Walkshop in Mathematical Physics
Constructor University of Bremen

UV Renormalization of Spin-Boson models with normal and 2-nilpotent interactions

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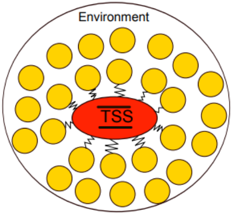
PADERBORN UNIVERSITY

1. Introduction: The Model



- Hilbert space of the (spin) system $\longrightarrow \mathcal{H}_s$.
- Hilbert space of the bosons $\longrightarrow \mathcal{F}_b(L^2(\mathbb{R}^d))$.

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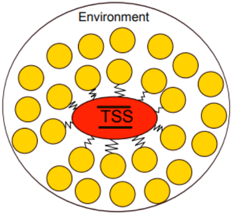

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- Hilbert space of the bosons $\longrightarrow \mathcal{F}_b(L^2(\mathbb{R}^d))$.

Description of the interaction: $\mathcal{H}_s \otimes \mathcal{F}_b(L^2(\mathbb{R}^d))$

$$H_{SB} = \underbrace{S \otimes \mathrm{Id} + \mathrm{Id} \otimes d\Gamma(\omega)}_{Free\ \ Energy} + \underbrace{B \otimes a(v)^* + B^* \otimes a(v)}_{Interaction\ \ Term}$$

- $\omega : \mathbb{R}^d \longrightarrow \mathbb{R}_{\geq 0}$ (Dispersion relation).
- $v : \mathbb{R}^d \longrightarrow \mathbb{C}$ (Form factor).
- $B \in \mathcal{B}(\mathcal{H}_s)$.

Image source: https://www.uni-ulm.de/fileadmin/website_uni_ulm/nawi.inst.260/Ãœbungen/6Spin_Boson_Modell.pdf

1. Introduction: Important Examples

- **Van-Hove model**. $S = 0$, $\mathcal{H}_s = \mathbb{C}$

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- **Van-Hove model**. $S = 0$, $\mathcal{H}_s = \mathbb{C}$

$$H_{VH} = d\Gamma(\omega) + a(v) + a(v)^*$$

- **Standard Spin-Boson model**. $\mathcal{H}_s = \mathbb{C}^2$, $S = \sigma_z$, $B = \sigma_x$

$$H_{SSB} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathrm{Id} + \mathrm{Id} \otimes d\Gamma(\omega) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes a(v) + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes a(v)^*$$

- **Rotating Wave Approximation**. $\mathcal{H}_s = \mathbb{C}^2$, $S = \sigma_z$, $B = \sigma_-$

$$H_{RWA} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \mathrm{Id} + \mathrm{Id} \otimes d\Gamma(\omega) + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes a(v) + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes a(v)^*$$

1. Introduction: Regularity of the model

- If $v \in L^2(\mathbb{R}^d)$:
  - H_{SB} is well defined.
  - If $\omega^{-1/2} v \in L^2(\mathbb{R}^d) \implies H_{SB}$ is self-adjoint + bounded from below (Kato-Rellich).

$$\inf \{\sigma(H_{VH})\} = -\int |v(q)|^2 \omega(q)^{-1} dq$$

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- If $v \notin L^2(\mathbb{R}^d)$:
  - $a(v)$ is not closable $\implies a(v)^*$ not densely defined.
  - If $\omega^{-1/2} v \in L^2(\mathbb{R}^d) \implies$ KLMN Theorem

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Questions:

- Can we say anything about the domain?
- If $\omega^{-s/2} v \in L^2(\mathbb{R}^d)$, renormalize the model? For which values of s?

