Magnetic Bernstein inequalities

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Motivation: Unique continuation

		6	5					
7		5			2	3		
	3						8	
	5			9	6		7	
1		4						8
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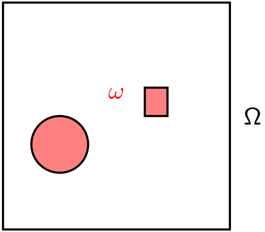
Unique continuation

Unique continuation principle

A class $\mathcal{F}$ of functions $f : \mathbb{R}^d \supset \Omega \to \mathbb{R}$ or $\mathbb{C}$ has the **unique continuation principle** on $\omega \subset \Omega$ if functions in $\mathcal{F}$ are uniquely determined by their values on ω.



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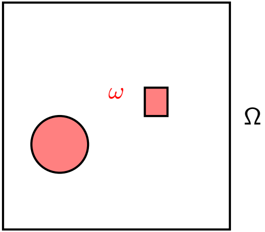

- analytic functions,
- entire holomorphic functions (identifying $\mathbb{C}$ with $\mathbb{R}^2$),
- Solutions of many PDEs (Kovalevskaya 1874).

Quantitative unique continuation

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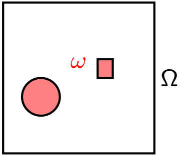
A class $\mathcal{F} \subset L^p(\Omega)$ has a **quantitative unique continuation principle** on $\omega \subset \Omega^d$ if there is $C > 0$ such that

$$\|f\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\omega)} \quad \text{for all } f \in \mathcal{F}.$$



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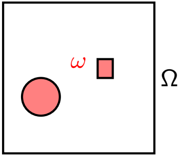

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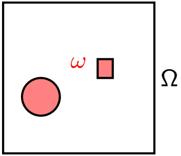

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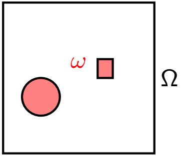

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- Ideally, want to understand C in terms of $\mathcal{F}$, ω, Ω.

Spectral subspaces

H non-negative self-adjoint differential operator in $L^2(\mathbb{R}^d)$,

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The **spectral subspace up to energy** $E \geq 0$ with respect to H, $\operatorname{Ran} \mathbf{1}_E(H)$, is the space of $f \in L^2(\mathbb{R}^d)$ such that $H^n f \in L^2(\mathbb{R}^d)$ for all $n \in \mathbb{N}$, and

$$\|Hf\|_{L^2(\mathbb{R}^d)} \leq E^n \|f\|_{L^2(\mathbb{R}^d)} \quad \text{for all } n \in \mathbb{N}.$$

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If H has discrete spectrum, $\operatorname{Ran} \mathbf{1}_E(H)$ is the **span of eigenfunctions with eigenvalues below** E.

QUCP for spectral subspaces: applications in Mathematical Physics.

Landau operator

Important object in mathematical Physics e.g. for **quantum Hall effect**.

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- For *Magnetic field strength* $B > 0$, the *Landau operator* is

$$H_B := \left(i\nabla + \frac{B}{2} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \right)^2 \quad \text{in } L^2(\mathbb{R}^2).$$

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Using a method called Periodicity, one has:

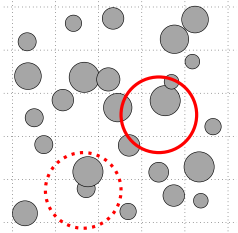
Theorem (Combes, Hislop, Klopp, Raikov)

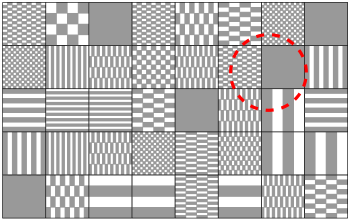
Let $B, E > 0$ and $S \subset \mathbb{R}^2$ be non-empty, open and periodic (+ technical assumptions). Then, there is $C = C(B, E, S)$ such that for all $f \in \operatorname{Ran} \chi_{(-\infty, E]}(H_B)$

$$\|f\|^2_{L^2(\mathbb{R}^2)} \leq C \|f\|^2_{L^2(S)}.$$

Interlude: Thick sets

$$\mathrm{Vol}\left(B_r(x) \cap S\right) \geq \rho \quad \text{for all } x \in \mathbb{R}^d.$$





