Spectral comparison results, now and then: An overview

based on joint work with J. Kerner (and C. Rose)

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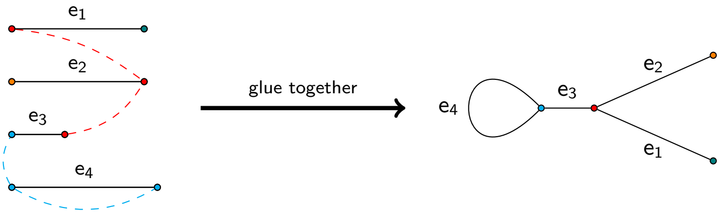
Setup – Metric graphs

Consider: Finite family $([0, \ell_e])_{e \in E}$ of intervals with $\ell_e \in (0, \infty)$, $e \in E$.

- A (compact finite) *metric graph* $\mathcal{G}$ is the metric measure space after gluing the endpoints of the intervals in a graph-like way:



- E is called *edge set*, V (= identified points on rhs) is called *vertex set* of $\mathcal{G}$.

Why metric? Define $d(x, y) := \inf_{p \text{ path from } x \text{ to } y} \ell(p)$, $x, y \in \mathcal{G}$ *path metric*

If $\ell_{e_1} < \ell_{e_2} + \ell_{e_3}$, then $d(x, y) = d(x, v) + \ell_{e_1} + d(w, y)$

If $\ell_{e_1} \geq \ell_{e_2} + \ell_{e_3}$, then $d(x, y) = d(x, v) + \ell_{e_2} + \ell_{e_3} + d(w, y)$

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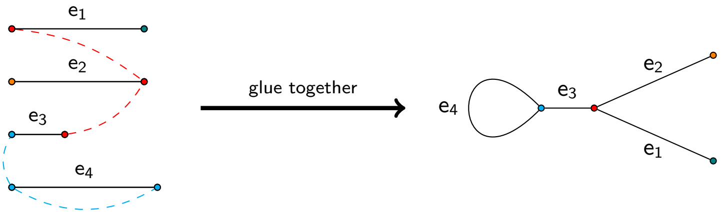

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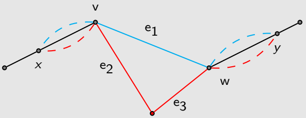

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Setup – Relevant spaces

- Each function $f \in L^2(\mathcal{G})$ can be uniquely identified with a function

$$(f_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}} \in \bigoplus_{\mathrm{e}\in\mathrm{E}} L^2(0,\ell_{\mathrm{e}}), \qquad \|f\|^2_{L^2(\mathcal{G})} = \sum_{\mathrm{e}\in\mathrm{E}} \|f_{\mathrm{e}}\|^2_{L^2(0,\ell_{\mathrm{e}})}$$

  where f_{e} denotes the restriction of f to $\mathrm{e} \simeq [0,\ell_{\mathrm{e}}]$.
- And the subspace $H^1(\mathcal{G})$ consisting of functions

$$f = (f_{\mathrm{e}})_{\mathrm{e}\in\mathrm{E}} \in \bigoplus_{\mathrm{e}\in\mathrm{E}} H^1(0,\ell_{\mathrm{e}})$$

  that are *continuous in the vertices* $\mathrm{v} \in \mathrm{V}$, i.e.,

$$f_{\mathrm{e}}(\mathrm{v}) = f_{\mathrm{f}}(\mathrm{v}) \quad \text{for all } \mathrm{E} \ni \mathrm{e},\mathrm{f} \sim \mathrm{v}.$$

- **Sometimes:** Impose additional *Dirichlet vertex conditions* at $\mathrm{V_D} \subset \mathrm{V}$, i.e.,

$$f(\mathrm{v}) = 0 \quad \text{for all } \mathrm{v} \in \mathrm{V_D}.$$

Setup – Schrödinger operators

Fix: *potential* $q \in L^\infty(\mathcal{G})$ and *parameters* $\sigma = (\sigma_{\mathsf{v}})_{\mathsf{v}\in\mathsf{V}} \in \mathbb{R}^{|\mathsf{V}|}$.