$$s \longrightarrow \text{UV regularity of } v\,.$$

$s = 1$	$s \in (1, 2)$	$s = 2$	$s > 2$
Good	Not that bad	Bad	Very bad

1. Introduction: Physical relevance

- Decoherence and non-Markovianity in open quantum systems.
- Superradiance and subradiance phenomena
  - R.H.Dicke 1954,M.G.Benedict and A.Ermolaev n.d.,M.Gross and S.Haroche 1982,Loo et al. 2013.
- Quantum optics $\rightsquigarrow$ Waveguides in quantum electrodynamics. (T.Tufarelli, F.Ciccarello, and M.S.Kim 2013) (1-Dimensional photonic waveguide)

$$\omega(k) \sim \omega_0 + \omega_1 k \quad \text{and} \quad v(k) \sim \sqrt{\omega(k)}\sin(ck)$$

$$v(k)\omega(k)^{-\frac{1}{2}} \sim \sin(ck) \notin L^2(\mathbb{R})$$

- Quantum information and simulation

2. What do we know?

- **Van-Hove** (J.Derezínski 2003, L.Van-Hove 1952)
  - Renormalization via dressing transformation (Weyl transform)
  - Including case $\omega^{-1}v \in L^2$ ($s = 2$).
  - Motivation to treat normal interactions $\rightsquigarrow$ Decoupling Hamiltonian in Van-Hove Hamiltonians.
- **Standard Spin-Boson Model** (T.N.Dam and J.S.Møller 2020)
  - Renormalizable iff $\omega^{-1}v \in L^2$
  - Transformation using symmetries of the model.
- **Rotating Wave Approximation** (D.Lonigro 2022)
  - Renormalization for $\omega^{-s/2}v \in L^2$, $s \in [1, 2]$. If $s < 2 \rightsquigarrow$ Norm resolvent convergence. If $s = 2 \rightsquigarrow$ Strong resolvent convergence
  - Explicit domain
- **Generalized Spin-Boson Model** (S.Lill and D.Lonigro 2023)

$$H_{INT} = \sum_{j=1}^{N} (B_j^* \otimes a(v_j) + B_j \otimes a(v_j)^*)$$

  - B_j normal + B_j commuting + Algebraic assumptions. Case $v_j\omega^{-1/2} \in L^2$.
  - Explicit domain

3. Generalized Spin-Boson Model

- Hilbert space:

$$\mathcal{H}_s \otimes \mathcal{F}_b(L^2(\mathbb{R}^d)) \cong \mathcal{H}_s \oplus \bigoplus_{n\geq 1} L^2_{sym}((\mathbb{R}^d)^n; \mathcal{H}_s) =: \mathcal{H}$$

$$\Psi \in \mathcal{H} \rightsquigarrow \Psi = (\Psi^{(0)}, \Psi^{(1)}(k_1), \Psi^{(2)}(k_1, k_2), \ldots)$$

- Hamiltonian

$$\boxed{H_{GSB}(S, V) = S + d\Gamma(\omega) + a(V) + a(V)^* =: S + d\Gamma(\omega) + \varphi(V)}$$

- $S \in \mathcal{B}(\mathcal{H})$ and self-adjoint
- $V : \mathbb{R}^d \longrightarrow \mathcal{B}(\mathcal{H}_s)$

$$(a(V)\Psi)^{(n)}(k_1, \ldots, k_n) = \sqrt{n+1} \int V(q)^* \Psi^{(n+1)}(q, k_1, \ldots, k_n) dq$$

and

$$(a(V)^*\Psi)^{(n)}(k_1, \ldots, k_n) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} V(k_j) \Psi^{(n-1)}(k_1, \ldots, \hat{k}_j, \ldots, k_n)$$

Relation to the previous setting: $V(q) = v(q)B$.

Definition 1

Let $s \in \mathbb{R}$

$$\mathfrak{b}_s := \left\{ V : \mathbb{R}^d \longrightarrow \mathcal{B}(\mathcal{H}_s) \ : \ \int \|V(q)\|^2_{\mathcal{B}(\mathcal{H}_s)} \omega(q)^{-s} dq < \infty \right\} =$$

$$= L^2(\mathbb{R}^d; \mathcal{B}(\mathcal{H}_s); \omega(q)^{-s} dq) \, .$$

For $V_1, V_2 \in \mathfrak{b}_s$

$$\langle V_1, V_2 \rangle_{\mathfrak{b}_s} := \int V_1(q)^* V_2(q) \omega(q)^{-s} dq \in \mathcal{B}(\mathcal{H}_s)$$