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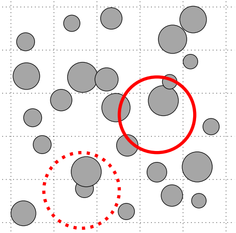

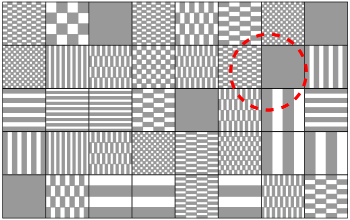

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- Also called **relatively dense**.

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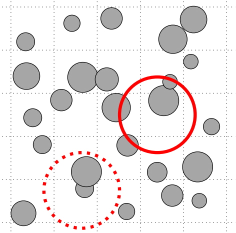

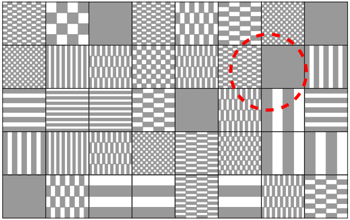

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- Also called **relatively dense**.
- Minimal criterion for quantitative quantitative unique continuation for functions with bounded Fourier support.

QUCP for the Landau operator in 2 dimensions

Theorem (Pfeiffer, Täufer, 23)
Let $S \subset \mathbb{R}^2$ *be* (r, ρ)*-thick. Then, there is* $C = C(\rho) > 0$ *such that for all* $E > 0$ *and all* $f \in \operatorname{Ran} \chi_{(-\infty, E]}(H_B)$

$$\|f\|^2_{L^2(\mathbb{R}^2)} \leq C \exp\left(C\sqrt{E}r + CBr^2\right) \|f\|^2_{L^2(S)}.$$

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+ Optimal (= minimal) assumption on geometry.
− Method only really works for very specific, explicit Hamiltonians.

Proof of QUCP Landau (1)

We follow the Logvinenko-Sereda strategy and first need to prove something similar to Paley Wiener. Need **Bernstein inequalities**.

$$\|\partial^\alpha f\|^2_{L^2(\mathbb{R}^2)} \leq E^n \|f\|^2_{L^2(\mathbb{R}^2)} \quad \text{for } \alpha \in \{1,2\}^n,\ f \in \operatorname{Ran} \mathbf{1}_E(H_B).$$

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For all $n \in \mathbb{N}$, and $\alpha \in \{1,2\}^n$, we can find $f \in \operatorname{Ran} \mathbf{1}_E(H_B)$ with

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New idea: H_B has a similar structure as $-\Delta = -\partial_1^2 - \partial_2^2$:

$$H_B = \tilde{\partial}_1^2 + \tilde{\partial}_2^2$$

for the **covariant derivatives** $\tilde{\partial}_1 = i\partial_1 + \frac{B}{2}x_2$, $\tilde{\partial}_2 = i\partial_2 - \frac{B}{2}x_1$.

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New problem: Magnetic derivative no longer commute.

$$\tilde{\partial}_1\tilde{\partial}_2 - \tilde{\partial}_2\tilde{\partial}_1 = iB \operatorname{Id}.$$

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To prove Paley-Wiener, one can use

$$\sum_{\alpha \in \{1,\ldots,d\}^n} (\partial^\alpha)^* \partial^\alpha f = (-\Delta)^n$$

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"Miracle"

For the magnetic derivatives, we still have that

$$\sum_{\alpha \in \{1,2\}^n} (\tilde{\partial}^\alpha)^* \tilde{\partial}^\alpha = P_n(H_B)$$

where P_n is a polynomial of n-th order satisfying $P_n(t) \leq (t + Bn)^n$. ☺

Proof of QUCP Landau (3)

With this, we can prove **magnetic Bernstein inequalities**

Theorem (Täufer, Pfeiffer 23)

For all $B > 0$, $E > 0$, $n \in \mathbb{N}$, we have

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Now, $4^n (E + Bn)^{n/2} \sim \sqrt{n!}$ still loses against $n!$ in Taylor series, which is needed for some complex analysis black magic known as Logvinenko-Sereda theorem ☺ $\square$