- We study the *Schrödinger operator* H_σ^q on $L^2(\mathcal{G})$ ass. with quadratic form a_σ^q:

$$a_\sigma^q(f) := \sum_{\mathsf{e}\in\mathsf{E}} \int_0^{\ell_{\mathsf{e}}} \left(|f_{\mathsf{e}}'(x)|^2 + q_{\mathsf{e}}(x)|f_{\mathsf{e}}(x)|^2\right) \mathrm{d}x + \sum_{\mathsf{v}\in\mathsf{V}} \sigma_{\mathsf{v}}|f(\mathsf{v})|^2, \quad f \in H^1(\mathcal{G}).$$

  - **Later**: If $\mathsf{v} \in \mathsf{V}_{\mathsf{D}} \neq \emptyset$, i.e. $f(\mathsf{v}) = 0$, we write "$\sigma_{\mathsf{v}} = \infty$".

- **In other words:** H_σ^q is the operator on $L^2(\mathcal{G})$ given by

$$H_\sigma^q f = (-f_{\mathsf{e}}'' + q_{\mathsf{e}} f_{\mathsf{e}})_{\mathsf{e}\in\mathsf{E}} \quad \text{for } f \in D(H_\sigma^q) \subset L^2(\mathcal{G})$$

  with operator domain given by

$$D(H_\sigma^q) := \left\{ f \in H^1(\mathcal{G}) \;\middle|\; \begin{array}{l} f_{\mathsf{e}}' \in H^1(0,\ell_{\mathsf{e}}) \text{ for all } \mathsf{e} \in \mathsf{E}, \\ \sum_{\mathsf{e}\sim\mathsf{v}} \partial_{\mathsf{e}} f(\mathsf{v}) = \sigma_{\mathsf{v}} f(\mathsf{v}) \text{ for all } \mathsf{v} \in \mathsf{V} \end{array} \right\}.$$

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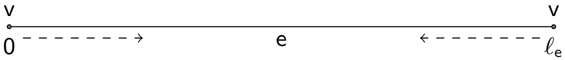
Setup – Schrödinger operators

- The condition

$$\sum_{\mathsf{e}\sim\mathsf{v}} \partial_{\mathsf{e}} f(\mathsf{v}) = \sigma_{\mathsf{v}} f(\mathsf{v}) \quad \text{for } \mathsf{v} \in \mathsf{V}$$

  is known as *δ-type vertex condition*.
  - or *Kirchoff vertex condition* if $\sigma_{\mathsf{v}} = 0$ (+ continuity at v *standard cond.*)
  - $\partial_{\mathsf{e}} f(\mathsf{v})$ is the *inward derivative* of f in $\mathsf{v} \in \mathsf{V}$ along $\mathsf{e} \sim \mathsf{v}$:

$$\partial_{\mathsf{e}} f(\mathsf{v}) := \begin{cases} f_{\mathsf{e}}'(0), & \text{if } \mathsf{v} \sim (\mathsf{e}, 0), \\ -f_{\mathsf{e}}'(\ell_{\mathsf{e}}), & \text{if } \mathsf{v} \sim (\mathsf{e}, \ell_{\mathsf{e}}). \end{cases}$$



- $(H^q_\sigma, D(H^q_\sigma))$ self-adjoint with *pure point spectrum*:

$$\lambda_1^q(\sigma) < \lambda_2^q(\sigma) \leq \lambda_3^q(\sigma) \leq \cdots \leq \lambda_n^q(\sigma) \leq \cdots \to +\infty.$$

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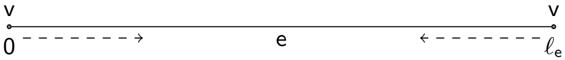

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Robin-Neumann gaps

Definition

For $n \in \mathbb{N} := \{1, 2, \dots\}$ we study the *Robin-Neumann gaps*

$$d_n^q(\sigma) := \lambda_n^q(\sigma) - \lambda_n^q(0), \qquad \hat{d}_n^q(\sigma) := \lambda_n^q(\sigma) - \lambda_n^0(0).$$

- $(|d_n^q(\sigma)|)_{n\in\mathbb{N}}$ and $(|\hat{d}_n^q(\sigma)|)_{n\in\mathbb{N}}$ are bounded (Kurasov, Suhr 2018).
- **But:** Not bounded away from 0 (**e.g.** a *2-star graph*).

Hence: $d_{2n}^0(\sigma) = \lambda_{2n}^0(\sigma) - \lambda_{2n}^0(0) = 0$ for all $n \in \mathbb{N}$.