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Lemma 1

Let $V \in \mathfrak{b}_0 \cap \mathfrak{b}_s$ *with* $s \geq 1$*, then*

$$\|\varphi(V)\Psi\| \leq 2(\|V\|_{\mathfrak{b}_s} + \|V\|_{\mathfrak{b}_0}) \|(d\Gamma(\omega) + 1)^{s/2} \Psi\|_{\mathcal{H}}$$

- If $V \in \mathfrak{b}_0 \cap \mathfrak{b}_s$, $s \in [1,2)$ or $V \in \mathfrak{b}_0 \cap \mathfrak{b}_2$ and $\|V\|_{\mathfrak{b}_s} + \|V\|_{\mathfrak{b}_0} < \frac{1}{2}$, then $H_{GSB}(V) : \mathscr{D}(d\Gamma(\omega)) \longrightarrow \mathcal{H}$ is a self-adjoint operator (Kato-Rellich).

- **Goal:** Given $V \in \mathfrak{b}_s$, $V \notin \mathfrak{b}_0$, can we find a sequence $(V_n) \subset \mathfrak{b}_s \cap \mathfrak{b}_0$, $V_n \xrightarrow{\mathfrak{b}_s} V$ such that

$$H_{GSB}(V_n) - E(V_n) \xrightarrow{r.s} H_{ren.}(V)?$$

$E(V_n) \in \mathcal{B}(\mathcal{H}_s)$ self-adjoint operator that plays the role of a renormalization energy.

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- Distinguish between the UV and the IR regimes:

$$V_{\leq} := V\chi_{\{\omega \leq \kappa\}} \qquad \text{and} \qquad V_{>} := V\chi_{\{\omega > \kappa\}}$$

$$\ldots \subset \mathfrak{b}_1^{\leq} \subset \mathfrak{b}_0^{\leq} \subset \ldots \quad \text{and} \quad \ldots \subset \mathfrak{b}_0^{>} \subset \mathfrak{b}_1^{>} \subset \mathfrak{b}_2^{>} \subset \ldots$$

Theorem 1 (B.Hinrichs, J. Lampart, JVM)

Let $S \in \mathcal{B}(\mathcal{H})$ be symmetric and $V = V_{\leq} + V_D + V_N : \mathbb{R}^d \longrightarrow \mathcal{B}(\mathcal{H})$ with $V_{\leq} \in \mathfrak{b}_1$, $V_D \in \mathfrak{b}_2$ and $V_N \in \mathfrak{b}_{s_N}$ for some $s_N \in [1,2]$, such that

i. $[V_\sharp, V_\star] = 0\,,\ \sharp, \star \in \{D, N\}$.
ii. $V_D(q)$ *is* ***normal****, i.e.* $[V_D, V_D^*](p,q) = V_D(p)V_D(q)^* - V_D(q)^* V_D(p) = 0$.
iii. $V_N(k)V_N(p) = 0$ *(****2-nilpotency****)*.

If $s_N \in [1,2)$ or if $\kappa > 0$ is large enough such that $\|V_N\|_{\mathfrak{b}_2} < \frac{1}{2}$ then there exists $H(S,V)$ a ***self-adjoint and lower semibounded operator*** *such that*

$$H_{GSB}(S, V_n) + \langle V_{n,>}, V_{n,>}\rangle_{\mathfrak{b}_1} \overset{s.r.s}{\longrightarrow} H(S,V)$$

for every $V_n = V_{n,\leq} + V_{n,D} + V_{n,N}$ with $V_{n,\star} \in \mathfrak{b}_0 \cap \mathfrak{b}_1$ ($\star \in \{\leq, N, D\}$) as above, with

$$V_{n,\leq} \overset{\mathfrak{b}_1}{\longrightarrow} V_{\leq}\,, \quad V_{n,\sharp} \overset{\mathfrak{b}_2}{\longrightarrow} V_\sharp\ , \sharp \in \{N, D\}\,.$$

The convergence is in ***norm resolvent sense*** *if either $V_{n,D} = 0$ or $s_N < 2$ and $V_{n,N} \overset{\mathfrak{b}_{s_N}}{\longrightarrow} V_N$.*

