2\Big). \quad (5)$$

$\Rightarrow$ $\dot{f}_t \leq -\frac{f_t^\gamma}{C}$ so f_t would be a subsolution to $\dot{g}_t = -\frac{g_t^\gamma}{C}$.

$\Rightarrow$ $f_t \leq C' \exp[-t]$ for $\gamma = 1$ or $f_t \leq C'(1+t)^{-\frac{1}{\gamma-1}}$.

$\Rightarrow$ Sufficient decay if $\gamma \in [1, 2)$.

Remark: Sadly, $\gamma = 2$ is again the critical case. Here, it is easy to prove

$$\|[A, B]\|_{\mathcal{L}^2}^4 \leq \|B\|_{\mathcal{L}^2}^2 \|[A, [A, B]]\|_{\mathcal{L}^2}^2.$$

Section 2

Iterated Commutator Estimates

General Observations

Consider inequalities of the form

$$\|[A,B]\|_{\mathcal{L}^2}^{2\gamma} \leq C(N)\Big(\|[A,[A,B]]\|_{\mathcal{L}^2}^2 + \|[B,[B,A]]\|_{\mathcal{L}^2}^2\Big), \tag{6}$$

for a fixed constant $C(N)$ and all $A, B \in \mathbb{C}^{N\times N}$.

- In general, $\gamma = \frac{3}{2}$ is necessary, even for A, B self-adjoint.
- For all $N \geq 2$ and $C(N) < \infty$ there are diagonalizable $A, B \in \mathbb{C}^{N\times N}$ that violate (6).
- There are self-adjoint $A, B \in \mathbb{C}^{N\times N}$ saturating (6) with $C(N) = (N/4)^{3/2}$. These matrices also saturate (6) if $\|\cdot\|_{\mathcal{L}^2}$ is changed to $\|\cdot\|_{op}$ and $C(N) \sim \sqrt{N}$.

Existence Results

Theorem (2, JG 25'+)

Let $N \in \mathbb{N}$, then there is a constant $C(N) < \frac{N^3}{2}$ such that for all self-adjoint Hilbert-Schmidt operators $A, B \in \mathcal{L}^2(\mathbb{C}^N)$ one has

$$\|[A, B]\|_{\mathcal{L}^2}^3 \le C(N) \cdot (\|[A, [A, B]]\|_{\mathcal{L}^2}^2 + \|[B, [A, B]]\|_{\mathcal{L}^2}^2). \quad (7)$$

- If $A = A^*$ or $B = B^*$ has just two eigenvalues, i.e., $\sigma(A) = \{a_1, a_2\}$ then $C(N) \le \sqrt{N}/4$.
- With spectral gap $\Delta := \min\{|a - a'| : a \neq a' \in \sigma(A)\}$, one has $\|[A, B]\|_{\mathcal{L}^2}^2 \le \frac{\|[A,[A,B]]\|_{\mathcal{L}^2}^2}{\Delta^2}$. $\Rightarrow$ In this case $\dot{B}_t = [A, [B_t, A]]$ has a unique solution, $B_t \to B_\infty$ exponentially and $\|B_\infty - B\|_{\mathcal{L}^2} \le \frac{\|[A,B]\|_{\mathcal{L}^2}}{\Delta}$.

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