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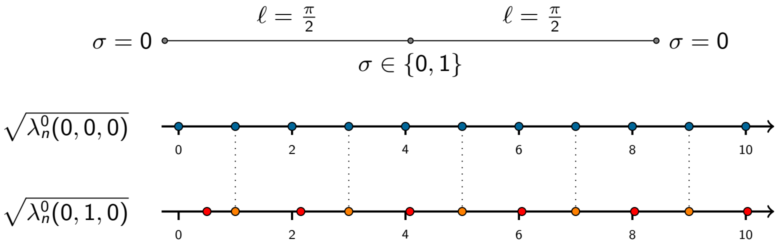

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The spectral comparison result

Theorem (Proc. Amer. Math. Soc. (2023))

Let $\mathcal{G}$ be compact finite quantum graph and $\mathcal{L} := \sum_{\mathsf{e}\in\mathsf{E}} \ell_{\mathsf{e}}$. Then the sequences consisting of Cesáro averages

$$\left(\frac{1}{N}\sum_{n=1}^{N} d_n^q(\sigma)\right)_{N\in\mathbb{N}} \quad \text{and} \quad \left(\frac{1}{N}\sum_{n=1}^{N} \hat{d}_n^q(\sigma)\right)_{N\in\mathbb{N}}$$

are convergent with respective limits

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} d_n^q(\sigma) = \frac{2}{\mathcal{L}}\sum_{\mathsf{v}\in\mathsf{V}} \frac{\sigma_{\mathsf{v}}}{\deg(\mathsf{v})},$$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \hat{d}_n^q(\sigma) = \frac{2}{\mathcal{L}}\sum_{\mathsf{v}\in\mathsf{V}} \frac{\sigma_{\mathsf{v}}}{\deg(\mathsf{v})} + \frac{1}{\mathcal{L}}\int_{\mathcal{G}} q(x)\,\mathrm{d}x.$$

Further related results

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- **(Riviére, Royer, JPA 2020)** For star-graphs with *Dirichlet vertex conditions* at the outer vertices and *δ-coupling condition* at the central vertex.
- **(Rudnick, Wigman, Yesha, CMP 2021)** For bounded domains $\Omega \subset \mathbb{R}^2$ (with piecewise smooth boundary $\partial\Omega$) with $\sigma > 0$ referring to associated *Robin boundary conditions*:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d_n^0(\sigma) = \frac{2|\partial\Omega|}{|\Omega|} \cdot \sigma.$$

  - **(Frank, Larson 2024)** Generalization to *higher* dimensions.
- **(Band, Schanz, Sofer, AHP 2023)** For the *free Laplacian* on metric graphs using different methods.

Our proof: Relies on properties of the *heat kernel* $p_t^{H_\sigma^q}$ associated with the *heat semigroup* $(\mathrm{e}^{-tH_\sigma^q})_{t\geq 0}$.

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Back to asymptotic result - proof ideas

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- **For simplicity**: Only consider the case $d_n^q(\sigma) = \lambda_n^q(\sigma) - \lambda_n^q(0)$.
- Let $f_n^{\sigma,q}$, $n \in \mathbb{N}$ refer to corresponding orthonormal eigenfunctions of H_σ^q.

The proof follows from **two** main ingredients...

Lemma ((IG_1) Ingredient 1)

There holds

$$d_n^q(\sigma) = \int_0^1 \sum_{\mathsf{v}\in\mathsf{V}} \sigma_{\mathsf{v}} |f_n^{\tau\sigma,q}(\mathsf{v})|^2 \,\mathrm{d}\tau \quad \textit{for every } n \in \mathbb{N}.$$

- Follows by the *Feynman-Hellman formula*: $[0,1] \ni \tau \mapsto \lambda_n^q(\tau\sigma)$ differentiable a.e. such that

$$\frac{\mathrm{d}\lambda_n^q(\tau\sigma)}{\mathrm{d}\tau} = \sum_{\mathsf{v}\in\mathsf{V}} \sigma_{\mathsf{v}} |f_n^{\tau\sigma,q}(\mathsf{v})|^2 \quad \text{and integrate over } \tau.$$

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Ingredient 2 – The local Weyl law

Proposition ((IG_2) Ingredient 2; local Weyl law)

There holds
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |f_n^{\sigma,q}(x)|^2 = \frac{2}{\mathcal{L}\deg(x)},$$
where $\deg(x) = 2$ *if* $x \in \mathcal{G}$ *is an inner edge point.*

The proof relies on:

- **(Weyl's law)** $\#\{n \in \mathbb{N} \mid \lambda_n^q(\sigma) \leq \lambda\} \sim \frac{\mathcal{L}}{\pi}\lambda^{\frac{1}{2}}$ as $\lambda \to \infty$,
- **(Heat kernel asymptotics)** $p_t^{H_\sigma^q}(x,x) \sim \frac{1}{\sqrt{4\pi t}}\frac{2}{\deg(x)}$ as $t \to 0^+$,
- **(Mercer's theorem)** One has for $x \in \mathcal{G}$ and $t > 0$

$$p_t^{H_\sigma^q}(x,x) = \sum_{n=1}^{\infty} \mathrm{e}^{-t\lambda_n^q(\sigma)}|f_n^{\sigma,q}(x)|^2 \text{ with uniformly convergent rhs.}$$

in combination with **Karamata's Tauberian theorem**.

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Proof of the asymptotic main result

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Want to prove: $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d_n^q(\sigma) = \frac{2}{\mathcal{L}} \sum_{\mathsf{v}\in\mathsf{V}} \frac{\sigma_{\mathsf{v}}}{\deg(\mathsf{v})}$; $d_n^q(\sigma) = \lambda_n^q(\sigma) - \lambda_n^q(0)$.

Proof of the main theorem (at least a sketch).

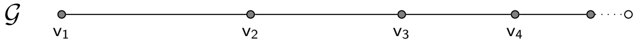
$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d_n^q(\sigma) \overset{(IG_1)}{=} \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \int_0^1 \sum_{\mathsf{v}\in\mathsf{V}} \sigma_{\mathsf{v}} |f_n^{\tau\sigma,q}(\mathsf{v})|^2 \,\mathrm{d}\tau$$

$$\underset{\text{conv.}}{\overset{\text{dominated}}{=}} \int_0^1 \sum_{\mathsf{v}\in\mathsf{V}} \sigma_{\mathsf{v}} \lim_{N\to\infty} \left(\frac{1}{N} \sum_{n=1}^{N} |f_n^{\tau\sigma,q}(\mathsf{v})|^2 \right) \mathrm{d}\tau$$

$$\overset{(IG_2)}{=} \int_0^1 \sum_{\mathsf{v}\in\mathsf{V}} \sigma_{\mathsf{v}} \frac{2}{\mathcal{L}\deg(\mathsf{v})} \,\mathrm{d}\tau = \frac{2}{\mathcal{L}} \sum_{\mathsf{v}\in\mathsf{V}} \frac{\sigma_{\mathsf{v}}}{\deg(\mathsf{v})}.$$

□

Consider: *Infinite* path graph $\mathcal{G}$ with *countably* many vertices $(\mathsf{v}_n)_{n\in\mathbb{N}}$ and edges $(\mathsf{e}_n)_{n\in\mathbb{N}}$ such that $\mathcal{L} = \sum_{n=1}^{\infty} \ell_{\mathsf{e}_n} < \infty$, i.e.



and operators $H_\sigma^{\mathcal{G}} = H_\sigma$, $\sigma = (\sigma_n)_{n\in\mathbb{N}} \subset [0,\infty)$ associcated with quadratic form

$$a_\sigma(f) := \int_{(0,\mathcal{L})} |f'(x)|^2 \,\mathrm{d}x + \sum_{n=1}^{\infty} \sigma_n |f(\mathsf{v}_n)|^2$$

on form domain $D(a_\sigma) := \{f \in H^1(0,\mathcal{L}) \,:\, a_\sigma(f) < \infty\}$ (have *pure* pp!).

Theorem (J. Math. Phys. (2024))

There holds: $$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d_n(\sigma) = \frac{2\sigma_1 + \sum_{n=2}^{\infty} \sigma_n}{\mathcal{L}}.$$

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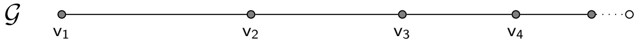

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Comments on the proof and limitations

1. If $(\sigma_n)_{n\in\mathbb{N}} \notin \ell^1(\mathbb{N})$, consider $\sigma_M = (\sigma_1, \dots, \sigma_M, 0, \dots)$, then

$$\frac{1}{N}\sum_{n=1}^{N} d_n(\sigma) \geq \frac{1}{N}\sum_{n=1}^{N} d_n(\sigma_M) \overset{N\to\infty}{\longrightarrow} \frac{\sigma_1 + \sum_{n=2}^{M}\sigma_n}{\mathcal{L}} \quad \forall M \in \mathbb{N}.$$