4. Main result (in a nutshell)

- Convergence summary table

Infrared ($V_{\leq}$)	Normal (V_D)	2-Nilpotent (V_N)	Convergence
$\mathfrak{b}_1$	✗	$s_N \leq 2$	*Norm resolvent*
$\mathfrak{b}_1$	$\mathfrak{b}_2$	$s_N < 2$	*Norm resolvent*
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- **Renormalization energies**

$$-\langle V, V \rangle_{\mathfrak{b}_1} = -\int V(q)^* V(q) \omega(q)^{-1} dq$$

Van-Hove	$-\int \lvert v(q)\rvert^2 \omega(q)^{-1} dq$
RWA	$-\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \int \lvert v(q)\rvert^2 \omega(q)^{-1} dq$
SSB	$-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \int \lvert v(q)\rvert^2 \omega(q)^{-1} dq$
GSB	$-\sum_{i,j=1}^{n} B_i^* B_j \int \overline{v_j(q)} v_i(q) \omega(q)^{-1} dq$

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$$H_{GSB}(S, V) = S + d\Gamma(\omega) + \varphi(V_N) + \varphi(V_\leq) + \varphi(V_D)$$

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- Step I. Nilpotent interaction: V_N
  - Interior Boundary Conditions method (IBC).
- Step II. Infrared interaction: $V_{\leq}$
  - Infinitesimal perturbation of the previous step.
- Step III. Normal interacction: V_D
  - Weyl transformations.

5. The 2-Nilpotent Interaction: Interior Boundary Conditions (IBC)

Goal: Rewrite the Hamiltonian to extract the divergent part.

- Introduce $\lambda > 0$ (we write $H_\lambda := d\Gamma(\omega) + \lambda$) and $T_\lambda : \mathscr{D}(T_\lambda) \longrightarrow \mathcal{H}$
  i. T_λ is H_λ-bounded with relative bound less than 1.
  ii. $T_\lambda + H_\lambda$ is invertible.
- Define $G_{V,\lambda} := a(V)(H_\lambda + T_\lambda)^{-1}$
- An algebraic computation gives:

$$H_{GSB}(0,V) = \underbrace{(1 + G_{V,\lambda})(H_\lambda + T_\lambda)(1 + G^*_{V,\lambda})}_{=:H_{IBC} \;\; (Self-adjoint)} \underbrace{-a(V)(H_\lambda + T_\lambda)^{-1}a(V)^* - T_\lambda}_{=E? \;\; , E \in \mathcal{B}(\mathcal{H})} - \lambda$$

- Self-Renormalization Energy $E \in \mathcal{B}(\mathcal{H})$ + First term in Neumann Series:

$$T_\lambda + E := -a(V)H_\lambda^{-1}a(V)^*$$

- We get

$$H_{GSB}(0,V) - E = H_{IBC} - \lambda + \underbrace{a(V)(H_\lambda^{-1} + (H_\lambda + T_\lambda)^{-1})a(V)^*}_{Error \;\; Term}$$

5. The 2-Nilpotent Interaction: The operator T_λ

$$((T_\lambda + E)\Psi)^{(n)}(k_1, ..., k_n) =$$
$$-\sum_{j=1}^{n} \int V(q)^* V(k_j) \frac{1}{\omega(q) + \sum_{i=1}^{n} \omega(k_i) + \lambda} \Psi^{(n)}(q, \hat{k}_j) dq -$$
$$-\int V(q)^* V(q) \frac{1}{\omega(q) + \sum_{i=1}^{n} \omega(k_i) + \lambda} dq \Psi^{(n)}(k_1, ..., k_n)$$

- Define E to remove the divergent contribution of the second term

$$E = E(V) := -\int V(q)^* V(q) \omega(q)^{-1} dq = -\langle V, V \rangle_{\mathfrak{h}_1}$$

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Proposition 1

Let $V_1, V_2 \in \mathfrak{b}_s$ with $s \in [1, 2]$, then $\mathscr{D}(H_\lambda^{s-1}) \subset \mathscr{D}(T_\lambda)$ and for every $\Psi \in \mathscr{D}(H_\lambda^{s-1})$.

$$\|(T_\lambda(V_1) - T_\lambda(V_2))\Psi\| \leq 2(\|V_1\|_{\mathfrak{b}_s} + \|V_2\|_{\mathfrak{b}_s})\|V_1 - V_2\|_{\mathfrak{b}_s} \|H_\lambda^{s-1}\Psi\|$$

5. The 2-Nilpotent Interaction: The H_{IBC} Hamiltonian

For $V_N \in \mathfrak{d}_0 \cap \mathfrak{d}_1$, such that $V_N(q)V_N(p) = 0$, define

$$G_{\lambda,V_N} := a(V_N)H_\lambda^{-1}$$

and

$$H_{IBC}(V_N) := (1 + G_{\lambda,V_N})(H_\lambda + T_\lambda(V_N))(1 + G^*_{\lambda,V_N}) .$$

Proposition 2

Let $V_N \in \mathfrak{d}_0 \cap \mathfrak{d}_1$ with $V_N(q)V_N(p) = 0$, then

$$H_{IBC}(V_N) = H_{GSB}(0, V_N) + \langle V_N, V_N \rangle_{\mathfrak{d}_1} - \lambda$$

Proposition 3

Let $s \in [1, 2]$ *and* $F \in \mathfrak{b}_s$, *then*

$$\|G_{\lambda,F}\Psi\| \leq \|F\|_{\mathfrak{b}_s} \|H_\lambda^{\frac{s-2}{2}}\Psi\| \leq \|F\|_{\mathfrak{b}_s} \lambda^{\frac{s-2}{2}} \|\Psi\| \, .$$

In particual, $G_{\lambda,F} \in \mathcal{B}(\mathcal{H})$ *and if* $F_n \xrightarrow{\mathfrak{b}_s} F$ *then* $G_{\lambda,F_n} \longrightarrow G_{\lambda,F}$ *in* $\mathcal{B}(\mathcal{H})$.

Lemma 2

Let $s \in [1, 2]$ *and* $F \in \mathfrak{b}_s$. *If* $F(q)F(r) = 0$ ***(2-nilpotency)***,

$$(1 + G_{\lambda,F})^{-1} = 1 - G_{\lambda,F} \quad \text{and} \quad (1 + G^*_{\lambda,F})^{-1} = 1 - G^*_{\lambda,F} \, .$$

This is, $1 + G_{\lambda,F}$ *is invertible and has bounded inverse.*

Proposition 4

Let $s \in [1,2]$ and $V_N \in \mathfrak{b}_s$ 2-Nilpotent (and $\|V_N\|_{\mathfrak{b}_2} < \frac{1}{2}$ if $\|V_N\|_{\mathfrak{b}_s} = +\infty$ for $s < 2$) then

$$H_{IBC}(V_N) := (1 + G_{\lambda,V_N})(H_\lambda + T_\lambda(V_N))(1 + G^*_{\lambda,V_N})$$

with

$$\mathscr{D}(H_{IBC}(V_N)) = (1 + G^*_{\lambda,V_N})^{-1}\mathscr{D}(d\Gamma(\omega)) \quad (IBC)$$

is self-adjoint and bounded from below. If $V_{N,n} \xrightarrow{\mathfrak{b}_s} V_N$ then

$$H_{IBC}(V_{N,n}) \xrightarrow{n.r.s} H_{IBC}(V_N)\,.$$

- $H_\lambda + T_\lambda$ self-adjoint (Kato-Rellich) + Isomorphisms Sandwich $\implies$ Self-adjointness.
- Continuity statements for $T_\lambda(V)$ and $G_{\lambda,V}$ $\implies$ Norm resolvent convergence.

Sketch proof of convergence:

$$\frac{1}{H_{IBC}(V_{N,n})+i} - \frac{1}{H_{IBC}(V_N)+i} =$$

$$= \underbrace{\frac{1}{(1+G_{\lambda,V_N})(H_\lambda + T_\lambda(V_{N,n}))(1+G_{\lambda,V_N})+i} - \frac{1}{H_{IBC}(V_N)+i}}_{(*)} +$$

$$+ \frac{1}{H_{IBC}(V_{N,n})+i} - \frac{1}{(1+G_{\lambda,V_N})(H_\lambda + T_\lambda(V_{N,n}))(1+G_{\lambda,V_N})+i}\,.$$