2. If $(\sigma_n)_{n\in\mathbb{N}} \in \ell^1(\mathbb{N})$, then *bracketing arguments* yield
   - The *standard* Weyl's law, and
   - the *small-time asymptotics* for the heat kernel

Obtain: $\frac{1}{\sqrt{4\pi t}} \overset{t\to 0^+}{\sim} p_t^{H^{\mathcal{I}}_{\sigma=(\infty,\infty)}}(x,x) \leq p_t^{H^{\mathcal{G}}_{\sigma}}(x,x) \leq p_t^{H^{\mathcal{G}}_{\sigma=0}}(x,x) \overset{t\to 0^+}{\sim} \frac{1}{\sqrt{4\pi t}}$.

- **Limitation**: Heat kernel small-time asmyptotics unknown for *infinite* graphs of *finite* total length with *infinitely* many vertices of degree ≥ 3

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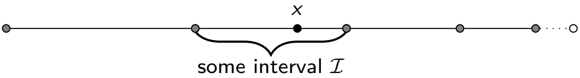
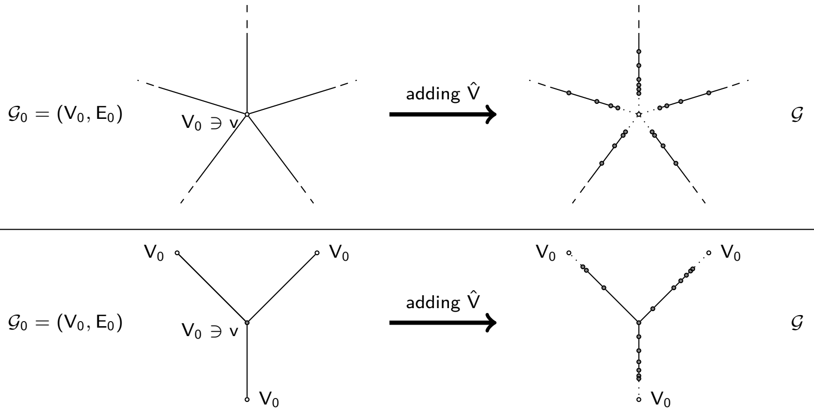


Obtain: $\frac{1}{\sqrt{4\pi t}} \overset{t\to 0^+}{\sim} p_t^{H^{\mathcal{I}}_{\sigma=(\infty,\infty)}}(x,x) \leq p_t^{H^{\mathcal{G}}_{\sigma}}(x,x) \leq p_t^{H^{\mathcal{G}}_{\sigma=0}}(x,x) \overset{t\to 0^+}{\sim} \frac{1}{\sqrt{4\pi t}}.$

- **Limitation**: Heat kernel small-time asmyptotics unknown for *infinite* graphs of *finite* total length with *infinitely* many vertices of degree ≥ 3

More general infinite graphs

1. Start with a *finite* metric graph $\mathcal{G}_0 = (\mathsf{V}_0, \mathsf{E}_0)$.
2. Add *countable* set of (isolated) vertices $\hat{\mathsf{V}} \subset \bigcup_{\mathsf{e}\in\mathsf{E}_0}(0, \ell_\mathsf{e})$ such that all cluster points of $\hat{\mathsf{V}}$ belong to V_0.



Infinite total length?

Theorem (Modified spectral comparison; J. Math. Phys. (2024))

*$\mathcal{G}$ infinite path graph **but** with $\ell_{e_n} = \frac{\pi}{n}$, $q \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$.*

Then:
$$\lim_{N\to\infty} \frac{1}{\sqrt{\lambda(N)}} \sum_{n=1}^{N} \left(\lambda_n^q(\infty) - \lambda_n^0(\infty)\right) = \frac{1}{\pi} \int_{\mathbb{R}_+} q \, \mathrm{d}x,$$

where $\lambda = \lambda(N)$ is the function implicitly defined via the relation

$$N = \frac{\sqrt{\lambda}}{2} \ln \lambda + (2\gamma - 1)\sqrt{\lambda} \quad \text{for large } N,$$

where $\gamma \approx 0.5572$ is the Euler–Mascheroni constant.