Apply Resolvent Identity

$$(*) = \underbrace{\frac{1}{(1+G_{\lambda,V_N})(H_\lambda + T_\lambda(V_N))(1+G_{\lambda,V_N})+i}(1+G_{\lambda,V_N})}_{(Self-adjointness)\Rightarrow \|\cdot\|\leq 2} \times$$

$$\times \underbrace{(T_\lambda(V_N) - T_\lambda(V_{N,n}))(d\Gamma(\omega)+1)^{-1}}_{(\text{Estimates on } T_\lambda)\Rightarrow \|\cdot\|\longrightarrow 0} \times$$

$$\times \underbrace{(d\Gamma(\omega)+1)(1+G^*_{\lambda,V_N})\frac{1}{H_{IBC}(V_N)+i}}_{\text{Closed Graph Theorem}\Rightarrow \|\cdot\|\leq C} \longrightarrow 0\,.$$

6. The infrared interaction

Proposition 5

Let $V_{\leq} \in \mathfrak{b}_1$ and $V_N \in \mathfrak{b}_2^{>}$ (V_N 2-nilpotent), then $\varphi(V_{\leq})$ is infinitesimally bounded with respect to $H_{IBC}(V_N)$.

Sketch of the Proof:

i. $\|d\Gamma(\omega_{\leq})\Psi\| \leq (1 + \|G_{\lambda,V_N}\|)^2 \left(1 + \left\|T_\lambda(V_N)H_\lambda^{-1}\right\|\right) \|H_{IBC}(V_N)\Psi\|$

ii. $\|\varphi(V_{\leq})\Psi\| \leq 2(\|V_{\leq}\|_{\mathfrak{b}_0} + \|V_{\leq}\|_{\mathfrak{b}_1}) \left\|(d\Gamma(\omega_{\leq}) + 1)^{1/2}\Psi\right\|$

Corollary 1

Let V_N and $V_{\leq}$ as above, then

$$H_{IBC}(V_N) + \varphi(V_{\leq}) : \mathscr{D}(H_{IBC}) \longrightarrow \mathcal{H}$$

is self-adjoint. Furthermore, for $(V_{N,n}), (V_{\leq,n}) \subset \mathfrak{b}_0 \cap \mathfrak{b}_1$ with $V_{N,n} \xrightarrow{\mathfrak{b}_2} V_N$ and $V_{n,\leq} \xrightarrow{\mathfrak{b}_1} V_{\leq}$ then

$$H_{IBC}(V_{N,n}) + \varphi(V_{n,\leq}) \xrightarrow{n.r.s} H_{IBC}(V_N) + \varphi(V_{\leq})$$

7. The Normal Interaction: Motivation with Van-Hove Model

- **Van-Hove Model:**

$$H_{VH} = d\Gamma(\omega) + a(v) + a(v)^*$$

- **Spin-Boson** (with $\mathcal{H}_s = \mathbb{C}^D$)

$$H_{SB} = S \otimes \mathrm{Id} + \mathrm{Id} \otimes d\Gamma(\omega) + B^* \otimes a(v) + B \otimes a(v)^*$$

7. The Normal Interaction: Motivation with Van-Hove Model

- **Van-Hove Model:**

$$H_{VH} = d\Gamma(\omega) + a(v) + a(v)^*$$

- **Spin-Boson** (with $\mathcal{H}_s = \mathbb{C}^D$)

$$H_{SB} = S \otimes \mathrm{Id} + \mathrm{Id} \otimes d\Gamma(\omega) + B^* \otimes a(v) + B \otimes a(v)^*$$

- B Normal $\implies$ $UBU^* = \mathrm{diag}(\lambda_1, ..., \lambda_D)$

$$UH_{SB}U^* = USU^* \otimes \mathrm{Id} + \\ + \begin{pmatrix} d\Gamma(\omega) + a(\lambda_1 v) + a(\lambda_1 v)^* & ... & 0 \\ ... & ... & ... \\ 0 & ... & d\Gamma(\omega) + a(\lambda_D v) + a(\lambda_D v) \end{pmatrix}$$

7. The Normal Interaction: The Weyl Transform

Lemma 3

Let $F \in \mathfrak{b}_0$, then the operator $\varphi(F)$ is essentially self-adjoint. We denote its clousure by $\Phi(F)$.

Definition 2 (Weyl Transform)

Let $F \in \mathfrak{b}_0$,

$$W(F) = e^{i\Phi(iF)},$$

which is a unitary operator.

Idea: For Van-Hove Model:

$$W(\omega^{-1}v)d\Gamma(\omega)W(\omega^{-1}v)^* = d\Gamma(\omega) + a(v) + a(v)^* + \int |v(q)|^2\omega(q)^{-1}dq$$

$\longrightarrow$ Replace v by an operator valued function.

7. The Normal Interaction: The Renormalized Hamiltonian

Definition 3 (Renormalized Hamiltonian)

Let $V = V_{\leq} + V_N + V_D$ (as in the main theorem) we define the ***self-adjoint*** *and* ***lower semibounded*** *operator*

$$H(S, V) := S + W(\omega^{-1} V_D)(H_{IBC}(V_N) + \varphi(V_{\leq}))W(\omega^{-1} V_D)^* - \lambda$$

with

$$\mathscr{D}(H(S, V)) = W(\omega^{-1} V_D)\mathscr{D}(H_{IBC}(V_N)) = W(\omega^{-1} V_D)(1 - G_{\lambda, V_N})\mathscr{D}(d\Gamma(\omega))\,.$$

Question: How does this Hamiltonian relates to the original one?

Lemma 4 (Algebraic properties of $W(F)$)

i. *Let* $F, G \in \mathfrak{b}_0$ *with* $[F, F] = [F, G] = [F, G^*] = [F, F*] = 0$, *then*

$$W(F)\Phi(G)W(F)^* = \Phi(G) + \langle F, G\rangle_{\mathfrak{b}_0} + \langle F, G\rangle_{\mathfrak{b}_0}$$

ii. *If* $F \in \mathfrak{b}_{-2} \cap \mathfrak{b}_0$

$$W(F)d\Gamma(\omega)W(F)^* = d\Gamma(\omega) + \varphi(\omega F) + \langle F, F\rangle_{\mathfrak{b}_{-1}} .$$

Proposition 6

Let $V = V_{\leq} + V_N + V_D$ *with* $V_\sharp \in \mathfrak{b}_0$ *for* $\sharp \in \{\leq, N, D\}$, *such that,* $[V_N, V_D] = [V_N, V_D^*] = 0$, *then*

$$H(S, V) = H_{GSB}(S, V) + \langle V_>, V_>\rangle_{\mathfrak{b}_1} .$$

Proof:

$$\begin{aligned} &H(S, V) = \\ &= S + W(\omega^{-1}V_D)(H_{GSB}(0, V_N) + \varphi(V_{\leq}) + \lambda + \langle V_N, V_N\rangle_{\mathfrak{b}_1})W(\omega^{-1}V_D)^* - \lambda = \\ &= S + W(\omega^{-1}V_D)(H_{GSB}(0, V_{\leq} + V_N) + \lambda)W(\omega^{-1}V_D)^* - \lambda + \langle V_N, V_N\rangle_{\mathfrak{b}_1} = \\ &= H_{GSB}(S, V) + \langle \omega^{-1}V_D, \omega^{-1}V_D\rangle_{\mathfrak{b}_{-1}} + \langle V_N, \omega^{-1}V_D\rangle_{\mathfrak{b}_0} + \langle \omega^{-1}V_N, V_D\rangle_{\mathfrak{b}_0} + \langle V_N, V_N\rangle_{\mathfrak{b}_1} \end{aligned}$$

7. The Normal Interaction: Convergence to $H(S,V)$

Generalization of result already used in Griesemer and Wünsch 2018 and O.Matte and J.S.Møller 2018

Lemma 5

Let $F, G \in \mathfrak{b}_0$ *with* $[F,F] = [F,G] = [F,G^*] = [F,F^*] = 0$ *and* $\psi \in \mathscr{D}(\Phi(F)) \cap \mathscr{D}(\Phi(G))$

$$\|(W(F) - W(G))\psi\| \leq \|\varphi(F-G)\psi\| + \frac{1}{2}\left\|(\langle F, F-G\rangle_{\mathfrak{b}_0} + \langle F-G, F\rangle_{\mathfrak{b}_0})\psi\right\|$$