- Based on a *different* Weyl's law (due to Egger and Steiner 2011):

$$\#\{n \in \mathbb{N} \,:\, \lambda_n^q(\sigma) \leq \lambda\} \overset{\lambda\to\infty}{\sim} \frac{\sqrt{\lambda}}{2} \ln \lambda + (2\gamma - 1)\sqrt{\lambda} + \mathcal{O}(\lambda^{\frac{1}{4}}).$$

Change of vertex conditions?

Question: What happens for *anti-Kirchhoff* and *δ'-coupling* conditions, i.e., the operators $H_\beta := (-\frac{\mathrm{d}^2}{\mathrm{d}x_\mathsf{e}^2})_{\mathsf{e}\in\mathsf{E}}$ with $\beta > 0$ and

$$D(H_\beta) := \left\{ f \in \bigoplus_{\mathsf{e}\in\mathsf{E}} H^2(0,\ell_\mathsf{e}) \;\middle|\; \begin{array}{l} (\partial_\mathsf{e} f(\mathsf{v}))_{\mathsf{e}\sim\mathsf{v}} \text{ equal to constant vector,} \\ \sum_{\mathsf{e}\sim\mathsf{v}} f_\mathsf{e}(\mathsf{v}) = \beta f(\mathsf{v}) \ \forall \mathsf{v}\in\mathsf{V} \end{array} \right\}.$$

- **(Band, Schanz, Sofer 2023)** Numerically averages seem to diverge (to ∞).

Theorem (J. Math. Phys. (2025))

For $\beta, \beta' \in [0,\infty)$ with $\beta + \beta' > 0$ there holds

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (\lambda_n(\beta) - \lambda_n(\beta')) = \frac{2}{\mathcal{L}} \left(\frac{1}{\beta} - \frac{1}{\beta'} \right) \sum_{\mathsf{v}\in\mathsf{V}} \deg(\mathsf{v}) = \begin{cases} \infty, & \beta = 0, \\ -\infty, & \beta' = 0. \end{cases}$$

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Thank you for the attention!

[1] R. Band, H. Schanz, G. Sofer, *Differences between Robin and Neumann eigenvalues on metric graphs*, Anal. Henri Poincaré (2023).

[2] B., J. Kerner, *Comparing the spectrum of Schrödinger operators on quantum graphs*, Proc. Amer. Math. Soc. (2023).

[3] B., J. Kerner, *A note on Ambazumian's theorem on quantum graphs*, Arch. Math. (2024).

[4] B., J. Kerner, *Some spectral comparison results on infinite quantum graphs*, J. Math. Phys. (2024).

[5] B., J. Kerner, *A modified local Weyl law and spectral comparison results for δ'-coupling conditions*, J. Math. Phys. (2025).

[6] B., J. Kerner, C. Rose *Spectral comparison results for Laplacians on discrete graphs*, arXiv:2412.15937.

Local Weyl law - Essential proof steps

- Put $\mu := \sum_{n=1}^{\infty} |f_n^{\sigma,q}(x)|^2 \delta_{\lambda_n^q(\sigma)}$ with $\delta_{\lambda_n^q(\sigma)}$ *Dirac measure* on $[0,\infty)$

$\overset{\text{Mercer's}}{\underset{\text{theorem}}{\Longrightarrow}}$ $\exp(-t\cdot) \in L^1([0,\infty),\mu)$ for all $t > 0$ with

$$\int_0^\infty \mathrm{e}^{-tx}\,\mathrm{d}\mu(x) = \sum_{n=1}^{\infty} \mathrm{e}^{-t\lambda_n^q(\sigma)} |f_n^{\sigma,q}(x)|^2 = p_t^{H_\sigma^q}(x,x) \underset{t\to 0^+}{\sim} \frac{1}{\sqrt{\pi}\deg(x)} t^{-\frac{1}{2}}$$

- **(Karamata, Tauber)** $\sum_{\lambda_n^q(\sigma)\le\lambda} |f_n^{\sigma,q}(x)|^2 = \mu[0,\lambda] \underset{\lambda\to\infty}{\sim} \frac{1}{\sqrt{\pi}\deg(x)} \frac{\lambda^{\frac{1}{2}}}{\Gamma(\frac{3}{2})}$
- Replace $\lambda_n^q(\sigma) \le \lambda$ by $1 \le n \le N \overset{\text{Weyl's}}{\underset{\text{law}}{\Longrightarrow}} N \sim \frac{\mathcal{L}}{\pi}\lambda^{\frac{1}{2}} \iff \frac{\pi N}{\mathcal{L}} \sim \lambda^{\frac{1}{2}}$

Thus: $\sum_{n=1}^{N} |f_n^{\sigma,q}(x)|^2 \underset{N\to\infty}{\sim} \frac{2N}{\mathcal{L}\deg(x)} \iff \frac{1}{N}\sum_{n=1}^{N} |f_n^{\sigma,q}(x)|^2 \underset{N\to\infty}{\sim} \frac{2}{\mathcal{L}\deg(x)}$.