Further, if $F, G \in \mathfrak{b}_0 \cap \mathfrak{b}_1$*, for all* $\theta \in [0,1]$

$$\left\|(W(F) - W(G))(1 + d\Gamma(\omega))^{-\theta/2}\right\|$$

$$\leq 2^{1-\theta}\left(4(\|F-G\|_{\mathfrak{b}_0} \vee \|F-G\|_{\mathfrak{b}_1}) + \frac{1}{2}\left\|\langle F, F-G\rangle_{\mathfrak{b}_0} + \langle F-G, F\rangle_{\mathfrak{b}_0}\right\|_{\mathcal{B}(\mathcal{H}_s)}\right)^{\theta}$$

- First inequality $\longrightarrow$ Strong convergence
- Second inequality (see S.G.Krein and Y.I.Petunin 1966)$\longrightarrow$ Norm convergence after regularizing with resolvents.

Theorem 2 (Convergence result)

Let $V = V_{\leq} + V_D + V_N$, $V_{\leq} \in \mathfrak{b}_1$, $V_D \in \mathfrak{b}_2$ *and* $V_N \in \mathfrak{b}_{s_N}$ *for some* $s_N \in [1,2]$ *satisfiying algebraic assumptions.*
For every $V_n := V_{\leq,n} + V_{N,n} + V_{D,n} \in \mathfrak{b}_0 \cap \mathfrak{b}_1$ *(+ algebraic assumptions) such that*

$$V_{n,\leq} \xrightarrow{\mathfrak{b}_1} V_{\leq}\,, \quad V_{n,\sharp} \xrightarrow{\mathfrak{b}_2} V_{\sharp}\ , \sharp \in \{N, D\}\,.$$

$$H_{GSB}(S, V_n) + \langle V_>, V_> \rangle_{\mathfrak{b}_1} \xrightarrow{s.r.s} H(S, V)\,.$$

Furthermore, if $V_D = 0$ *or* $s_N < 2$ *the convergence is in norm resolvent sense.*

Sketch of the proof:

- The operator S does not play any role. We can take $S = 0$
- Strong continuity of Weyl + Norm resolvent convergence IBC $\Longrightarrow$ Strong resolvent convergence.

- Norm resolvent convergence?

$$\frac{1}{H(0,V_n)+i}-\frac{1}{H(0,V)+i}=$$

$$= W(\omega^{-1}V_{D,n})^* \left(\frac{1}{H_{IBC}(V_{N,n})+\varphi(V_{\leq,n})+i} - \frac{1}{H_{IBC}(V_N)+\varphi(V_{\leq})+i} \right) W(\omega^{-1}V_D)+$$

$$+ \underbrace{W(\omega^{-1}V_{D,n})^* \frac{1}{H_{IBC}(V_{N,n})+\varphi(V_{\leq,n})+i} \left(W(\omega^{-1}V_{D,n}) - W(\omega^{-1}V_D) \right)}_{(\star)} +$$

$$+ \left(W(\omega^{-1}V_{D,n}) - W(\omega^{-1}V_D) \right)^* \frac{1}{H_{IBC}(V_N)+\varphi(V_{\leq})+i} W(\omega^{-1}F_{n,D})$$

  - Norm resolvent convergence IBC $\Longrightarrow$ First term $\longrightarrow 0$
  - Second and third term can be treated in the same way.

$$(\star)^* = \underbrace{\left(W(\omega^{-1}V_{D,n}) - W(\omega^{-1}V_D) \right)^* (d\Gamma(\omega)+1)^{\frac{s_N}{2}-1}}_{\text{Regularized continuity Weyl Transform} \Rightarrow \|\cdot\| \longrightarrow 0} \times$$

$$\times \underbrace{(d\Gamma(\omega)+1)^{1-\frac{s_N}{2}} \frac{1}{H_{IBC}(V_{N,n})+\varphi(V_{\leq,n})-i}}_{\text{Can be seen to be bounded.}} W(\omega^{-1}V_{D,n})$$

□

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Thanks!
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