A generalization on *infinite graphs*

- Impose *δ-coupling conditions* on $\hat{\mathsf{V}}$ with respect to some $\sigma = (\sigma_{\mathsf{v}})_{\mathsf{v}\in\hat{\mathsf{V}}} \in [0,\infty)^{\hat{\mathsf{V}}}$ (and *standard conditions* on V_0).
- **As in the finite case**: Consider self-adjoint operator H^q_σ associated with the quadratic form a^q_σ on $\mathrm{dom}(a^q_\sigma) = \{f \in H^1(\mathcal{G}_0) \mid a^q_\sigma(f) < \infty\}$.
  - Has again pure point spectrum!

Theorem (B., Kerner 2023)

If $(\sigma_{\mathsf{v}})_{\mathsf{v}\in\hat{\mathsf{V}}} \notin \ell^1(\hat{\mathsf{V}})$, *then* $\lim_N \frac{1}{N}\sum_{n=1}^N d_n(\sigma) = \lim_N \frac{1}{N}\sum_{n=1}^N \hat{d}_n(\sigma) = +\infty$. *Otherwise if* $(\sigma_{\mathsf{v}})_{\mathsf{v}\in\hat{\mathsf{V}}} \in \ell^1(\hat{\mathsf{V}})$, *then*

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N d_n(\sigma) = \frac{1}{\mathcal{L}}\sum_{\mathsf{v}\in\hat{\mathsf{V}}} \sigma_{\mathsf{v}}, \quad \textit{and} \quad \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^N \hat{d}_n(\sigma) = \frac{1}{\mathcal{L}}\left(\sum_{\mathsf{v}\in\hat{\mathsf{V}}} \sigma_{\mathsf{v}} + \int_{\mathcal{G}} q(x)\,\mathrm{d}x\right).$$

- **Idea**: Use bracketing for heat kernel; $p_t^{H^q_\infty} \le p_t^{H^q_\sigma} \le p_t^{H^q_0}$, $t > 0$.
- Can impose additional δ-coupl. cond. on $\{\mathsf{v} \in \mathsf{V}_0 \mid \mathsf{v} \text{ no cl.pt. of } \hat{\mathsf{V}}\}$!

Notes on the uniform boundedness

- **For simplicity**: $\sigma \in [0,\infty)^{|\mathsf{V}|}$, $q \equiv 0$ (otherwise one has to shift!).
- **One has** $|\mathrm{e}^{-tH^0_\sigma} f| \leq \mathrm{e}^{-tH^0_0}|f|$ for all $f \in L^2(\mathcal{G})$, *in particular*,

$$p_t^{H^0_\sigma}(x,y) \leq p_t^{H^0_0}(x,y) \quad \text{for every } x,y \in \mathcal{G} \text{ and every } t > 0.$$

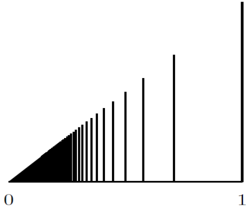
- **Moreover**: $p_t^{H^0_0}(\mathsf{v},\mathsf{v}) \leq \frac{C_\mathsf{v}}{\sqrt{t}}$ for some $C_\mathsf{v} \geq 0$ and $t > 0$ small enough.

$$\sum_{n=1}^{N}\sum_{\mathsf{v}\in\mathsf{V}} |f_n^\sigma(\mathsf{v})|^2 \leq \sum_{n=1}\sum_{\mathsf{v}\in\mathsf{V}} \mathrm{e}^{1-\frac{\lambda_n^0(\sigma)}{\lambda_N^0(\infty)}}|f_n^\sigma(\mathsf{v})|^2 = \mathrm{e}\sum_{\mathsf{v}\in\mathsf{V}}\sum_{n=1}^{\infty} \mathrm{e}^{-\frac{\lambda_n^0(\sigma)}{\lambda_N^0(\infty)}}|f_n^{0,\sigma}(\mathsf{v})|^2$$

$$\overset{\text{Mercer}}{=} \mathrm{e}\sum_{\mathsf{v}\in\mathsf{V}} p^{H^0_\sigma}_{\frac{1}{\lambda_N^0(\infty)}}(\mathsf{v},\mathsf{v}) \leq \mathrm{e}\sum_{\mathsf{v}\in\mathsf{V}} p^{H^0_0}_{\frac{1}{\lambda_N^0(\infty)}}(\mathsf{v},\mathsf{v})$$

$$\leq \widetilde{C}\sqrt{\lambda_N^0(\infty)} \underset{N\to\infty}{\overset{\text{Weyl}}{\sim}} \widetilde{C}\cdot\frac{\pi N}{\mathcal{L}} \quad \text{with } \widetilde{C} := \mathrm{e}\sum_{\mathsf{v}\in\mathsf{V}} C_\mathsf{v}.$$

- **In our case**: Only finitely many vertices of degree ≥ 3.
  - What about infinite graphs with an infinite number of *non-artificial vertices* (i.e. vert. of deg. ≥ 3), and of possibly *infinite length*?



- *Diagonal comb graphs* with edge lengths $\frac{1}{n^\alpha}$, $\alpha > 0$ (studied by Düfel, Kennedy, Mugnolo, Plümer, Täufer).
- For $\alpha > \frac{1}{2}$ + std cond.: Pure point spectrum!
- For $\alpha \geq 1$: Infinite graph length.

- Comparing discrete spectra of other Schrödinger operators (e.g., comparing *anti-Kirchhoff* with *δ'-coupling* conditions)?
- *Magnetic potentials* instead of/combined with electric potentials, i.e., considering the operator $\left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + a\right)^2 + q$?

Using main theorem: Derive *Ambartsumian's theorem* on metric graphs!

Theorem (Ambartsumian type theorem on metric graphs)

If $\lambda_n^q(0) = \lambda_n^0(0)$ for all $n \in \mathbb{N}$, then $q \equiv 0$.

- **(Davies)** For general metric graphs, using...

Lemma (Davies 2010)

If $\lambda_1^q(0) \geq 0$ and $\int_{\mathcal{G}} q(x)\,\mathrm{d}x \leq 0$, then $q \equiv 0$.

- **(Kurasov, Suhr)** $k_n^q(0) - k_n^0(0) \overset{n\to\infty}{\longrightarrow} 0$ enough to imply $q \equiv 0$.
- **In our setup**: If $\lambda_n^q(0) - \lambda_n^0(0) = d_n^q(0) \overset{n\to\infty}{\longrightarrow} 0$, then

$$0 = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} d_n^q(0) = \frac{1}{\mathcal{L}} \int_{\mathcal{G}} q(x)\,\mathrm{d}x \overset{\text{Davies}}{\Longrightarrow} q \equiv 0.$$

Question: What about the spectrum of H_σ^q?

Lemma (Davies 2010; modified)

If $\lambda_1^q(\sigma) \geq 0$ *and* $\sum_{v\in V} \sigma_v + \int_{\mathcal{G}} q(x)\,\mathrm{d}x \leq 0$, *then* $q \equiv 0$ *and* $\sigma = 0$.

Corollary (An Ambartsumian type theorem H_σ^q; some cases)

If $\lambda_n^q(\sigma) = \lambda_n^0(0)$ *for all* $n \in \mathbb{N}$ *s.t. one of the following conditions hold:*

(i) $\sum_{v\in V} \sigma_v \left(1 - \frac{2}{\deg(v)}\right) \leq 0$,

(ii) $\int_{\mathcal{G}} q\,\mathrm{d}x \geq 0$ *and* $\sigma \in [0,\infty)^{|V|}$,

(iii) $\left(\frac{\min_{v\in V} \deg(v)}{2} - 1\right) \int_{\mathcal{G}} q\,\mathrm{d}x \geq 0$ *and* $\sigma \in (-\infty, 0]^{|V|}$ *but* $q \neq 0$ *and* $\sigma \neq 0$

Then $q \equiv 0$ *and* $\sigma = 0$.

- **But**: Not true for *all* σ!
- **(Kurasov 2019)** $I = [0, \frac{1}{2}]$, $q(x) = \frac{2}{(1+x)^2}$ and $\sigma = (-1, \frac{2}{3})$, then $\lambda_n^q(\sigma) = \lambda_n^0(0)$ *but* $q \not\equiv 0$ and $\sigma \neq 0$.